# COMPARISON OF INTEGRAL STRUCTURES ON THE SPACE OF MODULAR FORMS OF FULL LEVEL $N$

by

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As members of the Dissertation Committee, we certify that we have read the dissertation prepared by: Anthony Kling
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Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Bryden R. Cais — Date: Aug 10, 2021
Mathematics

*"Watch;*

*as I combine*

*all the juice from the mind.*

*Heel up,*

*wheel up,*

*bring it back,*

*come,*

*rewind."*

— Trevor George Smith Jr.

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**Abstract**

Let $N \geq 3$ and $r \geq 1$ be integers and $p \geq 2$ be a prime such that $p \nmid N$. One can consider two different integral structures on the space of modular forms over $\mathbb{Q}$, one coming from arithmetic via $q$-expansions, the other coming from geometry via integral models of modular curves. Both structures are stable under the Hecke operators; furthermore, their quotient is finite torsion. Our goal is to investigate the exponent of the annihilator of the quotient. We will apply the tools in [BDP17] to the situation of weight 2 and level $\Gamma(Np^r)$ modular forms over $\mathbb{Q}_p(\zeta_{Np^r})$ to obtain an upper bound for the exponent. We also use Klein forms to construct explicit modular forms of level $p$, allowing us to compute a lower bound. When $r = 1$, both bounds agree, allowing us to compute the exponent precisely in this case.

# 1 Introduction

Let $k \geq 1$ be an integer and let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level $N$ *i.e.* a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing the kernel, $\Gamma(N)$, of the usual reduction mod $N$ map

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Consider the $\mathbb{Q}$-vector space $M_k(\Gamma, \mathbb{Q})$ consisting of modular forms of weight $k$ and level $\Gamma$ over $\mathbb{Q}$. We associate to each modular form $f$ in $M_k(\Gamma, \mathbb{Q})$ and each cusp $c$ of $\Gamma$, the $q$-expansion of $f$ at $c$, denoted $f_c$, which is a power series in $\mathbb{Q}[[q]]$.

We concern ourselves with $\mathbb{Z}$-structures on the vector space $M_k(\Gamma, \mathbb{Q})$ *i.e.* $\mathbb{Z}$-submodules $M'$ of $M_k(\Gamma, \mathbb{Q})$ such that the natural map

$$M' \otimes_{\mathbb{Z}} \mathbb{Q} \to M_k(\Gamma, \mathbb{Q})$$

is an isomorphism. The space $M_k(\Gamma, \mathbb{Q})$ naturally has two different $\mathbb{Z}$-structures. Define the first $\mathbb{Z}$-structure to be

$$M_k(\Gamma, \mathbb{Z}) = \{f \in M_k(\Gamma, \mathbb{Q}) : f_\infty \in \mathbb{Z}[[q]]\}$$

which consists of modular forms in $M_k(\Gamma, \mathbb{Q})$ whose $q$-expansion at the cusp $\infty$ has integral coefficients. One can show the Hecke operators of $M_k(\Gamma, \mathbb{Q})$ preserve integrality at the cusp $\infty$ by explicitly computing the $q$-expansion at $\infty$ under the Hecke operators (see [Kat04, 4.9.2]).

The second $\mathbb{Z}$-structure we consider is $M_{k,\mathbb{Z}}$, which consists of modular forms in $M_k(\Gamma, \mathbb{Q})$ whose $q$-expansions at *all* cusps have integral coefficients. This structure is also stable under the Hecke operators (cf. [Con07, Theorem 1.2.2]). There is an obvious containment $M_k(\Gamma, \mathbb{Z}) \subseteq M_{k,\mathbb{Z}}$ with quotient that is a torsion $\mathbb{Z}$-module. Our aim is to study/ determine the annihilator.

To better understand and work with $M_{k,\mathbb{Z}}$, we realize $M_{k,\mathbb{Z}}$ as the global sections of some line bundle $\underline{\omega}_{\mathfrak{X}}^{\otimes k}$ on $\mathfrak{X}$, the moduli space parameterizing $\Gamma$-level structures over $\mathbb{Z}$. More precisely, for an integer $N \geq 1$, [KM85, §3] considers four different moduli problem parameterizing $\Gamma(N)$-, $\Gamma_1(N)$-, balanced $\Gamma_1(N)$-, and $\Gamma_0(N)$-structures on elliptic curves. The definition of $\Gamma(N)$-structures is given in Section 2.2 since this will be our focus. When $N \geq 3$, the moduli problems parameterizing $\Gamma(N)$-, $\Gamma_1(N)$-, and balanced $\Gamma_1(N)$-structures are represented by a regular, flat two-dimensional scheme $\mathfrak{Y}(\Gamma)$ over $\mathbb{Z}$, by [KM85, 5.5.1]. For the rest of the introduction, we let $\Gamma$ denote one of these level structures. The scheme $\mathfrak{Y}(\Gamma)$ extends to an arithmetic surface $\mathfrak{X}(\Gamma)$ over $\mathbb{Z}$, known as the modular curve, so that $M_{2,\mathbb{Z}}$ is identified with $H^0(\mathfrak{X}(\Gamma), \underline{\omega}_{\mathfrak{X}(\Gamma)}^{\otimes 2})$. From this, we can show that $H^0(\mathfrak{X}(\Gamma), \underline{\omega}_{\mathfrak{X}(\Gamma)}^{\otimes 2})$ and $M_2(\Gamma, \mathbb{Z})$ are both finitely-generated $\mathbb{Z}$-modules of the same rank. Thus the quotient $M_2(\Gamma, \mathbb{Z})/H^0(\mathfrak{X}(\Gamma), \underline{\omega}_{\mathfrak{X}(\Gamma)}^{\otimes 2})$ is torsion.

Let $p$ be a prime. If $p$ does not divide the level $N$, then the $p$-adic valuation of the annihilator is zero (see Remark 3.8). Thus we focus on primes $p$ which divide the level and instead can work over $\mathbb{Z}_p$. Let $c$ be a cusp of $\mathfrak{X}(\Gamma)$. Define $\nu_p(f_c)$ to be the minimal $p$-adic valuation among all the coefficients of $f_c$. Then our general goal can be restated as follows: we seek to compute the smallest integer $e \geq 0$ such that $\nu_p(p^e f_c) \geq 0$ for all cusps $c$ of $\mathfrak{X}(\Gamma)$ not equal to $\infty$ and all $f \in M_2(\Gamma, \mathbb{Z}_p)$.

The problem of computing and bounding $e$ has a long history, which we briefly recall. The moduli problem representing $\Gamma_0(N)$-structures is represented by a regular, proper, flat Artin stack $\mathfrak{X}_0(N)$ over $\mathbb{Z}$ by [Con07, Theorem 1.2.1]. Computing a bound for $e$ for arbitrary weight $k$ and level $\Gamma_0(p)$ was done in [DR73,

3.19, 3.20], where they obtained an upper bound of

$$e \leq \frac{kp}{p-1}.$$

Their methods involved using intersection theory on $\mathfrak{X}_0(p)$ and the fact that $\mathfrak{X}_0(p)_{/\mathbb{F}_p}$ is reduced with two irreducible components.

In [Edi06], Edixhoven investigates the situation of weight $k = 2$ and level $\Gamma_0(N)$ *cusp forms* where $\mathrm{ord}_p(N) = 1$. In particular, he establishes the existence of a non-zero global section of $\Omega|_{C_0'}(-e \cdot S)$ where $\Omega$ is the relative dualizing sheaf of $\mathfrak{X}_0(N)$, $C_0'$ is essentially[1] the irreducible component containing the cusp 0, and $S$ is the divisor given by the sum of the supersingular points. By computing $\deg(\Omega|_{C_0'}(-e \cdot S))$ in terms of $e$ and using the inequality

$$0 < \deg(\Omega|_{C_0'}(-e \cdot S)),$$

Edixhoven is able to conclude $e < 1 + \frac{2}{p-1}$ which forces $e = 1$ whenever $p \geq 3$. This agrees with the bound provided by [DR73]. In the case $p = 2$, Edixhoven separately concludes $e = 2$.

Lastly, in [BDP17, Appendix B], Conrad investigates $e$ in the general situation $\mathrm{ord}_p(N) = r \geq 1$ and arbitrary weight $k$. He begins by developing intersection theory on the regular proper Artin stack $\mathfrak{X}_0(N)$ over $\mathbb{Z}$. Let $T$ denote the $r \times r$ matrix obtained by removing the column and row corresponding to the irreducible component containing the cusp $\infty$ from the intersection matrix of $\mathfrak{X}_0(N)$. He provides an upper bound in terms of $k, p, r$, and the entries of $T^{-1}$ . For $r = 2$, Conrad obtains an upper bound of

$$e \leq \frac{3k}{p-1}\left(1 + \frac{2}{p-1}\right)$$

and for $r = 3$ he obtains an upper bound of

$$e \leq \frac{2kp(2p-1)}{(p-1)^3}.$$

For general $r$, it is not clear how to obtain a uniform description of the entries of $T^{-1}$, and this prevents Conrad from establishing a similar bound for $r > 3$.

Let $N \geq 3$ and $r \geq 1$ be an integers and $p \geq 2$ be a prime such that $p \nmid N$. Fix a primitive root of unity $\zeta_{Np^r}$ and let $\pi$ be a uniformizer of $\mathbb{Z}_p[\zeta_{Np^r}]$. We consider the situation of the modular curve $\mathfrak{X}(Np^r)$ over $\mathbb{Z}_p[\zeta_{Np^r}]$, which parameterizes $\Gamma(Np^r)^{\mathrm{can}}$-structures (see the paragraph preceding Theorem 2.12, or [KM85, §9]). In this situation, the special fiber of $\mathfrak{X}(Np^r)$ consists of $p^r + p^{r-1}$ many irreducible components. We also restrict ourselves to weight $k = 2$ modular forms. Applying Conrad's method in this situation, we obtain a formula in terms of the entries of $T^{-1}$ and $\deg(\underline{\omega}^{\otimes 2}|_\Lambda)$, where $\Lambda$ is any irreducible component of the special fiber of $\mathfrak{X}(Np^r)$. We make the following two significant computations:

- We explicitly compute the entries of $T^{-1}$ for all $r$ and $p$.

[1] More precisely, since $\mathfrak{X}_0(N)$ is not necessarily regular, [Edi06] works instead with some finite cover $X' \to \mathfrak{X}_0(N)$ by appending extra level structure to $\Gamma_0(N)$. Let $C_0$ denote the irreducible component of $\mathfrak{X}_0(N)_{/\mathbb{F}_p}$ containing the cusp 0. Then $C_0'$ is the inverse image of $C_0$ under $X' \to \mathfrak{X}_0(N)$.

- We explicitly compute $\deg(\underline{\omega}^{\otimes 2}|_\Lambda)$. Note that Conrad in [BDP17, Appendix B] bounds $\deg(\underline{\omega}^{\otimes 2}|_\Lambda)$ by $\deg(\underline{\omega}^{\otimes 2})$. In our situation,

$$\begin{aligned}\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}) &= \frac{2}{24}\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z}) \\ &= \frac{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{12}(p^{3r}-p^{3r-2})\end{aligned}$$

  by [KM85, 10.13.12]. In Theorem 5.28, we show

$$\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda) = \frac{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{12}(p-1)p^{2r-1}$$

  independent of $\Lambda$, which is smaller than $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)})$ by a factor of $p^{r-1}(p+1)$. This savings leads to a marked improvement in our upper bound.

In Theorem 4.21, we calculate the entries of $T^{-1}$ for all $r$. First we describe the entries of $T$ by explicitly computing the intersection number between each irreducible component of the special fiber, using [KM85, 13.8.5], then we compute the self-intersection numbers.

We compute that

$$T = \deg \mathrm{S}(N)\begin{pmatrix} M(p^r)_{\hat{1},\hat{1}} & \mathbf{1}_{p^{r-1}\times p^{r-1}} \\ \mathbf{1}_{p^{r-1}\times p^{r-1}} & p^2M(p^{r-1}) \end{pmatrix}$$

where $M(p^r)$ is a $p^r \times p^r$ circulant matrix, dependent on $p$ and $r$, $M(p^r)_{\hat{1},\hat{1}}$ is the matrix $M(p^r)$ with the first row and column removed, $\mathbf{1}_{n\times m}$ is the $n \times m$ matrix consisting of all 1's, and $\deg \mathrm{S}(N)$ is the degree of supersingular locus in $\mathfrak{X}(Np^r)$. Applying the Woodbury Matrix Identity (see Appendix B) we obtain a formula for the entries of $T^{-1}$ involving the entries of $(M(p^r)_{\hat{1},\hat{1}})^{-1}$ and $p^{-2}M(p^{r-1})^{-1}$, as well as their row sums and the total sum of all entries.

Since $M(p^r)$ is circulant, we have a description of its eigenvalues and corresponding eigenvectors, allowing us to diagonalize $M(p^r)$. In turn, this allows us to explicitly compute the entries of $M(p^r)^{-1}$. Consequently, using Proposition 4.14, we also obtain the entries of the inverse of $M(p^r)_{\hat{1},\hat{1}}$. After more careful calculations, we obtain an exact expression for each entry of $T^{-1}$.

In Theorem 5.28, we compute $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda)$. Using the Kodaira-Spencer isomorphism (Theorem 2.34) and the adjunction formula (Theorem 5.5), we are able to identify $\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}$ with the relative dualizing sheaf $\Omega_{\mathfrak{X}(Np^r)/\mathbb{Z}_p[\zeta_{Np^r}]}(-\mathfrak{C}(Np^r))$ twisted by minus the cuspidal divisor $\mathfrak{C}(Np^r)$, tensored with relative dualizing sheaf $\Omega_{\mathfrak{X}(Np^r)/\mathfrak{X}(N)}$ of the map $\mathrm{pr} : \mathfrak{X}(Np^r) \to \mathfrak{X}(N)$ induced by forgetting the $\Gamma(p^r)$-level structure.

Investigating $\Omega_{\mathfrak{X}(Np^r)/\mathfrak{X}(N)}$ amounts to understanding the different of the morphism pr, which amounts to understanding the different of

$$\mathrm{pr}_x : \mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}_{\mathfrak{X}(Np^r),x}$$

where $x \in \mathfrak{X}(Np^r)$ is a codimension 1 point. Let $d_x$ denote the valuation of the different of $\mathrm{pr}_x$ in $\mathcal{O}_{\mathfrak{X}(Np^r),x}$. We split our analysis of $\mathrm{pr}_x$ into two cases. If $x$ is a closed point of the generic fiber, then we can compute $d_x$ over $\mathbb{C}$. This amounts to understanding the ramification of the analogous map of Riemann surfaces $X(Np^r) \to X(N)$, which is done in [Shi94, Prop. 1.37]. If $x$ is a generic point of the special fiber, then we show all $d_x$ contributions are the same which taken together contribute nothing to the different of pr. However, in Appendix C, we provide a method of computing $d_x$ in this situation, which may be useful in

the case of balanced $\Gamma_1(Np^r)$-structures which don't enjoy such uniformity. We ultimately conclude

$$\Omega_{\mathfrak{X}(Np^r)/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}(Np^r)}((p^r-1)\mathfrak{C}(Np^r))$$

and consequently

$$\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)} \simeq \Omega_{\mathfrak{X}(Np^r)/R}(\mathfrak{C}(Np^r)).$$

This identification allows us to directly compute $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda)$ by computing the quantities

$$\deg(\mathfrak{C}(Np^r)|_\Lambda) \text{ and } \deg(\Omega_{\mathfrak{X}(Np^r)/R}|_\Lambda).$$

The relative dualizing sheaf $\Omega_{\mathfrak{X}(Np^r)/R}$ enjoys strong functoriality properties, enabling us to identify

$$\Omega_{\mathfrak{X}(Np^r)/R}|_\Lambda \simeq \Omega_{\Lambda/\mathbb{F}_{p^{\varphi(N)}}} \otimes \mathcal{O}_{\mathfrak{X}(Np^r)}(-\Lambda)|_\Lambda.$$

This allows us to compute $\deg(\Omega_{\mathfrak{X}(Np^r)/R}|_\Lambda)$ in terms of the genus of an Igusa curve, and the self-intersection number of $\Lambda$. Combined with our calculation of the entries of $T^{-1}$, we arrive at the following upper bound.

