Active learning strategies with positive effects on students' achievements in undergraduate mathematics education[1]

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ABSTRACT

The paper describes active learning strategies used in undergraduate college algebra and business calculus courses. There are a variety of active learning strategies described in the literature. We wanted to implement a few that together, satisfy the key characteristics of active learning strategies. The active learning strategies described in this paper are, interactive presentation style, group-work with discussion and feedback, volunteer presentations of solutions by groups, raise students' learning interest towards specific topics, involve students in mathematical explorations, experiments, and projects, and last but not least, continuous motivation and engagement of students. We analyse the relationship between the average results of 27 college algebra and business calculus sections and the effects of the level of use of these active learning strategies. We demonstrate that the application of these strategies has a positive effect on the average results of the sections and the passing rates of the students.

Keywords: active learning, group-work, collaborative learning, linear regression, undergraduate mathematics education

1. Introduction

It was recognized over 30 years ago, that a traditional, or lecturing style of teaching is not always the best method to help students acquire knowledge or enable them to use that knowledge. Chickering and Gamson [2] gave an indisputable explanation of why a lecturing style is not the best way of teaching: 'Learning is not a spectator sport. Students do not learn much just sitting in classes listening to teachers.'

Interestingly, after 30 years, we still do not have a definition of active learning that is accepted and used by everyone. One often-used definition is recommended by Freeman et al. [1]: 'Active learning engages students in the process of learning through activities and discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group-work.'

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It is important to mention that the President's Council of Advisors on Science and Technology (PCAST) [3] that called for an approximately 33% increase over current production rates in STEM fields (science, technology, engineering, and mathematics) articulates that classroom approaches that engage students in active learning improve retention of information and critical thinking skills, compared with sole reliance on lecturing; and they further increase the persistence of students in STEM majors. Most of the studies on this topic agree that an essential key to this is to increase student engagement. Stanberry [4], for example, gives an example of increased student engagement among students majoring in the STEM disciplines while using the Strategic Engagement for Increased Learning (SEIL) model. As of her opinion, more class time for student engagement is beneficial for enhanced learning. Kitchens et al. [5] describe the CAPTIvatE scheme. In their scheme, interaction with fellow students and the instructor, team-based learning, team discussions, immediate feedback, 'doing' real experiments and real-world applications play important roles. Based on their study that included a student survey, they present preliminary evidence of a greater understanding of course content. A recent study of Theobald et al. [6] shows that active learning narrows achievement gaps for underrepresented students in undergraduate STEM education compared to those in traditional lecturing classrooms.

The above-mentioned positive results have shown that the application of active learning strategies can be a powerful tool in education. The essential goal with the application of AL-strategies in the college algebra course was to increase the passing rate of the students. Since we have found it beneficial for most of the students in that course, and since college algebra is one of the possible prerequisite courses of the business calculus course, we introduced the same strategies into the business calculus course, as well. The primary purpose of this paper is to give specific examples of applying six active learning strategies (AL-strategies in the following) we have been using in undergraduate mathematics education to enable instructors to apply these or develop similar AL-strategies in their classes. The examples described in section 2 could be used in most college algebra and calculus courses in colleges. These strategies were developed based on strategies described in the references, and have been continuously modified, based on our experiences while using them in classes. We also present a study that demonstrates that the AL-strategies described in this paper are associated with the achievement of students in undergraduate mathematics classes, which validate the main hypothesis of the study that the application of active learning strategies can improve the achievements of students.

AL-strategies help to achieve valuable learning objectives and outcomes stated in the syllabi of the courses. These are the following: improving basic mathematical and general academic skills, promoting problem-solving and critical thinking skills, enhancing learning and understanding of concepts through the integrated use of graphing tools, increasing the confidence of students in doing mathematics, and persevere in the face of difficulties and view mistakes as welcome opportunities to learn. In section 2, we describe how an AL-strategy contributes to these learning goals.

Finally, as many researchers in the literature have demonstrated it, the application of AL-strategies contributes to the achievement of important goals of undergraduate mathematics education. Here we list some of these goals:

(1) Make students be participants and not only spectators as it was articulated by Chickering and Gamson [2].
(2) Engage students in a mathematical investigation, communication, and group problem-solving, (CBMS [7], MAA Instructional Practices Guide [8]).
(3) Increase student performance in science, engineering, and mathematics. Freeman et al. [1] concluded that it is one of the essential consequences of using active learning.
(4) Teach students how to make better decisions, as Wieman [9] stated in his presentation.
(5) Demonstrate the importance and provide an opportunity for elaborative rehearsal (Haga [10]), which is, according to Thorne [11], effective both for transferring information from short-term memory to long-term memory and for storing information in long-term memory.
(6) Help students to develop sought-after skills (Association of American Colleges and Universities [12]), that employers value the most, such as the capability of solving problems, working effectively in a team, and be able to communicate about work-related topics, as discussed among other essential skills by Pierce [13].
(7) Reduce the mathematics anxiety of students (Ramirez et al. [14], Beilock & Willingham [15], Chen et al. [16]).
(8) Decrease the achievement gap between students from disadvantaged and non-disadvantaged educational backgrounds, between ethnic groups and between different genders – one of the many essential goals mentioned in the PCAST-report [3]. Haak et al. [17] give an example of reducing the achievement gap using structured course design and active learning practices.

Our experiences demonstrate that the application of AL strategies helps to achieve the following benefits: strengthen basic skills in mathematics, and general academic skills such as collaboration, communication, presentation, enhance learning and understanding of mathematical concepts, recognize that mathematics helps to understand the world around us, improve problem-solving and critical thinking capabilities in a real-life situation, enhance the ability to apply mathematical tools for decision-making, prepare students for their future careers, promote the use of computer technology in problem-solving, increase confidence in doing mathematics.

2. Six active learning strategies that help to achieve educational goals

In this section, we describe six AL-strategies to give examples of the application of specific AL-strategies. The first author of this paper has been using these strategies in undergraduate college algebra and business calculus classes. The college algebra course is an optional prerequisite course for majors such as biology, physics, general chemistry, statistics for the social sciences, business-related majors, agriculture, and life sciences. The business calculus course is a prerequisite course for most business majors. Most of the students enrolled in these courses are freshmen. First, the AL-strategies were introduced to college algebra course to increase the passing rate of the students. Then, since we have found them to be effective, the same strategies were applied in the business calculus course, as well. Through active learning activities, we want to inspire every student to learn mathematics; therefore, we develop questions in different contexts to arouse students' learning interest. The following examples have been used in college algebra, and business calculus classes, and in a few cases in both.

