On endowments and indivisibility: Partial ownership in the Shapley-Scarf model

Patrick Harless[*] William Phan[‡]

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Abstract

We introduce a parameterized measure of partial ownership, the α-*endowment lower bound*, appropriate to probabilistic allocation. Strikingly, among all convex combinations of *efficient* and *group strategy-proof* rules, only Gale's Top Trading Cycles is *sd efficient* and meets a positive α-*endowment lower bound* (Theorem 2); for efficiency, partial ownership must in fact be complete. We also characterize the rules meeting each α-*endowment lower bound* (Theorem 1). For each bound, the family is a semilattice ordered by strength of ownership rights. It includes rules where agents' partial ownership lower bounds are met exactly, rules conferring stronger ownership rights, and the full endowments of TTC. This illustrates the tradeoff between *sd efficiency* and flexible choice of ownership rights.

Keywords: object reallocation, Top Trading Cycles, α-endowment lower bound

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[*]University of Arizona. Correspondence: pharless@email.arizona.edu.

[†]North Carolina State University. Correspondence: wphan@ncsu.edu.

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1 Introduction

A public authority owns several housing units previously allocated to their current tenants. However, changes in demographics, preferences, and other circumstances necessitate reallocation. While existing tenants do not own their residences, their occupancy confers a limited entitlement which may be relevant to reallocation. New York's experience with reallocation is typical:[1]

> Ms. Jones, 70, moved into her apartment, in the Amsterdam Houses on West 63rd Street, with her son when she was 30 years old and has lived there ever since. But last month, she opened a letter that said it was time for her to go: Ms. Jones lives alone in a public housing development, taking up a two-bedroom apartment that she no longer requires... The (New York City) Housing Authority is trying to get tenants like Ms. Jones to move into smaller apartments, but there is *no guarantee* they will stay in their building, and many residents have been reluctant to comply.

Although unable to provide a guarantee, the housing authority recognizes and wishes to account for tenants' ties to their local communities. It lacks a systematic means for balancing private and collective interests. Absent a clear articulation of principle, the resulting policy increases tenants' anxiety and may appear to discount their residency entirely, contrary to the intentions of the authority. Even while recognizing that the circumstances of reallocation entail some disappoint, a desirable reallocation procedure will respect endowments, at least partially, and limit the chance that reshuffling leaves individuals worse off than initially. Our proposal codifies partial ownership in this way as a probabilistic guarantee.[2] Building on the familiar technique of achieving equity with lotteries, our notion of partial ownership can be readily explained to residents, assuaging some concerns and reassuring them that their interests are valued. Our formulation further allows all possible strengths of partial ownership, providing flexible tools by which the Housing authority may account for tenure, family conditions, occupation, and other relevant circumstances.

Beyond the public housing context that we emphasize, similar situations are common. For example, a department chair reallocating teaching assignment to account for enrollment or building availability or a manager who must reassign workers to branches, shifts, or tasks to meet the firm's needs. Like tenants' current residences, instructors' previous courses or workers' existing shifts function as effective endowments. And just as tenants establish ties to a community, instructors invest energy in planning and workers acclimate to routines which make these endowments salient references for comparison. As with tenants of public housing, instructors and workers do not own their endowments in the traditional sense, yet their original assignments confer some rights and welfare expectations.

To study these problems formally, we adopt a probabilistic version of the reallocation model of Shapley and Scarf (1974). In the standard model, each agent initially holds one object, his endowment, and has preferences over all available objects. A rule elicits those

[1] Harris, Elizabeth. "Alone in Public Housing, With a Spare Bedroom." *New York Times* March 11, 2012; italics added.

[2] Although we adopt the language of probabilities, our model can also be interpreted as time-sharing when applied to some of the examples we subsequently describe.

preferences and selects a new allocation. Because the setting is deterministic, respect for ownership is binary: An agent either is at least as well off as initially or he is not. For a more nuanced measure, we broaden the scope of rules to include rules that select lotteries over deterministic allocations.[3] We continue to suppose that agents report rankings over object and conservatively infer only those lottery comparisons induced by first-order stochastic dominance.[4]

To what extent can a probabilistic rule be said to respect ownership? First, given an allocation selected at a given report of preferences, we determine the probability for each agent that he receives an object at least as desirable as his endowment. Computing the minimum for each agent across all preference profiles establishes a set of lower bounds. These values measure the respect the rule accords to each agent's endowment, or the strength of his ownership claim. Given a vector α of probabilities for each agent, the *α-endowment lower bound* requires that these minimum values dominate α. The concept in fact parameterizes a family of axioms with individualized measures, each ranging from no endowment (value of zero) to full ownership (value of one). The *α-endowment lower bound* can be viewed as a special class of the fractional endowment lower bounds introduced by Athanassoglou and Sethuraman (2011). They allow arbitrary partial ownership across all objects and agents; in contrast, we restrict each agent to have shares of only *one* object, and no two agents (partially) own the same object. Subsequently, our family of axioms avoids the incompatibilities they identify; we discuss this more precisely in the conclusion.

In addition to respect for partial ownership, we seek rules with other desirable properties. Fundamentally, the goal of reallocation is to improve the welfare of the agents, albeit limited by scarce resources and competing interests. To this end, we require *sd efficiency*: No lottery selected by the rule should be dominated by another lottery according to the agents' (extended) preferences.[5] Although stronger than its deterministic counterpart, it is also conservative, declaring inadmissible only those lotteries which permit unambiguous improvements.

Next, because the designer must rely on reported preferences, we ask that rules provide incentives for agents to report their preferences truthfully. This is important both to prevent gaming the system as well as to ensure that the designer possesses accurate information by which to judge other properties. Formally, we require *strategy-proofness*: Each agent should find the lottery he receives when truthfully reporting his preferences at least as desirable as the lottery he could obtain with any other report. Since our extension of preferences to lotteries is incomplete, the requirement that the truthful lottery be comparable to other obtainable lotteries makes the property demanding. Among deterministic rules, however, comparability is trivial and *strategy-proofness* simply asks that truth telling be a dominant

[3]In the assignment problem without ownership, randomization is a common technique to introduce fairness (see Bogomolnaia and Moulin (2001)).

[4]That is, a lottery is preferred to another if it assigns higher probability to each subset of higher-ranked objects. The extension is conservative in the sense that one lottery is declared preferred to another if it is preferred by *all* von-Neumann–Morgenstern utilities compatible with the ranking over objects (McLennan, 2002; Manea, 2011).

[5]This notion is due to Bogomolnaia and Moulin (2001). A milder implied condition is *ex-post efficiency* which requires that each selected lottery be a convex combination of deterministic allocations which are themselves Pareto undominated.

strategy for each agent. Extending the logic beyond individuals, *group strategy-proofness* further requires that no group benefit by jointly misrepresenting their preferences.

Fairness calls randomization to mind, suggesting mixtures of deterministic rules, with Random Serial Priority/Dictatorship foremost in the mind of economists.[6] Several advantages contribute to their continued appeal. First, they are straightforward to implement: Randomly order the agents and run the corresponding priority rule. In addition to simplicity, the process is transparent and easily communicated to participants. Furthermore, those explaining the rules may rest assured that incentive properties of the component rules carry over to the mixture itself.[7] This two-fold incentive compatibility adds flexibility, allowing a designer to conduct the randomization either before or after agents report their preferences without worry about manipulation, mistakes, or new information.[8]

With Random Serial Priority as motivation, we expand the underlying set of deterministic rules to include all *efficient* and *group strategy-proof* rules.[9] Mixtures over the enlarged class define the Random Trading Cycles rules, all of which are *sd strategy-proof*.[10] The versatile family includes randomizations over Serial Priority rules, Gale's Top Trading Cycles rules, and Hierarchical Exchange rules to name a few.

Our first result characterizes, for each possible guarantee profile α, the subfamily of Random Trading Cycles rules that meet the *α-endowment lower bound*, (Theorem 1). Each includes a subset of "baseline" rules that provide agents exactly their guarantees as well as rules conferring stronger ownership rights. Initial ownership rights distinguish the family with the included rules forming a semilattice ordered by strength of guarantee. Investigating non-trivial guarantees, our second result reveals that TTC is the unique *sd efficient* rule that satisfies a strictly positive *α-endowment lower bound* (Theorem 2). Strikingly, the *sd efficiency* strengthens even the most limited partial ownership to a full-fledged endowment.

Altogether, we make two contributions. First, we introduce a parameterized notion of partial ownership and study its implications among probabilistic rules. With our two characterizations we identify a tradeoff between the efficiency of a rule and the ability of the policymaker to reallocate objects as it desires—more efficiency means less ability. Second, we provide a new characterization of TTC which elucidates the role of ownership in the reallocation model. This connects the characterizations by Ma (1994) with full ownership and Pycia and Ünver (2017) with mixed ownership and further justifies the centrality of TTC. In the process, we develop new techniques for the study of *sd efficiency* in the probabilistic model and elucidate features of the Trading Cycles rules. We postpone further discussion of related literature until we have formal concepts in hand.

In Section 2, we introduce the model and formalize our axioms in Section 3. We present our results in Section 4 and return to related work and extensions in Section 5. All proofs

[6] Most extensively studied in our environment are this rule and the Core from Random Endowments. See Abdulkadiroğlu and Sönmez (1998), Bade (2016a), Bogomolnaia and Moulin (2001), Carroll (2014), Che and Kojima (2010), Lee and Sethuraman (2011), and Sönmez and Ünver (2005).

[7] Formally, if each of the deterministic rules in the support of the mixture is *strategy-proof*, then the resulting rule is a *sd strategy-proof*.

[8] Pycia and Ünver (2015) advocate this two-fold incentive compatibility (and other properties) as a general principle.

[9] This is the family of Trading Cycles rules characterized by Pycia and Ünver (2017).

[10] They may not be *group sd strategy-proof*, however. See (Bade, 2016a).

appear in the appendix.

2 Model

Let N be a set of agents, and $\mathcal{O}$ be a set of objects with $|N| = |\mathcal{O}| \geq 4$. Each agent initially holds one object labeled so that agent i's **endowment** is $\omega_i \in \mathcal{O}$. An **allocation** is a bistochastic matrix $p = (p_{ik})_{i\in N,\, k\in\mathcal{O}}$ and the set of all probabilistic allocations is Z.[11] For each $i \in N$ and $a \in \mathcal{O}$, p_{ia} represents the probability that agent i receives object a.[12] An allocation is **deterministic** if each entry is either zero or one. By the Birkhoff–von-Neumann Theorem, each allocation can be represented as the convex combination of deterministic allocations although not uniquely (Birkhoff, 1946; von Neumann, 1953).