**Theorem 1.1.** *a. Let $N \geq 3$ and $r \geq 1$ be integers and $p \geq 2$ be a prime such that $p \nmid N$. The exponent $e$ of $\pi$ in the annihilator of $M_2(\Gamma(Np^r), \mathbb{Z}_p[\zeta_{Np^r}])/H^0(\mathfrak{X}(Np^r), \underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)})$ is bounded above by*

$$e \leq 2p^{r-1}(pr - r + 1).$$

For $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, let $\kappa_{(r_1,r_2)}(\tau)$ denote the Klein form associated to $(r_1, r_2)$ (see Definition 6.1). The lower bound is obtained by explicitly constructing a modular form of level $p$ out of a product of Klein forms. Let $\{m(t)\}_{t=1}^{N-1}$ be a family of integers. [EKS11, Thm. 2.6], which builds upon results in [KL81, §2.1, §3.4], provides a criterion on $\{m(t)\}_{t=1}^{N-1}$ for when a product of Klein forms

$$\kappa(\tau) = \prod_{t=1}^{N-1} \kappa_{(t/N,0)}(N\tau)^{m(t)}$$

is a nearly holomorphic modular form of level $\Gamma_1(N)$ and weight $-\sum_{t=1}^{N-1} m(t)$. We are able to choose a family $\{m(t)\}_{t=1}^{p-1}$ such that $\kappa(\tau)$ is a modular form of level $\Gamma_1(p)$ and weight 2 with integral $q$-expansion at $\infty$. The inclusion $\Gamma(Np^r) \leq \Gamma_1(p)$ induces a map $X(Np^r) \to X_1(p)$ between the corresponding modular curves over $\mathbb{C}$. Pulling back $\kappa(\tau)$ under this map, we can view $\kappa(\tau)$ as a modular form of level $\Gamma(Np^r)$. By explicitly computing the $q$-expansion of $\kappa(\tau)$ at the cusp 0, we obtain the following lower bound.

**Theorem 1.2.** *a. For $p \geq 5$, $e$ is bounded below by*

$$2p^r \leq e.$$

*b. For $p \geq 5$ and $r = 1$, $e$ is equal to $2p$.*

This lower bound agrees with the upper bound when $r = 1$, allowing us to compute the exponent precisely in this case. We are also able to obtain an upper bound in the situation of cusp forms. Although Edixhoven only considers $\mathfrak{X}_0(Np)$ with $p \nmid N$, his method can be adapted to yield an upper bound in our situation as well. However, this upper bound is worse than the one we obtain by a factor of $p^r/(pr - r + 1)$, yet coincides for $r = 1$ (see Remark 5.33).

We now briefly summarize the contents of this thesis. Section 2 builds necessary background material. We recall the formulation and properties of the modular curve $\mathfrak{X}(Np^r)$ as presented in [KM85]. We also provide material on intersection theory for arithmetic surfaces, most of which comes from [Liu02, §8, §9]. In Section 3, we formulate our problem of computing $e$ geometrically, and the resulting formula for it in terms of the intersection theory of $\mathfrak{X}(Np^r)$, and the degree of $\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda$. We follow [BDP17, Appendix B] to provide an expression that calculates the exponent in the situation of $\mathfrak{X}(Np^r)$.

Section 4 is devoted to describing the intersection matrix of $\mathfrak{X}(Np^r)$, and then computing the entries of $T^{-1}$. In Section 5, we compute $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda)$. Combined with the work in Section 4, this culminates in Theorem 5.31 where we provide an upper bound for the exponent. Lastly, we construct explicit modular forms of level $p$ in Section 6 which provides a lower bound for the exponent.

# 2 Background Material

## 2.1 Moduli Problems for Elliptic Curves

We will summarize the formulation of the regular integral model of the modular curve of full level, as presented in [KM85]. Let $R$ be a ring and let $\mathrm{Ell}_R$ denote the category whose objects are elliptic curves $E \to S$ where $S$ is an $R$-scheme, and whose morphisms $(E \to S) \to (E' \to S')$ are Cartesian squares

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \xrightarrow[f]{} & S' \end{array}$$

**Definition 2.1.** By a **moduli problem (for elliptic curves over $R$)** we mean a contravariant functor $\mathcal{F} : \mathrm{Ell}_R \to \mathrm{Sets}$ from $\mathrm{Ell}_R$ to the category of sets. For any elliptic curve $E \to S$, we call the set $\mathcal{F}(E \to S)$ a **level $\mathcal{F}$-structure** on $E \to S$.

**Definition 2.2.**

- We say $\mathcal{F}$ is **representable** if it is representable as a functor on $\mathrm{Ell}_R$ *i.e.* if there exists an elliptic curve $\mathcal{E} \to \mathfrak{M}(\mathcal{F})$, where $\mathfrak{M}(\mathcal{F})$ is an $R$-scheme, such that we have a bijection

  $$\mathcal{F}(E \to S) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Ell}_R}(E \to S, \mathcal{E} \to \mathfrak{M}(\mathcal{F}))$$

  functorial in $E \to S$ for every elliptic curve $E \to S \in \mathrm{Ell}_R$. We call $\mathcal{E}$ the **universal elliptic curve** over $\mathfrak{M}(\mathcal{F})$.

- We say $\mathcal{F}$ is **relatively representable** if for every elliptic curve $E \to S$, the induced functor $\mathrm{Sch}_S \to \mathrm{Sets}$ where

  $$(T \to S) \mapsto \mathcal{F}(E \times_S T \to T)$$

  is representable by an $S$-scheme, denoted $\mathcal{F}_{E/S}$.

- Let $P$ be a property of morphisms of schemes. We say $\mathcal{F}$ has **property $P$ over** $\mathrm{Ell}_R$ if it is relatively representable and $\mathcal{F}_{E/S} \to S$ has property $P$ for every elliptic curve $E \to S$.

We will concern ourselves with the representability of moduli problems $\mathcal{F} : \mathrm{Ell}_R \to \mathrm{Sets}$. If $\mathcal{F}$ is representable, then $\mathcal{F}$ is relatively representable (see [KM85, 4.3.2]). The functor $\mathcal{F}$ induces a functor $\tilde{\mathcal{F}} : \mathrm{Sch}_R \to \mathrm{Sets}$ defined by

$$S \mapsto \{[(E \to S, \gamma)] : \gamma \in \mathcal{F}(E \to S)\}$$

where $[(E \to S, \gamma)]$ denotes the isomorphism class of the pair $(E \to S, \gamma)$. If $\mathcal{F}$ is representable by $\mathcal{E} \to \mathfrak{M}(\mathcal{F})$, then $\tilde{\mathcal{F}}$ is representable by $\mathfrak{M}(\mathcal{F})$. Indeed, from the bijection

$$\Psi : \mathcal{F}(E \to S) \to \mathrm{Hom}_{\mathrm{Ell}_R}(E \to S, \mathcal{E} \to \mathfrak{M}(\mathcal{F}))$$

we can associate to $\gamma \in \mathcal{F}(E \to S)$ a morphism

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & \mathcal{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ S & \xrightarrow[f]{} & \mathfrak{M}(\mathcal{F}) \end{array}$$

in $\mathrm{Ell}_R$. We define the map $\tilde{\Psi} : \tilde{\mathcal{F}}(S) \to \mathrm{Hom}_{\mathrm{Sch}_R}(S, \mathfrak{M}(\mathcal{F}))$ by sending $[(E \xrightarrow{\pi} S, \gamma)]$ to the map $f : S \to \mathfrak{M}(\mathcal{F})$. This correspondence is bijective and functorial in $S$ by the properties of $\Psi$.

However, it is possible for $\tilde{\mathcal{F}}$ to be representable even when $\mathcal{F}$ is not, due to the presence of non-trivial automorphisms of $(E \to S, \gamma)$ (see Example 2.5). More precisely, let $\sigma \in \mathrm{Aut}(E/S)$ be an automorphism of the elliptic curve $E \to S$. Then $\sigma$ induces a bijection $\bar{\sigma} : \mathcal{F}(E \to S) \to \mathcal{F}(E \to S)$; we let $\sigma$ act on $\mathcal{F}(E \to S)$ by $\sigma \cdot \gamma := \bar{\sigma}(\gamma)$.

**Definition 2.3.** Let $\mathcal{F}$ be a moduli problem of elliptic curves. We say $\mathcal{F}$ is **rigid** if $\mathrm{Aut}(E/S)$ acts freely on $\mathcal{F}(E \to S)$ for all elliptic curves $E \to S$.

**Proposition 2.4.** *Let $\mathcal{F}$ be a moduli problem of elliptic curves. If $\mathcal{F}$ is representable, then $\mathcal{F}$ is rigid.*

*Proof.* Let $\mathcal{F} : \mathrm{Ell}_R \to \mathrm{Sets}$ be representable by an elliptic curve $\mathcal{E} \to \mathcal{Y}$. Let $\sigma \in \mathrm{Aut}(E/S)$. By functoriality, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(E \to S) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ell}_R}(E \to S, \mathcal{E} \to \mathfrak{M}(\mathcal{F})) \\ \downarrow{\scriptstyle \bar{\sigma}} & & \downarrow{\scriptstyle \bar{\sigma}} \\ \mathcal{F}(E \to S) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ell}_R}(E \to S, \mathcal{E} \to \mathfrak{M}(\mathcal{F})) \end{array}$$

Any $\gamma \in \mathcal{F}(E \to S)$ uniquely corresponds to a morphism $(E \to S) \to (\mathcal{E} \to \mathfrak{M}(\mathcal{F}))$ which is equivalent to the data $f : S \to \mathfrak{M}(\mathcal{F})$ and an isomorphism $\iota : E \to \mathcal{E} \times_{\mathfrak{M}(\mathcal{F})} S$. Using the diagram, $\sigma \cdot \gamma \in \mathcal{F}(E \to S)$ uniquely corresponds to $f$ and $\iota \circ \sigma$.

Thus if $\sigma \cdot \gamma = \gamma$, then $\iota \circ \sigma = \iota$ which forces $\sigma = \mathrm{id}_{E/S}$. Hence $\mathrm{Aut}(E/S)$ acts freely on $\mathcal{F}(E \to S)$. □

**Example 2.5.** We will construct an example of a functor $\mathcal{F} : \mathrm{Ell}_R \to \mathrm{Sets}$ such that $\tilde{\mathcal{F}}$ is representable, but $\mathcal{F}$ is not. Fix an elliptic curve $\mathbf{E} \to \mathrm{Spec}(R)$ and define $\mathcal{F}$ by

$$\mathcal{F}(E \to S) = \begin{cases} \{\mathbf{E}\} & \text{if } E \simeq \mathbf{E} \times_R S \\ \varnothing & \text{otherwise} \end{cases}$$

Then the usual involution $[-1]$ in $\mathrm{Aut}(E/S)$ acts trivially on $\mathcal{F}(E \to S)$. Hence $\mathcal{F}$ is not rigid and therefore not representable. On the other hand, $\mathrm{Spec}(R)$ represents the functor $\tilde{\mathcal{F}}$.

In light of this example, we have the following:

**Proposition 2.6.** *Let $\mathcal{F}$ be a relatively representable moduli problem affine over $\mathrm{Ell}_{\mathbb{Z}}$. Then $\mathcal{F}$ is representable if and only if $\mathcal{F}$ is rigid.*

*Proof.* This is [KM85, 4.7.0]. □

**Definition 2.7.** Let $\mathcal{F}, \mathcal{F}' : \mathrm{Ell}_R \to \mathrm{Sets}$ be two moduli problems for elliptic curves. A **morphism between moduli problems** (**over** $\mathrm{Ell}_R$) is a natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$.

Let $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$ be a morphism between two representable moduli problems over $\mathrm{Ell}_R$. We have an induced natural transformation $\tilde{\eta} : \tilde{\mathcal{F}} \Rightarrow \tilde{\mathcal{F}}$. The map $\eta$ induces a map $\eta : \mathfrak{M}(\mathcal{F}) \to \mathfrak{M}(\mathcal{F}')$ which on $T$-points coincides with the map $\tilde{\eta}_T$ in the following diagram:

$$\begin{array}{ccc} \mathfrak{M}(\mathcal{F})(T) & \xrightarrow{\quad\eta\quad} & \mathfrak{M}(\mathcal{F}')(T) \\ \| & & \| \\ \mathrm{Hom}_{\mathrm{Sch}_R}(T, \mathfrak{M}(\mathcal{F})) & & \mathrm{Hom}_{\mathrm{Sch}_R}(T, \mathfrak{M}(\mathcal{F}')) \\ \big| \simeq & & \big| \simeq \\ \tilde{\mathcal{F}}(T) & \xrightarrow{\quad\eta_T\quad} & \tilde{\mathcal{F}}'(T) \end{array}$$

**Definition 2.8.** Let $\mathcal{F}$ and $\mathcal{F}'$ be two moduli problems for elliptic curves over $R$. We define the **product**, or **simultaneous**, **moduli problem** $\mathcal{F} \times \mathcal{F}'$ by

$$(E \to S) \mapsto \mathcal{F}(E \to S) \times \mathcal{F}'(E \to S).$$

Suppose $\mathcal{F}$ is representable by an elliptic curve $\mathcal{E} \to \mathfrak{M}(\mathcal{F})$ and $\mathcal{F}'$ is relatively representable. According to [KM85, 4.3.4], $\mathcal{F} \times \mathcal{F}'$ is representable and $\mathfrak{M}(\mathcal{F} \times \mathcal{F}') = \mathcal{F}'_{\mathcal{E}/\mathfrak{M}(\mathcal{F})}$ so we naturally have a map

$$\mathfrak{M}(\mathcal{F} \times \mathcal{F}') \to \mathfrak{M}(\mathcal{F}) \tag{1}$$

Following the notation in [KM85], we will usually denote $\mathfrak{M}(\mathcal{F} \times \mathcal{F}')$ by $\mathfrak{M}(\mathcal{F}, \mathcal{F}')$.

## 2.2 Modular Curve of Full Level

In this section we will introduce the modular curve of full level. We will first recall the situation over $\mathbb{C}$. Define the principal congruence subgroup of level $N$ by

$$\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\mathrm{mod}\, N} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{matrix} a, d \equiv 1 \,(\mathrm{mod}\, N) \\ b, c \equiv 0 \,(\mathrm{mod}\, N) \end{matrix} \right\}$$

which is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Let $\mathcal{H}$ denote the complex upper half plane $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by Mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d}.$$

Let $Y(N)$ denote the quotient $\Gamma(N)/\mathcal{H}$ with the quotient space topology.

In order to compactify $Y(N)$, we adjoin the points in $\mathbb{P}^1(\mathbb{Q})$, also known as the cusps. More precisely, $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{P}^1(\mathbb{Q})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) = (ax + by, cx + dy)$$

so the action of $\mathrm{SL}_2(\mathbb{Z})$ extends to the extended complex upper half plane $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Let $X(N)$ denote the quotient $\Gamma(N)/\mathcal{H}^*$. One can define the structure of a Riemann surface on $X(N)$ (cf. §1.8 in [Miy89]).

