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AGE REPORTING OF VERY OLD SAMPLES

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ABSTRACT. Problems of the statistical interpretation of radiocarbon age measurements of old samples are discussed, based on the notion of fiducial probability distribution. A probability density function of age has been given. A detailed discussion of different facets of the probability distribution of age has led us to the confirmation of the use of 2σ as the best limiting value between the regions of finite and infinite dates. It has been proposed to make use of the principle of constant probability $P = 0.68$ in the regions of both finite and infinite ages instead of the criterion $N+k\sigma$.

INTRODUCTION

If the measurement of ^{14}C concentration in a sample yields several counts that differ from the number of background counts by the value that is comparable with the measuring error, we can say that the sample is too old to be dated by the ^{14}C method. The result of the measurement is consequently reported as: age $= T > T_0$. In a more general context, we address the problem of the detection of a weak signal in the presence of noise, caused by a counter background (Currie, 1968). Some implications in the case of radiocarbon dating have been discussed recently by Polach (1976), who tried to evaluate the inadequacy of information when results of age measurements of old samples are reported according to the procedure proposed by Callow and Hassal (1970). A statistical approach to the problem of very old samples has been presented by Gough (in press), who, using the Bayesian approach, developed a probability distribution of age. This treatment seems to be incorrect because it implies that there exists a probability distribution of ages of the samples, while, indeed, there is one real age (^{14}C concentration) which must be regarded as the unknown number we wish to estimate.

It is the purpose of this article to present an exact probability description of the results of age measurements of old samples, based on the notion of fiducial probability distribution. We hope our results will clarify some points of the probabilistic interpretation of "infinite" dates and support the use of 2σ limit proposed by Stuiver (1969), and advocated recently by Stuiver and Polach (1977).

Definitions of the limiting value T_0 differ in various laboratories, similar to the criteria of reporting age in the form "$>$". It is frequently required that the number of counts of the sample must exceed the background by at least twice the measurement error σ in order to quote the result of dating as $T = T' \pm \Delta T$. In some laboratories the criteria of 3σ and even 4σ are in use. In the case of too small counting rates the quantity T_0 is computed in the usual way, as T', but the value $k\sigma$ is added to the obtained net number of counts. When the sample counting rate is smaller than that of the background, its value may be ignored and $k\sigma$ forms the basis of the computations of T_0. The quantity k may be equal to 2, 1.5, 3, or 4.

The main component of errors in radiocarbon measurements is usually connected with counting statistics. Then, the value of the measurement error σ can be calculated with good accuracy, and the probability distribution of the age of the sample may be given. If the result of the measurement is reported as $T = T' \pm \Delta T$ the reader knows that the real age, or rather the age fit to the real concentration of ^{14}C, for example, lies in the interval $(T' - \Delta T, T' + \Delta T)$ with the probability 0.68. In many applications, knowledge of the correct value of ΔT, which has the sense of a standard deviation, 1σ, is very important. It is inadmissible to overestimate it "for security". Similarly, for old samples, the correct interpretation of the result of dating $T > T_0$ may be needed, the probability of the verity of this inequality.

STATISTICAL APPROACH

A correct statistical interpretation of age measurements should be based on the probability distribution of the measured age T. It is noteworthy that for both finite and infinite ages T should be characterized by the fiducial probability distribution (Hacking, 1965; Kendall and Stuart, 1966), which is the measure of confidence of any statement concerning T. Since the sample that has been dated has only one given concentration of ^{14}C, the fiducial probability distribution usually has no frequency interpretation.

The main quantities constituting the result of the measurement of any sample are: the sample counting rate N_s, background counting rate B, counting rate of the standard of modern ^{14}C activity M, and the mean standard errors σ_N and σ_B of N_s and B, respectively. It has been assumed here that σ_N, σ_B and M are known without errors.

The net sample counting rate is given by

$$N' = \gamma\,(N_s - B), \tag{1}$$

where γ symbolizes all the normalization constants and experimentally derived correction factors, whose errors are negligible. The value of N′ is known with the error

$$\sigma = (\sigma_N{}^2 + \sigma_B{}^2)^{1/2}. \tag{2}$$

The probability distribution of the counting rate N′ is

$$dP = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(N-N')^2}{2\sigma^2}\right]dN', \tag{3}$$

where N denotes the expected value of N′; then the fiducial probability distribution of N for experimentally determined N′ and σ, is given by

$$dP = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(N-N')^2}{2\sigma^2}\right]dN. \tag{4}$$