2.1. First AL-strategy: Interactive presentation style

Wieman [9] considers it beneficial to interrupt a lecture with challenging conceptual questions, which should be answered with student-student discussion. This, in turn, increases students' engagement, ensures immediate feedback, and provides an opportunity to practice expert thinking. This AL-strategy helps to achieve the learning objective: improving mathematical skills since it encourages students to apply previous knowledge and review basic mathematical skills. The interaction with the instructor enhances problem-solving and critical thinking skills.

Instructors regularly have to introduce new material and need to explain a new topic. While doing that, they need to make sure that students are actively involved. As of Stanberry [4], strategic engagement for increased learning begins with an interactive lecture in which, … students are given opportunities to get involved with the learning process. We can do this by asking students to advise the next step of the solution whenever it is appropriate. Instructors should minimize non-interactive lecture-style teaching. Instead, they need to communicate with the students as much as possible. Also, as Harper & Daane [18] pointed out, mathematics anxiety of students usually decreases when instructors talk to them instead of lecturing them. The following example demonstrates the interactive presentation style strategy while introducing the concept of derivation.

We tell students that we will discover an essential characteristic of a continuous function, *f(x)* at a certain point *A*, namely, the derivative of the function at point *A*, with other words, the slope of *f(x)* at point *A.* We ask students to approximate the slope of *f(x)* at point *A* by evaluating the slope of the line that connects point *A* with point B_1, another point of*f(x)* (Figure 1). This line is marked by y_1 in Figure 1. Students use their previous knowledge about lines to find the slope of the line, y_1.

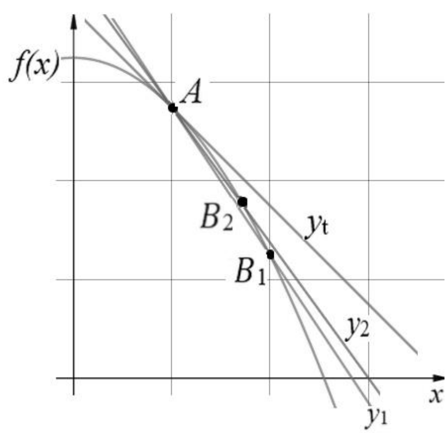


Figure 1. Introducing the concept of the derivative of a continuous function, *f(x)* at a point *A* by approximating the slope of the tangent line with the slopes of secant lines.

Then we advise them to choose another point, B_2, on the graph of *f(x)*, which is closer to A, than B_1, and draw the second secant line through A and B_2. This line is marked by y_2 in Figure 1. Encouraging students to draw additional secant lines choosing the second point B_i on the graph of *f(x)* closer and closer to A, students will figure out that for a continuous function, these secant lines will better and better approximate the tangent line, marked by y_t in the figure, at point A. Consequently, the slopes of the secant lines will better and better approximate the slope of the tangent line.

With the help of this interaction and discussion, students experience that they have discovered a basic concept of derivation, namely, that the derivative of a continuous function at point *A* is the limit of the average rate of change of the *f(x)* function at that point. Since they can evaluate the slope of the secant lines, they can also determine the derivative of the *f(x)* function at point *A* by finding the limit of the slopes of the secant lines when points B_i are closer and closer to point *A* as *i* goes to infinity. This increases their confidence in doing mathematics, which is a valuable learning objective. Bringing this otherwise difficult concept closer to the existing knowledge about lines makes the learning process easier for them. It further enhances their confidence, when together with their peers, they use graphing tools to display the function, a few secant lines, and the tangent line.

2.2. Second AL-strategy: Group-work with discussion and feedback

Collaborative learning has tremendous advantages. As Euclid stated, 'there is no royal road to geometry,' or as this statement was broadened later, 'there is no royal road to mathematics.' Students should think about the topic, figure out solutions to problems, and come up with ideas to answer questions.

In our experiences, we have found that most of the students considered working together with peers more encouraging and more relaxed than solving the problems alone. Team-based learning is a useful tool to help students overcome their anxieties as well (Lugosi [19, 20]). We have found group-work to be beneficial in helping to achieve the learning objectives listed in the introduction. While working with peers, students improve basic mathematical and general academic skills, problem-solving, and critical thinking skills. Sharing ideas, using graphing tools together enhances learning and understanding of concepts, increases the confidence of students in doing mathematics. Facing difficulties with peers teaches them to view mistakes as welcome opportunities to learn. When students work together with their peers on different tasks, then they will be less frustrated, as they will experience other students also struggling with certain parts of the material. On the other hand, they will also experience that if they discuss the problem with their peers, share their thoughts, then, in most cases, they will be able to find the way that leads to the solution. These experiences decrease students' mathematics anxiety, and prove to them that they can achieve better results by 'making their hands dirty with math.' Harper & Daane [18] reported that in their study, 60% of preservice elementary teachers indicated that factors such as working with a partner, working in cooperative learning groups, working with small groups, using manipulatives, or writing about mathematics, decreased their math anxiety.

Many papers deal with the application of team-based learning, group-work, collaborative learning, peer-instruction in education (Michaelsen et al. [21], Nanes [22], Freeman et al. [1], Davidson & Major [23], Orzolek [24], Lasry et al. [25]). Most of these publications articulate the importance of providing students an opportunity for joint work and discussion and consider team-based learning as an essential contributor in developing higher-order thinking skills.

Based on a detailed literature review of different disciplines, Howard [26] lists many vital benefits of discussion in his book, such as improvements in students' critical thinking, achieving higher exam scores, enhanced learning in the classroom, improved communication. Group-work provides an opportunity for discussion of the material with team members and with the instructor. We have found that walking around the room, encouraging students to participate, giving some points for participation in group-work enhances the learning process. In addition to that, as Tharayil et al. [27] state in their paper, with these interventions, instructors can mitigate the resistance of their students to active learning. It enables students to receive immediate feedback from peers and the instructor on the thought process involved in solving a problem.