Let $\zeta_N = e^{2\pi i/N}$ be a primitive $N$th root of unity. The points of $Y(N)$ correspond to the data of an elliptic curve $E$ over $\mathbb{C}$ along with a basis $(P, Q)$ of the $N$-torsion group $E[N]$, which is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$, with Weil pairing $e_N(P, Q) = \zeta_N$. Recall any elliptic curve over $\mathbb{C}$ can be realized as a quotient $\mathbb{C}/\Lambda$ where $\Lambda$ is some $\mathbb{Z}$-lattice of $\mathbb{C}$. Any $\mathbb{Z}$-lattice $\Lambda \subset \mathbb{C}$ is homothetic to the lattice $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$ for some $\tau \in \mathcal{H}$. Two homothetic lattices produce isomorphic elliptic curves.

**Proposition 2.9.** *We have a bijection*

$$Y(N) \to \left\{ [E, (P, Q)] : \begin{array}{c} E \text{ is an elliptic curve over } \mathbb{C} \\ (P, Q) \text{ is a basis of } E[N] \\ \text{such that } e_N(P, Q) = \zeta_N \end{array} \right\}$$

*where we make the identification* $[E, (P, Q)] = [E', (P', Q')]$ *if there is an isomorphism* $\phi : E \to E'$ *such that* $\phi(P) = P'$ *and* $\phi(Q) = Q'$. *The bijection is given by*

$$\Gamma(N)\tau \mapsto \left[ \mathbb{C}/\Lambda_\tau, \left( \frac{\tau}{N} + \Lambda_\tau, \frac{1}{N} + \Lambda_\tau \right) \right].$$

*Proof.* This is Theorem 1.5.1(c) in [DS05]. □

Note that a basis $(P, Q)$ of $E[N]$ is equivalent to a group isomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ where

$$\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = P \text{ and } \phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = Q.$$

The modular curve of level $\Gamma(N)$ over more general rings will arise by generalizing the above moduli data of a basis of $E[N]$. The following definition is from Section 3.1 of [KM85].

**Definition 2.10.** Let $E$ be an elliptic curve over a scheme $S$. A section $s \in E(S)$ corresponds to a morphism[2] $s : S \to E$ whose composition with the structural morphism $E \to S$ is the identity on $S$. Since $E \to S$ is separated, $s$ is a closed immersion. We denote by $[s]$ the scheme-theoretic image of $s$ in $E$ which is an effective Cartier divisor on $E$. Consider the multiplication-by-$N$ isogeny $[N] : E \to E$ whose kernel we denote by $E[N]$, an $S$-group scheme.

[2] In general, a section of a morphism $X \to Y$ will be a locally closed embedding $s : Y \to X$. When $X \to Y$ is separated, as in the case of an elliptic curve, the map $s : Y \to X$ becomes a closed immersion. The associated divisor $[s]$ is in general defined to be the closure of the scheme-theoretic image of $s$ in $E$.

A $\Gamma(N)$-**structure** on $E \to S$ is a group homomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N](S)$ such that we have an equality of effective Cartier divisors

$$\sum_{v \in (\mathbb{Z}/N\mathbb{Z})^2} [\phi(v)] = E[N].$$

In this case, we call $P = \phi(1,0)$ and $Q = \phi(0,1)$ the corresponding **Drinfeld basis** of $E[N]$.

We define $[\Gamma(N)] : \mathrm{Ell}_{\mathbb{Z}} \to \mathrm{Sets}$ to be the moduli problem which assigns to each elliptic curve $E \to S$ the set of all $\Gamma(N)$-structures on $E \to S$. Let $(\alpha, f) : (E \to S) \to (E' \to S')$ be a morphism in the category $\mathrm{Ell}_R$. Then

$$[\Gamma(N)](\alpha, f) : [\Gamma(N)](E'/S') \to [\Gamma(N)](E/S)$$

is defined by sending a Drinfeld basis $(P, Q)$ of $E'[N]$ to $(\alpha^* P, \alpha^* Q)$. Indeed, $(\alpha^* P, \alpha^* Q)$ is a Drinfeld basis of $E[N]$ (see the paragraph proceeding [KM85, 1.4.1.2]).

**Example 2.11.** Let $M \geq 3$ and $N \geq 3$ be two coprime integers. Let $E \to S$ be an elliptic curve and let $(P, Q)$ be a Drinfeld basis of $E[MN](S)$. According to [KM85, 3.5.1], we have an isomorphism of moduli problems $[\Gamma(MN)] \simeq [\Gamma(M)] \times [\Gamma(N)]$ where $(P, Q) \mapsto ((NP, NQ), (MP, MQ))$.

For ease of notation, denote $\mathcal{F}_L = [\Gamma(L)]$ for $L \geq 1$. Define the map $\mathrm{pr} : \mathcal{F}_{MN} \Rightarrow \mathcal{F}_N$ where on an elliptic curve $E \to S$, the morphism $\mathrm{pr}_{E/S} : \mathcal{F}_{MN}(E/S) \to \mathcal{F}_N(E/S)$ is induced by $(P, Q) \mapsto (MP, MQ)$. We show pr is a natural transformation.

Let $(\alpha, f) : (E \to S) \to (E' \to S')$ be a morphism in the category $\mathrm{Ell}_R$. Since $\alpha^*(MP) = M\alpha^*(P)$, the diagram

$$\begin{array}{ccc} \mathcal{F}_{MN}(E'/S') & \xrightarrow{\mathrm{pr}_{E'/S'}} & \mathcal{F}_N(E'/S') \\ {\scriptstyle \mathcal{F}_{MN}(\alpha,f)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_N(\alpha,f)} \\ \mathcal{F}_{MN}(E/S) & \xrightarrow[\mathrm{pr}_{E/S}]{} & \mathcal{F}_N(E/S) \end{array}$$

commutes. Hence pr is a morphism of moduli problems. In particular, we obtain a map $\mathrm{pr} : \mathfrak{Y}(NM) \to \mathfrak{Y}(N)$ from the morphism of moduli problems

$$\mathrm{pr} : [\Gamma(MN)] \to [\Gamma(N)]$$

induced by forgetting the $[\Gamma(M)]$-structure.

We moreover consider the full level $N$ moduli problem over the cyclotomic integers $\mathbb{Z}[\zeta_N] := \mathbb{Z}[X]/(\Phi_N(X))$, where $\Phi_N(X) \in \mathbb{Z}[X]$ is the $N$th cyclotomic polynomial, by constructing a new moduli problem $[\Gamma(N)]^{\mathrm{can}}$ from $[\Gamma(N)]$ as follows (see also [KM85, 9.4.3.1]). For a $\mathbb{Z}[\zeta_N]$-algebra $A$, let $\zeta_N$ denote the image of $X$ mod $\Phi_N(X)$ under the map $\mathbb{Z}[\zeta_N] \to A$, following the convention in [KM85, 9.1.5]. Section 2.8 of [KM85] associates to the isogeny $[N]$ a bilinear pairing

$$e_N : E[N] \times E[N] \to \mu_N.$$

Define the moduli problem $[\Gamma(N)]^{\mathrm{can}} : \mathrm{Ell}_{\mathbb{Z}[\zeta_N]} \to \mathrm{Sets}$ which assigns to each elliptic curve $E \to S$ the set of all Drinfeld bases $(P, Q)$ such that $e_N(P, Q) = \zeta_N$.

**Theorem 2.12.** *Suppose $N \geq 3$. The moduli problem $[\Gamma(N)]^{\mathrm{can}}$ is represented by a regular scheme $\mathfrak{Y}(N)$ which is flat over $\mathrm{Spec}(\mathbb{Z}[\zeta_N])$ of dimension 2. Moreover, $\mathfrak{Y}(N)$ is smooth over $\mathbb{Z}[\zeta_N, 1/N]$.*

*Proof.* The moduli problem $[\Gamma(N)]$ is relatively representable and finite, flat over $\mathrm{Ell}_{\mathbb{Z}}$ by [KM85, 5.1.1]. Furthermore, $[\Gamma(N)]$ is rigid whenever $N \geq 3$ by [KM85, 2.7.2]. Therefore, by Proposition 2.6 and [KM85, 5.1.1], $[\Gamma(N)]$ is representable by a regular scheme flat over $\mathbb{Z}$ of dimension 2. By [KM85, 9.1.8, 9.1.9], the same things hold true for the associated canonical moduli problem $[\Gamma(N)]^{\mathrm{can}}$. Using [KM85, 5.1.1] again, $[\Gamma(N)]$ is étale over $\mathrm{Ell}_{\mathbb{Z}[1/N]}$, which, along with [KM85, 4.7.1], implies smoothness. □

*Remark* 2.13. Theorem 2.12 also holds for the moduli problem $[\Gamma(N)]^{\mathrm{can}}$ over $\mathrm{Ell}_R$, for any ring extension $\mathbb{Z}[\zeta_N] \to R$. For any moduli problem $\mathcal{F} : \mathrm{Ell}_{\mathbb{Z}} \to \mathrm{Sets}$ and ring $R$, let $\mathcal{F}_R : \mathrm{Ell}_R \to \mathrm{Sets}$ denote the moduli problem obtained by composing $\mathcal{F}$ with the forgetful functor $\mathrm{Ell}_R \to \mathrm{Ell}_{\mathbb{Z}}$. According to [KM85, 4.13], if $\mathcal{F}$ is relatively representable, then $\mathcal{F}_R$ is also relatively representable by the same morphism $\mathcal{F}_{E/S} \to S$ for any $R$-scheme $S$. Furthermore, if $\mathcal{F}$ is representable by $\mathcal{E} \to \mathfrak{M}(\mathcal{F})$, then $\mathcal{F}_R$ is representable by the base change $\mathcal{E}_R \to \mathfrak{M}(\mathcal{F})_R$. In other words, we have $\mathfrak{M}(\mathcal{F}_R) = \mathfrak{M}(\mathcal{F})_R$.

## 2.3 Compactified Modular Curve

In order to make $\mathfrak{Y}(N)$ proper, we will "compactify" our modular curve via a normalization process as described in [KM85, 8.6].

**Definition 2.14.** Let $R$ be a ring and let $\mathcal{G} : \mathrm{Sch}_R \to \mathrm{Sets}$ be a functor. For an $R$-scheme $S$, let $h_S : \mathrm{Sch}_R \to \mathrm{Sets}$ denote the functor $T \mapsto \mathrm{Hom}_{\mathrm{Sch}_R}(T, S)$. A **coarse moduli space** for $\mathcal{G}$ is an $R$-scheme $\mathbf{M}(\mathcal{G})$ along with a morphism $\phi : \mathcal{G} \Rightarrow h_{\mathbf{M}(\mathcal{G})}$ such that

1. For all algebraically closed fields $k$,
$$\phi(\mathrm{Spec}(k)) : \mathcal{G}(\mathrm{Spec}(k)) \to \mathbf{M}(\mathrm{Spec}(k))$$
is a bijection.
2. For all $R$-schemes $S$, and morphisms $\psi : \mathcal{G} \Rightarrow h_S$, there exists a unique map $\chi : \mathbf{M}(\mathcal{G}) \to S$ such that $\psi = h_\chi \circ \phi$, where $h_\chi : h_M \Rightarrow h_S$ is induced by postcomposing by $\chi$.

If $\mathcal{G}$ is representable by an $R$-scheme $\mathfrak{M}(\mathcal{G})$, then $\mathbf{M}(\mathcal{G})$ coincides with $\mathfrak{M}(\mathcal{G})$.

Let $R$ be a noetherian ring which is regular and excellent. For example, $R$ could be a finite type ring extension of $\mathbb{Z}$, a field, or a mixed characteristic DVR. Let $\mathcal{F} : \mathrm{Ell}_R \to \mathrm{Sets}$ be a moduli problem which is relatively representable and finite over $\mathrm{Ell}_R$. Existence of the coarse moduli space for $\tilde{\mathcal{F}}$ is given by [KM85, 8.1.1], while uniqueness can be deduced from the universal property. By §8.2 of [KM85], we have a unique morphism of moduli problems $\mathcal{F} \Rightarrow [\Gamma(1)]$ which defines a morphism of coarse moduli spaces $\mathbf{M}(\tilde{\mathcal{F}}) \to \mathbb{A}^1_R := \mathrm{Spec}(R[j])$. Moreover, by [KM85, 8.2.2], this map is finite due to the finiteness of $\mathcal{F}$.

Furthermore, we suppose the map $\mathbf{M}(\tilde{\mathcal{F}}) \to \mathbb{A}^1_R$ makes $\mathbf{M}(\tilde{\mathcal{F}})$ "normal near infinity" *i.e.* that there exists a monic polynomial $f(j) \in R[j]$ such that $\mathbf{M}(\tilde{\mathcal{F}})$ is normal over the open subset $U \subset \mathbb{A}^1_R$ where $f(j) \neq 0$. We will now extend $\mathbf{M}(\tilde{\mathcal{F}})$ to a scheme $\overline{\mathbf{M}}(\tilde{\mathcal{F}})$ over $\mathbb{P}^1_R$ via "normalizing near infinity" as described in [KM85,

8.6.3]. We define $\overline{\mathbf{M}}(\tilde{\mathcal{F}})$ by gluing together, over $\bar{U} := \mathbb{P}^1_R - V(f)$, the scheme $\mathbf{M}(\tilde{\mathcal{F}})$ with the normalization of $\bar{U}$ in $\mathbf{M}(\tilde{\mathcal{F}})|_U$. In particular, $\overline{\mathbf{M}}(\tilde{\mathcal{F}})$ is the unique scheme finite over $\mathbb{P}^1_R$ which agrees with $\mathbf{M}(\tilde{\mathcal{F}})$ over $\mathbb{A}^1_R$ and is normal over an open neighborhood of the $\infty$-section of $\mathbb{P}^1_R$.

We will make a quick comment on the finiteness of $\overline{\mathbf{M}}(\tilde{\mathcal{F}}) \to \mathbb{P}^1_R$. Since $R$ is excellent, $\mathbb{P}^1_R$ is an excellent scheme. Therefore $\bar{U} \subseteq \mathbb{P}^1_R$ is a Nagata scheme by [Liu02, Theorem 2.39(d)]. Because the map $\mathbf{M}(\tilde{\mathcal{F}})|_U \to \bar{U}$ is finite and $\mathbf{M}(\tilde{\mathcal{F}})|_U$ is reduced, we can use [Sta21, Tag03GR], to conclude the normalization of $\bar{U}$ in $\mathbf{M}(\tilde{\mathcal{F}})|_U$ is finite. In particular, $\overline{\mathbf{M}}(\tilde{\mathcal{F}}) \to \mathbb{P}^1_R$ is proper hence $\overline{\mathbf{M}}(\tilde{\mathcal{F}}) \to \mathrm{Spec}(R)$ is proper.

The moduli problem $[\Gamma(N)]^{\mathrm{can}}$ is relatively representable and finite by [KM85, 5.11, 9.1.7, 9.18] and normal near infinity by [KM85, 8.4.7, 5.1.1]. Let $\mathfrak{X}(N)$ denote the normalization of $\mathfrak{Y}(N)$ near infinity.

**Definition 2.15.** Let $\mathfrak{C}(\mathfrak{X}(N))$ denote the closed subscheme $\mathfrak{X}(N) - \mathfrak{Y}(N)$ of $\mathfrak{X}(N)$ endowed with the reduced scheme structure, called the **cuspidal locus** of $\mathfrak{X}(N)$. If the modular curve is clear from context, we denote $\mathfrak{C}(N) = \mathfrak{C}(\mathfrak{X}(N))$.