An obvious confinement arising from the fact that $N \geqslant 0$ can be accounted for by calculating the conditional probability

$$P\,(N < \hat{N} < N + dN \mid \hat{N} \geqslant 0) =$$

$$= \frac{P\,(N < \hat{N} < N + dN \text{ and } \hat{N} \geqslant 0)}{P\,(\hat{N} \geqslant 0)} = \tag{5}$$

$$= \frac{g\,(N)\,dN}{\dfrac{1}{\sqrt{2\pi}\,\sigma} \displaystyle\int_0^\infty \exp\left[-\frac{(x - N')^2}{2\sigma^2}\right] dx},$$

where $\hat{N}$ is a random variable (while N is a number) and

$$g\,(N) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\dfrac{(N - N')^2}{2\sigma^2}\right] & \text{for } N \geqslant 0 \\ 0 & \text{for } N < 0 \end{cases} \tag{6}$$

is the cut-off normal probability distribution. After substituting to (5)

$$N = M\,e^{-T/\tau}, \tag{7}$$

where τ is the mean lifetime of radiocarbon, we obtain the probability density function of age

$$f(T) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-T/\tau - \frac{(M\,e^{-T/\tau} - N')^2}{2\sigma^2}\right] / P(N'/\sigma), \tag{8}$$

where

$$P(x) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt. \tag{9}$$

The mode of this probability distribution, the most probable value is given by the equation

$$T_M = -\tau \ln \left[\frac{N' + (N'^2 + 4\sigma^2)^{1/2}}{2M}\right], \tag{10}$$

which in the case of $N' > 0$ can be written as

$$T_M = T' - \tau \ln\,[1/2 + (1/4 + \sigma^2/N'^2)^{1/2}], \tag{11}$$

where

$$T' = -\tau \ln N'/M. \tag{12}$$

The logarithmic term appearing in eq (11) has the meaning of a correction to the value T′, which in the case of finite dates is quoted as the measured age. The value of this correction is negligible for $N' >> \sigma$, for $N' \approx 3\sigma$ its value constitutes ca 1/3 of the age error ΔT. The expected value, or mean, of the probability distribution (8) is based on the frequency interpretation and is not very important in this case.

PRACTICAL INFERENCES

Except for the complicated form of the probability distribution (8) it is easy to calculate the probability of the relation $T > T_\alpha$, based on the tables of normal probability distribution. Denoting the probability that the relation $T < T_\alpha$ is true by α, we have

$$\text{Prob}\,(T > T_\alpha) = 1 - \alpha =$$
$$= \frac{P\left(\dfrac{M \exp\,(-\,T_\alpha/\tau) - N'}{\sigma}\right) - P\,(-\,N'/\sigma)}{1 - P\,(-\,N'/\sigma)}\,. \tag{13}$$

This equation is valid irrespective of the sign of N′. Figure 1 presents the plots of the function $\alpha = \alpha\,(T_\alpha)$ for some values of the parameters N'/M and σ/M. The curves A and B correspond to the case of finite age, quoted as "±". The greater slope of curve A is caused by a smaller value of the measurement error σ. The curves D and E correspond to $N' = 0$, the limiting value T_0 of age in this case obviously depends on the measurement error σ. A comparison of curve D with F, for which $N' < O$, indicates an increase of the limiting age T_0 related only by obtaining a counting rate below the background level (both D and F have been plotted for the same value of the measurement error σ). It looks like a paradox, but even at a low value of the factor of merit, for a high value of the measurement error, the limiting value of age can be arbitrarily high, due to chance (by incidental detection of, for example, $N' = -\,3\sigma$). Assuming correct performance of the measuring apparatus, which means that the estimated value of σ is equal to the real error of measurement, such a conclusion is justified. It is for the experimenter to decide whether he has observed a rare random event, which may occur, on the average, once in a series of 740 measurements (the 3σ level) of an inactive sample, or if the instability of the background counting rate is greater, σ has been underestimated. If the sample does not contain ^{14}C, the probability of obtaining $N' < 0$ is equal to 0.5. Consequently, if the estimated value of σ is accepted, it is unfounded to discriminate half the results.