Neuroscientists (Davidow et al. [28], Somerville [29]) established that, in most cases, the development of the human brain continues until the age of 22. Furthermore, Davidow et al. [28] affirm that the greater complexity of learning demands elucidates the continued developmental gains in strategy and optimization of learning. Based on the previously mentioned findings (Freeman et al. [1], Michaelsen et al. [21] Nanes [22], Davidson & Major [23], Davidow et al. [28]), furthermore, on the work of Johnson et al. [30], we can conclude that providing challenging cognitive demands, such as trying newly learned skills, discussing different alternatives, constructing the path of solutions to different problems is beneficial for healthy brain development. Group-work is an adequate tool to provide all of these to students during classes.

The following question can be used both in college algebra and business calculus classes as a discussion topic in groups. This question connects mathematics to nature that most students are interested in. Showing the shadow-photo of the Catalina Mountains of Figure 2, we can ask students if they can identify the highest peak in this mountain range with certainty using only this contour function.

Figure 2. Shadow-photo of the Catalina Mountains

This question is surprising for most of the students. It is an ideal topic to discuss the difference between reality and a model of that. Students should utilize critical thinking while discussing this question with their peers. Students will figure out that we get different functions by taking the photo from different locations and get to the conclusion that absolute maximum point of the contour function of Figure 2 is not necessarily the same as the highest point in this mountain range. This exercise demonstrates that reality is usually more complex than a simple mathematical model and shows that there is a need for critical thinking skills. Most of the time, when students discuss this question, some students make a mistake. However, talking with their peers, everyone will get to the right conclusion; therefore, this question teaches them to view mistakes as welcome opportunities to learn.

2.3. Third AL-strategy: Volunteer presentations of solutions by groups

The situation summarized by the quote, 'You don't really understand something unless you can explain it to your grandmother,' attributed to Albert Einstein, was investigated by many researchers in connection to education. We consider giving an opportunity for students to explain their solution on a volunteer basis to the whole class very educational. During these short presentations, they practice sharing their thought processes with others using mathematical terminology.

Coleman [31] states that requiring students to explain scientific phenomena in their own words while utilizing the formal knowledge learned in class may force them to draw inferences, integrate and understand the material, and provide reasons for their responses. When we asked students why explanation to others is beneficial for the explainer, Ziah Patrick said that you master your learning when you teach others because you are practicing the concept, and passing on the knowledge that you know well enough to answer questions about.

Students who present their work benefit a lot since presenting a solution, explaining the thought process to others is an engagement in an activity that will enhance the consolidation of the new to-be-learned information in long-term memory (Thore [11]). Therefore, this AL-strategy contributes to the achievement of the learning objective of improving basic mathematical and general academic skills. Also, presenters gain the appreciation of the other students, which makes them feel proud of their accomplishments, increasing their positive attitude towards mathematics, which in turn is essential in mathematics learning as of Chen et al. [16]. We consider this AL-strategy essential in achieving the learning objective, increasing the confidence of students in doing mathematics. On the other hand, students' presentations, due to the use of a simplified language in the explanation, often happens to be helpful, especially for those students who struggle with the material. Bednarz [32] states that language has an important role already in children's mathematical thought and problem-solving. Talking about mathematics, developing the capability of logical reasoning is essential in the development of mathematical thinking.

Another advantage of volunteer presentation by the whole group together is that it lessens the mathematics anxiety of students. Students feel that together with their group members, they can accomplish more than individually.

2.4. Fourth AL-strategy: Raise students' learning interest towards specific topics

Depending on the students' interests, we can develop different types of exercises that attract their attention. These can be real-life applications, connecting mathematics to nature, or other disciplines, providing 'ownership' of the problems.

This is a vital strategy to demonstrate to students why a particular topic is essential, how it relates to different fields outside of mathematics, and why it is relevant in real life. Attention-attracting exercises help to demonstrate that a given topic is relevant (Eggen & Schellenberg [33]). Bowers et al. [34] emphasize that dealing with real-world situations plays an essential role in enhancing student engagement. Most of these exercises are open-ended problems that are beneficial in achieving the learning objective of promoting problem-solving skills (Sole [35]). Whenever students are interested in a topic, they get more engaged in problem-solving, since they are inspired to analyse the question and solve the problem.

Here we give two examples of evoking students' interest. The first exercise is from the semester-long Art-Math project in a college algebra class (Lugosi [20]).