The following definition captures many of the desirable properties of $\mathfrak{X}(N)$.

**Definition 2.16.** Let $R$ be a Dedekind domain. We call a regular, integral, projective, flat $R$-scheme $X$ of dimension 2 an **arithmetic surface**.

The intersection theory on arithmetic surfaces is well-developed, for example, in Chapter 9 of [Liu02]. We will make heavy use of it on our modular curve.

**Theorem 2.17.** *Let $N \geq 3$. The scheme $\mathfrak{X}(N)$ is an arithmetic surface over $\mathbb{Z}[\zeta_N]$. Moreover, $\mathfrak{X}(N)$ is geometrically connected with reduced closed fibers and smooth over $\mathbb{Z}[\zeta_N, 1/N]$.*

*Proof.* We have already established that $\mathfrak{Y}(N)$ is regular and flat over $\mathbb{Z}[\zeta_N]$ of dimension 2. By [KM85, 10.9.1(2)], there exists an open neighborhood of $\mathfrak{C}(N)$ which is smooth over $\mathbb{Z}[\zeta_N]$. In particular, $\mathfrak{C}(N)$ is regular and flat over $\mathbb{Z}[\zeta_N]$ so $\mathfrak{X}(N)$ is regular and flat over $\mathbb{Z}[\zeta_N]$. Since $\mathfrak{X}(N)$ is connected and regular, $\mathfrak{X}(N)$ is integral.

By construction, $\mathfrak{X}(N)$ is proper over $\mathbb{Z}[\zeta_N]$, so by [Liu02, Theorem 8.3.16], we can conclude $\mathfrak{X}(N)$ is projective over $\mathbb{Z}[\zeta_N]$ hence is an arithmetic surface. By [KM85, 10.9.2(2)], $\mathfrak{X}(N)$ is geometrically connected and by [KM85, 13.8.4], $\mathfrak{X}(N)$ has reduced closed fibers. Smoothness follows from the smoothness of $\mathfrak{Y}(N)$ over $\mathbb{Z}[\zeta_N, 1/N]$. □

## 2.4 Intersection Theory on Arithmetic Surfaces

To better understand the geometry of the modular curve $\mathfrak{X}(N)$, we will summarize some properties of arithmetic surfaces, as presented in [Liu02, §8, §9]. We will also develop intersection theory on arithmetic surfaces. Let $R$ be a Dedekind domain, $K = \mathrm{Frac}(R)$, and $S = \mathrm{Spec}(R)$. The following lemma, which is [Liu02, 8.3.3], offers some properties of the fibers of an arithmetic surface.

**Lemma 2.18.** *Let $\pi : \mathfrak{X} \to S$ be an arithmetic and $\eta \in S$ the generic point. Then the generic fiber $\mathfrak{X}_\eta$ is a normal, integral curve over $k(\eta) = K$. For any $s \in S$, the fiber $\mathfrak{X}_s$ is a projective curve over $k(s)$.*

Note that if $x \in \mathfrak{X}$ is a codimension 1 point of an arithmetic surface, then $\mathcal{O}_{\mathfrak{X},x}$ is a normal, noetherian local ring of dimension 1. Hence $\mathcal{O}_{\mathfrak{X},x}$ is a DVR.

Recall a Weil divisor is a formal $\mathbb{Z}$-linear sum of irreducible closed subsets of codimension 1. The following, which is [Liu02, 8.3.4], characterizes the irreducible closed subsets of an arithmetic surface.

**Proposition 2.19.** *Let $\pi : \mathfrak{X} \to S$ be an arithmetic surface.*

*a. Let $x$ be a closed point of $\mathfrak{X}_\eta$. Then $\overline{\{x\}}$ is an irreducible closed subset of $X$ which is finite and surjective onto $S$.*

*b. Let $D$ be an irreducible closed subset of $\mathfrak{X}$ of dimension 1. Then either $D$ is an irreducible component of a closed fiber or $D = \overline{\{x\}}$ where $x$ is a closed point of $\mathfrak{X}_\eta$.*

*c. Let $x_0 \in \mathfrak{X}$ be closed. Then $\dim \mathcal{O}_{\mathfrak{X},x_0} = 2$.*

*Proof.* a. Let $D = \overline{\{x\}}$. Since $\pi$ is proper, it is a closed morphism so $\pi(D)$ is closed in $S$. Furthermore, because $D$ is irreducible and $\pi$ is dominant, $\pi(D)$ contains the generic point of $S$ so $\pi(D) = S$.

Since $x$ is closed in $\mathfrak{X}_\eta$, $D \neq \mathfrak{X}$ so $\dim D \leq 1$. By Lemma 2.18, $\dim \mathfrak{X}_s = 1$ for all $s \in S$. If $\dim(D \cap \mathfrak{X}_s) \neq 0$, then $D$ contains an irreducible component of $\mathfrak{X}_s$ and is therefore equal to such a component. This is a contradiction because $x \notin \mathfrak{X}_s$ for any closed point $s \in S$. Hence $\dim(D \cap \mathfrak{X}_s) = 0$ so $D \cap \mathfrak{X}_s$ is a finite set. In other words, $\pi|_D : D \to S$ is quasi-finite. Since $\pi$ is already projective, $\pi|_D$ is finite.

b. $\pi(D)$ is an irreducible closed subset of $S$. If $\pi(D) = \{s\}$, then $D \subseteq \mathfrak{X}_s$. Since $\dim D = \dim \mathfrak{X}_s = 1$, $D$ is an irreducible component of $\mathfrak{X}_s$. If $\pi(D)$ is not a single point, then it must be all of $S$ because $\pi(D)$ is irreducible. Therefore $D$ contains some point in $\mathfrak{X}_\eta$. Since $D \cap \mathfrak{X}_\eta \neq \emptyset$ and $D \cap \mathfrak{X}_\eta$ is quasi-compact, it contains a closed point $x$. By (a), $\dim \overline{\{x\}} = 1$ so $D = \overline{\{x\}}$.

c. Let $s = \pi(x_0)$ which is closed in $S$. Since $x_0$ is closed and $\mathfrak{X}_s$ is a curve, we have $\dim \mathcal{O}_{\mathfrak{X}_s,x_0} = 1$. Note that $\pi$ is a flat morphism between locally noetherian schemes, so by [Liu02, 4.3.12], we have

$$\dim \mathcal{O}_{\mathfrak{X},x_0} = \dim \mathcal{O}_{\mathfrak{X}_s,x_0} + \dim \mathcal{O}_{S,s} = 1 + 1 = 2$$

as desired. □

Since $\mathfrak{X} \to S$ is regular, we have, by [Liu02, 7.2.16], an isomorphism between Cartier divisors and Weil divisors

$$\mathrm{Div}(\mathfrak{X}) \to Z^1(\mathfrak{X})$$

given by

$$D \mapsto [D] := \sum_{\substack{x \in \mathfrak{X} \\ \dim \mathcal{O}_{\mathfrak{X},x} = 1}} \mathrm{mult}_x(D)[\overline{\{x\}}] \in Z^1(\mathfrak{X})$$

where $\mathrm{mult}_x(D) := \mathrm{mult}_{\mathcal{O}_{\mathfrak{X},x}}(D_x)$. Furthermore, this map respects principal divisors and effective divisors.

Let $Z$ be a closed subscheme of $\mathfrak{X}$ and let $\xi_1, \ldots, \xi_n$ be the generic points of $Z$. The Weil divisor associated to $Z$ is given by

$$[Z] := \sum_{i=1}^{n} \text{length}(\mathcal{O}_{\mathfrak{X},\xi_i})[\overline{\{\xi_i\}}].$$

**Definition 2.20.** If $Z$ is irreducible with generic point $\xi$, we call $\text{length}(\mathcal{O}_{\mathfrak{X},\xi})$ the **multiplicity** of $Z$ in $\mathfrak{X}$, denoted $\text{mult}_{\mathfrak{X}}(Z)$.

Note that $\text{mult}_{\mathfrak{X}}(Z) \geq 1$ and $\text{mult}_{\mathfrak{X}}(Z) = 1$ if and only if $\mathcal{O}_{\mathfrak{X},\xi}$ is reduced. Indeed, if $\text{length}(\mathcal{O}_{\mathfrak{X},\xi}) = 1$, then the nilradical ideal is forced to be trivial. Conversely, if $\mathcal{O}_{\mathfrak{X},\xi}$ is reduced, then $\mathcal{O}_{\mathfrak{X},\xi}$ is a field since $\xi$ corresponds to a minimal prime ideal. Hence $\text{length}(\mathcal{O}_{\mathfrak{X},\xi}) = 1$.

Let $\pi : \mathfrak{X} \to S$ be an arithmetic surface and let $D$ be an irreducible Weil divisor of $\mathfrak{X}$. Since $\mathfrak{X}$ is of dimension 2 and $D$ is of codimension 1, $D$ is of dimension 1. We can use Proposition 2.19 to characterize $D$.

**Definition 2.21.** If $D$ is an irreducible component of a closed fiber, or equivalently $\pi(D)$ is a point, then $D$ is called **vertical**. If $D$ is instead the closure of a closed point of $X_\eta$, or equivalently $\pi(D) = S$, then $D$ is called **horizontal**.

In general, a Weil divisor is horizontal (resp. vertical) if all its irreducible components are horizontal (resp. vertical).

Before defining the intersection number between two divisors globally, we will first establish the local intersection number. Let $D$ and $E$ be two effective divisors of $\mathfrak{X}$ with no common irreducible component. Let $x \in \mathfrak{X}$ be a closed point. In a small enough open neighborhood of $x$, we have

$$\text{Supp}(D) \cap \text{Supp}(E) = \{x\} \text{ or } \emptyset.$$

Since $\mathcal{O}_{\mathfrak{X},x}$ is a local regular ring of dimension 2, the only prime ideals of $\mathcal{O}_{\mathfrak{X},x}$ fit in the chain

$$\{0\} \subsetneq \mathfrak{m}_x \subsetneq \mathcal{O}_{\mathfrak{X},x}.$$

If $\text{Supp}(D) \cap \text{Supp}(E) = \{x\}$, then $\sqrt{\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x}$ is a non-zero prime ideal which must be equal to $\mathfrak{m}_x$. If $\text{Supp}(D) \cap \text{Supp}(E) = \emptyset$, then $\sqrt{\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x} = \mathcal{O}_{\mathfrak{X},x}$. Regardless, we always have

$$\mathfrak{m}_x \subseteq \sqrt{\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x}.$$

Therefore $\mathcal{O}_{\mathfrak{X},x}/(\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x)$ is a finite $\mathcal{O}_{\mathfrak{X},x}/\mathfrak{m}_x$-algebra so is Artinian and therefore of finite length.

**Definition 2.22.** We define the **(local) intersection number** of $D$ and $E$ at a point $x \in \mathfrak{X}$ by

$$i_x(D, E) = \text{length}_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{O}_{\mathfrak{X},x}/(\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x).$$

By definition, we immediately have that $i_x(D,E)$ is symmetric and bilinear. If $x \notin \mathrm{Supp}(D) \cap \mathrm{Supp}(E)$, then $i_x(D,E) = 0$.

**Example 2.23.** This example is from [Liu02, 9.1.2]. Let $k$ be a field, and let $X = \mathrm{Spec}(k[u,v])$. Let $x \in X$ correspond to the maximal ideal $(u,v)$ in $k[u,v]$. Consider the Cartier divisors $D = V(u)$ and $E = V(u+v^r)$ for some integer $r \geq 1$. Then we have

$$\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x = (u, u+v^r)_{(u,v)}.$$

Therefore

$$\begin{aligned} i_x(D,E) &= \mathrm{length}_{k[u,v]_{(u,v)}} \left(k[u,v]/(u,u+v^r)\right)_{(u,v)} \\ &= \mathrm{length}_{k[u,v]}(k[u,v]/(u,u+v^r)). \end{aligned}$$

Now $k[u,v]/(u,u+v^r) \simeq k[v]/(v^r)$ which is of length $r$, where $i_x(D,E) = r$.

We will now establish the intersection number between a general divisor of $\mathfrak{X}$ and a vertical divisor of a fixed closed fiber. For a fixed closed point $s \in S$, let $\mathrm{Div}_s(\mathfrak{X})$ denote the set of divisors of $\mathfrak{X}$ with support in $\mathfrak{X}_s$; such divisors are vertical divisors. A $\mathbb{Z}$-basis of $\mathrm{Div}_s(\mathfrak{X})$ consists of all irreducible components of $\mathfrak{X}_s$.

**Theorem 2.24.** *Let $s \in S$ be a closed point. Then there exists a unique bilinear map of $\mathbb{Z}$-modules*

$$i_s : \mathrm{Div}(\mathfrak{X}) \times \mathrm{Div}_s(\mathfrak{X}) \to \mathbb{Z}$$

*such that:*

*a. If $D \in \mathrm{Div}(\mathfrak{X})$ and $E \in \mathrm{Div}_s(\mathfrak{X})$ have no common component, then*

$$i_s(D,E) = \sum_{x \in \mathfrak{X}_s} i_x(D,E)[k(x) : k(s)]$$

*where the sum is over all closed points $x \in \mathfrak{X}_s$.*

*b. $i_s$ restricted to $\mathrm{Div}_s(\mathfrak{X}) \times \mathrm{Div}_s(\mathfrak{X})$ is symmetric.*

*c. $i_s(D,E) = i_s(D',E)$ if $D \sim D'$ are linearly equivalent.*

*d. If $0 < E \leq \mathfrak{X}_s$, then $i_s(D,E) = \deg_{k(s)} \mathcal{O}_{\mathfrak{X}}(D)|_E$.*

*e. If $D$ is principal, then $i_s(D,E) = 0$.*

*Proof.* This is proved in [Liu02, Theorem 9.1.12] besides part (e), which we prove. By [Liu02, Corollary 9.1.10], there exists a principal divisor $F$ such that $D+F$ and $E$ have no common component. Thus we may assume $D$ and $E$ have no common component. By (a), it suffices to show if $D$ is principal, then the local intersection $i_x(D,E)$ is zero. By definition,

$$i_x(D,E) = \mathrm{length}_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{O}_{\mathfrak{X},x}/(\mathcal{O}_{\mathfrak{X}}(-D)_x + \mathcal{O}_{\mathfrak{X}}(-E)_x).$$

Since $D$ is principal, $\mathcal{O}_{\mathfrak{X}}(-D) \simeq \mathcal{O}_{\mathfrak{X}}$. Thus

$$\begin{aligned} i_x(D,E) &= \text{length}_{\mathcal{O}_{\mathfrak{X},x}} \mathcal{O}_{\mathfrak{X},x}/(\mathcal{O}_{\mathfrak{X},x} + \mathcal{O}_{\mathfrak{X}}(-E)_x) \\ &= \text{length}_{\mathcal{O}_{\mathfrak{X},x}}(0) = 0. \end{aligned}$$

□

**Definition 2.25.** Let $\pi : \mathfrak{X} \to S$ be an arithmetic surface, and let $s \in S$ be closed. For any $(D,E)$ in $\text{Div}(\mathfrak{X}) \times \text{Div}_s(\mathfrak{X})$, we call $i_s(D,E)$ the **intersection number** as defined in Theorem 3.4. More generally, if $E$ is a vertical divisor, we define

$$D.E := \sum_{s \in S} i_s(D,E)[s]$$

where the sum is over all closed points of $S$, which is a 0-cycle on $S$. We call $E^2 = E.E$ the **self-intersection number** of $E$. If $D.E$ is only concentrated at a single point $s$, as in the case when $S$ is the spectrum of a DVR, we identify $D.E$ with the integer $i_s(D,E)$.