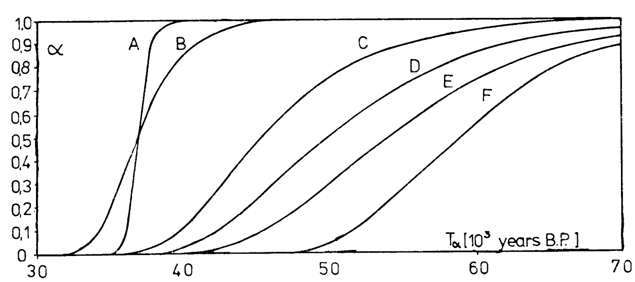

Fig 1. Probability α of the verity of the relation $T < T_\alpha$ as a function of T_α for some values of N'/M and σ/M M: modern counting rate, N′ and σ: measured sample counting rate and measuring error. N'/M and σ/M are equal, respectively, for the curves A: 0.01, 0.001; B: 0.01, 0.003; C: 0.003, 0.003; D: 0, 0.003; E: 0, 0.001; F: −0.003, 0.003.

Using eq (13), we may express T_α as a function of α

$$T_\alpha = -\tau \ln \frac{N'}{M}\left[+ \frac{\sigma}{M} P^{-1}\left(1 - \alpha P\left(\frac{N'}{\sigma}\right)\right)\right], \quad (14)$$

and assuming a certain value of α, we are able to calculate the limiting age T_α of a sample.

For finite dates, quoted as $T = T' \pm \Delta T$, the probability that the inequality $T' - \Delta T < T < T' + \Delta T$ is true, is equal to 0.68. If we want to minimize the differences in the interpretation of dates reported as "±" or ">" we should retain the value of the probability, Prob $(T > T_0) = 0.68$, choose $\alpha = 0.32$.

For $N' > 0$, formula (14) can be written as $T_\alpha = T' + \Delta T_\alpha$, where

$$\Delta T_\alpha = -\tau \ln \left[1 + \frac{\sigma}{N'} P^{-1}\left(1 - \alpha P\left(\frac{N'}{\sigma}\right)\right)\right]. \quad (15)$$

The value of correction ΔT_α depends only on the quotient N'/σ and it has been plotted in figure 2 for some values of α. $\Delta T_{0.5}$ denotes the difference of T' and the median of the probability distribution of age, its value is significant only for $N' < 2\sigma$. The quantities $\Delta T_{0.84}$ and $\Delta T_{0.16}$ correspond to the errors ΔT_1 and ΔT_2 defined by

$$\Delta T_{1,2} = \mp \tau \ln (1 \mp \sigma/N'). \quad (16)$$

The differences between ΔT_1 and $\Delta T_{0.84}$ and between ΔT_2 and $\Delta T_{0.16}$ caused by neglecting the inequality $N \geqslant 0$ became significant for low values of N'/σ. The curve ΔT_M in figure 2 shows a difference between the value of T' and the mode of distribution, defined by eq (10). For

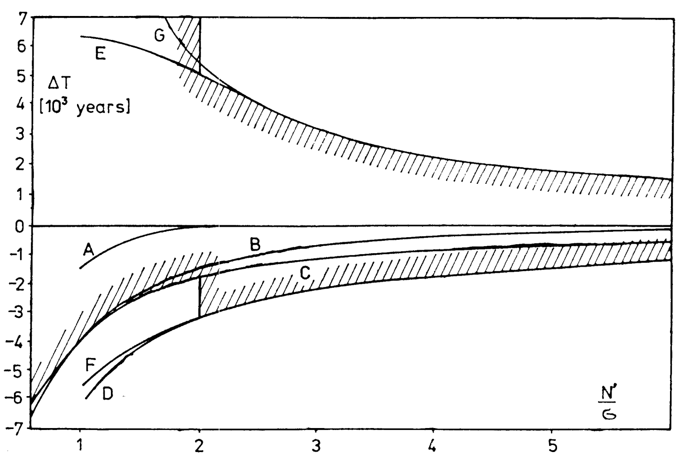

Fig 2. Statistical characteristics of the probability distribution of age. The dashed area denotes age intervals corresponding to the probability 0.68 (for more detailed explanations cf the text). G and F: age errors ΔT_1 and ΔT_2, respectively, according to the eq (16); E, A, C, and D: age differences between $T' = -\tau \ln N'/M$ and T_α defined by eq (14) for the value of probability α equal to 0.84, 0.5 (median), 0.32, and 0.16, respectively; B: difference ΔT_M of the value T' and the mode (eq 10) of the probability distribution of age.

$N' \approx \sigma$, owing to the high asymmetry of the probability distribution (8), we have $\Delta T_M = \Delta T_{0.32}$, while for $-\sigma < N' < 2\sigma$, the difference of these quantities is small in comparison with ΔT_1 or ΔT_2. This property may be used as an additional argument for the acceptance of $T_0 = T_{0.32}$; the reported limiting age would be, at the same time, the most probable age of the sample.