Each agent i has a **preference relation** R_i over $\mathcal{O}$ that strictly ranks distinct objects and $\mathcal{R}$ is the set of all such relations. For each $a \in \mathcal{O}$, the **upper contour set of a in R_i** is $U(a, R_i) \equiv \{b \in \mathcal{O} : b \ R_i \ a\}$. Since allocations involve lotteries, we extend agents' preferences by first-order stochastic dominance:[13] For each $R_i \in \mathcal{R}$ and each pair $p, q \in Z$, $p \ R_i \ q$ if for each $a \in \mathcal{O}$, $\sum_{b\in U(a,R_i)} p_{ib} \geq \sum_{b\in U(a,R_i)} q_{ib}$. An **economy** consists of a preference profile $R \in \mathcal{R}^N$. For each $S \subseteq N$, R_S denotes the preferences of S and R_{-S} denotes the preferences of $N\backslash S$. A **rule** is a mapping $\varphi\colon \mathcal{R}^N \to Z$. Among deterministic rules, we abuse notation slightly and also write $\varphi_i(R) = a$ for $\varphi_{ia}(R) = 1$. Convex combinations (or simply, combinations) of rules are themselves rules. As with allocations, each rule can be written as the combination of deterministic rules. That is, φ is a **combination** of rules $\{\varphi^1, \ldots, \varphi^L\}$ if there are weights $(\lambda^1, \ldots, \lambda^L) \in (0, 1]^L$ with $\sum \lambda^l = 1$ such that for each $R \in \mathcal{R}^N$, and each $i \in N$,

$$\varphi_i(R) = \sum_{l\in\{1,\ldots,L\}} \lambda^l \varphi_i^l(R).$$

Each of the rules $\varphi^1, \ldots, \varphi^L$ are **component rules**. Note that a combination requires that the weight for each component rule be strictly positive.

Rules

As the formal definitions are complicated, we limit our description and notation to that required to formalize our results.[14] Instead, we provide an intuitive description of the primary rules appearing in our analysis. Our interest in these rules derives from their properties (See Remark 1).

As in the seminal work of Shapley and Scarf (1974), Gale's **Top Trading Cycles (TTC)** is central to our analysis. Given a profile of preferences, TTC simulates rounds of trading described by pointing among objects and agents. Initially, each agent points at his most preferred object and each agent's endowment points at him. This creates at least one cycle

[11] That is, $p \in [0, 1]^{N\times\mathcal{O}}$ and for each $i \in N$ and each $a \in \mathcal{O}$, $\sum_{b\in\mathcal{O}} p_{ib} = 1$ and $\sum_{j\in N} p_{ja} = 1$.

[12] When comparing vectors, we use $\geq$, $>$, and $\gg$ to include equality, indicate inequality in at least one component, and indicate inequality in all components.

[13] With slight abuse of notation, we use R_i to represent both agent i preference relation over objects and its extension to lotteries.

[14] See Shapley and Scarf (1974), Pápai (2000), and Pycia and Ünver (2017) for formal definitions.

and each agent in a cycle is assigned the object at which he points. These agents and objects are removed and each remaining agent points at his most preferred among the remaining objects, again forming at least one cycle. Agents and objects in a cycle are assigned and the algorithm continues until each agent receives one object.

Extending this idea leads to the family of **Trading Cycles** rules. A subclass of this family, first identified by Pápai (2000), allows for ownership of multiple objects and prescribes fixed rules for inheritance. Previewing the novelty required to describe the full family, we call these *Trading Cycles rules without brokers*. Each such rule is described by an initial vector of endowments, μ, which assigns initial ownership of each object to one agent.[15] Each object points at its owner, each agent points at his most preferred object, and assignments are made along cycles as with TTC. As agents leave, some objects may become unowned. Remaining agents inherit these objects as prescribed by the rule and remaining agents retain their existing endowments. Inheritance patterns may depend on assignments made in previous rounds, but may not otherwise condition on preferences. The algorithm continues until each agent receives one object.

In addition to varying ownership and inheritance structures, Pycia and Ünver (2017) generalize "control rights" to allow "brokers" as well as owners. A broker functions similarly to an owner in that his object points at him. In contrast, however, a broker may not point at the object he brokers and instead points to his most preferred among the other objects. Also in contrast, a broker may not control additional objects. Like a Trading Cycles rule without brokers, a *Trading Cycles rule with brokers* prescribes inheritance of control rights, now both ownership and brokerage, which preserves control of objects between rounds. When four or more agents remain, there may be at most one broker, and a broker may be present only when at least two other agents are owners. To accommodate the second requirement, inheritance rules for brokered objects are modified when fewer than two owners remain. If there is one owner, then he becomes the owner of the previously brokered object. If instead the broker is the last remaining agent, then he becomes the owner of the object. Finally, if exactly three agents remain, it is possible for all three agents to be brokers. In this case, the pointing restrictions continue to apply whenever consistent with efficiency and otherwise each agent receives his brokered object. Generalizing the notation for ownership we write $a \in \mu_i$ or $a^* \in \mu_i$ to indicate respectively that agent i owns or brokers object a. We denote the set of owners (not brokers) by $\mu(N) \equiv \{i \in N : \exists a \in \mathcal{O},\, a \in \mu_i\}$. To track conditional inheritance, we write $\mu(i \to a)$ and $\mu((i,j) \to (a,b))$ to indicate the updated control rights when agent i receives a and when additionally agent j receives b.

A rule φ is a **Random Trading Cycles** rule if it is a combination of Trading Cycles rules. Random Serial Priority and Core from Random Endowments are examples of Random Trading Cycles rules. The family is a natural generalization of these, containing mixtures over all currently known attractive deterministic rules.

3 Axioms

Given our interest in agents' welfare, we ask first that a rule not pass up opportunities to make everyone better off. An allocation p is **Pareto undominated at $\boldsymbol{R}$** if there is no

[15] Therefore, for each pair $i, j \in N$, $\mu_i, \mu_j \subseteq \mathcal{O}$, $\mu_i \cap \mu_j = \emptyset$, and $\bigcup_{k \in N} \mu_k = \mathcal{O}$.

allocation q such that for each $i \in N$, $q_i \; R_i \; p_i$, and for some $j \in N$, $q_j \; P_j \; p_j$. The rule should always select a Pareto undominated allocation.[16] This requirement applies to the general case when assignments can be lotteries. We also define a weaker ex post variant: The allocation recommended by the rule can be written as a convex combination of Pareto undominated deterministic allocations.[17]

sd Efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R)$ is Pareto undominated at R.

ex-post Efficiency: For each $R \in \mathcal{R}^N$, $\varphi(R)$ can be written as a convex combination of deterministic allocations that are Pareto undominated at R.

Like *sd efficiency* and its variant, our next axioms are standard. The first says that no agent should benefit by misreporting his preferences and the second extends the requirement to groups. Importantly, the first axiom require that the lotteries obtained by truth-telling and lying be comparable, itself a significant restriction due to the incompleteness of the stochastic dominance extension. The final property also considers how a rule responds to a change in one agent's preferences: If that agent's assignment does not change, then neither should the assignments of the other agents.

sd Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R) \; R_i \; \varphi_i(R'_i, R_{-i})$.

sd Group-strategy-proofness: For each $R \in \mathcal{R}^N$, there is no $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ such that for each $i \in S$, $\varphi_i(R'_S, R_{-S}) \; R_i \; \varphi_i(R)$, and for some $j \in S$, $\varphi_j(R'_S, R_{-S}) \; P_j \; \varphi_j(R)$.

Non-bossiness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, if $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$, then $\varphi(R) = \varphi(R'_i, R_{-i})$.

If a rule is deterministic, then the individual and group incentive compatibility properties reduces to their respective standard versions (without the "sd" preface) as the image of a rule consists only of deterministic allocations. Thus, there is no confusion if we drop the preface in these cases. A deterministic rule is *group strategy-proof* if and only if it is *strategy-proof* and *non-bossy* (Pápai, 2000), although this equivalence does not apply to the probabilistic rules (Bade, 2016a).

Remark 1. [Pycia and Ünver (2017)] A deterministic rule is *group strategy-proof* and *efficient* if and only if it is a Trading Cycles rule.

It is straightforward that since each Trading Cycles rule is *strategy-proof*, each Random Trading Cycles rule is *sd strategy-proof*. Furthermore, mixtures of *strategy-proof* and *non-bossy* deterministic rules are *non-bossy*, so a Random Trading Cycles rule is *non-bossy* as well (Bade, 2016a). *Group strategy-proofness* is not preserved by combinations (Bade, 2016a).

We now introduce our new family of axioms which measure the extent to which a rule respects endowments. The standard requirement takes the strongest position, as is appropriate when agents own their objects: Each agent should be assigned an object he finds

[16] Bogomolnaia and Moulin (2001) introduce this axiom with the name "ordinal efficiency."

[17] See Bogomolnaia and Moulin (2001) and Abdulkadiroğlu and Sönmez (2003a) for comparison of the two properties.

at least as desirable as his endowment. In our model, ownership rights are partial, so we weaken the condition, instead requiring that it hold with some probability. More precisely, given a vector of probabilities representing individual lower bounds, we ask that each agent be assigned a lottery which includes objects he finds as least as desirable as his endowment with probability at least equal to his individual lower bound, independent of the preferences of the agents. Let $\alpha \in [0,1]^N$.

$\boldsymbol{\alpha}$-endowment lower bound: For each $R \in \mathcal{R}^N$ and each $i \in N$, $\sum_{a \in U(\omega_i, R_i)} \varphi_{ia}(R) \geq \alpha_i$.