**Proposition 2.26.** *Let $\pi : \mathfrak{X} \to S$ be an arithmetic surface and fix a closed point $s \in S$. Let $\Gamma_1, \dots \Gamma_n$ be the irreducible components of $\mathfrak{X}_s$. We have an equality of Weil divisors*

$$[\mathfrak{X}_s] = \sum_{i=1}^{n} \text{mult}_{\mathfrak{X}_s}(\Gamma_i)\Gamma_i = [\pi^* s]$$

*where $[\pi^* s]$ is the Weil divisor associated to the Cartier divisor $\pi^* s$.*

*Proof.* The first equality follows from the definition of $[\mathfrak{X}_s]$. Let $t \in \mathcal{O}_{S,s}$ and $t_i \in \mathcal{O}_{\mathfrak{X},\xi_i}$ be uniformizers where $\xi_i$ is the generic point of $\Gamma_i$. Let $\nu_i$ denote the normalized valuation of $K(\mathfrak{X})$ associated to $\mathcal{O}_{\mathfrak{X},\xi_i}$ and let $\xi$ denote the generic point of $\mathfrak{X}$.

In $K(\mathfrak{X}) = \mathcal{O}_{\mathfrak{X},\xi}$, we can write $t = t_i^{\nu_i(t)} u_i$ for some $u_i \in \mathcal{O}_{\mathfrak{X},\xi_i}^{\times}$. Then

$$[\pi^* s] = \sum_{i=1}^{n} \nu_i(t)\Gamma_i.$$

Note that

$$\begin{aligned} \text{mult}_{\mathfrak{X}_s}(\Gamma_i) &= \text{length}(\mathcal{O}_{\mathfrak{X}_s,\xi_i}) \\ &= \text{length}(\mathcal{O}_{\mathfrak{X},\xi_i}/(t)) \\ &= \text{length}(\mathcal{O}_{\mathfrak{X},\xi_i}/(t_i^{\nu_i(t)})) \\ &= \nu_i(t). \end{aligned}$$

□

The next proposition, which is [Liu02, 9.1.21] illustrates how to compute the self-intersection of an irreducible component of a closed fiber.

**Proposition 2.27.** *Let $\pi : \mathfrak{X} \to S$ be an arithmetic surface and let $s \in S$ be a closed point.*

*a. For any $E \in \mathrm{Div}_s(\mathfrak{X})$, we have $E.\mathfrak{X}_s = \mathfrak{X}_s.E = 0$.*

*b. Let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of $\mathfrak{X}_s$ with multiplicities $\mu_1, \ldots, \mu_n$ respectively. For any $i = 1, \ldots, n$, we have*

$$\Gamma_i^2 = -\frac{1}{\mu_i} \sum_{i \neq j} \mu_j (\Gamma_j.\Gamma_i).$$

*Proof.* a) By [Liu02, 9.1.13], the intersection number remains the same after base change over $\mathrm{Spec}(\mathcal{O}_{S,s}) \to S$ so we may assume $S$ is local. Then $s$ is a principal Cartier divisor, so the pullback $\pi^* s$ is also principal. By Proposition 2.26, $\mathfrak{X}_s$ is principal so by Theorem 2.24, we have $E.\mathfrak{X}_s = 0$.

b) By (a), we have

$$0 = \Gamma_i.\mathfrak{X}_s = \Gamma_i.\left(\sum_{j=1}^{n} \mu_j \Gamma_j\right) = \sum_{j=1}^{n} \mu_j (\Gamma_i.\Gamma_j) = \mu_i \Gamma_i^2 + \sum_{j \neq i} \mu_j (\Gamma_j.\Gamma_i). \qquad \square$$

Let $\mathrm{Div}_s(\mathfrak{X})_{\mathbb{R}} := \mathrm{Div}_s(\mathfrak{X}) \otimes_{\mathbb{Z}} \mathbb{R}$, which is a finite dimensional vector space over $\mathbb{R}$. Then the intersection pairing gives a symmetric $\mathbb{R}$-bilinear form

$$\langle \cdot, \cdot \rangle_s : \mathrm{Div}_s(\mathfrak{X})_{\mathbb{R}} \times \mathrm{Div}_s(\mathfrak{X})_{\mathbb{R}} \to \mathbb{R}.$$

The following is from [Liu02, 9.1.23] (see also [Lan88, III, 3.4])

**Theorem 2.28.** *Let $\mathfrak{X} \to S$ be an arithmetic surface and let $s \in S$ be a closed point. Then $\langle \cdot, \cdot \rangle_s$ is negative semi-definite. Moreover, if $\mathfrak{X}_s$ is connected, then $\langle v, v \rangle_s = 0$ if and only if $v \in \mathfrak{X}_s \mathbb{R}$.*

*Proof.* Let $\Gamma_1, \ldots, \Gamma_n$ be the irreducible components of $\mathfrak{X}_s$ with multiplicities $\mu_1, \ldots, \mu_n$ respectively. We may assume $n \geq 2$ as the $n = 1$ case can be deduced from Proposition 2.27(b).

Set $b_{ij} := \mu_i \mu_j \Gamma_i.\Gamma_j$. Note that $b_{ij} \geq 0$ if $i \neq j$ and $b_{ij} = b_{ji}$. Also note

$$\sum_{j=1}^{n} b_{ij} = \mathfrak{X}_s.(\mu_i \Gamma_i) = 0 \tag{2}$$

and similarly $\sum_{i=1}^{n} b_{ij} = 0$. Consider a general element $v = \sum x_i \Gamma_i \in \mathrm{Div}_s(\mathfrak{X})_{\mathbb{R}}$ and set $y_i = x_i / \mu_i$. Then we have

$$\begin{aligned} \langle v, v \rangle_s &= \sum_{1 \leq i,j \leq n} (\Gamma_i.\Gamma_j) x_i x_j \\ &= \sum_{1 \leq i,j \leq n} b_{ij} y_i y_j \\ &= 2 \sum_{1 \leq i < j < n} b_{ij} y_i y_j + \sum_{1 \leq i \leq n} b_{ii} y_i^2 \end{aligned} \tag{3}$$

Note that by the above equality (2), we have

$$0 = \sum_{1\le j\le n} b_{ij}y_i^2 = b_{ii}y_i^2 + \sum_{\substack{1\le j\le n\\ j\neq i}} b_{ij}y_i^2.$$

Substituting $b_{ii}y_i^2$, in equation (3) with $-\sum_{\substack{1\le j\le n\\ j\neq i}} b_{ij}y_i^2$, we have

$$\begin{aligned}\langle v,v\rangle_s &= 2\sum_{1\le i<j<n} b_{ij}y_iy_j - \sum_{1\le i\le n}\sum_{\substack{1\le j\le n\\ j\neq i}} b_{ij}y_i^2\\ &= 2\sum_{1\le i<j<n} b_{ij}y_iy_j - \sum_{1\le i<j\le n} b_{ij}y_i^2 - \sum_{1\le j<i\le n} b_{ij}y_i^2.\end{aligned}$$

Using the fact $b_{ij} = b_{ji}$ and swapping the indices $i$ and $j$ in the last sum, we get

$$\begin{aligned}&= 2\sum_{1\le i<j<n} b_{ij}y_iy_j - \sum_{1\le i<j\le n} b_{ij}y_i^2 - \sum_{1\le i<j\le n} b_{ij}y_j^2\\ &= -\sum_{1\le i<j\le n} b_{ij}(y_i-y_j)^2 \le 0. \qquad (4)\end{aligned}$$

so $\langle\cdot,\cdot\rangle_s$ is negative semi-definite.

Suppose $\langle v,v\rangle_s = 0$. By (4), we have $y_i = y_j$ whenever $b_{ij}\neq 0$ or $\Gamma_i.\Gamma_j\neq 0$. This implies $y_1 = y_i$ for all $i$: indeed as $\mathfrak{X}_s$ is connected there exists a sequence of irreducible components $\Gamma_1 = \Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_m} = \Gamma_i$ with $\Gamma_{i_k}.\Gamma_{i_{k+1}} > 0$, whence $y_{i_k} = y_{i_{k+1}}$ for $k = 1,\ldots,m-1$. Therefore $y_1 = y_i$ as claimed so $v = y_1\mathfrak{X}_s$. □

We end this section with some results on the intersection between a horizontal divisor and a closed fiber. The following is [Liu02, 9.1.30].

**Proposition 2.29.** *Let $\pi : \mathfrak{X}\to S$ be an arithmetic surface. Let $\eta$ be the generic point of $S$ and $s\in S$ a closed point. Then for any closed point $P\in\mathfrak{X}_\eta$, we have*

$$\overline{\{P\}}.\mathfrak{X}_s = [K(P):K(S)]$$

*where $\overline{\{P\}}$ is the Zariski closure of $\{P\}$ in $\mathfrak{X}$, endowed with the reduced closed subscheme structure.*

**Corollary 2.30.** *Let $\pi : \mathfrak{X}\to S$ be an arithmetic surface and let $P\in\mathfrak{X}_\eta$ be a $K(S)$-rational point. Then $\overline{\{P\}}\cap\mathfrak{X}_s$ is reduced to a single point $z\in\mathfrak{X}_s(k(s))$ and $\overline{\{P\}}.\mathfrak{X}_s = 1$.*

*In particular, $\overline{\{P\}}$ intersects exactly one irreducible component $\Gamma$ of $\mathfrak{X}_s$. Moreover, $\Gamma$ has multiplicity 1 and $\overline{\{P\}}.\Gamma = 1$.*

*Proof.* Since $P$ is $K(S)$-rational, by Proposition 2.29 we have $\overline{\{P\}}.\mathfrak{X}_s = 1$. Let $\Gamma_1,\ldots,\Gamma_n$ denote the

irreducible components of $\mathfrak{X}_s$ with multiplicities $\mu_1, \ldots, \mu_n$ respectively. Then

$$1 = \overline{\{P\}}.\mathfrak{X}_s = \sum_{i=1}^{n} \mu_i \overline{\{P\}}.\Gamma_i.$$

Since $\overline{\{P\}}$ and $\Gamma_i$ are effective divisors with no common components, $\overline{\{P\}}.\Gamma_i \geq 0$ which forces $\mu_i \overline{\{P\}}.\Gamma_i = 1$ for some $i$. Hence $\mu_i = \overline{\{P\}}.\Gamma_i = 1$. Moreover, $\mu_j \geq 1$ so $\overline{\{P\}}.\Gamma_j = 0$ for $j \neq i$ i.e. $\overline{\{P\}}$ does not intersect $\Gamma_j$. □

## 2.5 Modular Sheaf

Let $N \geq 3$ and let $R$ be a noetherian, regular, excellent $\mathbb{Z}[\zeta_N]$-algebra. We will construct an invertible sheaf on $\mathfrak{X}(N)_{/R}$ whose global sections will be defined as the space of modular forms. Let $f : \mathcal{E} \to \mathfrak{Y}(N)_{/R}$ denote the universal elliptic curve. We define $\underline{\omega} := f_*\Omega^1_{\mathcal{E}/\mathfrak{Y}(N)_{/R}}$ to be the pushforward of the sheaf of Kahler differentials on $\mathcal{E}$. If $e : \mathfrak{Y}(N)_{/R} \to \mathcal{E}$ is the identity section of our elliptic curve, then we have $\underline{\omega} \simeq e^*\Omega^1_{\mathcal{E}/\mathfrak{Y}(N)_{/R}}$. Hence $\underline{\omega}$, being the pullback of an invertible sheaf, is an invertible sheaf on $\mathfrak{Y}(N)_{/R}$.

According to [KM85, 10.13.2], there is a canonical way to extend $\underline{\omega}^{\otimes 2}$ to an invertible sheaf on $\mathfrak{X}(N)_{/R}$, which we denote by $\underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R}}$ or simply $\underline{\omega}^{\otimes 2}$ if the modular curve is clear from context.

**Proposition 2.31.** *The invertible sheaf $\underline{\omega}^{\otimes 2}$ canonically extends to an invertible sheaf $\underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R}}$ on $\mathfrak{X}(N)_{/R}$.*

**Definition 2.32.** We call the invertible sheaf $\underline{\omega}^{\otimes 2}$ on $\mathfrak{X}(N)_{/R}$ the **modular sheaf** (**of weight 2**). The global sections $H^0(\mathfrak{X}(N)_{/R}, \underline{\omega}^{\otimes 2})$ are known as **modular forms of weight** 2 **and level** $\Gamma(N)$. The global sections $H^0(\mathfrak{X}(N)_{/R}, \underline{\omega}^{\otimes}(-\mathfrak{C}(N))$ are known as **cusp forms** of weight 2 and level $\Gamma(N)$.

The formation of the modular sheaf $\underline{\omega}^{\otimes 2}$ behaves well under base change. The following proposition is [KM85, 10.13.6].

**Proposition 2.33.** *Let $R \to R'$ be an extension of noetherian, regular, excellent rings and let*

$$j : \mathfrak{X}(N)_{/R'} \to \mathfrak{X}(N)_{/R}$$

*denote the induced base change map. Then we have an isomorphism of invertible sheaves*

$$j^*\underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R}} \simeq \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R'}}.$$

Let $f : E \to S$ be an elliptic curve over a smooth $R$-scheme $S$. According to [KM85, 10.13.10] (see also [Kat73, A1.4]), we have map $(f_*\Omega^1{}_{E/S})^{\otimes 2} \to \Omega^1_{S/R}$ of $\mathcal{O}_S$-modules, known as the Kodaira-Spencer map, which becomes an isomorphism precisely when $E \to S$ represents a moduli problem which is étale over $\mathrm{Ell}_R$. In particular, if $\mathfrak{Y}(N)_{/R}$ is smooth over $R$ (*e.g.* if $N$ is a unit in $R$), we get an isomorphism $\underline{\omega}^{\otimes 2}_{\mathfrak{Y}(N)_{/R}} \simeq \Omega^1_{\mathfrak{Y}(N)_{/R}/R}$.

**Theorem 2.34.** *Let $R$ be a noetherian, regular, excellent $\mathbb{Z}[\zeta_N]$-algebra containing $1/N$. The Kodaira-Spencer isomorphism*

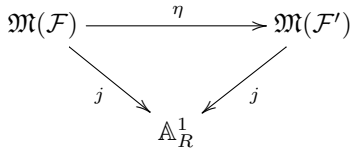
$$\underline{\omega}^{\otimes 2}_{\mathfrak{Y}(N)_{/R}} \simeq \Omega^1_{\mathfrak{Y}(N)_{/R}/R}$$

*on $\mathfrak{Y}(N)_{/R}$ extends to an isomorphism on $\mathfrak{X}(N)_{/R}$*

$$\underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R}} \simeq \Omega^1_{\mathfrak{X}(N)_{/R}/R}(\mathfrak{C}(N)_{/R}).$$

*Proof.* By [KM85, 5.1.1], the moduli problem $[\Gamma(N)]^{\mathrm{can}}$ is finite étale over $\mathrm{Ell}_R$. The result then follows from [KM85, 10.13.11]. □

Suppose $\mathcal{F}, \mathcal{F}'$ are two representable moduli problems that are finite over $\mathrm{Ell}_R$ and normal near infinity. Any morphism $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$ of moduli problems is compatible with the usual morphism $\mathcal{F}, \mathcal{F}' \Rightarrow [\Gamma(1)]$. Thus the induced map $\eta : \mathfrak{M}(\mathcal{F}) \to \mathfrak{M}(\mathcal{F}')$ fits in the commutative diagram




Hence $\eta$ extends to a map $\bar{\eta} : \overline{\mathfrak{M}}(\mathcal{F}) \to \overline{\mathfrak{M}}(\mathcal{F}')$ on the normalizations near infinity.