The choice of the value of probability α, at which the statement $T > T_0 = T_\alpha$ is not true, is in fact a question of agreement, as well as the choice of the limiting value of the quotient N'/σ, which marks the boundaries between the regions of "finite" and "infinite" ages. It should be noted that even in the case of $N' < 0$ the age might be quoted as $T' + \Delta T_{0.16} < T < T' + \Delta T_{0.84}$ since $\Delta T_{0.84}$ always has a finite value. From a purely statistical point of view, the procedure of quoting finite age intervals at low values of N'/σ is not justified, since the probability distribution of age is highly asymmetrical. As shown in figure 1, the curves C, D, E, and F have a very low slope in their upper parts, which means that in order to be sure that $T < T' + \Delta T_1$ not at the level 0.84 but at 0.9987, much more than 3 ΔT_1 should be added to T', as may be expected from a normal probability distribution.

It is difficult to find such a value of the quotient N'/σ, which might be used as a natural boundary between "±" and ">". The curve $\Delta T_{0.84}$ has an inflection at $N'/\sigma = 1.72$ (see fig 2), which means that starting from this point, the value of the upper age limit, $T' + \Delta T_{0.84}$, reveals a slower increase with an increasing measurement error α. Intuition, however, should predict $\Delta T_{0.84}$ to be faster and faster, increasing to infinity. As the value 1.72 does not differ significantly from 2.0, those accepted in most radiocarbon laboratories, and recommended by Stuiver and Polach (1977), and the argument based on the point that inflection is somewhat arbitrary, it seems that 2 should be the best limiting value for the quotient N'/σ. The assumption of greater values is inherently related to a significant loss of information which has been obtained in the measurement. The probability that an inactive sample will produce a counting rate $N' > 2\sigma$ is equal to 0.023.

DISCUSSION AND CONCLUSION

The procedure of treating and reporting results of dating old samples, described above and deduced from rigorous statistical analysis, seems to confirm and supplement the recommendations of Stuiver and Polach (1977). This may be summarized as follows: when the net sample counting rate is greater than twice the standard deviation σ, the age should be reported as $T = T' \begin{smallmatrix} +\Delta T_1 \\ -\Delta T_2 \end{smallmatrix}$ or $T = T' \pm \Delta T$. Otherwise, the result should be reported as $T > T_0$, where T_0 is given by eq (14) for $\alpha = 0.31731$.

The choice of such a value of α is equivalent to accepting the principle of constant probability Prob = 0.68, for the regions of both finite and infinite radiocarbon ages.

The values of ΔT_1 and ΔT_2 should be, in principle, calculated from eq (14) for $\alpha = 0.84$ and $\alpha = 0.16$, respectively, in order to keep constant the probability that $T' - \Delta T_2 < T < T' + \Delta T_1$. Since the difference between the value of $\Delta T_{1,2}$, calculated from eq (16), and $\Delta T_{0.84,0.16}$ does not exceed 10 percent (*cf* fig 2), the simpler equation (16) can be used. In calculations of T_0 eq (14) can be approximated by eq (10); in the interval $-\sigma < N' < 2\sigma$ the error of this approximation does not exceed 300 years (see fig 2).

The use of any value of α is a question of choice, which should be made in agreement with some commonly accepted recommendations. The proposed value of $\alpha = 0.32$ warrants a uniform interpretation of both finite and infinite radiocarbon dates. The procedure of calculating the limiting value of age by assuming $N'' = N' + k\sigma$, where $k = 2$, 3 or 4 is highly asymmetrical and inherently associated with a significant loss of information. If, for example, the 4σ criterion is used, according to the recommendation of Callow and Hassal (1970), and the measured sample counting rate is close to the limiting value $N' \approx 4\sigma$, only the minimum age should be quoted, which in fact corresponds to the 8σ level! There is no reason for using such high values of the significance level. This has been clearly recognized by Stuiver and Polach (1977) who recommend that when the measured sample activity is between 1σ and 2σ, an "apparent age" can be added. As we see it, the procedure of calculating the limiting age by assuming $N'' = N' + k\sigma$ is connected with a risk of the contamination of the sample material with recent carbon. This, however, is quite a different problem which must be solved by means of careful laboratory and field work. As has been indicated by Polach (1976), dating chemical and/or physical fractions of a sample seems to be best for evaluating the possibility and degree of contamination. It should be noted that the problem of sample contamination occurs also for finite radiocarbon ages, where the use of 1σ limits of age, corresponding to a probability of 0.68, has been commonly accepted.

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