Consistent with our interpretation of partial ownership, each *α-endowment lower bound* considers all possible preference profiles and therefore represents a probabilistic guarantee. The axioms in the family provide nuanced and individualized measures of the strength of ownership rights that a rule provides. As the components of α increase, the set of rules satisfying the *α-endowment lower bound* shrinks, and distinct sets of rules satisfy the property for distinct values of α. The maximal $\alpha = (1)_{i \in N}$ corresponds to the **endowment lower bound** whereas the minimal $\alpha = (0)_{i \in N}$ models an environment without endowments. Of course, our parametrization has force only in the probabilistic setting. As summarized by Remark 2, the force of the axioms differ only according to which components of α are positive or zero, and the family collapses to individualized versions of the *endowment lower bound*.

Remark 2. A deterministic rule satisfies the *α-endowment lower bound* for i with $\alpha_i > 0$ if and only if it satisfies the *α-endowment lower bound* for i with $\alpha_i = 1$.

4 Results

As we are interested in combinations of deterministic rules, we first review which of our properties this operation preserves. Most importantly, combinations preserve *strategy-proofness* and the *α-endowment lower bound*. Remark 3 summarizes these facts and well-known results.

Remark 3. Combinations preserve *strategy-proofness*, *ex-post efficiency*, each *α-endowment lower bound*, and the combination of *strategy-proofness* and *non-bossiness*. Combinations do not preserve *sd efficiency*, *group strategy-proofness*, and *non-bossiness*.

With the Random Trading Cycles rules as a starting point, we now study the implications of our properties.

4.1 A Family of Rules Ordered by Guarantees

We state implications of our new axiom, the *α-endowment lower bound*, within the family of Random Trading Cycles rules. First, we find that ownership is required in order to achieve a positive *α-endowment lower bound*.

Lemma 1. *Let φ be a combination of Trading Cycles rules. If $i \in N$ does not own an object in any component of φ, then φ satisfies no* α-endowment lower bound *with $\alpha_i > 0$.*

The simplest way for a rule to satisfy a positive *α-endowment lower bound* for a given agent is for that agent to own his endowment in one of the components. Indeed, this is the only possibility among deterministic rules and even combinations of Trading Cycles rules without brokers. Yet surprisingly, this is not required generally as brokerage introduces new possibilities.

Example 1. Achieving a positive α-endowment lower bound without ownership of own endowment. Let $N = \{1, 2, 3, 4\}$, $\mathcal{O} = \{a, b, c, d\}$, and $\omega = (a, b, c, d)$. Let φ^1, φ^2, and φ^3 be Trading Cycles rules with initial control rights $\mu^1 = \big(\{a\}, \{b\}, \{d^*\}, \{c\}\big)$, $\mu^2 = \big(\{c\}, \emptyset, \{d^*\}, \{a, b\}\big)$, and $\mu^3 = \big(\{a\}, \{b\}, \{c, d\}, \emptyset\big)$ and let $\varphi = \frac{1}{4}\varphi^1 + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi^3$. Then φ satisfies the *α-endowment lower bound* with $\alpha = \big(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\big)$. For agents 1, 2, and 3, these values represent the probabilities assigned to rules in which they own their endowments. Although agent 4 never owns d, φ nevertheless satisfies the stated bound for agent 4, the guarantee arising from the combination of φ^1 and φ^2. Even though agent 3 controls d in these rules, he initially points at a different object. When agent 3 points at c, agent 4 receives an object at least as desirable at d under φ^1; when agent 3 instead points at a or b, agent 4 receives an object at least as desirable at d under φ^2. The smaller of the probabilities assigned to these rules is $\frac{1}{4}$, which establishes the guarantee.

As a consequence, given a Random Trading Cycles rule and the *α-endowment lower bounds* satisfied by each of its components, the rule may satisfy an *α-endowment lower bound* which exceeds the convex combination of those *α-endowment lower bounds* (with respective weights, of course). Essential for agent 4's guarantee in Example 1 is that he own each remaining object in rules across which d is brokered by a single agent. If even one object were omitted or brokerage of d were divided between two agents, then the guarantee would not be met. The two ideas illustrated in Example 1 – ownership of one's endowment and brokerage of one's endowment by another agent – allow us to precisely identify the greatest *α-endowment lower bound* satisfied by a rule.

Theorem 1. *A Random Trading Cycles rule with components $\{\varphi^1, \ldots, \varphi^L\}$ and weights $\{\lambda^1, \ldots, \lambda^L\}$ satisfies the* α-endowment lower bound *if and only if for each $i \in N$,*

$$\alpha_i \leq \sum_{l \in \{1,\ldots,L\}} \left\{\lambda^l : \omega_i \in \mu^l_i\right\} + \sum_{j \in N\setminus\{i\}} \min_{a \in \mathcal{O}\setminus\{\omega_i\}} \sum_{l \in \{1,\ldots,L\}} \left\{\lambda^l : \omega^*_i \in \mu^l_j \text{ and } a \in \mu^l_i\right\}.$$

Note that Theorem 1 identifies an entire class of rules for *each* possible guarantee profile α.

Theorem 1 also shows that whether a combination of Trading Cycles rules satisfies a given *α-endowment lower bound* depends only on the structure of initial control rights among the components. Beyond describing the *α-endowment lower bound*, Theorem 1 formalizes a novel feature of brokerage. This nuance provides a new equity interpretation for broker arrangements which complements the original equity motivation of Pycia and Ünver (2017),[18] reinforcing interest in the Trading Cycles rules outside previously identified families.

[18]Whereas our example shows that brokerage by one agent allows us to make *another* agent better off according to our measure, Pycia and Ünver (2017) show that brokerage may improve the welfare of the *broker* in a setting where restrictions frown on him consuming a particular object.

For each α, the set of allowable rules forms a semilattice with a natural order based on agents' guarantees. For each Random Trading Cycles rule, the level of guarantee provided to each agent is simply the respective summation defined in Theorem 1; this defines a partial order on the set of rules. Aside from when $\alpha = (1)$ there are many rules that satisfy the same level of guarantee. For example, starting from a combination of rules simply augment secondary, tertiary... ownership structures in component rules. Thus, for some subsets of rules there may be no minimum element. TTC, however, is the only rule that provides the guarantee of $\alpha = (1)$, and is subsequently greater than all other rules.

With regards to our NYC Housing Authority application, Theorem 1 provides a spectrum of rules representing the possible compromises between NYC and tenants. These start from those rules that provide tenants *exactly* their guarantee and no further—allowing the policymaker maximum power to reshuffle apartments as it wishes—and, progress by conferring more and more ownership rights to tenants.

4.2 Adding *sd Efficiency* Implies TTC

With an entire menu of rules available with regards to the *α-endowment lower bound*, we now additionally impose *sd efficiency*. If each agent receives a positive guarantee, then out of many possible randomizations, we are ultimately left with a single deterministic rule: TTC.

Theorem 2. *For each $\alpha \in (0,1]^N$, Top Trading Cycles is the unique Random Trading Cycles rule satisfying* sd efficiency *and the* α-endowment lower bound.

The implication of adding *sd efficiency* is surprising and stark. If we drop either property or the assumption that $\alpha >> 0$, then many rules that are not TTC are satisfactory. We describe these further in the following section.

Theorems 1 and 2 together illustrate the tradeoff to a policymaker between efficiency and the ability/flexibility to reshuffle tenants. Strengthening *ex-post efficiency* to *sd efficiency* requires the policymaker to also strengthen partial guarantees to *full* ownership.

The proof takes Lemma 1 as a starting point and identifies collections of Trading Cycles rules consistent with its requirements. Much of the remaining work involves careful analysis of combinations of Trading Cycles rules which satisfy *sd efficiency*. The techniques and results we develop will therefore be of independent interest for researchers interested in the implications of *sd efficiency* in the probabilistic assignment model or properties of Trading Cycles rules generally.

To give a flavor of the approach, we present here a preliminary result which we apply extensively in subsequent arguments. Lemma 2 generalizes a typical case demonstrating incompatibility so that we may often identify inefficiencies by considering only top preferences.

Lemma 2. *Let φ and φ' be* group strategy-proof *deterministic rules. If there are $R \in \mathcal{R}^N$, $i, j, k, l \in N$, and $a, b \in \mathcal{O}$ such that R_i and R_j rank a at the top, R_k and R_ℓ rank b at the top, $\varphi_i(R) = \varphi'_j(R) = a$, and $\varphi_k(R) = \varphi'_\ell(R) = b$, then no combination of φ and φ' is* efficient.

Example 2 illustrates the intuition behind Lemma 2 whose formal proof appears in the appendix.

Example 2. Illustrating a canonical case of incompatibility. Let $N = \{1, 2, 3, 4\}$, $\mathcal{O} = \{a, b, c, d\}$. Let φ and φ' be *group strategy-proof* deterministic rules. In the table below, (partial) allocations under φ are boxed and those under φ' are circled. Since *sd efficiency* of a combination implies *ex-post efficiency* of each component, we may suppose that φ and φ' are Trading Cycles rules. Let $R \in \mathcal{R}^N$ be such that R_1 and R_2 rank a at the top, R_3 and R_4 rank b at the top as below. Suppose that $\varphi_1(R) = \varphi'_2(R) = a$ and $\varphi_3(R) = \varphi'_4(R) = b$. Then agents 2 and 4 receive c and d at $\varphi(R)$ while agents 1 and 3 receive c and d at $\varphi'(R)$. Now consider $R' \in \mathcal{R}^N$ as in the table. A consequence of *group strategy-proofness* is that each rule assigns a and b to the same agents at R' as at R, so agents 2 and 4 receive c and d at $\varphi(R')$ and agents 1 and 3 receive c and d at $\varphi'(R')$. After possibly relabeling the agents, we may suppose that $\varphi'_1(R') = c$ and $\varphi_2(R') = d$. Although both $\varphi(R')$ and $\varphi'(R')$ are *efficient*, no combination of these assignments is *sd efficient* because $\varphi_4(R') = c \; P'_3 \; d = \varphi'_3(R')$ while $\varphi'_3(R') = d \; P'_4 \; c = \varphi_4(R')$.

R			
$\boxed{a}$	$\textcircled{a}$	$\boxed{b}$	$\textcircled{b}$

R'			
$\boxed{a}$	$\textcircled{a}$	$\boxed{b}$	$\textcircled{b}$
$\textcircled{c}$	$\boxed{d}$	c	d
d	c	$\textcircled{d}$	$\boxed{c}$
b	b	a	a

An immediate corollary of Lemma 2 illustrates its usefulness: If two Trading Cycles rules assign ownership of two objects to two distinct pairs of agents, then they cannot be combined without sacrificing *sd efficiency*.

