ON THE BEHAVIOR OF NULL RAYS IN STATIONARY SPHERICALLY SYMMETRIC SPACETIMES

By

ADAM MICHAEL BAUER

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Dr. Samuel Gralla

Contents

Abstract

In this thesis, we investigate the various properties of null rays in the Schwarzschild and Reissner-Nördstrom geometries. We first provide the general construction of the Penrose limit metric in an arbitrary spacetime. We then show that the Penrose limit metric for a general ingoing geodesic in the Schwarzschild geometry is flat. We further show a mathematical construction which extends null rays with $L = E = 0$ through the Schwarzschild singularity at $r = 0$ via a "wormhole" structure. Also presented is the Penrose limit metric of the Schwarzschild event horizon, which is shown to be flat. We then use perturbation theory to derive a rate at which event horizon null generators escape/fall into the black hole in the Schwarzschild, non-extremal Reissner-Nördstrom and extremal Reissner-Nördstrom spacetimes.

1 Introduction

The study of null rays in general relativity has always been a primary focus of theoretical physicists, and has led to numerous interesting discoveries. Such an examples of this include the discovery of the Kerr metric and further analysis of its stability, as well as gravitational lensing [9]. Light has also been shown to generate null hypersurfaces (also known as "horizons," e.g., the event horizon), which play a major role in characterizing the spacetime in which they reside, especially with respect to the stability of black holes. This is especially true as of late, as in 2011 Aretakis showed that *all extremal black holes are unstable to gravitational perturbation along their event horizon* [1, 2]. In doing so, Aretakis showed that scalar fields blow up along the event horizon of an extremal Reissner-Nördstrom black hole. While Aretakis first showed this "Aretakis instability" by considering scalar fields, recent work has implied

that this same phenomenon can be captured by analyzing null ray behaviour near the horizon, see [6] for example in the extremal BTZ spacetime.

It is in the spirit of this past work, particularly in regards to the stability of black hole horizons, that we present the contents of this thesis. We use both a geometric method (Penrose limit metric) and a dynamical method (perturbation theory) to probe the stability of stationary spherically symmetric spacetimes, namely the Schwarzschild and Reissner-Nördstrom spacetimes. Penrose limit metrics have been used in the past to analyze spacetime singularities [4], and we continue work in this vein herein. We also provide a mathematical construction involving the Penrose limit metric with respect to a special null ray (referred to as a principle null ray), and show that it can traverse the singularity in the Schwarzschild spacetime by virtue of a smooth connection procedure. We also further elucidate the findings of Aretakis through our analysis using perturbation theory, finding exponential (algebraic, resp.) growth of radial perturbations off the horizon in the Schwarzschild and non-extremal Reissner-Nördstrom spacetimes (extremal Reissner-Nördstrom spacetime, resp.).

The remainder of this thesis is structured as follows. In §2, we present a brief introduction to pp-wave (and plane wave more generally) spacetimes. In §3, we construct the Penrose limit metric. We then use this Penrose limit metric in §4 to show that the ingoing null ray with $L = E = 0$ can be extended through the Schwarzschild singularity at $r = 0$ via a smooth connection. We also provide the Penrose limit of the Schwarzschild event horizon in this section. In §5, we perturb the event horizon generators in the Schwarzschild, non-extremal and extremal Reissner-Nördstrom spacetimes. Concluding remarks are made in §6, and the formal technicalities of the construction in §3 are carried out in detail in Appendix A. Throughout, we use the conventional units of $c = G = 1$; Greek indices run from 0 to 3, while Latin indices run from 1 to 3.

2 A Brief Introduction to pp-wave Spacetimes

A major breakthrough in gravitational physics occurred when it was shown that, to linear order, a perturbed Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

must necessarily satisfy a scalar wave equation [9]. This finding is the fundamental result behind the recent explosion of gravitational wave science, expecially through collaborations such as LIGO and Virgo [5]. One also finds that these waves must be linearly polarized [3]. A typical solution to Einstein's equations in such a case is

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + (\delta_{ij} + h_{ij}(U))\,\mathrm{d}x^i\,\mathrm{d}x^j;$$

we can also drop the requirement that $h_{ij}(U)$ is small and simply write

$$\mathrm{d}\bar{s}^2 = 2\,\mathrm{d}U\,\mathrm{d}V + \bar{g}_{ij}(U)\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{2.1}$$

Metrics that take this form are said to be in *Rosen coordinates* (these coordinates are not often used, as they can have spurious coordinates singularities [3]). We will denote any such "wave" metrics herein by $\mathrm{d}\bar{s}^2$.

Note that (2.1) is not the only solution to Einstein's equations that satisfies a wave equation. Indeed, there are many other solutions, one of the most famous being *plane-fronted waves with parallel rays*, or *pp-waves* for short. These metrics take the form of

$$\mathrm{d}\bar{s}^2 = 2\,\mathrm{d}u\,\mathrm{d}v + K(u,x^j)\,\mathrm{d}u^2 + 2A_i(u,x^j)\,\mathrm{d}u\,\mathrm{d}x^i + \mathrm{d}x_i\,\mathrm{d}x^i. \tag{2.2}$$

"Plane-fronted" refers to the fact that the wave fronts $u = \textit{constant}$ are planar (flat) and "parallel rays" refers to the existence of a parallel null vector field [3]. It is easy to see that plane wave spacetimes are a special case of pp-wave spacetimes; indeed, a plane wave spacetime is a pp-wave spacetime with $A_i = 0$ and $K(u,x^j)$ quadratic in x^a. One common coordinate system to analyze pp-wave spacetimes are *Brinkmann coordinates*; these metrics have the form of

$$\mathrm{d}\bar{s}^2 = 2\,\mathrm{d}u\,\mathrm{d}v + A_{ab}(u)x^a x^b\,\mathrm{d}u^2 + \mathrm{d}x_i\,\mathrm{d}x^i. \tag{2.3}$$

In [3], they show that one pragmatic strength of the Penrose limit is that it provides a straightforward coordinate transformation between Rosen and Brinkmann coordinates.