**Proposition 2.35.** *Let $R$ be an excellent, noetherian, regular ring and let $\mathcal{F}$ and $\mathcal{F}'$ be two representable moduli problems, both finite over $\mathrm{Ell}_R$ and both normal near infinity. Let $\eta : \mathcal{F} \Rightarrow \mathcal{F}'$ be a morphism of moduli problems over $\mathrm{Ell}_R$. Under the induced map $\bar{\eta} : \overline{\mathfrak{M}}(\mathcal{F}) \to \overline{\mathfrak{M}}(\mathcal{F}')$ we have*

$$\bar{\eta}^*(\underline{\omega}^{\otimes 2}_{\overline{\mathfrak{M}}(\mathcal{F}')}) = \underline{\omega}^{\otimes 2}_{\overline{\mathfrak{M}}(\mathcal{F})}.$$

*Proof.* This is [KM85, 10.13.5(2)]. □

## 2.6 Cusps

In this section we will investigate the cusps of $\mathfrak{X}(N)$ and recall the theory of $q$-expansions for modular forms. First we establish that the formation of the cuspidal locus $\mathfrak{C}(N)$ and the formation of $\mathfrak{X}(N)$ behaves well under base change. The following is [KM85, 8.6.6, 8.6.7].

**Proposition 2.36.** *Let $R$ and $R'$ be excellent, noetherian, regular $\mathbb{Z}[\zeta_N]$-algebras. For any extension of scalars $R \to R'$, we have*

$$\mathfrak{Y}(N)_{/R} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \simeq \mathfrak{Y}(N)_{/R'}$$

$$\mathfrak{C}(\mathfrak{X}(N)_{/R}) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \simeq \mathfrak{C}(\mathfrak{X}(N)_{/R'})$$

$$\mathfrak{X}(N)_{/R} \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R') \simeq \mathfrak{X}(N)_{/R'}$$

For two groups $G_1$ and $G_2$, let $\mathrm{HomSurj}(G_1, G_2)$ denote the set of surjective group homomorphisms from $G_1$ to $G_2$. Any subgroup $\Gamma \leq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts on the set

$$\mathrm{HomSurj}((\mathbb{Z}/N\mathbb{Z})^2, \mathbb{Z}/N\mathbb{Z})$$

via $\gamma \cdot \Lambda = \Lambda \circ \gamma$ for any $\gamma \in \Gamma$. Let

$$\mathrm{HS}(N) = \mathrm{HomSurj}((\mathbb{Z}/N\mathbb{Z})^2, \mathbb{Z}/N\mathbb{Z}) / \{\pm I\}$$

where $I$ is the identity matrix in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

According to [KM85, 10.9.1], the cusps $\mathfrak{C}(N)$ of $\mathfrak{X}(N)$ is a disjoint union of $\#\mathrm{HS}(N)$ many sections of $\mathfrak{X}(N)$. Furthermore, the formal completion of $\mathfrak{X}(N)$ along $\mathfrak{C}(N)$ is the disjoint union of $\#\mathrm{HS}(N)$ copies of the formal spectrum $\mathrm{Spf}(\mathbb{Z}[\zeta_N]\,[\![q^{1/h_\Lambda}]\!])$ for some integer $h_\Lambda \geq 1$ dividing $N$, dependent on the index $\Lambda \in \mathrm{HS}(N)$. Refer to [KM85, 10.2.5] which provides a canonical bijection between a cuspidal section and the corresponding label in $\mathrm{HS}(N)$.

**Proposition 2.37.** *Let $R$ be a Dedekind domain with fraction field $K$. As Weil divisors, we have*

$$\mathfrak{C}(\mathfrak{X}(N)_{/R}) = \sum_x \overline{\{x\}}$$

*where the sum is indexed over $x \in \mathfrak{C}(\mathfrak{X}(N)_{/K})$.*

*Proof.* As a divisor, $\mathfrak{C}(\mathfrak{X}(N)_{/R})$ consists only of horizontal components since $\mathfrak{C}(\mathfrak{X}(N)_{/R})$ is a finite disjoint union of sections of $\mathfrak{X}(N)_{/R}$. By Proposition 2.19, these horizontal components are necessarily the closure, in $\mathfrak{X}(N)_{/R}$, of points closed in the generic fiber $\mathfrak{X}(N)_{/K}$. By Proposition 2.36, $\mathfrak{C}(\mathfrak{X}(N)_{/R})_{/K} \simeq \mathfrak{C}(\mathfrak{X}(N)_{/K})$, hence these points are precisely the cusps of $\mathfrak{X}(N)_{/K}$. □

Now we recall how the $q$-expansion map

$$H^0(\mathfrak{X}(N)_{/R}, \underline{\omega}^{\otimes 2}) \to R[[q^{1/N}]]$$

is defined for a modular form over a $\mathbb{Z}[\zeta_N]$-algebra $R$ at a specified cusp. Let $c$ be a cusp of $\mathfrak{X}(N)$, which corresponds to the image of a section $\mathrm{Spec}(\mathbb{Z}[\zeta_N]) \to \mathfrak{X}(N)$ and is a connected component $\overline{\{c'\}}$ of $\mathfrak{C}(N)$ where $c'$ is a cusp of the generic fiber. According to [DR73, §VII, 2.3.2], the cusp $c$ corresponds to a Tate curve $\mathrm{Tate}(q^{1/N})$ over $\mathbb{Z}[\zeta_N][[q^{1/N}]$ (see [DR73, VII, Définition 1.16]) and a closed immersion $\mathrm{Spec}(\mathbb{Z}[\zeta_N][[q^{1/N}]]) \to \mathfrak{X}(N)$ fitting in the diagram

$$\begin{array}{ccc} \mathrm{Tate}(q^{1/N}) & & \\ \downarrow & & \\ \mathrm{Spec}(\mathbb{Z}[\zeta_N][[q^{1/N}]] & \longrightarrow & \mathfrak{X}(N) \end{array}$$

By [DR73, §VII, Corollaire 2.4] (see also [KM85, 10.9.1]), this closed immersion induces an isomorphism of formal schemes

$$\mathrm{Spf}(\mathbb{Z}[\zeta_N][[q^{1/N}]]) \xrightarrow{\sim} \hat{\mathfrak{X}}(N)_c$$

where $\hat{\mathfrak{X}}(N)_c$ denotes the formal completion of $\mathfrak{X}(N)$ along the cusp $c$. According to [KM85, 8.8(T.2)] (see also [DR73, VII, 1.16.1]) we have an isomorphism

$$\phi : \hat{\mathrm{Tate}}(q^{1/N}) \xrightarrow{\sim} \hat{\mathbb{G}}_m$$

between formal Lie groups. Let $\omega_{\mathrm{can},c} := \phi^*(dX/X)$, where $dX/X$ is the standard invariant differential on $\mathbb{G}_m$. Pulling back by $\mathrm{Tate}(q^{1/N}) \to \mathfrak{X}(N)$ allows us to identify $\hat{\omega}_c^{\otimes 2}$ with $\mathbb{Z}[\zeta_N][[q^{1/N}]] \cdot \omega_{\mathrm{can},c}$.

**Definition 2.38.** Let $f \in H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ and let $c$ be a cusp of $\mathfrak{X}(N)$. Viewing $f$ in the completed stalk $\hat{\underline{\omega}}_c^{\otimes 2}$, we write $f = f_c \cdot \omega_{\mathrm{can},c}$ for some $f_c \in \hat{\mathcal{O}}_{\mathfrak{X}(N),c}$. We call $f_c$ the **$q$-expansion of $f$ at $c$**. This gives a map

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) \to \mathbb{Z}[\zeta_N][[q^{1/N}].$$

More generally, for a $\mathbb{Z}[\zeta_N]$-algebra $R$, we obtain the $q$-expansion map

$$H^0(\mathfrak{X}(N)_{/R}, \underline{\omega}^{\otimes 2}) \to \mathbb{Z}[\zeta_N][[q^{1/N}] \otimes_{\mathbb{Z}[\zeta_N]} R \subset R[[q^{1/N}]]$$

by tensoring with $R$. We will always use $\omega_{\mathrm{can},c}$ as a local generator for $\omega^{\otimes 2}$ to obtain the $q$-expansion at $c$.

The following result, known as the $q$-expansion principle, essentially says $q$-expansions can detect the "ring of definition".

**Proposition 2.39.** *Let $A$ be a $\mathbb{Z}[\zeta_N]$-algebra, let $B$ be a subalgebra of $A$, and let $f \in H^0(\mathfrak{X}(N)_{/A}, \underline{\omega}^{\otimes k})$. If the $q$-expansion of $f$ at every cusp of $\mathfrak{X}(N)_{/A}$ lies in $B[[q^{1/N}]]$, then $f$ is a modular form over $B$ i.e. $f \in H^0(\mathfrak{X}(N)_{/B}, \underline{\omega}^{\otimes k})$.*

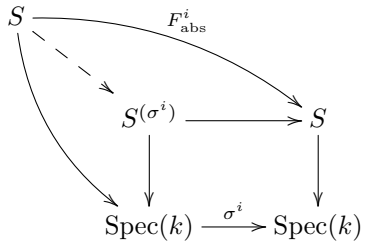
*Proof.* This is [DR73, VII, Théorème 3.9]. □

## 2.7 Special Fiber of $\mathfrak{X}(Np^r)$

Let $N \geq 3$ and $r \geq 0$ be integers and let $p \geq 2$ be a prime such that $p \nmid N$. In this section, we describe the special fiber of $\mathfrak{X}(Np^r)$. For simplicity, we consider the modular curve $\mathfrak{X}(Np^r)$ over a $\mathbb{Z}[\zeta_{Np^r}]$-algebra $R$ which is a DVR of mixed characteristic $(0, p)$ in which $N$ is invertible and with fraction field $K$ and perfect residue field $k$. Let $\bar{\mathfrak{X}}(Np^r)$ denote the special fiber of $\mathfrak{X}(Np^r)$.

We will recall the theory of Igusa curves, as developed in Section 12 of [KM85], to describe the irreducible components of $\bar{\mathfrak{X}}(Np^r)$.

For $i \in \mathbb{Z}$, let $\sigma^i : \mathrm{Spec}(k) \to \mathrm{Spec}(k)$ denote the map induced by the $i$th power of the Frobenius automorphism on $k$. For any $k$-scheme $S$, we let $S^{(\sigma^i)}$ denote the pullback under $\sigma^i$ from which we also obtain a map $F^i_{\mathrm{abs}} : S \to S$ called the absolute Frobenius. The maps $\sigma^i$ and $F^i_{\mathrm{abs}}$ sit in the Cartesian diagram




More generally, for any scheme $X \to S$, we let $X^{(p^i)}$ denote the pullback under $F^i_{\mathrm{abs}}$ i.e. $X^{(p^i)}$ is defined by the Cartesian diagram

$$\begin{array}{ccc} X^{(p^i)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{F^i_{\mathrm{abs}}} & S \end{array}$$

from which we obtain a morphism $F_{X/S} : X \to X^{(p^i)}$ of $S$-schemes, called the $i$th-fold relative Frobenius.

Let $\mathcal{F}$ be a moduli problem on $\mathrm{Ell}_k$. We define the moduli problem $\mathcal{F}^{(\sigma^i)}$ on $\mathrm{Ell}_k$ by extending scalars via $\sigma^i$ so that

$$\mathcal{F}(E \to S) = \mathcal{F}^{(\sigma^i)}(E^{(\sigma^i)} \to S^{(\sigma^i)}).$$

If $\mathcal{F}$ is representable and finite over $\mathrm{Ell}_k$ and normal near infinity, then the same holds for $\mathcal{F}^{(\sigma^i)}$ and we have

$$\overline{\mathfrak{M}}(\mathcal{F}^{(\sigma^i)}) = \overline{\mathfrak{M}}(\mathcal{F})^{(\sigma^i)}.$$

**Definition 2.40.** Let $E$ be an elliptic curve over an $k$-scheme $S$. An **Igusa structure of level** $p^r$ on $E \to S$ is a point $P \in E^{(p^r)}(S)$ which generates in the sense of [KM85, 1.4.1] the kernel of Vershiebung

$$V^r : E^{(p^r)} \to E.$$

Let $[\mathrm{Ig}(p^r)] : \mathrm{Ell}_k \to \mathrm{Sets}$ denote the moduli problem which assigns to each elliptic curve $E \to S$ the set of all Igusa structures of level $p^r$ on $E \to S$.

According to [KM85, 12.7.1] , if $\mathcal{F}$ is a representable moduli problem finite over $\mathrm{Ell}_k$ which is normal near infinity, the simultaneous moduli problem $[\mathrm{Ig}(p^r)] \times \mathcal{F}$ is representable over $\mathrm{Ell}_k$ and normal near infinity.

The following result is [KM85, 12.7.2] applied to the simultaneous moduli problem $[\mathrm{Ig}(p^r)] \times [\Gamma(N)]^{\mathrm{can},(\sigma^{-i})}$ over $\mathrm{Ell}_k$ for any $i \in \mathbb{Z}$. We denote $\mathrm{Ig}(p^r, N) = \overline{\mathfrak{M}}([\mathrm{Ig}(p^r)], [\Gamma(N)]^{\mathrm{can},(\sigma^{-i})})$.

**Proposition 2.41.** *a.* $\mathrm{Ig}(p^r, N)$ *is a proper smooth curve over* $k$.

*b. The usual projection* $\mathrm{Ig}(p^r, N) \to \mathfrak{X}(N)^{(\sigma^{-i})}_{/k}$ *is finite and étale outside the supersingular points of* $\mathfrak{X}(N)^{(\sigma^{-i})}_{/k}$ *for all* $i \in \mathbb{Z}$.

Next we define another moduli problem closely related to Igusa structures, which will directly appear in the description of $\bar{\mathfrak{X}}(Np^r)$.

**Definition 2.42.** Let $E$ be an elliptic curve over an $k$-scheme $S$ and fix $1 \leq i \leq r$. An **exotic Igusa structure of level** $(p^r, i)$ on $E \to S$ is a point $P \in E(S)$ such that $(O, P)$ is a Drinfeld $p^i$-basis of $E \to S$ along with a point $Q \in E^{(p^{r-i})}(S)$ such that $V^{r-i}(Q) = P$. Let

$$[\mathrm{ExIg}(p^r, i)] : \mathrm{Ell}_k \to \mathrm{Sets}$$

denote the moduli problem which assigns to each elliptic curve $E \to S$ the set of all exotic Igusa structures of level $(p^r, i)$ on $E \to S$.

The following result, which is [KM85, 12.10.6] in the situation $\mathcal{F} = [\Gamma(N)]^{\mathrm{can},(\sigma^{-i})}$, relates Igusa structures with exotic Igusa structures and also establishes the representability of $[\mathrm{ExIg}(p^r, i)]$. We denote

$$\mathrm{ExIg}(p^r, i, N) = \overline{\mathfrak{M}}([\mathrm{ExIg}(p^r, i)], [\Gamma(N)]).$$

**Theorem 2.43.** *For $1 \leq i \leq r$, we have a canonical isomorphism $\mathrm{Ig}(p^r, N) \simeq \mathrm{ExIg}(p^r, i, N)$ sitting in the commutative diagram*

$$\begin{array}{ccc} \mathrm{Ig}(p^r, N) & \xrightarrow{\simeq} & \mathrm{ExIg}(p^r, i, N) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ \mathfrak{X}(N)_{/k}^{(\sigma^{-i})} & \xrightarrow{F^i} & \mathfrak{X}(N)_{/k} \end{array}$$

*where $F^i$ is the ith-fold relative Frobenius $F^i_{\mathfrak{X}(N)/k}$.*

We introduce the following definition from [KM85, §13.1] which will allow us to describe the special fiber of $\mathfrak{X}(Np^r)$.