Corollary 1. *Let φ and φ' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b \in \mathcal{O}$ such that $a \in \mu_i \cap \mu'_k$ and $b \in \mu_j \cap \mu'_l$, then no combination of φ and φ' is* sd efficient.

4.3 Independence of Assumptions

Without the assumption of φ being a Random Trading Cycles rule, many rules satisfy *sd efficiency* and the *α-endowment lower bound*: If $\alpha \in [0, 1]^N$ is such that for each $i \in N$, $\alpha_i \in (0, \frac{1}{n}]$, then the Serial rule suffices (Bogomolnaia and Moulin, 2001).[19] For arbitrary α, the Competitive Equilibrium wherein each agent i is endowed with probability shares of objects with at least α_i share of ω_i suffices (Hylland and Zeckhauser, 1979).

If we drop *sd efficiency*, then we are back to the case of Theorem 1.

Without the *α-endowment lower bound*, various combinations of Trading Cycles rules satisfy *sd efficiency*.[20] In fact, Example 3 shows that this is true when just one agent has a trivial guarantee while all other agents' are strengthened to unity.

Example 3. Relaxing the requirement that $\boldsymbol{\alpha \gg 0}$. Let φ and φ' be Trading Cycles rules each with initial control rights $\mu = (\{a\}, \{b\}, \{c\}, \{d, f\}, \{e\}, \emptyset)$ which also prescribe the

[19] In fact, all "envy-free" and *efficient* rules satisfy the axioms for these α, as do various (though not all) combinations of Trading Cycles rules in which TTC appears as a component.

[20] Further developing these ideas, Harless and Phan (2018) identify several subfamilies among which combinations satisfy *sd efficiency*.

same inheritance with one difference: When each of the first three agents receives his endowment and agent 4 receives d, then agent 5 inherits f under φ while agent 6 inherits f under φ'. Combinations of φ and φ' satisfy the *α-endowment lower bound* with $\alpha = (1, 1, 1, 1, 1, 0)$. They are also *sd efficient* because their allocations differ by at most reassigning e and f among agents 5 and 6 when both prefer f to e.

The intuition behind Example 3 also applies when there are extra agents or extra objects or an outside option (null object) which need not be ranked uniformly last, so $|N| = |\mathcal{O}|$ is also essential to the characterization. On the other hand, even in these cases, our conclusions about the initial control structures continue to apply. The potential for combinations is necessarily limited so that all components are "close" to TTC with respect to those agents with positive guarantees and further handle control rights of all objects similarly until the final stages of inheritance.

Finally, our modelling assumption that $|N| \geq 4$ is also essential. When $|N| \leq 3$, *sd efficiency* is equivalent to *ex-post efficiency* (Bogomolnaia and Moulin, 2001). Consequently, combinations of Serial Priority rules are *efficient*. With equal weights assigned to orders in which each agent appears first, these combinations satisfy an *α-endowment lower bound* with $\alpha = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) >> 0$.

5 Conclusion

We briefly relate our work to earlier literature and highlight remaining open questions.

Within the diverse and vibrant literature spawned by Shapley and Scarf (1974), the previously mentioned characterizations of Pápai (2000) and Pycia and Ünver (2017) play an essential role in our analysis. Hylland and Zeckhauser (1979) introduce the probabilistic assignment model; Abdulkadiroğlu and Sönmez (1998) and Knuth (1996) subsequently show that the Core from Random Endowments is equivalent to Random Serial Priority.[21] Bogomolnaia and Moulin (2001) introduce the notion of *sd efficiency* that we adopt, and show that no *sd efficient* and *strategy-proof* rule treats agents symmetrically.[22] Relating the deterministic and probabilistic models, Pycia and Ünver (2015) study decomposability and ask for which properties of probabilistic rules those rules can be decomposed as rules with similar properties.

Closest to our approach, Athanassoglou and Sethuraman (2011) introduce fractional endowment lower bounds, leading to a family of axioms which includes the *α-endowment lower bounds* as a subclass. More specifically, they allow for arbitrary fractional ownership, while we restrict each agent to (partially) own only one object, and each object to be (partially) owned by only one agent. They show that there is no rule satisfying *sd efficiency*, the *fractional endowment lower bound*, and *sd strategy-proofness*. This result does not carry over to

[21] This equivalence has since been extended to all Trading Cycles rules as well as to particular lottery systems in school choice (Bade, 2016b; Carroll, 2014; Lee and Sethuraman, 2011; Pathak and Sethuraman, 2011).

[22] A similar incompatibility holds in a cardinal setting where agents report utility functions rather than ordinal rankings and efficiency is now judged with respect to these utilities (Zhou, 1990).

our environment as our lower bound requirement is substantially weaker.[23] Starting from the impossibility, their and our results comprise complementary studies: They keep *sd efficiency*, drop *strategy-proofness*, and emphasize fairness by identifying a rule that satisfies further fairness conditions (*no justified envy*). We keep *sd efficiency*, consider a special class of endowment lower bounds, and emphasize incentive compatibility and practical implementation by considering the Random Trading Cycles family.

The import of our second result is best understood by comparison to the original characterization of TTC among deterministic rules by *ex-post efficiency*, *strategy-proofness*, and the *endowment lower bound* (Ma, 1994).[24] Restricted to deterministic rules, our axioms are the same with *strategy-proofness* strengthened to *group strategy-proofness*, the difference being the redundant requirement of *non-bossiness*. Consequently, Theorem 2 effectively generalizes this result to the probabilistic assignment model. More importantly, because the probabilistic model allows us to parameterize and substantially weaken the *endowment lower bound*, our results elucidate the role of this axiom in the original result: Introducing even a modicum of respect for endowments, the requirement $\alpha \gg 0$ in the *α-endowment lower bound*, characterizes TTC. By contrast, even when *strategy-proofness* is strengthened to *group strategy-proofness*, removing the *endowment lower bound* permits the full range of Trading Cycles rules with combinations of these rules yielding an unfathomable diversity of probabilistic rules. While not all of these combinations are *sd efficient*, many are, and all of the Trading Cycles rules themselves satisfy *ex-post efficiency* in the enriched model. Viewed in one light, the starkness of our characterization has a negative flavor, as we might hope partial ownership would provide additional flexibility. On the other hand, our results underscore the desirability of TTC and strongly reinforce its centrality in both the literature and practice.

Turning to extensions, an important open question remains on whether there are rules outside of the Random Trading Cycles family that satisfy the three main axioms—*sd efficiency*, *strategy-proofness*, and the *α-endowment lower bound*. Thinking more broadly, we believe that adaptations of our *α-lower bound* will provide a similarly nuanced understanding of partial satisfaction of axioms in other models where probabilistic assignments are possible. For example, in the school choice model, an interesting exploration would identify those rules satisfying *α-stability* together with other desirable properties.

A Appendix

A.1 Proof of Lemma 1

Proof. Let φ be a combination of Trading Cycles rules $\{\varphi^1, \ldots, \varphi^L\}$, $\alpha \in [0,1]^N$, and $i \in N$. Suppose that φ satisfies the *α-endowment lower bound* and $i \notin \bigcup_{l=1}^{L} \mu^l(N)$. Let $R \in \mathcal{R}^N$ be

[23] For example, in their proof of the impossibility, they use the equal-division endowment profile—a profile ruled out by our restrictions.

[24] Sönmez (1999) realizes the result as an implication of a more general theorem and recent work provides simple and intuitive proofs of the original result (Anno, 2015; Sethuraman, 2016). Further work adapts TTC environments with mixed ownership Sönmez and Ünver (2010), and to the school choice setting of (Abdulkadiroğlu and Sönmez, 2003b) where TTC is characterized by properties based on respecting priorities (Abdulkadiroğlu and Che, 2010; Dur, 2012; Dur and Morrill, 2018; Morrill, 2013, 2015).

such that each agent ranks ω_i at the top, and $l \in \{1, \ldots, L\}$. If there is $j \in N$ such that $\omega_i \in \mu^l_j$, then $\varphi^l_j(R) = \omega_i$. Since $j \in \bigcup_{l=1}^L \mu^l(N)$, $j \neq i$. Suppose instead there is $j \in N$ such that $\omega^*_i \in \mu^l_j$ and let $b \in \mathcal{O}$ be the object ranked second in R_j. Since there are at least four agents present, there is at most one broker and so there is $k \in N$ such that $b \in \mu^l_k$. Then $\varphi^l_k(R) = \omega_i$. Again $k \in \bigcup_{l=1}^L \mu^l(N)$, so $k \neq i$ and $\varphi^l_i(R) \neq \omega_i$. This is true for each component rule, so $\varphi_{ia}(R) = 0$ which implies $\alpha_i = 0$. □

A.2 Proof of Theorem 1

Proof. Remark 1 implies an *ex-post efficient* and *decomposable* rule is a combination of Trading Cycles rules. Let φ be a combination of Trading Cycles rules $\{\varphi^1, \ldots, \varphi^L\}$ with weights $\{\lambda^1, \ldots, \lambda^L\}$. For each $i \in N$, let $\bar{\alpha}_i$ be the RHS of the weak inequality of Thereom 1. For each $i \in N$, and each $j \in N\backslash\{i\}$, let $a_j \in \text{argmin}_{a \in \mathcal{O}\backslash\{\omega_i\}} \sum\{\lambda^l : \omega^*_i \in \mu^l_j \text{ and } a \in \mu^l_i\}$. Also let $A(i) \equiv \{l \in \{1, \ldots, L\} : \omega_i \in \mu^l_i\}$ and $B(i, j, a) \equiv \{l \in \{1, \ldots, L\} : \omega^*_i \in \mu^l_j \text{ and } a \in \mu^l_i\}$. Then $\bar{\alpha}_i = \sum_{l \in A(i)} \lambda^l + \sum_{j \in N\backslash\{i\}} \sum_{l \in B(i,j,a_j)} \lambda^l$.