One way to obtain a plane wave metric from a standard metric (i.e., not a solution to a scalar wave equation) is through a Penrose limit. The Penrose limit metric is a plane-wave metric that results from taking the large volume limit of an asymmetrically scaled set of "adapted coordinates." Penrose describes the physics of a Penrose limit as the following [8]:

> *We envisage a succession of observers traveling in the space-time $\mathcal{M}$ whose world lines approach the null geodesic γ more and more closely; so we picture these observers as traveling with greater and greater speeds, approaching that of light. As their speeds increase they must correspondingly recalibrate their clocks to run faster and faster (assuming that all space-time measurements are referred to clock measurements in the standard way), so that in the limit the clocks measure the affine parameter x_0 along γ. (Without clock recalibration a degenerate space-time metric would result). In the limit the observers measure the space-time to have the plane wave structure W_γ.*

We now turn our attention to the construction of the Penrose limit metric.

3 Construction of Penrose Limit Metric

Consider metrics of the form

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + a(U,V,Y^k)\,\mathrm{d}V^2 + 2b_i(U,V,Y^k)\,\mathrm{d}V\,\mathrm{d}Y^i + g_{ij}(U,V,Y^k)\,\mathrm{d}Y^i\,\mathrm{d}Y^j. \tag{3.1}$$

One immediately notices that this class of metrics are characterized by $g_{UU} = 0 = g_{Ui} = 0$ and $g_{UV} = 1$, and it has been shown that any metric can take this form in an arbitrary open neighborhood by Penrose [8]. Penrose also claims that for U an affine parameter and $V = Y^k = 0$, the metric is in "adapted coordinates" around a particular null geodesic, which we prove explicitly in Appendix A. We will refer to such a coordinate system as *Penrose coordinates* for the remainder of this thesis, denote metrics in this form about a null geodesic γ as $\mathrm{d}s^2_\gamma$.

We can construct the Penrose limit metric of any metric in the form of (3.1) using the following prescription. First, we apply an asymmetric scaling to the coordinates

$$(U, V, Y^k) \to (U, \lambda^2 V, \lambda Y^k), \tag{3.2}$$

for $\lambda \in \mathbb{R}$ and $k = 1, 2$. This puts the metric (3.1) in the form

$$\mathrm{d}s_\gamma^2 = 2\lambda^2\,\mathrm{d}U\,\mathrm{d}V + \lambda^4 a(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}V^2 + 2\lambda^3 b_i(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}V\,\mathrm{d}Y^i + \lambda^2 g_{ij}(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}Y^i\,\mathrm{d}Y^j. \tag{3.3}$$

We then apply an overall λ^{-2} scaling to the metric and take $\lambda \to 0$ (formally referred to as the "large volume limit") to obtain the general Penrose limit metric, i.e.,

$$\begin{aligned}
\mathrm{d}\bar{s}_\gamma^2 &= \lim_{\lambda\to 0} \lambda^{-2}\,\mathrm{d}s_\gamma^2 \\
&= \lim_{\lambda\to 0} \left(2\,\mathrm{d}U\,\mathrm{d}V + \lambda^2 a(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}V^2 + 2\lambda b_i(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}V\,\mathrm{d}Y^i + g_{ij}(U, \lambda^2 V, \lambda Y^k)\,\mathrm{d}Y^i\,\mathrm{d}Y^j\right) \\
&= 2\,\mathrm{d}U\,\mathrm{d}V + g_{ij}(U, 0, 0)\,\mathrm{d}Y^i\,\mathrm{d}Y^j.
\end{aligned} \tag{3.4}$$

This completes the construction of the Penrose limit metric. One notices that this metric is indeed a plane-wave metric in Rosen coordinates by comparison to (2.1). Geometrically speaking, one can think of the recovered metric as an infinitesimally small neighborhood of a geodesic that has been blown up to cover the entire spacetime.

4 Penrose Limits in the Schwarzschild Geometry

In this section, we derive the Penrose limit metric of a generic ingoing null ray in the Schwarzschild geometry, as well as construct the Penrose limit about the event horizon in the same spacetime. Special attention is paid to the principal ingoing null ray, where we show the mathematical consistency of the ray traversing the singularity at $r = 0$, revealing a "wormhole" structure in the Schwarzschild spacetime.