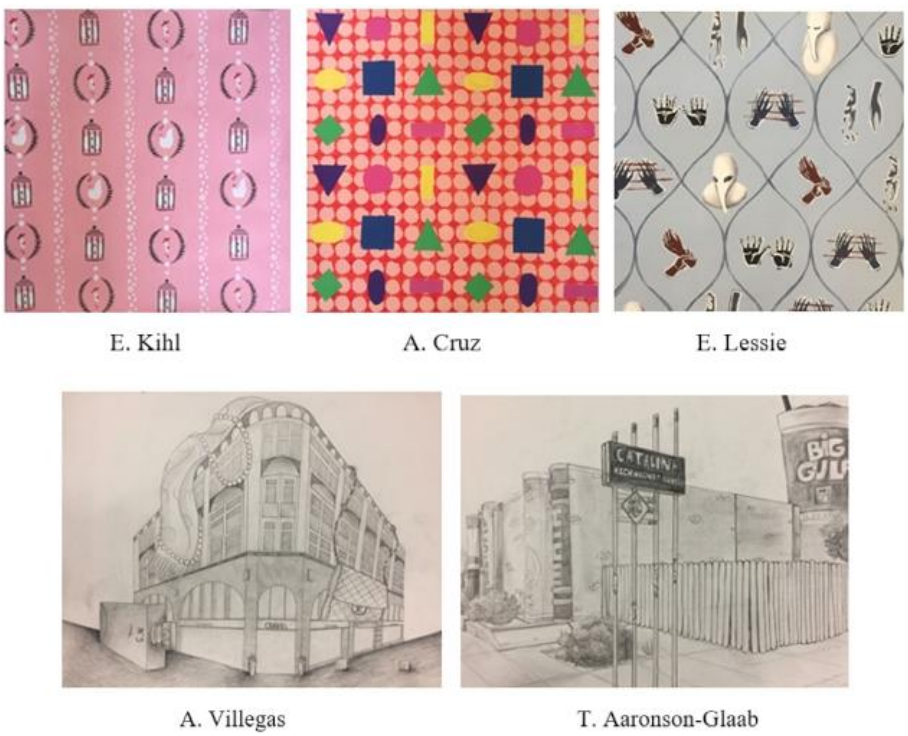


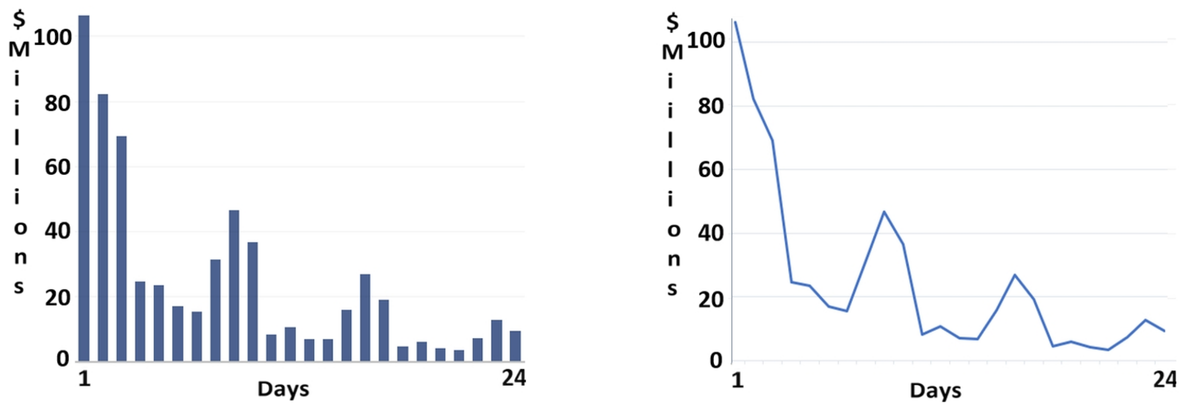
Figure 3: Artworks of the students of the School of Art

Students were asked to draw a coordinate system on the copies of the drawings shown in Figure 3. Then "find" and draw two characteristic lines on the copies, set up the equations of those lines, and determine the transformation that transforms the first line into the other one. These types of exercises are more interesting for students than pure mathematics questions. Also, it demonstrates for students a connection of mathematics to another discipline.

With the second example, we demonstrate for students that mathematics can be applied to solve real-life questions. This exercise connects mathematics to business, specifically, to the movie industry. Box Office Mojo publishes daily box office values of movies. After the opening weekend of the Avengers: Infinity War movie last year, we have constructed the following exercises for a business calculus class.

On the website of Box Office Mojo [36], you can find daily domestic box office values of the movie Avengers: Infinity War. Construct a chart that shows the daily box office, with other words revenue, in millions of dollars as a function of days in release. Using the chart, approximate the total revenue for the first 24 days.

Students usually construct either the chart in Figure 4a or Figure 4b, and they can quickly approximate the total revenue during a given time. Then we ask if they can construct a function whose integral is equal to the total revenue for the first 24 days. The following question can be to estimate the total future revenue of the movie.



Figures 4a and 4b. The daily domestic revenue of the movie Avengers: Infinity War displayed in two different ways.

This exercise provides an opportunity to investigate how data can be displayed, how a particular chart should be interpreted, and how integration can be used to find the total domestic revenue. Students also experience that there are essential details they need to consider to have a good approximation. A unique advantage of a problem like this is that a few weeks later, we can compare the estimates of the students to the real value.

Providing ownership of the problem further increases students' engagement. Placing them into the role of a significant position makes them the owner of the situation. The following question provides this type of ownership of the previous problem. Suppose that you are a financial advisor, and you need to estimate the total revenue of this movie. You need to give a formula of a function that can be used for the approximation. How do you set up the formula?

With this question, we provide opportunities to discuss the assumptions of a mathematical model and the accuracy in the prediction. Solving this exercise, students experience that by making different approximations, or choosing different functions for modelling, the results will be different. Giving them ownership of the problem, placing them into the role of a financial advisor, they feel that they have a responsibility for the accuracy of their approximations. This exercise draws attention to the importance and difficulty of making predictions for future values.

2.5 Fifth AL-strategy: Involve students in mathematical explorations, experiments, and projects

A Chinese proverb tells that 'not hearing is not as good as hearing, hearing is not as good as seeing, seeing is not as good as knowing, knowing is not as good as acting; true learning continues until it is put into action.' Bot et al. [37] use the expression 'Learning by doing' to emphasize the importance of hands-on activities. We have found that providing an opportunity to do an experiment or work on a project related to a specific topic (Lugosi [20]) are effective ways to increase understanding of the material. As Mikhaylov [38] summarized in her paper, getting students physically involved with their lessons, the more likely they are to grasp the deeper meaning of the concepts. Harper & Daane [18] has found that 'doing something' in most cases decreases mathematics anxiety, and manipulatives enable students to 'see' mathematics and better understand how it works. Here we give two examples of mathematics explorations.

2.5.1. Maximizing the volume of a box

Many students have difficulty solving problems connected to area or volume. The following exploration helps them visualize what is happening. For many students, hands-on activities and visualization are essential in gaining an in-depth understanding of a problem. Therefore, this AL-strategy is useful in achieving the learning objective, increasing the confidence in doing mathematics.

Suppose that we want to send a box as checked-in luggage on an airplane. The base of the box is a square. What is the shape of the largest volume box?

To visualize this question, we prepared two boxes satisfying the requirements. One of the boxes was a cube, the other box had a larger base, and the sum of the three dimensions was the same for both. We encouraged the groups to look at the boxes, measure the dimensions, discuss the dimension limitation on checked-in luggage, and come up with ideas to decide which box has a larger volume. Then we asked them to set up the formula for the volume of the boxes they have touched, measured, investigated, and find the maximum of the volume-function. After the groups solved the question, we asked them to design a demonstration to show that the solution is correct using the two boxes and foam packing peanuts. Then a volunteer group made a demonstration.

2.5.2. Transformation of the absolute value function

We use an elastic stretch rope for this experiment and ask three students to participate. One of them holds the middle of the elastic rope, which will be the origin of the coordinate system in the experiment. We ask the other two students to make the shape of the $f(x) = |x|$ absolute value function by holding the ends of the rope. Then we ask them to transform this function into the $f(x) = 2|x|$ function and describe their perceptions. The student at the origin usually tells that he had to hold the origin firmly, and the other two students tell that they had to stretch the elastic rope.

Then students can demonstrate other transformations such as $f(x) = 2\left|x - 3\right|$, and $f(x) = 2\left|x - 3\right| + 1$.