To see that $\bar{\alpha}$ is an upper bound, let $i \in N$ and let $R \in \mathcal{R}^N$ be such that all agents rank ω_i at the top and for each $j \neq i$, R_j ranks a_j second. For each $l \in A(i)$, $\varphi^l_i(R) = \omega_i$. Also, for each $j \in N\backslash\{i\}$ and each $l \in B(i, j, a_j)$, $\varphi^l_i(R) = \omega_i$. On the other hand, for each $l \in \{1, \ldots, L\}$, if $\omega_i \in \mu_j$, then $\varphi^l_j(R) = \omega_i$ and if $\omega^*_i \in \mu_j$, then there is $k \in N\backslash\{i, j\}$ such that $a_j \in \mu^l_k$ and $\varphi^l_k(R) = \omega_i$. Therefore, agent i receives ω_i with probability $\bar{\alpha}_i$ at R. Thus, for each $\alpha \in [0, 1]^N$, if $\alpha \not\leq \bar{\alpha}$, then φ violates the *α-endowment lower bound*.

To see that φ satisfies the *α-endowment lower bound* for $\alpha = \bar{\alpha}$, let $i \in N$ and $R \in \mathcal{R}^N$. By definition of the pointing algorithm, for each $l \in A(i)$, $\varphi^l_i(R)\ R_i\ \omega_i$. Now let $j \in N\backslash\{i\}$ and $b_j \in \mathcal{O}\backslash\{\omega_i\}$ be the object ranked highest in R_j. For each $\varphi^l \in B(i, j, b_j)$, agent j begins by pointing at b_j which points at agent i, so $\varphi^l_i(R)\ R_i\ \omega_i$. Then by definition of a_j, $\sum_{l \in B(i,j,b_j)} \lambda^l \geq \sum_{l \in B(i,j,a_j)} \lambda^l$. Therefore, agent i receives an object at least as desirable as ω_i with probability at least $\bar{\alpha}_i$ at $\varphi(R)$. □

A.3 Proof of Lemma 2 and Corollary 1

Proof. Without loss of generality, suppose that $N = \{1, 2, 3, 4\}$, and $\mathcal{O} = \{a, b, c, d\}$. Let φ and φ' be *group strategy-proof* deterministic rules.

Lemma 2. Let $R, R' \in \mathcal{R}^N$ be as in Example 2 and suppose that $\varphi_1(R) = \varphi'_2(R) = a$ and $\varphi_3(R) = \varphi'_4(R) = b$. Then $\{\varphi_2(R), \varphi_4(R)\} = \{\varphi'_1(R), \varphi'_3(R)\} = \{c, d\}$. By *group strategy-proofness*, $\{\varphi_1(R'_1, R_2, R'_3, R_4), \varphi_3(R'_1, R_2, R'_3, R_4)\} = \{\varphi'_2(R_1, R'_2, R_3, R'_4), \varphi'_4(R_1, R'_2, R_3, R'_4)\} = \{a, b\}$ so $\{\varphi_2(R'_1, R_2, R'_3, R_4), \varphi_4(R'_1, R_2, R'_3, R_4)\} = \{\varphi'_1(R_1, R'_2, R_3, R'_4), \varphi'_3(R_1, R'_2, R_3, R'_4)\} = \{c, d\}$. Then again by *group strategy-proofness*, $\{\varphi_1(R'), \varphi_3(R')\} = \{\varphi'_2(R'), \varphi'_4(R')\} = \{a, b\}$ so $\{\varphi_2(R'), \varphi_4(R')\} = \{\varphi'_1(R'), \varphi'_3(R')\} = \{c, d\}$. Then there are $i, j \in N$ such that $c\ P_i\ d$, $d\ P_j\ c$, and $\{\varphi_i(R').\varphi_j(R')\} = \{c, d\}$. Suppose that $\varphi(R') = (a, d, b, c)$ and $\varphi'(R') = (c, a, d, b)$. Since $c\ P_3\ d$ while $d\ P_4\ c$, no combination of φ and φ' is *efficient*.

Corollary 1. Further suppose that φ and φ' are Trading Cycles rules and let $a, b \in \mathcal{O}$ be such that $a \in \mu_1 \cap \mu'_2$ and $b \in \mu_3 \cap \mu'_4$. Let $R \in \mathcal{R}^N$ be such that R_1 and R_2 rank a at the top and R_3 and R_4 rank b at the top. Then $\varphi_1(R) = \varphi'_2(R) = a$ and $\varphi_3(R) = \varphi'_4(R) = b$. This satisfies the conditions of Lemma 2, so no combination of φ and φ' is *efficient*. □

A.4 Proof of Theorem 2

We begin with additional notation to track inheritance of control among Trading Cycles rules. We denote by μ, μ', and μ'' the initial control rights under rules φ, φ', and φ'' respectively. Given φ with initial control rights μ, we write $\mu(1 \to a)$ to indicated the updated control rights conditional on agent 1 receiving a. Similarly, we write $\mu(1 \to a, 2 \to b)$ to indicated the updated control rights conditional on agent 1 receiving a and agent 2 receiving b.

We suppose that $N = \{1, 2, 3, 4\}$, $\mathcal{O} = \{a, b, c, d\}$, and $\omega = (a, b, c, d)$. This is without loss of generality because each of our examples can be embedded in problems with additional agents and our results therefore apply directly to these settings as well.

Since $\alpha \gg 0$, the *α-endowment lower bound* requires that each agent be an owner in at least one rule (Lemma 1). We conclude that the components must include rules whose initial ownership structures match one of a small number of patterns.

Remark 4. If agents $\{1, 2, 3, 4\}$ are each owners in at least one rule in a collection, then the collection includes a subset of rules whose ownership structures restricted to these agents match one of the following nine patterns:

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)
$\mu^1(N)$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3,4\}$
$\mu^2(N)$	$\{2\}$	$\{3\}$	$\{1,3\}$	$\{1,3\}$	$\{3,4\}$	$\{4\}$	$\{1,4\}$	$\{1,2,4\}$	
$\mu^3(N)$	$\{3\}$	$\{4\}$	$\{4\}$	$\{1,4\}$					
$\mu^4(N)$	$\{4\}$								

The following series of results develop the tools to consider each of these patterns. We show that combinations of rules matching the first eight patterns are incompatible with *sd efficiency.*

Lemma 3 postulates two rules, one with three owners and a second which reassigns ownership of one object to a fourth agent.

Lemma 3. *Let φ and φ' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b, c \in \mathcal{O}$ such that $a \in \mu_i$, $b \in \mu_j$, and $c \in \mu_k \cap \mu'_l$, then no combination of φ and φ' is* efficient.

Proof. Let φ and φ' be Trading Cycles rules and $a, b, c \in \mathcal{O}$ be such that $a \in \mu_1$, $b \in \mu_2$, $c \in \mu_3 \cap \mu'_4$. Let $R^1, R^2, R^3, R^4, R^5 \in \mathcal{R}^N$ be as partially specified in the table:

R^1				R^2				R^3				R^4				R^5			
b	c	b	c	c	a	a	c	b	b	c	c	a	c	a	c	b	a	b	c
d	a	a	d	b	d	b	d	d	a	a	d	d	d	b	b	c	d	a	d
a	d	d	a	d	b	d	b	a	d	d	a	b	b	d	d	a	b	c	b

Then $\varphi(R^1) \in \{(a, c, b, d), (d, c, b, a)\}$, $\varphi(R^2) \in \{(c, b, a, d), (c, d, a, b)\}$, $\varphi(R^3) \in \{(d, b, c, a), (a, b, c, d)\}$, $\varphi(R^4) = (a, c, b, d)$, and $\varphi(R^5) = (b, a, c, d)$. Also, $\varphi'_4(R^1) = \varphi'_4(R^2) = \varphi'_4(R^3) = \varphi'_4(R^4) = \varphi'_4(R^5) = c$. We distinguish cases according to control of a after agent 4 receives c. This may be by initial control at μ' or, if the object is initially owned by agent 4, inheritance conditional on agent 4 receiving c. In the first three cases, a is owned and in the final three cases a is brokered. When a is brokered, because three agents remain, each agent controls one object and there are either three brokers or one broker and two owners. In those cases with three brokers, we consider a profile with a unique efficient allocation in which no agent receives the object he brokers and this is the allocation under φ'.

Case 1: $a \in \mu'_1(4 \to c)$. We distinguish subcases according to which agent controls b. Since $a \in \mu'_1(4 \to b)$, $b^* \notin \mu'_1(4 \to b)$.

Subcase 1.1: $b \in \mu'_1(4 \to c)$. Then $\varphi_2(R^1) = \varphi'_4(R^1) = c$ and $\varphi_3(R^1) = \varphi'_1(R^1) = b$. This satisfies the conditions of Lemma 2 applied at R^1.

Subcase 1.2: $b \in \mu'_2(4 \to c)$ or $b^* \in \mu'_2(4 \to c)$. Then $\varphi_1(R^2) = \varphi'_4(R^2) = c$ and $\varphi_3(R^2) = \varphi'_2(R^2) = a$. This satisfies the conditions of Lemma 2 applied at R^2.

Subcase 1.3: $b \in \mu'_3(4 \to c)$ or $b^* \in \mu'_3(4 \to c)$. Then $\varphi_3(R^3) = \varphi'_4(R^3) = c$ and $\varphi_2(R^3) = \varphi'_1(R^3) = b$. This satisfies the conditions of Lemma 2 applied at R^3.

Case 2: $a \in \mu'_2(4 \to c)$. Then $\varphi_1(R^2) = \varphi'_4(R^2) = c$ and $\varphi_3(R^2) = \varphi'_2(R^2) = a$. This satisfies the conditions of Lemma 2 applied at R^2.

Case 3: $a \in \mu'_3(4 \to c)$. Then $\varphi_2(R^4) = \varphi'_4(R^4) = c$ and $\varphi_1(R^4) = \varphi'_3(R^4) = a$. This satisfies the conditions of Lemma 2 applied at R^4.

Case 4: $a^* \in \mu'_1(4 \to c)$. The possibilities for $\mu'(4 \to c)$ are: (i) $(\{a^*\},\{b\},\{d\})$, (ii) $(\{a^*\},\{d\},\{b\})$, (iii) $(\{a^*\},\{b^*\},\{d^*\})$, (iv) $(\{a^*\},\{d^*\},\{b^*\})$. In cases (i) and (iv), $\varphi'(R^4) = (d,b,a,c)$. In cases (ii) and (iii), $\varphi'(R^4) = (b,d,a,c)$. In each case, the assignments of a and c satisfy the conditions of Lemma 2 applied at R^4.