4.1 Principal Ingoing Null Ray

We begin by writing the Schwarzschild spacetime in ingoing Eddington-Finkelstein coordinates,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta + \vartheta_0)\,\mathrm{d}\varphi^2\right), \tag{4.1}$$

where we have noted that shifting ϑ by an arbitrary constant, ϑ_0, leaves the metric invariant. Note that $r > 0$, $-\infty < v < \infty$, $0 \le \vartheta \le \pi$, and $0 \le \varphi \le 2\pi$. One can show by direct comparison that (4.1) has the same form as the metric in (3.1), and therefore it stands to reason that the Penrose limit can be taken directly. By noting that there is no $\mathrm{d}r^2$ term in (4.1) (and that $\Gamma^r_{rr} = 0$ in this spacetime [7]), this metric must be in adapted coordinates for the principal geodesics that are affinely parameterized by r and satisfy $v = \vartheta = \varphi = 0$.

Before we consider a principal ingoing null geodesic for which this is true and proceed in taking the limit, one subtlety must be discussed. Above we stated that we will consider geodesics for which

$v = \vartheta = \varphi = 0$ is satisfied. However, this restricts us to *a unique* principal null geodesic in the spacetime, namely at $v = 0$. This restriction is immediately lifted by exploiting the isometry $v \to v + \bar{v}$, for a constant $\bar{v}$; this transformation results in the same metric (no metric components are dependent on v), and allows us to arbitrarily choose a null ray around which to take the Penrose limit. Thus, these adapted coordinates really can describe *any* principal ingoing null geodesic in the spacetime, not just the spacetime for which $v = 0$.

Now we consider a principal ingoing null geodesic, γ, for which $v = \vartheta = \varphi = 0$ is satisfied, and is affinely parameterized by r. We begin as formulated in §3; we apply the asymmetric scaling to the coordinates,

$$r = r, \qquad v = \lambda^2 v', \qquad \vartheta = \lambda \vartheta', \qquad \varphi = \lambda \varphi', \tag{4.2}$$

which gives

$$\mathrm{d}s^2{}_{\gamma,\lambda} = -\lambda^4 \left(1 - \frac{2M}{r}\right) \mathrm{d}v'^2 + 2\lambda^2 \,\mathrm{d}v'\,\mathrm{d}r + r^2\lambda^2 \left(\mathrm{d}\vartheta'^2 + \sin^2(\lambda\vartheta' + \vartheta_0)\,\mathrm{d}\varphi'^2\right). \tag{4.3}$$

We can now take the large volume limit to obtain the Penrose limit metric,

$$\begin{aligned} \mathrm{d}\bar{s}^2_\gamma &= \lim_{\lambda\to 0} \lambda^{-2} \left(-\lambda^4 \left(1 - \frac{2M}{r}\right) \mathrm{d}v'^2 + 2\lambda^2 \,\mathrm{d}v'\,\mathrm{d}r + r^2\lambda^2 \left(\mathrm{d}\vartheta'^2 + \sin^2(\lambda\vartheta' + \vartheta_0)\,\mathrm{d}\varphi'^2\right)\right) \\ &= 2dv'dr + r^2 \left(d\vartheta'^2 + \sin^2(\vartheta_0) d\varphi'^2\right). \end{aligned} \tag{4.4}$$

We can determine the ranges of the newly found coordinates v', ϑ' and φ' by using (4.2), which leads to coordinate ranges $-\infty < v' < \infty$, and $\vartheta', \varphi' \in [0, \infty)$. However, note that (4.4) is invariant under the parity transformation $\vartheta' \to -\vartheta'$ and $\varphi' \to -\varphi'$. Therefore, we can immediately extend the coordinate ranges of ϑ' and φ' to $-\infty < \vartheta', \varphi' < \infty$.

Under the coordinate transformation,

$$r = r, \qquad v' = \tilde{v} + \frac{\alpha^2 + \beta^2}{2r}, \qquad \vartheta' = \frac{\alpha}{r}, \qquad \varphi' = \frac{\beta}{r}, \tag{4.5}$$

(4.4) is put in double-null flat form,

$$\mathrm{d}\bar{s}^2_{\gamma,\mathrm{in}} = 2\,\mathrm{d}\tilde{v}\,\mathrm{d}r + \mathrm{d}\alpha^2 + \mathrm{d}\beta^2. \tag{4.6}$$

Note that $\tilde{v}, \alpha, \beta$ take on the same coordinate range as v', ϑ', φ'. We have also attached "in" to the metric to clearly denote the ingoing Penrose limit metric, to be differentiated from the outgoing Penrose limit metric, which will become relevant soon.

This completes the construction of the Penrose limit metric of an arbitrary ingoing principal geodesic in the Schwarzschild geometry. We can now utilize this metric to uncover information about the null ray. Recall that the Penrose limit metric (4.6) was adapted to the ingoing geodesic γ for $v = \vartheta = \varphi = 0$. Using the coordinate transformations (4.2) and (4.5), we see that the limit metric (4.6) is now in adapted coordinates for $\alpha = \beta = \tilde{v} = 0$. Therefore consider the slice where $(\alpha, \beta) = (0, 0)$, and transform $r, \tilde{v}$ by the following,