If time allows for a semester-long project, then more parts of the material can be connected to a particular topic, making it even more interesting for the class. For instance, students of college algebra classes applied the material learned about linear functions in a semester-long Art-Math project that connected mathematics to art (Lugosi [20]). One of the exercises of the Art-Math project was described in section 2.4.

The benefit of mathematical explorations, experiments is that students who struggle with the material understand mathematical concepts easier with the help of hands-on activities, and through visualization. Therefore, this AL-strategy is especially crucial in achieving the learning objective, improving basic mathematical and general academic skills. Our experiences are in agreement with those of Harper & Daane [18] regarding mathematics anxiety. We have found that most of the students enjoy doing mathematics experiments, explorations, and projects that contribute to lessening their mathematics and social anxiety.

2.6. Sixth AL-strategy: Continuous motivation and engagement of students

We believe that the famous quote, which is attributed to both Abraham Lincoln and Winston Churchill, 'Success is going from failure to failure without losing your enthusiasm,' is valid for the learning process as well. Therefore, it is essential to encourage students to continue to work on a problem when they feel like giving up. Providing appropriate hints when they are stuck or struggling will guide them to find the next step towards the solution. Students will experience that they can find the way leading to the answer, and this helps them to improve their thought processes.

Motivation and engagement play an important role in decreasing the achievement gap between students from disadvantaged and non-disadvantaged educational backgrounds, between ethnic groups and between different genders (PCAST-report [3]). Fakayode et al. [39] list strategies of different institutions that have been developed to motivate and improve students' success. George [30] summarizes different ways an instructor may attempt to motivate the whole class together or students individually. We have found that encouraging, inspiring the groups, and strengthening the importance of collaborative work is extremely beneficial in providing equity and in decreasing the achievement gap between students.

Encouragement and motivation are also essential to develop a positive attitude towards learning. Chen et al. [16] elucidated neurocognitive mechanisms by which positive attitude influences learning and academic achievement. They claim that a positive attitude toward mathematics has a unique and significant effect on mathematics achievement independent of general cognitive abilities. Deslauriers et al. [41] have also found that during group work, poor attitudes or low engagement of a few students can have negative effects on other students in their groups. We believe that instructors have an essential role in developing a positive attitude by strengthening belief in oneself, inspiring to keep trying, promoting self-confidence. In many cases, telling only a few words to students as an affirmation on what they are doing, such as keep trying, or make sure you understand the question, is sufficient to help them to move on. We should emphasize that being stuck at problem-solving is often part of the process. One of the most beneficial capabilities that we can teach them is not to give up, but rather ask questions from themselves about the problem, discuss it with their peers, come up with a new idea, and try it again. One of our students in a business calculus class, Abraham Cho, summarized this the following way: Thank you for always pushing me with ‘you know this’ and ‘just try.’ This AL-strategy is essential in achieving the learning objective, increasing the confidence of students in doing mathematics.

It is essential to engage every student to participate in class-work, and in group-work, to demonstrate that it is possible for every one of them to come up with new ideas and to make them feel that they can solve problems together and then individually as well.

Stanberry [4] mentions that active learning activities can be awarded by some extra credit points since this typically gives the students extra motivation to do their best. We have also found in our experiences that letting students earn some credit (5-10% of the total possible points) toward their final grades for completing active learning activities improves their engagement. As of our experiences, another encouraging, motivating tool is creating a habit of clapping after a group presentation to express appreciation towards the presenters.

3. Study on the effects of the application of active learning components

In this paper, we compare the achievement of students under the various level of active learning use in the sections. The Institutional Review Board (IRB) of the Human Subjects Protection Program (HSPP) approved this research. In the study, we have investigated how the section average and the grade distribution of the sections depends on the level of use of active learning components (AL-components in the following) in undergraduate college algebra and business calculus courses. Our hypothesis was that the application of active learning strategies improves the achievements of students. In the following, we give a detailed description of the research, the analysis, and the results.

3.1. Description of the research design

In the study, every instructor who taught college algebra and business calculus courses in the fall 2017 and spring 2018 semesters was contacted in emails and asked to participate in a survey that aimed to learn on what scale instructors use AL-components. The ISO 27001 certified online survey software package, Qualtrics was used to execute the survey. According to the IRB policy, instructors who gave their consent to participate in the research could submit their responses to the survey questions. The instructors who teach these courses represent broad demographics; faculty members, instructional staff, graduate teaching assistants. The survey was fully anonymous; therefore, there is no specific information about the instructors who filled out the survey for the 27 sections.

We list the survey questions, the possible responses, and the name of the AL-components used in the research in Table 1. The survey questions were clearly defined for the respondents, and instructors were encouraged to ask for clarification if needed. The first question was included because we were interested in how important is a class taking place in a Collaborative Learning Space (CLS), with other words an Active Learning Classroom (Phillipson et al. [42]). In this paper, we consider the classroom a Collaborative Learning Space if it is designed to provide an opportunity for three to six students to sit together, preferably around a table, and discuss the material, solve problems together with teammates.

The other six questions were included in the survey to get feedback about the application of the six AL-strategies described in section 2. The terms CLS, Interactive style, Group-work, Present the solution, Real-life connection, Explorations, and Motivation do not have exact definition in literature, and indeed, instructors could interpret them slightly differently. We will address this problem among the limitations of the study in section 4.

Table 1. List of the survey questions, the response choices to the survey questions, and the AL-component names

Survey-Questions	Response choices to the survey questions	AL-component names
Q1. Did you teach in a collaborative learning space?	1.No 2.Yes	CLS
Q2. When you presented new material or explained the solution of an exercise, how often did you use an interactive presentation style? (E.g. asking for students' advice on the next step of the solution.)	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week) D. Very often (every class meeting) E. Always (multiple times in every class	Interactive style
Q3. How often did you encourage students to do group-work? (2 to 5 students are working together to solve a problem or discuss a given subject.)	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week) D. Very often (every class meeting) E. Always (multiple times in every class	Group-work
Q4. How often did you ask students to present their work, explain a solution to the whole class?	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week) D. Very often (every class meeting) E. Always (multiple times in every class	Present the solution
Q5. How often did you connect the material to real-life and/or other disciplines to arouse interest in the topic?	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week) D. Very often (every class meeting) E. Always (multiple times in every class	Real-life connection

Q6. How often did you involve students in math-explorations?	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week)	Exploration
Q7. How often did you motivate and advise students to actively participate in class-work or group-work?	A. Rarely (once or twice in the semester) B. Seldom (3 to 5 times in the semester) C. Often (every week) D. Very often (every class meeting) E. Always (multiple times in every class)	Motivation

We sent out the emails with the survey questions to all instructors who taught college algebra and business calculus courses and received responses for 27 sections. The total number of students in the 19 collage algebra and 8 business calculus sections was 886. The class-sizes ranged from 26 to 58. Most of the students in the college algebra and business calculus classes are freshmen.