Case 5: $a^* \in \mu'_2(4 \to c)$. The possibilities for $\mu'(4 \to c)$ are: (i) $(\{b\},\{a^*\},\{d\})$, (ii) $(\{d\},\{a^*\},\{b\})$, (iii) $(\{b^*\},\{a^*\},\{d^*\})$, (iv) $(\{d^*\},\{a^*\},\{b^*\})$. In case (i), $\varphi'(R^3) = (b,d,a,c)$. The assignments of b and c satisfy the conditions of Lemma 2 at R^3. In case (iv), $\varphi'(R^1) = (b,d,a,c)$. The assignments of b and c satisfy the conditions of Lemma 2 at R^1. In cases (ii) and (iii), $\varphi'(R^5) = (a,d,b,c)$. Since $c\ P_1\ a$ while $a\ P_3\ c$, no combination of φ and φ' preserves *sd efficiency*.

Case 6: $a^* \in \mu'_3(4 \to c)$. The possibilities for $\mu'(4 \to c)$ are: (i) $(\{b\},\{d\},\{a^*\})$, (ii) $(\{d\},\{b\},\{a^*\})$, (iii) $(\{b^*\},\{d^*\},\{a^*\})$, (iv) $(\{d^*\},\{b^*\},\{a^*\})$. In cases (i) and (iv), $\varphi'(R^3) = (b,a,d,c)$. The assignments of b and c satisfy the conditions of Lemma 2 at R^3. In cases (ii) and (iii), $\varphi'(R^2) = (d,a,b,c)$. The assignments of a and c satisfy the conditions of Lemma 2 applied at R^2. □

Lemma 4 and Lemma 5 consider changes in initial control structures which replace ownership with brokerage rather than ownership. Lemma 4 considers cases in which an agent who brokers an object in one rule loses control of that object in a second rule.[25] Lemma 5 considers case in which an agent who owns an object in one rule has his control rights weakened to brokerage in a second rule.

[25] Although the cases in Lemma 4 are not compatible with an *α-endowment lower bound* that is positive for all agents, we include them for completeness and to permit a simple summary statement in Corollary 2.

Lemma 4. *Let φ and φ' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b, c, d \in \mathcal{O}$ such that $a^* \in \mu_i$, $c \in \mu_k \cap \mu'_k$, and $d \in \mu_l \cap \mu'_l$ and either (i) $a^* \in \mu'_j$ and $b \in \mu_k \cap \mu'_k$ or (ii) $a \in \mu'_j$ and $b \in \mu_j \cap \mu'_j$, then no combination of φ and φ' is* efficient.

Proof. Let φ, φ', and φ'', and φ''' be Trading Cycles rules such that $\mu = (\{a^*\}, \emptyset, \{b, c\}, \{d\})$, $\mu' = (\emptyset, \{a^*\}, \{b, c\}, \{d\})$, $\mu'' = (\{a^*\}, \{b\}, \{c\}, \{d\})$, and $\mu''' = (\emptyset, \{a, b\}, \{c\}, \{d\})$. Then φ and φ' match the conditions with (i) and φ'' and φ''' match the conditions with (ii). Let $R, R' \in \mathcal{R}^N$ be as partially specified in the table:

R					R'			
b	d	a	a		c	a	a	c
c	c	d	b		d	b	d	b
		b	d		b			d

Then $\varphi(R) = (b, c, a, d)$, $\varphi'(R) = (c, d, b, a)$, $\varphi''(R') = (c, b, a, d)$, and $\varphi'''(R') = (b, a, d, c)$. Since $d\ P_3\ b$ while $b\ P_4\ d$, no combination of φ and φ' preserves *sd efficiency*. Similarly, since $d\ P'_1\ b$ while $b\ P'_4\ d$, no combination of φ'' and φ''' preserves *sd efficiency.* □

Lemma 5. *Let φ and φ' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b, c, d \in \mathcal{O}$ such that $a \in \mu_i$, $a^* \in \mu'_i$, $b \in \mu'_j$, $c \in \mu_k \cap \mu'_k$, $d \in \mu_l \cap \mu'_l$, and either (i) $b \in \mu_j$ or (ii) $b^* \in \mu_j$, then no combination of φ and φ' is* efficient.

Proof. Let φ, φ', and φ'', be such that $\mu = (\{a\}, \{b\}, \{c\}, \{d\})$, $\mu' = (\{a^*\}, \{b\}, \{c\}, \{d\})$, and $\mu'' = (\{a\}, \{b^*\}, \{c\}, \{d\})$. Then φ and φ' match the conditions with (i) and φ'' and φ' match the conditions with (ii). Let $R, R' \in \mathcal{R}^N$ be as partially specified in the table:

R					R'			
a	a	b	b		a	b	a	b
d	c	d	c		c	d	d	c
		c	d				c	d

Then $\varphi(R) = (a, c, b, d)$, $\varphi'(R) = (d, a, c, b)$, $\varphi'(R') = (c, b, a, d)$, and $\varphi''(R') = (a, d, c, b)$. Since $d\ P_3\ c$ while $c\ P_4\ d$, no combination of φ and φ' preserves *sd efficiency.* Similarly, since $d\ P'_3\ c$ while $c\ P'_4\ d$, no combination of φ' and φ'' preserves *sd efficiency.* □

As a consequence of Lemma 3, if one component rule includes four or more owners, then no additional owners may appear in other component rules, nor may other component rules reallocate ownership of the objects among these agents. Lemmas 4 and 5 extend this logic to brokers: No other component may reallocate control of an object to a different agent, nor even change one agent's control from ownership to brokerage. Because the presence of the broker with four or more agents implies that there are simultaneously at least two owners, the cases covered by these lemmas are exhaustive, yielding the formal statement in Corollary 2.

Corollary 2. *Let φ and φ' be Trading Cycles rules. If $|\mu(N)| \geq 4$ and there is $i \in N$ such that $\mu_i \neq \mu'_i$, then no combination of φ and φ' is* efficient.

Lemma 6 considers combinations of three rules which collectively assign initial ownership of two objects to four different agents.

Lemma 6. *Let φ, φ', and φ'' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b \in \mathcal{O}$ such that $a \in \mu_i$, $b \in \mu_j$, and $\{a, b\} \subseteq \mu'_k \cap \mu''_l$, then no combination of φ, φ', and φ'' is* efficient*.*

Proof. Let φ, φ', and φ'' be Trading Cycles rules and $a, b \in \mathcal{O}$ be such that $a \in \mu_1$, $b \in \mu_2$, $\{a, b\} \subseteq \mu'_3 \cap \mu''_4$. Let $R \in \mathcal{R}^N$ be as partially specified in the table:

R			
a	b	a	b
$\{c, d\}$	$\{c, d\}$		
b	a		

Then $\varphi_1(R) = \varphi'_3(R) = a$ and $\varphi_2(R) = \varphi''_4(R) = b$. By *ex-post efficiency* of Trading Cycles rules, $\varphi'_1(R) \neq b$ and $\varphi''_2(R) \neq a$. If $\varphi'_4(R) = b$, then the conditions of Lemma 2 apply at R with respect to φ and φ'. If $\varphi''_3(R) = a$, then the conditions of Lemma 2 apply at R with respect to φ and φ''. If $\varphi'_4(R) \neq b$ and $\varphi''_3(R) \neq a$, then $\varphi'_2(R) = b$ and $\varphi''_1(R) = a$. The conditions of Lemma 2 now apply at R with respect to φ' and φ''. Instead, no combination of φ, φ', and φ'' preserves *sd efficiency*. □

Lemma 6 postulates two agents who are simultaneously owners in one of the rules, but this is not necessary. Rather than be an initial owner, it suffices that this agent inherit one of the objects, perhaps only in the narrow case where the initial owner receives the other object. Corollary 3 formalizes this observation.

Corollary 3. *Let φ, φ', and φ'' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b \in \mathcal{O}$ such that $\{a, b\} \subseteq \mu_i \cap \mu'_k \cap \mu''_l$ and $b \in \mu_j(i \rightarrow a)$, then no combination of φ, φ', and φ'' is* efficient*.*

Corollary 3 carries implications for combinations of rules which initially endow a single agent with all of the objects. In particular, if four rules each initially endow a different agent with all of the objects, then combinations of these rules will not preserve *sd efficiency*. The reason is that the objects are inherited: In the first rule, after the first agent receives an object, at some point one of the other special agents will inherit an object. This agent plays the role of agent j and the remaining special agents play the roles of agents k and l with the inherited object and the object received by the first agent playing the roles of b and a. Corollary 4 reiterates this conclusion.

Corollary 4. *Let φ, φ', φ'', and φ''' be Trading Cycles rules. If $\mu_i = \mu'_j = \mu''_k = \mu'''_l = \mathcal{O}$, then no combination of φ, φ', φ'', and φ''' is* efficient*.*

Lemma 7 considers additional permutations of initial ownerships which collectively assign two objects to each of four distinct agents.

Lemma 7. *Let φ, φ', and φ'' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b \in \mathcal{O}$ such that $\{a, b\} \subseteq (\mu_i \cup \mu_j) \cap (\mu'_i \cup \mu'_k) \cap \mu''_l$ with $\mu_i \cap \{a, b\}, \mu_j \cap \{a, b\}, \mu'_i \cap \{a, b\}, \mu'_k \cap \{a, b\} \neq \emptyset$, then no combination of φ, φ', and φ'' is* efficient*.*

Proof. Let φ, φ', and φ'' be Trading Cycles rules and $a, b \in \mathcal{O}$ be such that $a \in \mu_1$, $b \in \mu_2$, $\{a, b\} \subseteq \mu''_4$, and either (i) $a \in \mu'_1$ and $b \in \mu'_3$ or (ii) $b \in \mu'_1$ and $a \in \mu'_3$. Let $R \in \mathcal{R}^N$ be as partially specified in the table:

R			
b	b	a	a
$\{c,d\}$		$\{c,d\}$	
a		b	

Then $\varphi_2(R) = \varphi'_1(R) = b$ and $\varphi'_3(R) = \varphi''_4(R) = a$. By *ex-post efficiency* of Trading Cycles rules, $\varphi_1(R) \neq a$ and $\varphi''_3(R) \neq b$. If $\varphi_4(R) = a$, then the conditions of Lemma 2 apply at R with respect to φ and φ'. If $\varphi''_2(R) = b$, then the conditions of Lemma 2 apply at R with respect to φ' and φ''. If $\varphi_4(R) \neq a$ and $\varphi''_2(R) \neq b$, then $\varphi_3(R) = a$ and $\varphi''_1(R) = b$. The conditions of Lemma 2 now apply at R with respect to φ and φ''. Instead, no combination of φ, φ', and φ'' preserves *sd efficiency*. □

Lemma 7 carries implications for combinations which include a rule with a single initial owner. In particular, if one rule initially endows an agent with all of the objects and two additional rules together include three new agents as owners, then no combination of these rules preserves *sd efficiency*. Corollary 5 states this formally.