$$r = \frac{x - t}{\sqrt{2}}, \qquad \tilde{v} = \frac{x + t}{\sqrt{2}}.$$

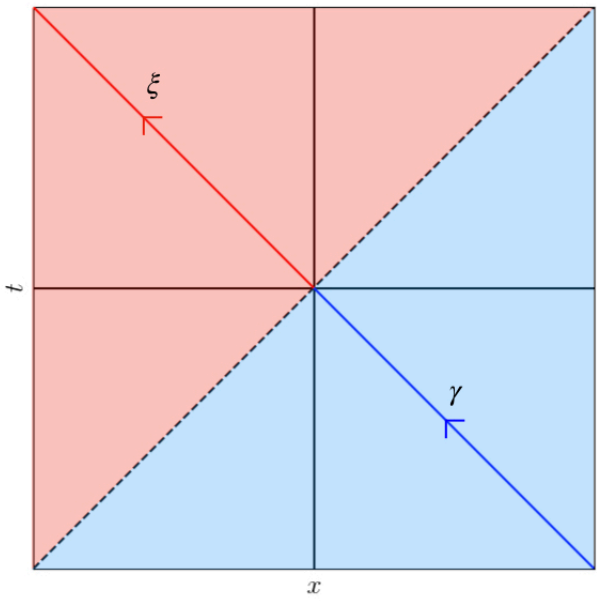

Figure 1: Shown is the (t, x) plane of the Penrose limit metrics (4.6) and (4.8) for the $(\alpha, \beta) = (0, 0)$ slice. The shaded blue region is described by ingoing Eddington-Finkelstein coordinates with blue null ray γ, and the shaded red region is the extension of the blue region using outgoing Eddington-Finkelstein coordinates with red geodesic ξ. Observe directly that γ naturally continues as ξ through $r = 0$. The union of γ (blue line) and ξ (red line) is defined as ζ, the complete ingoing geodesic.

Setting $\tilde{v} = 0$ reveals that γ is represented by the line $t = -x$ in the (t, x) plane. Further, noting that $r > 0$ uncovers a restriction on the validity of this metric: $x > t$. See the blue region of Figure 1 for an illustration.

The restriction $x > t$ on the Penrose limit metric (4.6) is problematic. Indeed, the Penrose limit claims to blow up a selected geodesic over the *entire spacetime*. If this is the case, why is the limit metric only valid for a subset of the entire manifold? It therefore stands to reason that the Penrose limit metric (4.6) must be only a section of the ingoing geodesic, rather than the entire trajectory. It is in this spirit we search for an extension of the spacetime such that this restriction is a coordinate restriction rather than a physical constraint on the spacetime.

We therefore search for such an extension using the outgoing Eddington-Finkelstein coordinates, given by,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}r\,\mathrm{d}u + r^2\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta + \vartheta_0)\,\mathrm{d}\varphi^2\right). \tag{4.7}$$

This metric is in adapted coordinates for the outgoing principal null ray ξ, which is affinely parameterized by r when $u = \vartheta = \varphi = 0$. By a similar procedure as the ingoing coordinates, one can show that the Penrose limit metric for outgoing Eddington-Finkelstein coordinates is given by

$$\mathrm{d}\bar{s}^2_{\xi,\mathrm{out}} = -2\,\mathrm{d}\tilde{u}\,\mathrm{d}r + \mathrm{d}\alpha^2 + \mathrm{d}\beta^2, \tag{4.8}$$

where α, β are defined as in (4.5), and

$$u' = \tilde{u} - \frac{\alpha^2 + \beta^2}{2r},$$

where u' is formulated in the same way as v' in the ingoing case. Further, $-\infty < \tilde{u}, \alpha, \beta < \infty$, as before.

It is easy to check that if we use the same transformations for $r, \tilde{u}$ as done for $r, \tilde{v}$, we recover a metric with the incorrect metric signature. We therefore negate the transformation for r; this changes the restriction of r from $r > 0$ to $r < 0$. We can now use similar transformations as before, i.e.,

$$r = \frac{x - t}{\sqrt{2}}, \qquad \tilde{u} = \frac{t + x}{\sqrt{2}},$$

to analyze the outgoing geodesic. Note that again, since the geodesic lies on the slice $\alpha = \beta = \tilde{u} = 0$, the geodesic appears as the line $t = -x$ on the (t, x) plane for the $(\alpha, \beta) = (0, 0)$ slice. We now invoke the requirement that $r < 0$ and see that $t > x$ for (4.8). This is represented by the shaded red region in Figure 1.

We now can use the fact that (4.4) is exactly equal to (4.8) under the parity transformation $r \to -r$ to fully extend the coordinate range of r to $-\infty < r < \infty$. This frees us of the coordinate restriction on the metric imposed by our analysis of the ingoing case only. We can therefore identify the ingoing principal null ray, γ, with the outgoing principal null ray, ξ, the union of which forms the complete principal null ray ζ (see Figure 1 for reference). The Penrose limit metric for ζ is therefore asserted to be

$$\mathrm{d}\bar{s}^2_\zeta = 2\,\mathrm{d}\tilde{v}\,\mathrm{d}r + \mathrm{d}\alpha^2 + \mathrm{d}\beta^2, \tag{4.9}$$

where the coordinates $(r, \tilde{v}, \alpha, \beta)$ have ranges $-\infty < r, \tilde{v}, \alpha, \beta < \infty$. From this extension we can conclude that the geodesic ζ must traverse the singularity at $r = 0$ and appear in the universe parallel to its original, i.e., travel from Region I to Region II in Figure 2. The notion that ζ traverses the singularity at $r = 0$ proves explicitly that one can construct a "wormhole" structure in the Schwarzschild geometry using the Penrose limit metric and analytic continuations of it.

4.2 Schwarzschild Event Horizon

In §4.1, we found the Penrose limit metric for an arbitrary ingoing geodesic in the Schwarzschild spacetime. But how does this compare to finding the Penrose limit metric for a specific null geodesic? A different approach is required than that in §4.1. We will construct the Penrose limit metric of the event horizon of a Schwarzschild black hole to highlight the difference in construction.