The survey responses were fully anonymized and de-identified, and any identifiable data, such as name, and ID number of individuals and sections were removed; therefore, no specific information is available about the students. We have done the statistical analysis of the anonymized and de-identified data of 27 sections.

3.2. Description of the analysis

We developed an active learning metric (AL-metric) of the sections to compare the sections to each other regarding the level of the use of the AL-components. Therefore, we assigned numerical values, called AL-weights in the analysis, to possible response choices. We have used a Likert scale (Allen & Seaman [43]). In case of a high-level usage of an AL-component, the AL-weight is 100; at a minimal application (and at question Q1, when the class took place in a regular classroom), it is 20, and it is a linear scale between.

We used SAS 9.04 University Edition for the analysis. We analysed the relationship between the level of the use of the AL-components and the section averages. We defined two AL-metrics, the AL-score and the AL-base-score, to measure the level of the use of the AL-components.

Definition. The AL-score is the arithmetic average of the AL-weights of all AL-components listed in Table 1.

Definition. The AL-base-score is the arithmetic average of the AL-weights of four AL-components, Interactive style, Group-work, Present the solution, and Motivation listed in Table 1.

We determined two types of section averages. We analysed the effects of applying AL-components on the Grade-Average, which measures the average final score, and on the Exams'-Average, which is the average of the exam scores only. In the sections we analysed, there were three midterm exams during the semester and one cumulative final exam at the end of the semester.

Definition. The Grade-Average is the average of students' end-of-semester percentage score in a section.

Definition. The Exams'-Average is the average of students' percentage scores on all exams, that is on three midterm exams and the final exam, in a section.

While planning the study, we considered using the AL-score metric only. We have done the Linear Regression Analysis and compared the effects of the different AL-components on the Grade-Average. The analysis resulted in positive coefficients having the p-values displayed in Table 2.

Table 2. Comparing the effects of the AL-components on the Grade-Average. p-values of the different AL-components

AL-components	p-values
CLS	0.14
Interactive style	0.007
Group-work	0.01
Present the solution	0.013
Real-life connection	0.55
Exploration	<0.0001
Motivation	0.012

The analysis gave large p-values for Real-life connection and CLS. Investigating the reason for this, we have found that only 6 of the 27 sections took place in a CLS, meaning that the sample size was minimal. Regarding the Real-life connection, it is likely that subjectivity played a significant role while answering this question. For Exploration, the analysis returned a very small positive p-value.

To make sure that some effects that we are not aware of will not have a significant influence on the results of the study, we decided to construct another metric as well, where we did not include these three AL-components. This resulted in constructing the AL-base-score metric, which is the average of four AL-components: Interactive style, Group-work, Present the solution, and Motivation.

3.3. Results of the statistical analysis

The hypothesis was that the section average depends linearly on the AL-metric, and a larger value of the AL-metric results in larger section average value. We expected that the linear regression would show dependence between the AL-metric and the section average. Therefore, we have used linear regression with the usual hypothesis and p-values.

The results of the statistical analysis support this hypothesis. In the linear regression, the estimates of the slopes (displayed in Table 3. and 4.) are larger than 0.1 at all scenarios, and the p-values are less than 0.01. This indicates the existence of an association between the AL-metric and the section-average with positive, larger than 0.1 slopes (displayed in Table 3. and 4.) at every scenario.

Comparing Tables 3a to 3b and 4a to 4b, we can see that the slopes are smaller at AL-base-score than at AL-score, meaning that the effect of the AL-base-score on the section averages is smaller than that of the AL-score.

Table 3. Summary of the coefficients for the Grade-Average as a function of the AL-score, (Table 3a), and that of the AL-base-score, (Table 3b)

Variable	AL-score	
	Intercept	Slope
Parameter Estimate	62.335	0.159
Standard Error	2.986	0.043
t-value	20.873	3.665
p-value	2.5E-17	0.001

Variable	AL-score	
	Intercept	Slope
Parameter Estimate	64.359	0.119
Standard Error	2.784	0.037
t-value	23.116	3.208
p-value	2.2E-18	0.004

Table 4. Summary of the coefficients for the Exams'-Average as a function of the AL-score, (Table 4a), and that of the AL-base-score, (Table 4b)

Variable	AL-score	
	Intercept	Slope
Parameter Estimate	63.160	0.138
Standard Error	2.657	0.039
t-value	23.775	3.578
p-value	1.1E-18	0.001

Variable	AL-score	
	Intercept	Slope
Parameter Estimate	65.101	0.101
Standard Error	2.496	0.033
t-value	26.084	3.031
p-value	1.2E-19	0.006

The Fit Plot for the Grade-Average as a function of the AL-score is shown in Figure 5a, and as a function of the AL-base-score in Figure 5b. The small circles in the figure correspond to the section-averages of the 27 sections examined in this study.

Comparing the parameters obtained for Grade-Average to Exams'-Average, we can notice that the slopes are slightly smaller at the Exams'-Average. However, the Fit Plots for Exams'-Average are very similar to those for Grade-Average. Therefore, we did not include the Fit Plots for Exams'-Average in this paper.