Corollary 5. *Let φ, φ', and φ'', be Trading Cycles rules. If $\mu_i \cup \mu_j = \mu'_i \cup \mu'_k = \mu''_l = \mathcal{O}$ and $\mu_i, \mu_j, \mu'_i, \mu'_k \neq \emptyset$, then no combination of φ, φ', and φ'' is* efficient.

Proof. We show that these conditions imply that a pair of objects satisfy Lemma 7. Let φ, φ', and φ'' be Trading Cycles rules such that $\mu_i \cup \mu_j = \mu'_i \cup \mu'_k = \mu''_l = \mathcal{O}$ and $\mu_i, \mu_j, \mu'_i, \mu'_k \neq \emptyset$. Let $a, b \in \mathcal{O}$ be such that $a \in \mu_i$ and $b \in \mu_j$. If (i) $a \in \mu'_i$ and $b \in \mu'_k$ or (ii) $b \in \mu'_i$ and $a \in \mu'_k$, then $\{a, b\}$ satisfies the conditions of Lemma 7. Otherwise, either (iii) $\{a, b\} \subseteq \mu'_i$ or (iv) $\{a, b\} \subseteq \mu'_k$. Suppose (iii) and let $c \in \mu'_k$. If $c\mu_i$, then $\{b, c\}$ satisfies the conditions of Lemma 7. Otherwise, $c\mu_j$ and $\{a, c\}$ satisfies the conditions of Lemma 7. Finally suppose (iv) and let $c \in \mu'_i$. If $c\mu_i$, then $\{b, c\}$ satisfies the conditions of Lemma 7. Otherwise, $c\mu_j$ and $\{a, c\}$ satisfies the conditions of Lemma 7. □

Lemma 8 considers a further possibility for shared ownership of two objects by four agents. This time, one agent owns one of the objects in each of three rules and each remaining agent owns the other object in one rule.

Lemma 8. *Let φ, φ', and φ'' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b \in \mathcal{O}$ such that $\{a, b\} \subseteq (\mu_i \cup \mu_j) \cap (\mu'_i \cup \mu'_k) \cap (\mu''_i \cup \mu''_l)$ with $\mu_i \cap \{a, b\}, \mu_j \cap \{a, b\}, \mu'_i \cap \{a, b\}, \mu'_k \cap \{a, b\}, \mu''_i \cap \{a, b\}, \mu''_l \cap \{a, b\} \neq \emptyset$, then no combination of φ, φ', and φ'' is* efficient.

Proof. Let φ, φ', and φ'' be Trading Cycles rules and $a, b \in \mathcal{O}$ be such that $a \in \mu_1 \cap \mu'_1$, $b \in \mu_2 \cap \mu'_3$, and either (i) $a \in \mu''_1$ and $b \in \mu''_4$ or (ii) $b \in \mu''_1$ and $a \in \mu''_4$.[26] Let $R \in \mathcal{R}^N$ be as partially specified in the table:

R			
b	b	a	a
$\{c,d\}$			
a			

[26] The conditions of the lemma imply that agent 1 owns one of the objects in at least two of the rules. After possibly relabeling the agents and objects, these cases are exhaustive.

Then $\varphi_2(R) = \varphi'_1(R) = \varphi''_1(R) = b$ and $\varphi'_3(R) = \varphi''_4(R) = a$. By *ex-post efficiency* of Trading Cycles rules, $\varphi_1(R) \neq a$. If $\varphi_3(R) = a$, then the conditions of Lemma 2 apply at R with respect to φ and φ''. If $\varphi_4(R) = a$, then the conditions of Lemma 2 apply at R with respect to φ and φ'. Instead, no combination of φ, φ', and φ'' preserves *sd efficiency*. □

Lemma 9 considers a final way for four agents to each be owners among three rules which involves three objects. One agent owns two of the objects in each of three rules and each remaining agent owns a different one of the three objects in one rule.

Lemma 9. *Let φ, φ', and φ'' be Trading Cycles rules. If there are $i, j, k, l \in N$ and $a, b, c \in \mathcal{O}$ such that $a \in \mu_i \cap \mu'_k \cap \mu''_i$, $b \in \mu_j \cap \mu'_i \cap \mu''_i$, and $c \in \mu_i \cap \mu'_i \cap \mu''_l$, then no combination of φ, φ', and φ'' is* efficient.

Proof. Let φ, φ', and φ'' be Trading Cycles rules and $a, b, c \in \mathcal{O}$ be such that $a \in \mu_1 \cap \mu'_3 \cap \mu''_1$, $b \in \mu_2 \cap \mu'_1 \cap \mu''_1$, and $c \in \mu_1 \cap \mu'_1 \cap \mu''_4$. We trace implications of *sd efficiency* for inheritance structures for $\{a, b, c\}$ (Step 1) and d (Step 2). Let $R^0, \ldots, R^6 \in \mathcal{R}^N$ be as partially specified in the table:

R^0			
c	a	a	c
b		b	b
d			

R^1			
b	b	d	a
a	d	a	d
		c	c

R^2			
b	b	d	d
a	c	c	a
		a	c

R^3			
a	d	a	d
b	b	c	c
	c	b	

R^4			
c	d	d	c
a	b	a	b
	a	b	a

R^5			
c	d	d	c
b	b	a	a
	a	b	b

R^6			
a	d	a	d
c			

Step 1: $a \in \mu_3(1 \to c) \cap \mu''_3(1 \to b)$, $b \in \mu'_2(1 \to c) \cap \mu''_2(1 \to a)$, and $c \in \mu_4(1 \to a) \cap \mu'_4(1 \to b)$. We argue that no other cases are compatible with *sd efficiency*. Consider first ownership of a in $\mu(1 \to c)$.
At R^0, $\varphi_1(R^0) = \varphi'_1(R^0) = \varphi''_4(R^0) = c$ and $\varphi'_3(R^0) = a$. By *ex-post efficiency* of Trading Cycles rules, $\varphi''_1(R^0) \neq a$. If $\varphi''_2(R^0) = a$, then the conditions of Lemma 2 apply at R^0 with respect to φ' and φ''. Instead, $\varphi''_3(R^0) = a$. If $a \in \mu_2(1 \to c) \cup \mu_4(1 \to c)$ or $a^* \in \mu_3(1 \to c) \cup \mu_4(1 \to c)$, then $\varphi_2(R^0) = a$ and the conditions of Lemma 2 apply at R^0 with respect to φ and φ''. Also, $b \in \mu_2$, so $a^* \notin \mu_2(1 \to c)$. Instead, $a \in \mu_3(1 \to c)$.

By relabeling the agents and objects, symmetric arguments show that $a \in \mu''_3(1 \to b)$, $b \in \mu'_2(1 \to c) \cap \mu''_2(1 \to a)$, and $c \in \mu_4(1 \to a) \cap \mu'_4(1 \to b)$.

Step 2: $d \in \mu_2(1 \to a) \cap \mu_2(1 \to c) \cap \mu'_3(1 \to b) \cap \mu'_3(1 \to c) \cap \mu''_4(1 \to a) \cap \mu''_4(1 \to b)$. We argue that no other cases are compatible with *sd efficiency*. Consider first $\mu'(1 \to b)$.

Case 2.1: $d \in \mu'_2(1 \to b) \cup \mu''_2(1 \to b) \cup \mu_3(1 \to a \cup \mu''_3(1 \to a) \cup \mu_4(1 \to c) \cup \mu'_4(1 \to c)$.

Suppose that $d \in \mu'_2(1 \to b)$. Then $\varphi'(R^1) = (b, d, a, c)$ and $\varphi'(R^2) = (b, c, a, d)$. Consider φ. If $d \in \mu_4(1 \to a)$, then $\varphi(R^1) = (a, b, c, d)$. Since $d \; P^1_3 \; a$ while $a \; P^1_4 \; d$, no combination of φ and φ' preserves *efficiency*. If $d \in \mu_3(1 \to a)$, then $\varphi(R^2) = (a, b, d, c)$. Since $c \; P^2_3 \; a$ while $a \; P^2_4 \; c$, no combination of φ and φ' preserves *efficiency*. Suppose instead that $d \in \mu_2(1 \to a)$ and consider $\mu(1 \to a, 2 \to b)$. If $d \in \mu_4(1 \to a, 2 \to b)$, then $\varphi(R^1) = (a, b, c, d)$. If

$d \in \mu_3(1 \to a, 2 \to b)$, then $\varphi(R^2) = (a, b, d, c)$. Again, no combination of φ and φ' preserves *efficiency*. Instead, $d \notin \mu'_2(1 \to b)$.

By relabeling the agents and objects, symmetric arguments show that $d \notin \mu''_2(1 \to b) \cup \mu_3(1 \to a) \cup \mu''_3(1 \to a) \cup \mu_4(1 \to c) \cup \mu'_4(1 \to c)$.

Case 2.2: $d \in \mu'_4(1 \to b) \cup \mu''_3(1 \to b) \cup \mu_4(1 \to a) \cup \mu''_2(1 \to a) \cup \mu_3(1 \to c) \cup \mu'_2(1 \to c)$. Suppose that $d \in \mu'_4(1 \to b)$. Then $\varphi'(R^3) = (b, c, a, d)$. Consider φ''. By Case 2.1, $d \notin \mu''_3(1 \to a)$. If $d \in \mu''_2(1 \to a)$, then $\varphi''(R^3) = (a, d, b, c)$. Since $b\ P^3_2\ c$ while $c\ P^3_3\ b$, no combination of φ' and φ'' preserves *sd efficiency*. Finally suppose that $d \in \mu''_4(1 \to a)$. By Case 2.1, $d \notin \mu''_2(1 \to b)$. By the same arguments, $d \notin \mu''_3(1 \to a, 4 \to c)$ and $d \notin \mu''_2(1 \to b, 4 \to c)$. Instead, $d \in \mu''_2(1 \to a, 4 \to c)$ and $d \in \mu''_3(1 \to b, 4 \to c)$. Then $\varphi''(R^4) = (a, d, b, c)$ and $\varphi''(R^5) = (b, a, d, c)$.