Before we proceed in constructing the Penrose limit, one may ask why we can not use outgoing Eddington-Finkelstein coordinates and construct the Penrose limit metric in the way outlined in §4.1. The reason is the following: in outgoing Eddington-Finkelstein, the future event horizon in the Schwarzschild geometry is at $u = \infty$ (u plays the role of v in outgoing Eddington-Finkelstein coordinates). Thus, outgoing Eddington-Finkelstein coordinates are ill-equipped to construct the Penrose limit of the future event horizon. This further highlights the need for an alternative approach for the event horizon.

We seek a coordinate transformation that puts the metric in (4.1) in Penrose form (3.1) in the neighborhood of the event horizon. It has been shown that any metric can be put into Penrose form [8], and thus it is reasonable to expect that the Taylor expansion of the relevant coordinates near the event

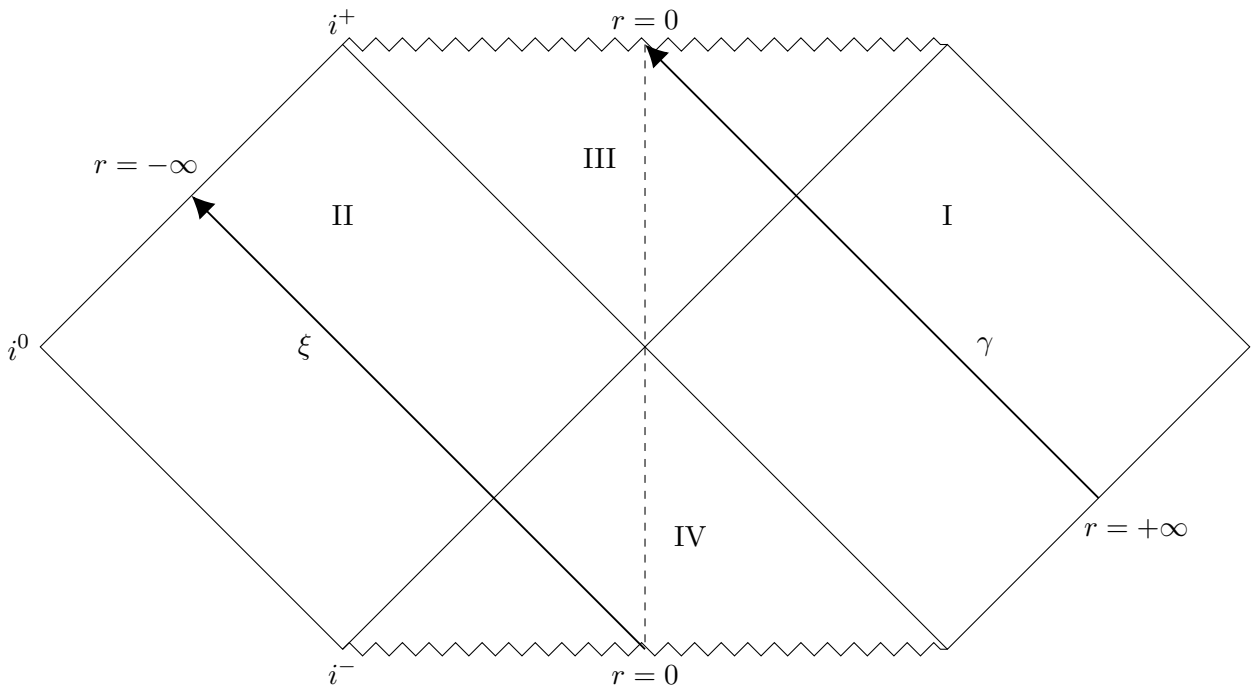

Figure 2: Shown are the ingoing and outgoing null rays (γ and ξ, respectively) on the Penrose diagram of the maximally extended Schwarzschild solution. Observe that γ and ξ are connected via a dashed line to clearly showcase the null ray traversing the singularity at $r = 0$. The union of γ and ξ yields the complete geodesic ζ.

horizon ($r = 2M$) is well defined. We can therefore assume the following ansatz,

$$U(r, v) = 2Me^{v/4M} + (r - 2M)f(v) + \mathcal{O}\left((r - 2M)^2\right) \tag{4.10}$$

$$V(r, v) = g(v) + (r - 2M)h(v) + \mathcal{O}\left((r - 2M)^2\right), \tag{4.11}$$

where the leading term of $U(r, v)$ has been assumed to have the same form as the affine parameter at the event horizon, $\tau \propto e^{v/4M}$. After some algebra, it can be shown that for free functions

$$f(v) = -e^{-v/4M}, \qquad g(v) = -4Me^{v/4M}, \qquad h(v) = 0,$$

the metric (4.1) is put into the form

$$\mathrm{d}ds^2_\gamma = 2\,\mathrm{d}U\,\mathrm{d}V + \mathrm{d}V^2 + 4M^2\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right) + \mathcal{O}(r - 2M), \tag{4.12}$$

which has the same form as (3.1), as desired. Rescaling $\vartheta \to \vartheta + \vartheta_0$ and then applying the asymmetric scaling (3.2), the metric takes the form

$$\mathrm{d}s^2_{\gamma,\lambda} = 2\lambda^2\,\mathrm{d}U\,\mathrm{d}V + \lambda^4\,\mathrm{d}V^2 + 4M^2\lambda^2\left(\mathrm{d}\vartheta^2 + \sin^2(\lambda\tilde{\vartheta} + \vartheta_0)\,\mathrm{d}\varphi^2\right) + \mathcal{O}(r - 2M). \tag{4.13}$$

Before taking the limit $\lambda \to 0$ we must examine the $\mathcal{O}(r - 2M)$ terms to understand how they behave under the rescaling (3.2). Solving (4.10) for $r - 2M$ and (4.11) for v, it can be shown that $r - 2M \propto V$.