Consider the general situation with $k$ a field, $Y$ a smooth scheme over $k$, and $X \to Y$ a finite flat morphism of schemes. Suppose there exists a nonempty finite set $S$ of $k$-rational points of $Y$ such that for each $y_0 \in S$ there exists a unique closed $k$-rational point $x_0 \in X$ over $y_0$ such that $\hat{\mathcal{O}}_{X,x_0} \simeq k[[x, y]]/(f)$ for some $f \in k[[x, y]]$. The points of $S$ are referred to as the *supersingular points*; indeed in the situation $X$ and $Y$ are modular curves, $S$ will be taken to be the supersingular points of $Y$ which correspond to supersingular elliptic curves.

Furthermore, we assume there is a finite collection of $k$-schemes $\{Z_i\}_{i \in I}$ with a morphism $\coprod_{i \in I} Z_i \to X$ such that

- for each $i \in I$ and $y_0 \in S$ there exists a unique closed, $k$-rational point $z_{i,0} \in Z_i$ over $y_0$.
- $Z_i$ is finite flat over $Y$ and $(Z_i)^{\mathrm{red}}$ is smooth over $k$.

- $Z_i \to X$ is a closed immersion and $\coprod_{i\in I} Z_i \to X$ is an isomorphism over the complement of $S$ in $Y$.

By [KM85, 13.1.3], known as the "Crossings Theorem", if $Y$ is connected, then the $Z_i$ are the irreducible components of $X$. Furthermore, if each $Z_i$ is reduced, then $X$ is also reduced.

**Definition 2.44.** In the situation just discussed, we say $X$ is the **disjoint union of the $Z_i$'s with crossings at the supersingular points**.

*Remark* 2.45. Following [KM85, 13.1.7], we can relax the $k$-rationality of $x_0$ and $y_0$ and instead require $k$-rationality after extending scalars to a separable closure of $k$. In this situation, we still say $X$ is the disjoint union of the $Z_i$'s with crossings at the supersingular points.

Next we describe $\bar{\mathfrak{X}}(Np^r)$. We identify $(\mathbb{Z}/p^r\mathbb{Z})^\times$ as the subgroup

$$\left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in (\mathbb{Z}/p^r\mathbb{Z})^\times \right\}$$

of $\mathrm{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$. According to [KM85, 13.7.1], the moduli problem $[\Gamma(p^r)]^{\mathrm{can}} \otimes_{\mathbb{Z}[\zeta_{p^r}]} k$ on $\mathrm{Ell}_k$ assigns to each elliptic curve $E \to S$ the set of all Drinfeld $p^r$-bases $(P,Q)$ with $e_{p^r}(P,Q) = 1$. Let $\phi : (\mathbb{Z}/p^r\mathbb{Z})^2 \to E[p^r]$ denote the homomorphism of $S$-schemes corresponding to $(P,Q)$. Consider the diagram

$$\begin{array}{ccccccccc} & & & & (\mathbb{Z}/p^r\mathbb{Z})^2 & & & & \\ & & & & \downarrow{\scriptstyle \phi} & \searrow^{\Lambda} & & & \\ 0 & \longrightarrow & \ker F^r & \longrightarrow & E[p^r] & \xrightarrow{F^r} & \ker(V^r) & \longrightarrow & 0 \end{array}$$

where $\Lambda = F^r \circ \phi$. By [KM85, 13.7.2(3)], a choice of $\mathbb{Z}/p^r\mathbb{Z}$-basis of $(\mathbb{Z}/p^r\mathbb{Z})^2/\ker\Lambda$ defines an isomorphism $\mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\sim} \ker(V^r)$, allowing us to view $\Lambda$ as a surjective homomorphism $(\mathbb{Z}/p^r\mathbb{Z})^2 \to \mathbb{Z}/p^r\mathbb{Z}$.

**Definition 2.46.** The **component label** of $\phi$ is the class of $\Lambda$ in

$$(\mathbb{Z}/p^r\mathbb{Z})^\times / \mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z}).$$

By [KM85, 13.7.4, 13.7.5], this establishes a canonical bijection between the irreducible components of the special fiber and the set of component labels. The following is [KM85, 13.7.6].

**Theorem 2.47.** *The special fiber of $\mathfrak{X}(Np^r)$ is the disjoint union, with crossings at the supersingular points of $\mathfrak{X}(N)_{/k}$, of the exotic Igusa curves $\mathrm{ExIg}(p^r, r, N)$ over $\mathfrak{X}(N)_{/k}$ indexed by*

$$(\mathbb{Z}/p^r\mathbb{Z})^\times / \mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z}).$$

*Furthermore, $\bar{\mathfrak{X}}(Np^r)$ is reduced.*

Note that the claim $\bar{\mathfrak{X}}(Np^r)$ is reduced comes from the fact that $\mathrm{ExIg}(p^r, r, N)$ is reduced, together with the "Crossings Theorem" of [KM85, 13.1.3] and [KM85, 13.1.4].

Recall the closed subscheme of cusps $\mathfrak{C}(Np^r)$ of $\mathfrak{X}(Np^r)$ is a disjoint union of $\#\mathrm{HS}(Np^r)$ many sections of $\mathfrak{X}(Np^r)$ which, by Proposition 2.37, can be viewed as the closure $\overline{\{x\}}$ of points $x \in \mathfrak{C}(\mathfrak{X}(N)_{/K})$. By Corollary 2.30, $\overline{\{x\}}$ intersects precisely one irreducible component of $\bar{\mathfrak{X}}(Np^r)$. The following, which is [KM85, 13.9.3], tells us precisely the component label of the irreducible component which $\overline{\{x\}}$ intersects with, given the index of $\overline{\{x\}}$ in $\mathrm{HS}(Np^r)$ (see the paragraph preceding Proposition 2.37).

**Theorem 2.48.** *The natural projection*

$$\begin{array}{c} \mathrm{HS}(Np^r) \\ \| \\ \{\pm I\}\backslash\mathrm{HomSurj}((\mathbb{Z}/N\mathbb{Z})^2, \mathbb{Z}/N\mathbb{Z}) \times \mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z}) \\ \downarrow \\ (\mathbb{Z}/p^r\mathbb{Z})^\times\backslash\mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z}) \end{array}$$

*assigns to a component $\overline{\{x\}}$ of $\mathfrak{C}(Np^r)$ indexed by $\mathrm{HS}(Np^r)$ the irreducible component of $\bar{\mathfrak{X}}(Np^r)$ that $\overline{\{x\}}$ intersects with.*

We immediately have the following Corollary:

**Corollary 2.49.** *Each irreducible component of $\bar{\mathfrak{X}}(Np^r)$ intersects with the same number of cuspidal components of $\mathfrak{C}(Np^r)$.*

Every group homomorphism $\Lambda : (\mathbb{Z}/p^r\mathbb{Z})^2 \to \mathbb{Z}/p^r\mathbb{Z}$ is uniquely determined by its image on the basis vectors $(1,0)$ and $(0,1)$. Let $\Lambda_{(a,b)} : (\mathbb{Z}/p^r\mathbb{Z})^2 \to \mathbb{Z}/p^r\mathbb{Z}$ denote the map defined by

$$\Lambda_{(a,b)}(1,0) = a \text{ and } \Lambda_{(a,b)}(0,1) = b.$$

A complete list of representatives in $(\mathbb{Z}/p^r\mathbb{Z})^\times/\mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z})$ is given by

$$\begin{cases} \Lambda_{(1,-a)} & a \in \mathbb{Z}/p^r\mathbb{Z} \\ \Lambda_{(-pb,1)} & b \in \mathbb{Z}/p^{r-1}\mathbb{Z} \end{cases}.$$

With this labeling, the following is clear:

**Corollary 2.50.** *The special fiber $\bar{\mathfrak{X}}(Np^r)$ has $p^r + p^{r-1}$ many irreducible components.*

We will record the number of cuspidal components of $\mathfrak{C}(Np^r)$ an irreducible component of $\bar{\mathfrak{X}}(Np^r)$ intersects with. For convenience, we let $C(Np^r) = \mathfrak{C}(\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})})$, the set of all cusps of the generic fiber. The following is [Miy89, 4.2.10].

**Lemma 2.51.** *For $M \geq 3$, we have*

$$\#C(M) = \frac{1}{2}M^2 \prod_{p|M}(1 - 1/p^2) = \frac{1}{2M}\#\mathrm{SL}_2(\mathbb{Z}/MZ).$$

We will also need to know how to compute $\#\mathrm{SL}_2(\mathbb{Z}/M\mathbb{Z})$. The following is from [Miy89, 4.2.3, 4.2.4].

**Lemma 2.52.** *a) Let $A$ and $B$ be coprime integers. Then*

$$\mathrm{SL}_2(\mathbb{Z}/AB\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z}/A\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/B\mathbb{Z}).$$

*b) Let $p$ be prime and $r \geq 1$ an integer. Then*

$$\#\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z}) = p^{3r} - p^{3r-2}.$$

**Proposition 2.53.** *Each irreducible component of $\bar{\mathfrak{X}}(Np^r)$ intersects precisely*

$$\frac{\#C(Np^r)}{p^r + p^{r-1}} = \varphi(p^r)\#C(N)$$

*many cuspidal components of $\mathfrak{C}(Np^r)$.*

*Proof.* By Corollary 2.49, $\deg(\mathfrak{C}(Np^r)|_\Lambda)$ is independent of irreducible component $\Lambda$. Hence $\deg(\mathfrak{C}(Np^r)|_\Lambda)$ is equal to the total number of cusps of the generic fiber divided by the total number of irreducible components. By Corollary 2.50, the number of irreducible components is $p^r + p^{r-1}$.

Using Lemma 2.51 and Lemma 2.52a, we compute:

$$\begin{aligned}\frac{\#C(Np^r)}{p^r + p^{r-1}} &= \frac{\#\mathrm{SL}_2(\mathbb{Z}/Np^r\mathbb{Z})}{2Np^r(p^r + p^{r-1})} \\ &= \frac{\#\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})}{p^r(p^r + p^{r-1})} \cdot \frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{2N} \\ &= \frac{\#\mathrm{SL}_2(\mathbb{Z}/p^r\mathbb{Z})}{p^r(p^r + p^{r-1})}\#C(N)\end{aligned}$$

Lastly, using Lemma 2.52b, we have

$$\begin{aligned}&= \frac{p^{3r} - p^{3r-2}}{p^r(p^r + p^{r-1})}\#C(N) \\ &= \varphi(p^r)\#C(N).\end{aligned}$$

□

# 3 The Exponent

## 3.1 Formulation of the Exponent

We will consider two different sub $\mathbb{Z}[\zeta_N]$-modules of the space of modular forms $H^0(\mathfrak{X}(N)_{/\mathbb{Q}(\zeta_N)}, \underline{\omega}^{\otimes 2})$ over $\mathbb{Q}(\zeta_N)$. The first structure is $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$, the space of modular forms over $\mathbb{Z}[\zeta_N]$ as introduced in Section 2.5, which has the following description in terms of $q$-expansions.

**Lemma 3.1.** *We have*

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) = \left\{ f \in H^0(\mathfrak{X}(N)_{/\mathbb{Q}(\zeta_N)}, \underline{\omega}^{\otimes 2}) : f_c \in \mathbb{Z}[\zeta_N][[q^{1/N}]] \text{ for all } c \in \mathfrak{C}(N) \right\}.$$

*Proof.* This comes from the $q$-expansion principle, as stated in Proposition 2.39. □

Let $R$ be a $\mathbb{Z}[\zeta_N]$-algebra contained in $\mathbb{Q}(\zeta_N)$. The second structure is defined as

$$M_2(N, R) := \left\{ f \in H^0(\mathfrak{X}(N)_{/\mathbb{Q}(\zeta_N)}, \underline{\omega}^{\otimes 2}) : f_\infty \in R[[q^{1/N}]] \right\}$$

which are modular forms over $\mathbb{Q}(\zeta_N)$ whose $q$-expansion at the cusp $\infty$ has coefficients in $R$.

**Lemma 3.2.** *The usual map*

$$M_2(N, \mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N, 1/N] \to M_2(N, \mathbb{Z}[\zeta_N, 1/N])$$

*is an isomorphism.*

*Proof.* By Proposition 2.33, we have

$$H^0(\mathfrak{X}(N)_{/\mathbb{Z}[\zeta_N]}, \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/\mathbb{Z}[\zeta_N]}}) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Q}(\zeta_N) = H^0(\mathfrak{X}(N)_{/\mathbb{Q}(\zeta_N)}, \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/\mathbb{Q}(\zeta_N)}}).$$

In particular, the coefficients of any $q$-expansion of a modular form over $\mathbb{Q}(\zeta_N)$ have bounded denominators. Therefore we can write the $q$-expansion of any $f \in M_2(N, \mathbb{Z}[\zeta_N, 1/N])$ at $\infty$ as

$$f_\infty = \frac{1}{N^m} f'$$

for some $f' \in \mathbb{Z}[\zeta_N][[q^{1/N}]]$ and integer $m \geq 0$, *i.e.* $f' \in M_2(N, \mathbb{Z}[\zeta_N])$. Then $f$ is the image of $f' \otimes \frac{1}{N^m}$. □

**Definition 3.3.** Let $M$ be a module over a ring $R$. Then **annihilator** of $M$ is the ideal

$$\mathrm{Ann}_R(M) := \{ r \in R : rm = 0 \text{ for all } m \in M \}.$$

The **annihilator** of an element $m \in M$ is the ideal

$$\mathrm{Ann}_R(m) := \{r \in R : rm = 0\}.$$

The next proposition showcases some properties of our two $\mathbb{Z}[\zeta_N]$-modules necessary to discuss the annihilator of the quotient $M_2(N, \mathbb{Z}[\zeta_N])/H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$.

**Proposition 3.4.** *a. Both $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ and $M_2(N, \mathbb{Z}[\zeta_N])$ are finitely generated $\mathbb{Z}[\zeta_N]$-modules of the same rank.*

*b. We have*

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N, 1/N] = M_2(N, \mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N, 1/N].$$

*Proof.* Since $\mathfrak{X}(N)$ is projective over $\mathbb{Z}[\zeta_N]$, $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ is a finitely generated $\mathbb{Z}[\zeta_N]$-module by [Liu02, Theorem 5.3.2]. Since $M_2(N, \mathbb{Z}[\zeta_N])$ is a submodule of a finitely generated module over a noetherian ring, namely of $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$, it is also finitely generated.

Now we show the second claim. By Proposition 2.33, we have

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N, 1/N] = H^0(\mathfrak{X}(N)_{/\mathbb{Z}[\zeta_N, 1/N]}, \underline{\omega}^{\otimes 2})$$

while, by Lemma 3.2, we have

$$M_2(N, \mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}[\zeta_N, 1/N] = M_2(N, \mathbb{Z}[\zeta_N, 1/N]).$$

Since modular forms over $\mathbb{Z}[\zeta_N, 1/N]$ are determined by their $q$-expansion at $\infty$ (see [Kat73, 1.6.2]) , we get an equality

$$H^0(\mathfrak{X}(N)_{/\mathbb{Z}[\zeta_N, 1/N]}, \underline{\omega}^{\otimes 2}) = M_2(N, \mathbb{Z}[\zeta_N, 1/N])$$

as desired. In particular, we have an equality of finite dimensional vector spaces

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Q}(\zeta_N) = M_2(N, \mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Q}(\zeta_N)$$

which shows both our $\mathbb{Z}[\zeta_N]$-modules have the same rank. □

Our description of $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ in Lemma 3.1 shows that it is contained in $M_2(N, \mathbb{Z}[\zeta_N])$. Having established that $H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ and $M_2(N, \mathbb{Z}[\zeta_N])$ have the same rank, the quotient

$$M_2(N, \mathbb{Z}[\zeta_N])/H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$$

is a torsion $\mathbb{Z}[\zeta_N]$-module. Our goal will be to investigate the annihilator of this quotient.