Now consider φ'. By Case 2.1, $d \notin \mu'_4(1 \to c)$. If $d \in \mu'_2(1 \to c)$, then $\varphi'(R^5) = (c, d, a, b)$. Since $b\ P^5_2\ a$ while $a\ P^5_4\ b$, no combination of φ and φ' preserves *sd efficiency*. If $d \in \mu'_3(1 \to c)$, then $\varphi'(R^4) = (c, b, d, a)$. Since $a\ P^5_3\ b$ while $b\ P^5_4\ a$, no combination of φ and φ' preserves *sd efficiency*. Instead, $d \notin \mu'_4(1 \to b)$.

By relabeling the agents and objects, symmetric arguments show that $d \notin \mu''_3(1 \to b) \cup \mu_4(1 \to a) \cup \mu''_2(1 \to a) \cup \mu_3(1 \to c) \cup \mu'_2(1 \to c)$.

Case 2.3: d is brokered in $\mu'(1 \to b)$, $\mu''(1 \to b)$, $\mu(1 \to a)$, $\mu''(1 \to a)$, $\mu(1 \to c)$, or $\mu'(1 \to c)$. Suppose that d is brokered in $\mu'(1 \to b)$. By Step 1, $c \in \mu'_4(1 \to b)$. Also, $a \in \mu'_3$, so $d^* \notin \mu'_3(1 \to b) \cup \mu'_4(1 \to b)$. If $d^* \in \mu'_2(1 \to b)$, then $\varphi'(R^3) = (b, c, a, d)$ and subsequent arguments from Case 2.2 apply. Instead, d is not brokered in $\mu'(1 \to b)$.

By relabeling the agents and objects, symmetric arguments show that d is not brokered in $\mu''(1 \to b)$, $\mu(1 \to a)$, $d\mu''(1 \to a)$, $\mu(1 \to c)$, or $\mu'(1 \to c)$. Altogether, $d \in \mu_2(1 \to a) \cap \mu_2(1 \to c) \cap \mu'_3(1 \to b) \cap \mu'_3(1 \to c) \cap \mu''_4(1 \to a) \cap \mu''_4(1 \to b)$.

Step 3: No combination preserves *efficiency*. Consider φ' and inheritance of d in $\mu'_2(1 \to c, 3 \to a)$. If $d \in \mu'_2(1 \to c, 3 \to a)$, then $\varphi'(R^6) = (c, d, a, b)$. But $\varphi''_1(R^6) = a$ and $\varphi''_4(R^6) = d$, so the conditions of Lemma 2 apply at R^6 with respect to φ' and φ''. If $d \in \mu'_4(1 \to c, 3 \to a)$, then $\varphi'(R^6) = (c, b, a, d)$. But $\varphi_1(R^6) = a$ and $\varphi_2(R^6) = d$, so the conditions of Lemma 2 apply at R^6 with respect to φ' and φ. Instead, no combination of φ, φ', and φ'' preserves *sd efficiency*. □

The cases considered by Lemma 7, Lemma 6, Lemma 8, and 9 exhaust the possible arrangements of initial ownership for collections of rules, each with two owners, across which one special agent appears as one of the initial owners. Corollary 6 presents this implication.

Corollary 6. *Let φ, φ', and φ'', be Trading Cycles rules. If $\mu_i \cup \mu_j = \mu'_i \cup \mu'_k = \mu''_i \cup \mu''_l = \mathcal{O}$ with $\mu_i, \mu_j, \mu'_i, \mu'_k, \mu''_i, \mu''_l \neq \emptyset$, then no combination of φ, φ', and φ'' is* efficient.

Proof. Let φ, φ', and φ'' be Trading Cycles rules and $a, b, c \in \mathcal{O}$ be such that $\mu_1 \cup \mu_2 = \mu'_1 \cup \mu'_3 = \mu''_1 \cup \mu''_4 = \mathcal{O}$ with $\mu_1, \mu_2, \mu'_1, \mu'_3, \mu''_1, \mu''_4 \neq \emptyset$. Label the objects so that $a \in \mu_1$, $b \in \mu_2$, and $\{a, b, c\} \cap \mu'_3 \cap \mu''_4 \neq \emptyset$.[27]

[27] To see that this labeling is always possible, begin with a labeling such that $a \in \mu_1$ and $b \in \mu_2$ and $\mu'_3 \cap \{a, b, c\} \neq \emptyset$. If $\mu''_4 \cap \{a, b, c\} = \emptyset$, then $\mu''_4 = \{d\}$. If $\{a, b\} \cap \mu'_3 \neq \emptyset$, relabel c and d. If instead $\{a, b\} \cap \mu'_3 = \emptyset$, then $\mu'_3 \cap \{c, d\} \neq \emptyset$. In this case, relabel (b, c, d) as (c, a, b) and reverse the names of agent 2 and agent 4.

Consider ownership of a and b in μ' and μ''. If $\mu'_1 \cap \{a,b\} = \emptyset$ and $\mu''_1 \cap \{a,b\} = \emptyset$, then the setting corresponds to Lemma 6. If exactly one of these conditions holds, then the setting corresponds to Lemma 7. Instead, suppose that $\mu'_1 \cap \{a,b\} \neq \emptyset$ and $\mu''_1 \cap \{a,b\} \neq \emptyset$.

If $\mu'_3 \cap \{a,b\} \neq \emptyset$ and $\mu''_4 \cap \{a,b\} \neq \emptyset$, then the setting corresponds to Lemma 8 with the conditions met by a and b. If $\mu'_3 \cap \{a,b\} = \emptyset$ and $\mu''_4 \cap \{a,b\} = \emptyset$, then the setting corresponds to Lemma 8: If $c \in \mu_1$, then the conditions met by b and c, and if $c \in \mu_2$, then the conditions met by a and c. Instead, suppose that exactly one of these conditions holds. Relabeling agent 3 and agent 4 if necessary, suppose that $\mu'_3 \cap \{a,b\} \neq \emptyset$ and $\mu''_4 \cap \{a,b\} = \emptyset$. Then $\{a,b\} \subseteq \mu''_1$ and $c \in \mu''_4$.

Consider ownership of b and c in μ and μ'. By labeling, $b \in \mu_2$. Suppose also that $c \in \mu_2$. If $\mu'_3 \cap \{b,c\} = \{b,c\}$, then the setting corresponds to Lemma 6 with the conditions met by b and c. If $\mu'_3 \cap \{b,c\} = \{b\}$ or $\mu'_3 \cap \{b,c\} = \{c\}$, then the setting corresponds to Lemma 7 with the conditions met by b and c. If $\mu'_3 \cap \{b,c\} = \emptyset$, then $a \in \mu'_3$ and the setting corresponds to Lemma 8 with the conditions met by a and c.

Finally suppose that $c \notin \mu_2$ so that $c \in \mu_1$. If $\mu'_3 \cap \{b,c\} = \{b,c\}$, then the setting corresponds to Lemma 7 with the conditions are met by b and c. If $\mu'_3 \cap \{b,c\} = \{b\}$ or $\mu'_3 \cap \{b,c\} = \{c\}$, then the setting corresponds to Lemma 8 with the conditions are met by b and c. If $\mu'_3 \cap \{b,c\} = \emptyset$, then $a \in \mu'_3$ and the setting corresponds to Lemma 9 with the conditions are met by a, b, and c.

Altogether, all combinations of initial ownership which meet the conditions of the statement match the conditions of (at least) one of Lemma 7, Lemma 6, Lemma 8, or Lemma 9. □

With all preliminaries in hand, we conclude the proof of Theorem 2 by considering each of the patterns described in Remark 4 in turn.

Proof. Let φ be a Random Trading Cycles rule and satisfy the *α-endowment lower bound* for some $\alpha \gg 0$. Let $|N| = 4$ By Lemma 1, $\alpha \gg 0$ implies that each agent is an owner in at least one component. Therefore, it suffices to consider collections of rules following one of the patterns of initial ownership structures described in Remark 4. If a collection of rules includes a subset following one of patterns (i) through (viii), then its combinations are incompatible with *sd efficiency*: Corollary 4 applies to pattern (i); Lemma 6 to pattern (ii); Corollary 5 to pattern (iii); Corollary 6 to pattern (iv); Corollary 1 to pattern (v); and Lemma 3 to patterns (vi), (vii), and (viii).

It remains to consider collections of rules based on pattern (ix). Each such collection includes a rule with four owners, so by Corollary 2, *sd efficiency* implies that the components share a common initial control structure. The *α-endowment lower bound* now implies that each agent in fact owns his endowment, so the common initial control structure is ω. This initial control structure defines a unique rule: TTC. Therefore, all components are themselves TTC and φ =TTC as well.

We now consider economies consisting of more than four agents. Let $M \supsetneq N$ and $\mathcal{Q} \supsetneq \mathcal{O}$ be the agents and objects of a larger economy. For each $i \in N$, let R_i be such that for each $x \in \mathcal{O}$, and each $y \in \mathcal{Q} \backslash \mathcal{O}$, $x \; P_i \; y$. Similarly, for each $i \in M \backslash N$, let R_i be such that for each $x \in \mathcal{O}$, and each $y \in \mathcal{Q} \backslash \mathcal{O}$, $y \; P_i \; x$. By *efficiency*, agent in N receive objects $\mathcal{O}$ and agents in $M \backslash N$ receive objects $\mathcal{Q} \backslash \mathcal{O}$. Moreover, the allocation on objects $\mathcal{O}$ among N is independent of the preferences $M\backslash N$ and allocation of objects $\mathcal{Q}\backslash\mathcal{O}$. Therefore, the arguments for each of the previous lemmas apply. Thus, for each $i \in N$, each component rule of φ gives initial ownership of ω_i to i in the economy $(M, \mathcal{Q})$. Repeating this argument, we conclude that all components are themselves TTC and $\varphi = TTC$. □

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