Thus, under the rescaling (3.2), the $\mathcal{O}(r-2M)$ terms will be

$$\mathcal{O}(r-2M)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \begin{cases} 0, & \mu=\nu=U \\ \propto \lambda^n, n>3, \text{ for } \mu,\nu \text{ otherwise.} \end{cases} \tag{4.14}$$

Thus, any $\mathcal{O}(r-2M)$ terms in the metric will vanish after the overall rescaling by λ^{-2} and limiting $\lambda \to 0$.

Now applying the aforementioned overall rescaling and limit, we find that the Penrose limit metric for the Schwarzschild event horizon is,

$$\begin{aligned} \mathrm{d}\bar{s}^2_\gamma &= \lim_{\lambda\to 0} \lambda^{-2}\left(2\lambda^2\,\mathrm{d}U\,\mathrm{d}V + \lambda^4\,\mathrm{d}V^2 + 4M^2\lambda^2\left(\mathrm{d}\vartheta^2 + \sin^2(\lambda\tilde{\vartheta}+\vartheta_0)\,\mathrm{d}\varphi^2\right) + \mathcal{O}(r-2M)\right) \\ &= 2\,\mathrm{d}U\,\mathrm{d}V + 4M^2\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta_0)\,\mathrm{d}\varphi^2\right). \end{aligned} \tag{4.15}$$

This metric is manifestly flat and in double null coordinates. Differing from the Penrose limit metric found in §4.1, there is no degeneracy in this metric for any given parameter value. This agrees with our intuition; a photon orbiting at $r = 2M$ should never encounter a singularity for any value of U. Thus, in this construction, the singularity has, in a sense, "disappeared" from the spacetime entirely.

5 Perturbing Event Horizon Generator Trajectories

We now turn our attention to the instability null rays at the event horizon of stationary spherically symmetric spacetimes (i.e., Schwarzschild and Reissner-Nördstrom). We investigate this via perturbation theory; solving explicitly for the evolution of a radial perturbation sheds light on the behavior of null rays in the neighborhood of the event horizon.

5.1 Schwarzschild and Non-Extremal Reissner-Nördstrom

We begin by considering the Schwarzschild spacetime and the non-extremal Reissner-Nördstrom spacetime. The general metric that describes such spacetimes is that of a stationary spherically symmetric singularity present in flat Minkowski space, expressed here in ingoing Eddington-Finkelstein coordinates,

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\,\mathrm{d}\varphi^2, \tag{5.1}$$

where for Schwarzschild we have $f(r) = f_S(r) := 1 - 2M/r$ and for Reissner-Nördstrom we have $f(r) = f_{RN}(r) := 1 - 2M/r + Q^2/r^2$, where Q is the black hole charge.

We are interested in perturbing the event horizon generator geodesics and investigating the first-order behaviour of the perturbation in r. Thus, consider a horizon in the spacetime described by (5.1), given by the radial coordinate $r = r^*$ (in Schwarzschild, $r^*_S := 2M$ and in Reissner-Nördstrom $r^*_{RN} := M + \sqrt{M^2 - Q^2}$). We can also choose $\vartheta = \pi/2$, and $\varphi = \varphi_0 \in [0, 2\pi)$ freely for the unperturbed case. We now seek to solve the geodesic equation in v (see (A.1) for general formula). Noting that the only

nonzero Christoffel symbol corresponds to $\Gamma^v_{vv} = f'(r^*)/2$, the geodesic equation yields,

$$v(\lambda) = \kappa^{-1} \ln(\kappa\lambda), \quad \kappa := \frac{1}{2} f'(r^*), \tag{5.2}$$

where $\lambda \in \mathbb{R}$ is an affine parameter, a prime indicates a derivative with respect to r, and κ is referred to as the "surface gravity." It is well known that, in the unperturbed case, the conserved quantities on the horizon, i.e., the energy, E, and the angular momentum, L, are both zero.

We now can perturb the geodesic to first order using the following prescription,

$$\begin{aligned} E &= \varepsilon E^{(1)}, \qquad v = \kappa^{-1} \ln(\kappa\lambda) + \varepsilon v^{(1)}(\lambda), \qquad r = r^* + \varepsilon r^{(1)}(\lambda), \\ L &= \varepsilon L^{(1)}, \qquad \varphi = \varphi_0 + \varepsilon \varphi^{(1)}(\lambda), \qquad \vartheta = \frac{\pi}{2}. \end{aligned} \tag{5.3}$$