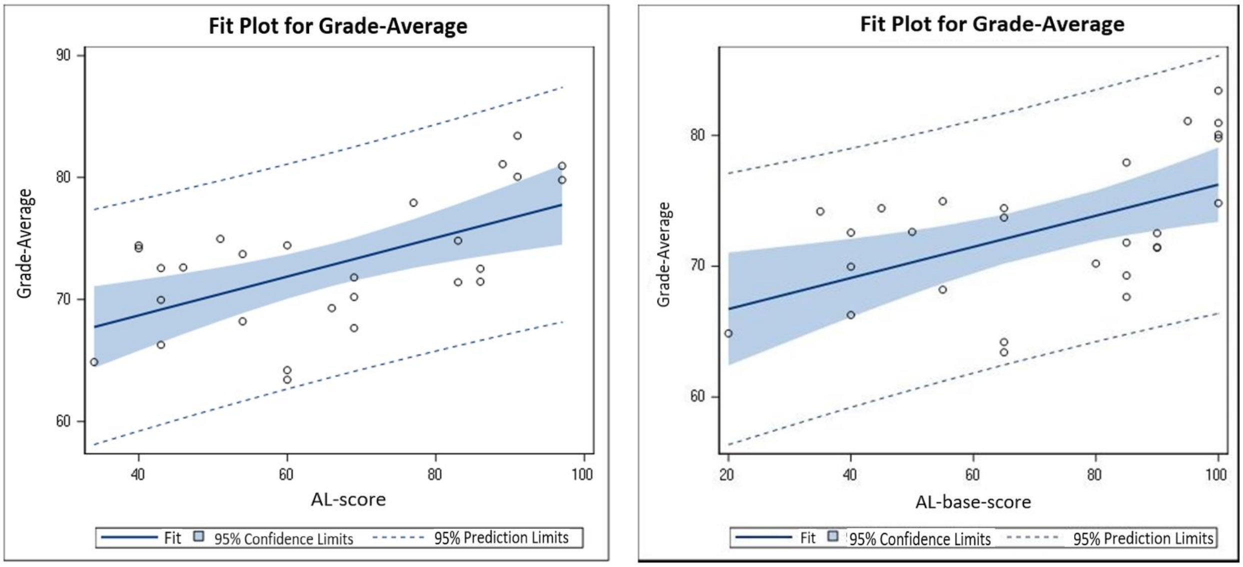


Figure 5. Fit Plot for Regression of the Grade-Average as a function of AL-score, (Figure 5a), and that of AL-base-score, (Figure 5b), respectively

Based on the results of the statistical analysis (summarized in Tables 3a and 3b for Grade-Average and Tables 4a and 4b for Exams'-Average), we can state that the level of use of AL-components is associated with positive effects on both the Grade-Average and the Exams'-Average. A larger AL-score or AL-base-score results in a larger Grade-Average and Exams'-Average.

Using the regression line with the parameters of Table 3a, we can evaluate the difference between the theoretical values of the Grade-Average at large AL-score and low AL-score. Specifically, the theoretical difference between the Grade-Averages when the AL-score equals to 90 and 20 is approximately 11.1%. The evaluation using the parameters of Table 3b results in a difference of 8.3% in the theoretical values of the Grade-Averages when the AL-base-score equals to 90 and 20.

A similar evaluation can be done for the Exams'-Average using the regressions line with the parameters of Table 4a to determine the difference between the theoretical values of the Exams'-Average at large AL-score and low AL-score. The theoretical difference between the Exams'-Averages when the AL-score equals to 90 and 20, respectively, is approximately 9.7%. The evaluation using the parameters of Table 4b results in a difference of 7% in the theoretical values of the Exams'-Averages.

3.4. Grade distributions in the sections

To measure the influence of AL-components on specific grades, we divided the sections into three groups. Group 1 includes nine sections which AL-scores are the lowest, Group 3 consists of nine sections which AL-scores are the largest, and Group 2 includes the remaining sections. Altogether, 272 students belonged to the first group, 302 to the second group, and 312 to the third group.

Figure 6 displays the achievements of students in Group 1 and Group 3. Figure 6a shows the percentage of students achieving a Grade-Average larger or equal to 0 and less than 60%, larger or equal to 60% and less than 70%, and so on. Figure 6b shows the percentage of students regarding the Exams'-Average.

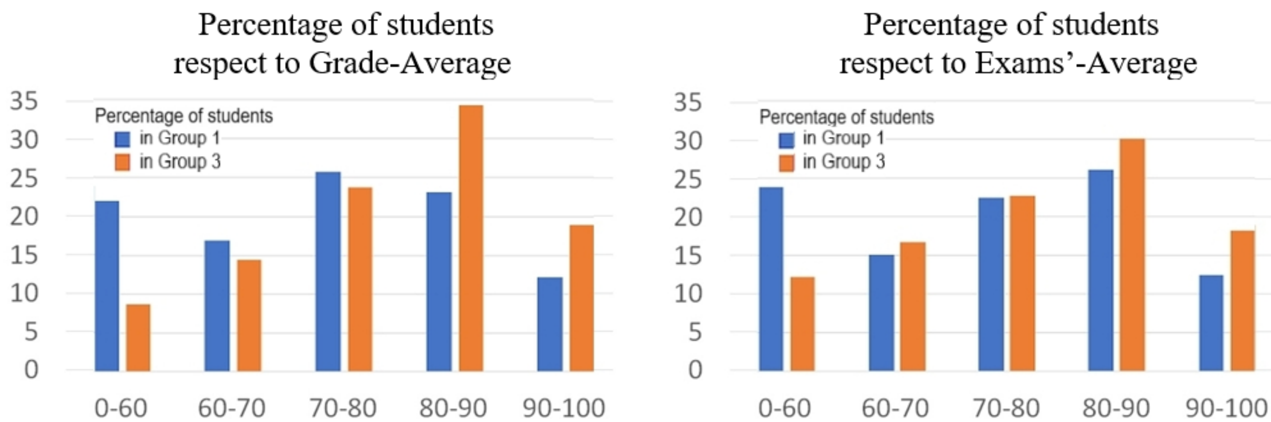


Figure 6. The percentage of students achieving a certain Grade-Average (Figure 6a), and Exams'-Average (Fig. 6b), respectively. Figure 6a shows the percentage of students in Group 1 and Group 3, achieving Grade-Averages 0-60%, 60-70%, 70-80%, 80-90%, and 90-100%. Figure 6b shows the percentage of students in Group 1 and Group 3, achieving an Exams'-Averages 0-60%, 60-70%, 70-80%, 80-90%, and 90-100%

Comparing the percentage of students with a Grade-Average below 60% in Group 1 and Group 3, we can see that it is less by 13.4% ($p = 0.009$, ANOVA) in Group 3. Regarding Exams'-Average, the difference is 11.7% ($p = 0.034$, ANOVA). This shows that the failure rate in sections with more active learning content is by more than 11% less than at the limited usage of AL-strategies.