We begin first with the perturbed null condition. Using the perturbations given in (5.3), we have that,

$$\begin{aligned} g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} &= -f\left(r^* + \varepsilon r^{(1)}(\lambda)\right) \left(\frac{\mathrm{d}v}{\mathrm{d}\lambda}\right)^2 + 2 \frac{\mathrm{d}v}{\mathrm{d}\lambda} \frac{\mathrm{d}r}{\mathrm{d}\lambda}, \\ &= -\frac{2\varepsilon}{\kappa\lambda} r^{(1)} + \frac{2\varepsilon}{\kappa\lambda} \frac{\mathrm{d}r^{(1)}}{\mathrm{d}\lambda} + \mathcal{O}(\varepsilon^2) = 0, \end{aligned} \tag{5.4}$$

where in the second line we Taylor expanded the first term. Therefore, to first order, (5.4) tells us that $r^{(1)}$ evolves like

$$\frac{\mathrm{d}r^{(1)}}{\mathrm{d}\lambda} = \frac{r^{(1)}}{\lambda}. \tag{5.5}$$

Note that the above implies that $r = 2M + \varepsilon\lambda + \mathcal{O}(\varepsilon^2)$, which implies that, to first order, r is also an affine parameter on null rays. This, however, is not an issue; as we saw in our discussion of Penrose limit metrics, we know that (5.1) is in an adapted coordinate system for some set of null geodesics that are affinely parameterized by r. Given that we showed that there were null rays passing through the horizon, we can say that in the neighborhood of the horizon we are "close" to these ingoing rays, i.e., our perturbed horizon generator is in the congruence of null rays near the horizon which includes these ingoing null geodesics.

We now note that λ has a vague physical meaning along the geodesic, and it is therefore useful to express (5.5) in terms of physical coordinates. Using (5.2), we find that $r^{(1)}$ evolves like,

$$\frac{\mathrm{d}r^{(1)}}{\mathrm{d}v} = \kappa r^{(1)},$$

with respect to v, which has the straightforward solution of,

$$r^{(1)}(v) = Ce^{\kappa v}, \qquad C \in \mathbb{R}. \tag{5.6}$$

We now turn to the energy expression,

$$E = -g_{\mu\nu} \partial^\mu_v \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}, \tag{5.7}$$

where $\partial^\mu_v = \mathrm{col}(1, 0, 0, 0)$ is the timelike Killing vector, to constrain the value of C. We once again use

the values in (5.3), showing that

$$E = \varepsilon E^{(1)} = f\left(r^* + \varepsilon r^{(1)}(\lambda)\right)\frac{\mathrm{d}v}{\mathrm{d}\lambda} - \frac{\mathrm{d}r}{\mathrm{d}\lambda}, \\ = \varepsilon\left(\frac{2r^{(1)}}{\lambda} - \frac{\mathrm{d}r^{(1)}}{\mathrm{d}\lambda}\right) + \mathcal{O}(\varepsilon^2). \tag{5.8}$$

Therefore, to first order, we have that

$$\frac{\mathrm{d}r^{(1)}}{\mathrm{d}v} = 2\kappa r^{(1)} - E^{(1)}e^{\kappa v}, \tag{5.9}$$

where (5.2) has again been used. Plugging in (5.6) and solving for C, we find that

$$C = \frac{E^{(1)}}{\kappa}. \tag{5.10}$$

Therefore, in both the Schwarzschild and the non-extremal Reissner-Nördstrom geometries, radial perturbations for null trajectories behave like,

$$r^{(1)}(v) = \frac{E^{(1)}}{\kappa}e^{\kappa v}. \tag{5.11}$$

One immediately notices that if $E^{(1)} > 0$, then the null ray falls off the horizon and tends towards spatial infinity (i.e., $r \to \infty$), whereas if $E^{(1)} < 0$, the null ray falls into the black hole; this finding agrees with our physical intuition from rudimentary orbital analysis in classical mechanics.

We also notice a pathology in the result (5.11); if the surface gravity goes to zero (i.e., $\kappa \to 0$), then the radial perturbation becomes pathological, leading to a nonphysical result. No such extremal result is possible in Schwarzschild, but one does exist for Reissner-Nördstrom for the case, when the black hole mass equals the black hole charge. This, in fact, was the case in which Aretakis proved his famous result originally [1]. Therefore, this case demands further study, and we will do so using a second order perturbation approach. This is the contents of the following subsection.

5.2 Perturbing Extremal Reissner-Nördstrom

Consider the Reissner-Nördstrom metric in ingoing Eddington-Finkelstein coordinates,

$$\mathrm{d}s^2 = -f_{RN}(r)\,\mathrm{d}v^2 + 2\,\mathrm{d}r\,\mathrm{d}v + r^2\,\mathrm{d}\Omega^2, \tag{5.12}$$

where $\mathrm{d}\Omega^2$ is the area element of the unit sphere. We aim to perturb the null horizon generators in the extremal Reissner-Nördstrom case, i.e., Reissner-Nördstrom black holes for which $M = Q$. It is important to note that in this case, $\kappa = 0$, and so v is an affine parameter in the spacetime. Noting that the horizon in the extremal case is at $r = M$, we have the following perturbations to second order,

$$v = \lambda + \varepsilon v^{(1)}(\lambda) + \varepsilon^2 v^{(2)}(\lambda), \qquad r = M + \varepsilon r^{(1)}(\lambda) + \varepsilon^2 r^{(2)}(\lambda), \qquad \vartheta = \frac{\pi}{2}, \\ \varphi = \varphi_0 \in [0, 2\pi), \qquad E = \varepsilon E^{(1)} + \varepsilon^2 E^{(2)}, \qquad L = 0. \tag{5.13}$$