4. Impact of AL strategies on educational goals

As of our experiences, the active learning strategies described in this paper contribute to the achievement of the educational goals listed in the introduction. The necessity of participation is imperative at the application of any active learning strategy. Engagement in a mathematical investigation, communication, and group problem-solving, is achieved through group-work activities, presentations to the class, doing experiments, explorations, and projects. The results of the statistical analysis show that the average achievement of students in the courses is better under active learning; therefore, it is likely that the future average performance of these students will also be better. Group-discussions about the problems while getting the appropriate hint from the instructor if needed, encouragement and motivation to keep trying, and feedback on the work done, will improve their decision-making capabilities.Through the active learning activities, students can apply the knowledge they acquired recently and connect it to previous knowledge, which helps to store the new knowledge in long-term memory. Students experience the importance of elaborative rehearsal. The basic sought-after skills, such as problem-solving, critical thinking, working effectively in a team, coordinating with others, negotiation, and be able to communicate about work-related topics can be practiced and improved during group-work, presentation, and while doing explorations and experiments. An interactive presentation style enhances communication capabilities. Motivation, encouragement, and feedback during group activities can improve emotional intelligence. Discussion with fellow-students, participation in experiments, explorations, and group-projects, connecting mathematics to art and nature provide a relaxed environment for learning where students can gain positive experiences. As of our experiences, these positive experiences reduce mathematics anxiety. Continuous motivation and encouragement also mitigate mathematics anxiety. As the results of the statistical analysis demonstrated in section 3.3, the passing rate of students under active learning in this study is by more than 11% larger than under lecturing type of teaching. We did not make a study specifically for underrepresented students. However, Theobald et al. [6] have found in their studies on active learning that, on average, active learning reduced achievement gaps for underrepresented students in passing rates by 45%.

5. Limitations of the study and potentials for future work

The primary purpose of this paper is to enable instructors to apply specific AL-strategies in teaching that we have found useful in some undergraduate mathematics courses. Our intention with the study described in section 3 was to provide some confirmation of the hypothesis that the level of use of AL-components summarized in section 2 of this paper increased the achievements of students in undergraduate mathematics classes.

This study has certain limitations, such as the sample size, data collection process, and the methods used in the analysis. In this section, we give a summary of these. We suggest potential future work to mitigate these limitations and to survey the student participants to understand their perception of active learning better.

5.1. Data collection process: survey questions and responses

This study belongs to the Human Subjects Protection Program, meaning that we had no personal information on any person involved in the study. Therefore, we asked all instructors who taught the sections to answer pre-designed survey questions (Q1 to Q7 in Table 1), if they gave their consent to participate in the research. We have got responses regarding 27 sections. The email with a request to participate in the study was sent out multiple times to every instructor. After sufficient waiting time, we have closed the survey and worked with 27 sections.

The survey questions were designed to measure the level of use of the AL-components described in this paper and listed in Table 1. The survey questions were clearly defined for the respondents, and instructors were encouraged to ask for clarification if needed. Questions Q2 to Q7 were included to collect information on the perception of the instructors regarding the AL-strategies. Question Q1 asked about the environment itself if the classroom was a collaborative learning space or not. The terminology used in the questions were familiar to the instructors.

However, the questions might have been interpreted slightly differently by the responders. Regarding Real-life connection, it is likely that question Q5 was not specific enough.

A possible reason for a large p-value regarding CLS (Table 2) is that only 6 of the 27 sections took place in a CLS, which is certainly a small sample size.

Regarding Exploration, we could not identify a specific reason for the p-value being this small. Therefore, we defined the AL-base-score as well to make sure that the influence of CLS, Real-life connection, and Exploration on the results will not cause significant discrepancies. Literature lists many advantages of hands-on activities (Bot et al. [37], Harper & Daane [18], Lugosi [20], Mikhaylov [38]). It would certainly be an important topic of further research to measure the positive effects of mathematics experiments, explorations, and projects on students' achievements.

Another limitation regarding the data is that we have collected self-reported data from the instructors. Therefore, we have learned the instructors' perception of their use of specific strategies, which is certainly not the same as measuring the exact level of the application of AL-strategies objectively. Future studies on the implementation of AL-strategies could be more objective using the same design of the application of the strategies in every section that would be included in the study.

Further essential work could be, as one of our reviewers pointed it out, surveying the student participants based on their experiences under the active learning strategies at the beginning and the end of the semester. Deslauriers et al. [41] measured students' actual learning versus the feeling of learning and found that the perception of learning of students under active learning was not as high as in a passive environment. Analysing how students' opinion changes throughout the semester while participating in a collaborative environment might lead to interventions that would increase the appreciation of students toward active learning.

5.2. Methods and metrics used in the analysis

To identify the level of the use of AL-components in the sections, we used a Likert scale (Allen & Seaman [43]) and then evaluated two AL-metrics described in section 3. It could be an interesting research topic to develop additional metrics of the level of application of AL-strategies and compare the results.

To measure the average achievement of students in the sections, we calculated two types of section averages. The first one is the Grade-Average that measures the average of students' end-of-semester percentage score in a section. The end-of-semester letter grade is determined based on that. We considered it important to analyse the effects of the level of use of the AL-components on Grade-Average because students' achievement is usually compared using the end-of-semester letter grade. However, the Grade-Average can contain elements that depend on the personality of the instructors; for example, how the written homework and in-class quizzes are graded. To decrease this effect, we have evaluated the Exams'-Average as well. It is the average of students' percentage scores on all exams in a section. The Exams'-Average is more independent of the instructors since 78% of the exams are common exams with automatic grading. Regarding the grading of the not-common exams, instructors have to follow specific rules that make the grading unified for these exams as well.

6. Conclusion

Literature proves the importance of applying AL-strategies, especially in undergraduate education. On the other hand, in the opinion of Carl Wieman [44], it appears that mathematics as a discipline is highly traditional in its teaching and more resistant than other STEM fields to adopt research-based teaching methods. With this paper on specific AL-strategies, our primary purpose was to give examples on the application of AL-strategies in college algebra and business calculus courses. The results of the study described in section 3 support the original hypothesis that the application of active learning strategies improves the achievements of students. Our findings show that in the 27 sections of the study, the higher-level use of AL-components resulted in better average scores of students. Specifically, the failure rate in sections with more active learning content is by more than 10% less than at limited usage of active learning.

Through education, we prepare students to overcome professional challenges in life by providing them the necessary skills, knowledge, and experience that enables them to solve problems and to be ready to adapt as the changing environment requires. We consider AL-strategies an adequate tool to achieve this goal and an appropriate tool to contribute to the achievement of the educational goals listed in the introduction.

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