We first look to the null condition, where we find the following,

$$\varepsilon\frac{\mathrm{d}r^{(1)}}{\mathrm{d}\lambda}+\varepsilon^2\left(\frac{\mathrm{d}r^{(2)}}{\mathrm{d}\lambda}+\frac{\mathrm{d}v^{(1)}}{\mathrm{d}\lambda}\frac{\mathrm{d}r^{(1)}}{\mathrm{d}\lambda}-\frac{(r^{(1)})^2}{2M^2}\right)=0. \tag{5.14}$$

The first order expression clearly tells us that

$$r^{(1)}(v)=C\in\mathbb{R}, \tag{5.15}$$

whereas the second order expression easily yields

$$r^{(2)}(v)=\frac{C^2}{2M^2}v+D,\qquad D\in\mathbb{R}, \tag{5.16}$$

where (5.15) has been taken into account.

We now look to the energy expression to constrain the constants of integration found in (5.15) and (5.16). In doing so, we find that

$$\varepsilon E^{(1)}+\varepsilon^2E^{(2)}=\varepsilon^2\left(\frac{C^2}{M^2}-\frac{\mathrm{d}r^{(2)}}{\mathrm{d}\lambda}\right). \tag{5.17}$$

Using (5.17), the first order term clearly indicates that $E^{(1)}=0$, and that C takes the value of

$$C=M\sqrt{2E^{(2)}}. \tag{5.18}$$

Plugging this into (5.15) and (5.16), we find that the radial perturbation behaves like,

$$r=M+\varepsilon M\sqrt{2E^{(2)}}+\varepsilon^2\left(E^{(2)}v+D\right). \tag{5.19}$$

Note that we still have some freedom in how we choose D, and, in principal, we could choose D such that it cancels the erroneous shift in horizon position given by the constant first order term. However, we have yet to find an avenue that would allow us to do this while remaining physically justified, and therefore (5.19) is our final result.

Note again that the sign of $E^{(2)}$ determines whether the null ray tends towards spatial infinity or towards the black hole, agreeing with the result found in the non-extremal case. Also, comparing that the rate at which (5.19) expands from/falls into the black hole to (5.11) shows that it is, in fact, slower than in the non-extremal case. This aligns with the findings of Gralla *et al.* [6], where they found that generators of the extremal horizon diverged from/fell into the black hole at a slower rate than in the non-extremal case.

6 Conclusion

In this thesis, we presented a brief introduction to pp-wave spacetimes, and showed a methodology for obtaining a plane wave spacetime from an arbitrary spacetime using a Penrose limit. We then used the Penrose limit metric to show that, in the Schwarzschild geometry, the ingoing null ray with $L=E=0$

traverses the Schwarzschild singularity at $r = 0$. We also showed that the Penrose limit metric of the Schwarzschild event horizon is flat. We then used perturbation theory to find that the event horizon generators in the Schwarzschild, non-extremal and extremal Reissner-Nördstrom spacetimes grow from/fall into the singularity. We also find that the rate of this growth/decay is slower in the extremal Reissner-Nördstrom case than in the non-extermal case.

An important continuation of the work carried out in this thesis is an extension of the coordinate dependent formulations presented to coordinate invariant results. Such results are the hallmark of general relativity, and once achieved, depict truly fundamental results about the true nature of the relevant physics. Such a coordinate invariant formulation would undoubtedly shed light on the growth/decay results found in §5 in both the non-extremal and extremal cases. This would also allow for a more direct comparison to Aretakis's results, which would be a powerful and significant finding. This will be the direction of future work.

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A Technical Details of the Penrose Limit Metric Construction

Consider metrics of the form (3.1), and the general geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{A.1}$$

where s is an affine parameter. If we allow $V = Y^k = 0$, then the geodesic equation is trivially satisfied for V and Y^k, for any affine parameter. It remains to show that (A.1) is satisfied for $x^\mu = s = U$.

Substituting U into (A.1) for both s and x^μ, one finds that showing the affinely parametrized geodesic equation holds is equivalent to showing that

$$\Gamma^U_{UU} = 0. \tag{A.2}$$

Now consider the general form of the Christoffel symbols,

$$\Gamma^\beta_{\alpha\nu} = \frac{g^{\mu\beta}}{2}\left(\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right). \tag{A.3}$$

Allowing $\beta = \alpha = \nu = U$ in (A.3), and noting that $g_{UU} = g_{Uk} = 0$ and $g_{UV} = 1$ by construction is sufficient to show that (A.2) is satisfied. Thus we can conclude that for $V = Y^k = 0$ and U an affine parameter that the coordinates in (3.1) indeed describe the spacetime about a given geodesic.

We will now build on our description of metrics that take the form (3.1) by showing that the geodesics described by $V = Y^k = 0$ with U an affine parameter are null. Consider the tangent vector to the geodesic γ, $\xi^\mu = (1, 0, 0, 0)$. Then the null condition,

$$\begin{aligned} g_{\mu\nu}\xi^\mu\xi^\nu &= g_{UU}\xi^U\xi^U \\ &= g_{UU} \\ &= 0, \end{aligned}$$

is satisfied, since $g_{UU} = 0$ was taken as true in the original construction of (3.1). This completes the proof of Penrose's assertion discussed in §3. One can also show that expressing a metric of the form (3.1) in Penrose coordinates about a null geodesic γ is equivalent to embedding γ into a twist-free null geodesic congruence [3].