By localizing, we can focus our attention on investigating a single prime in the annihilator. The following lemma, which is [Sta21, Tag080S], shows how the annihilator behaves under flat base change.

**Lemma 3.5.** *Let $R \to S$ be a flat ring map. Let $M$ be an $R$-module and $m \in M$. Then*

$$\mathrm{Ann}_S(m \otimes 1) = \mathrm{Ann}_R(m)S$$

*for any $m \otimes 1 \in M \otimes_R S$. If $M$ is finite over $R$, then*

$$\mathrm{Ann}_S(M \otimes_R S) = \mathrm{Ann}_R(M)S.$$

*Proof.* Let $I = \mathrm{Ann}_R(m)$. Then we have an exact sequence

$$0 \to I \to R \to M$$

where $R \to M$ is the map $r \mapsto r \cdot m$. Since $R \to S$ is flat, tensoring this exact sequence with $S$ yields an exact sequence

$$0 \to I \otimes_R S \to S \to M \otimes_R S.$$

Note that the map $S \to M \otimes_R S$ is given by $s \mapsto s \cdot (m \otimes 1)$; hence its kernel is precisely $\mathrm{Ann}_S(m \otimes 1)$. By exactness, this coincides with $I \otimes_R S = \mathrm{Ann}_R(m) \otimes_R S$. Again by flatness of $R \to S$, we have $\mathrm{Ann}_R(m) \otimes_R S = \mathrm{Ann}_R(m)S$.

Let $m_1, \ldots, m_n$ be a finite set of generators of $M$ over $R$. Then we have $\mathrm{Ann}_R(M) = \bigcap \mathrm{Ann}_R(m_i)$ and similarly $\mathrm{Ann}_S(M \otimes_R S) = \bigcap \mathrm{Ann}_S(m_i \otimes 1)$. Since $R \to S$ is flat, we have $IS \cap JS = (I \cap J)S$ for any ideals $I, J \subseteq R$. Thus

$$\mathrm{Ann}_S(M \otimes_R S) = \bigcap_{i=1}^{n} \mathrm{Ann}_S(m_i \otimes 1) = \bigcap_{i=1}^{n} \mathrm{Ann}_R(m_i)S$$

$$= \left( \bigcap_{i=1}^{n} \mathrm{Ann}_R(m_i) \right) S = \mathrm{Ann}_R(M)S$$

as desired. □

**Corollary 3.6.** *Let $M$ be a finitely generated torsion module over a Dedekind domain $R$ and let $\mathfrak{p}$ be a prime in $R$. The exponent of $\mathfrak{p}$ appearing in $\mathrm{Ann}_R(M)$ is equal to the exponent of $\mathfrak{p}$ appearing in $\mathrm{Ann}_{R_\mathfrak{p}}(M \otimes_R R_\mathfrak{p})$.*

*Proof.* We factor the annihilator as a product of distinct primes $\mathrm{Ann}_R(M) = \mathfrak{p}^e \mathfrak{q}_1^{e_1} \ldots \mathfrak{q}_m^{e_m}$. By Lemma 3.5, we have

$$\mathrm{Ann}_{R_\mathfrak{p}}(M \otimes_R R_\mathfrak{p}) = \mathrm{Ann}_R(M)R_\mathfrak{p} = \mathfrak{p}^e \mathfrak{q}_1^{e_1} \ldots \mathfrak{q}_m^{e_m} R_\mathfrak{p} = \mathfrak{p}^e R_\mathfrak{p}.$$

Hence the exponent of $\mathfrak{p}$ appearing in $\mathrm{Ann}_{R_\mathfrak{p}}(M \otimes_R R_\mathfrak{p})$ is precisely $e$. □

**Proposition 3.7.** *Let $\mathfrak{p} \in \mathbb{Z}[\zeta_N]$ be a prime lying over the prime $p \in \mathbb{Z}$. The exponent of $\mathfrak{p}$ appearing in the annihilator of $M_2(N, \mathbb{Z}[\zeta_N])/H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}))$ and $M_2(N, \mathbb{Z}_p[\zeta_N])/H^0(\mathfrak{X}(N)_{/\mathbb{Z}_p[\zeta_N]}, \underline{\omega}^{\otimes 2})$ are the same.*

*Proof.* For convenience, we let

$$M = M_2(N, \mathbb{Z}[\zeta_N]),\ N = H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$$

and

$$M_p = M_2(N, \mathbb{Z}_p[\zeta_N]),\ N_p = H^0(\mathfrak{X}(N)_{/\mathbb{Z}_p[\zeta_N]}, \underline{\omega}^{\otimes 2}).$$

Consider the exact sequence

$$0 \to N \to M \to M/N \to 0.$$

Since $\mathbb{Z}_p[\zeta_N]$ is flat over $\mathbb{Z}[\zeta_N]$, the above sequence remains exact after tensoring with $\mathbb{Z}_p[\zeta_N]$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes \mathbb{Z}_p[\zeta_N] & \longrightarrow & M \otimes \mathbb{Z}_p[\zeta_N] & \longrightarrow & (M/N) \otimes \mathbb{Z}_p[\zeta_N] & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N_p & \longrightarrow & M_p & \longrightarrow & M_p/N_p & \longrightarrow & 0 \end{array}$$

By Proposition 2.33, $f$ and $g$ are isomorphisms; consequently $h$ is an isomorphism. By Corollary 3.6, the exponent of $\mathfrak{p}$ appearing in $M_p/N_p \simeq (M/N) \otimes \mathbb{Z}_p[\zeta_N]$ agrees with that of $M/N$. □

*Remark* 3.8. If $p \nmid N$, then $N$ is invertible in $\mathbb{Z}_p[\zeta_N]$ so by Proposition 3.4 we have

$$H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2}) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}_p[\zeta_N] = M_2(N, \mathbb{Z}[\zeta_N]) \otimes_{\mathbb{Z}[\zeta_N]} \mathbb{Z}_p[\zeta_N].$$

Hence the $p$-adic valuation of the annihilator of $M_2(N, \mathbb{Z}[\zeta_N])/H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ is trivial in this case. Therefore we restrict our attention to primes $p$ dividing the level.

Let $N \geq 3$ and $r \geq 0$ be integers and let $p \geq 2$ be a prime such that $p \nmid N$. Let $\pi$ be a uniformizer of $\mathbb{Z}_p[\zeta_{Np^r}]$. We seek to compute the smallest integer $e \geq 0$ such that

$$\pi^e H^0(\mathfrak{X}(Np^r)_{/\mathbb{Z}_p[\zeta_{Np^r}]}, \underline{\omega}^{\otimes 2}) \subseteq M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}]).$$

## 3.2 Geometric Interpretation of the Exponent

For convenience, we let $\mathfrak{X} = \mathfrak{X}(Np^r)_{/\mathbb{Z}_p[\zeta_{Np^r}]}$. We will formulate an explicit, geometric description of the exponent $e$ by first providing an algebraic description coming from $q$-expansions.

Let $c$ be a cusp of $\mathfrak{X}$, and let $f \in H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2})$ be a non-zero modular form. Let $f_c \in \mathbb{Z}_p[\zeta_{Np^r}][[q^{1/N}]]$ denote the $q$-expansion of $f$ at the cusp $c$. Define $\nu_\pi(f_c)$ to be the minimal $\pi$-adic valuation among all the coefficients of $f_c$ *i.e.*

$$\nu_\pi\left(\sum_{n=0}^{\infty} a_n q^{n/N}\right) = \min_{n \geq 0} \{\nu_\pi(a_n)\}.$$

Note that this minimum exists since the denominators of the coefficients $a_n$ are bounded (see proof of Lemma 3.2). Thus, by Lemma 3.1, we may describe

$$H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2}) = \left\{ f \in H^0(\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})}, \underline{\omega}^{\otimes 2}) : \nu_\pi(f_c) \geq 0 \text{ for all cusps } c \text{ of } \mathfrak{X} \right\}.$$

Consequently, we seek to compute the smallest integer $e \geq 0$ such that $\nu_\pi(\pi^e f_c) \geq 0$ for all cusps $c$ of $\mathfrak{X}$ not equal to $\infty$ and all $f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$.

Next we will provide a geometric interpretation of $\nu_\pi(f_c)$ which does not rely on $q$-expansions. Let $\eta_\Lambda$ denote the generic point of an irreducible component $\Lambda$ of the special fiber $\bar{\mathfrak{X}}$. The stalk $\mathcal{O}_{\mathfrak{X},\eta}$ is a DVR so it has a valuation which we denote by $\nu_\eta$. Viewing $f$ inside the stalk $\underline{\omega}_\eta^{\otimes 2}$, we can write $f = f_\eta \cdot \omega_{\mathrm{can},\eta}$ where $\omega_{\mathrm{can},\eta}$ is the canonical generator of $\underline{\omega}_\eta^{\otimes 2}$ (see Definition 2.38). We define

$$\nu_\Lambda(f) := \nu_\eta(f_\eta).$$

Indeed, $\nu_\Lambda$ is a valuation, independent of the choice of local generator. The following result is stated in [DR73, Théorème 3.10(ii)].

**Proposition 3.9.** *Let $c \in \mathfrak{X}$ be a cusp let $\Lambda$ be an irreducible component of $\bar{\mathfrak{X}}$ with generic point $\eta$ on which $c$ lies. For any non-zero $f \in H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$, we have $\nu_\pi(f_c) = \nu_\Lambda(f)$. Furthermore, $\pi$ is a uniformizer of $\mathcal{O}_{\mathfrak{X},\eta}$.*

*Proof.* Since $c \in \overline{\{\eta\}}$, the stalk $\mathcal{O}_{\mathfrak{X},\eta}$ is a localization of $\mathcal{O}_{\mathfrak{X},c}$. Furthermore, the map $\mathcal{O}_{\mathfrak{X},c} \to \mathcal{O}_{\mathfrak{X},\eta}$ is injective since all stalks are regular local rings so are integral domains in particular. The induced map on completions $\hat{\mathcal{O}}_{\mathfrak{X},c} \to \hat{\mathcal{O}}_{\mathfrak{X},\eta}$ is also injective by [Sta21, Tag00MB, Tag07N9].

We will show $\pi$ is also a uniformizer of $\mathcal{O}_{\mathfrak{X},\eta}$. Consider the exact sequence

$$0 \to \mathfrak{m}_\eta \to \mathcal{O}_{\mathfrak{X},\eta} \to \kappa(\eta) \to 0$$

where $\mathfrak{m}_\eta$ is the maximal ideal of $\mathcal{O}_{\mathfrak{X},\eta}$ and $\kappa(\eta)$ is the residue field of $\eta$ which is of characteristic $p$. Since $(\pi^k) = (p)$ when $k = p^{r-1}(p-1)$, we have $\pi^k = 0$ in $\kappa(\eta)$ so $\pi = 0$ in $\kappa(\eta)$. Therefore $\pi \in \mathfrak{m}_\eta$. It remains to show $(\pi)$ is a maximal ideal in $\mathcal{O}_{\mathfrak{X},\eta}$. Consider the quotient $\mathcal{O}_{\mathfrak{X},\eta}/\pi\mathcal{O}_{\mathfrak{X},\eta} = \mathcal{O}_{\bar{\mathfrak{X}},\eta}$. Since $\eta$ corresponds to a minimal prime and $\bar{\mathfrak{X}}$ is reduced, the stalk $\mathcal{O}_{\bar{\mathfrak{X}},\eta}$ is a field. Therefore $(\pi)$ is a maximal ideal in $\mathcal{O}_{\mathfrak{X},\eta}$ hence $\pi$ is a uniformizer in $\mathcal{O}_{\mathfrak{X},\eta}$.

Let $f \in H^0(\mathfrak{X}(N), \underline{\omega}^{\otimes 2})$ be non-zero. Using the local generator $\omega_{\mathrm{can},c}$ of $\hat{\underline{\omega}}^{\otimes 2}$ (see Definition 2.38), we write $f = f_c \omega_{\mathrm{can},c}$ where $f_c \in \hat{\mathcal{O}}_{\mathfrak{X},c}$ is the $q$-expansion of $f$ at $c$. Write $f_c = \pi^{\nu_\pi(f_c)} f_c'$ where $f_c' \notin \pi\hat{\mathcal{O}}_{\mathfrak{X},c}$. The map

$$\mathcal{O}_{\mathfrak{X},c}/\pi\mathcal{O}_{\mathfrak{X},c} \to \mathcal{O}_{\mathfrak{X},\eta}/\pi\mathcal{O}_{\mathfrak{X},\eta}$$

is the same as the map $\mathcal{O}_{\bar{\mathfrak{X}},c} \to \mathcal{O}_{\bar{\mathfrak{X}},\eta}$, which is also a localization map as $c \in \overline{\{\eta\}}$. By [KM85, 10.9.1(2)], $\mathfrak{X}$ is smooth at the cusps so, in particular, $\mathcal{O}_{\bar{\mathfrak{X}},c}$ is a domain. Therefore $\mathcal{O}_{\bar{\mathfrak{X}},c} \to \mathcal{O}_{\bar{\mathfrak{X}},\eta}$ is injective so the map on completions

$$\hat{\mathcal{O}}_{\mathfrak{X},c}/\pi\hat{\mathcal{O}}_{\mathfrak{X},c} \to \hat{\mathcal{O}}_{\mathfrak{X},\eta}/\pi\hat{\mathcal{O}}_{\mathfrak{X},\eta}$$

is injective. Since $f_c' \notin \pi\hat{\mathcal{O}}_{\mathfrak{X},c}$, we can conclude $f_c' \notin \pi\hat{\mathcal{O}}_{\mathfrak{X},\eta}$ *i.e.* $f_c'$ is a unit in $\hat{\mathcal{O}}_{\mathfrak{X},\eta}$. Thus $\nu_\pi(f_c)$ coincides with $\nu_\Lambda(f)$, the valuation in $\mathcal{O}_{\mathfrak{X},\eta}$. □

With this geometric description of the valuation, we can reformulate our integrality condition as follows. Let $\Lambda_0, \ldots, \Lambda_n$ be the irreducible components of the special fiber $\bar{\mathfrak{X}}$ where $\Lambda_0$ contains the cusp $\infty$. Let $\mu_{\Lambda_i}$ denote the multiplicity of $\Lambda_i$. We desire to find the smallest integer $e \geq 0$ such that $\nu_{\Lambda_i}(\pi^e f) \geq 0$ for all $i = 1, \ldots, n$ and all $f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$.

By 3.9, $\pi$ is a uniformizer of $\mathcal{O}_{\mathfrak{X},\eta_i}$, where $\eta_i$ is the generic point of $\Lambda_i$. Hence

$$\begin{aligned} \nu_{\Lambda_i}(\pi^e f) &= e \cdot \nu_{\Lambda_i}(\pi) + \nu_{\Lambda_i}(f) \\ &= e \cdot \mu_{\Lambda_i} + \nu_{\Lambda_i}(f). \end{aligned}$$

Since $\bar{\mathfrak{X}}$ is reduced, $\mu_{\Lambda_i} = 1$. Hence the condition $\nu_{\Lambda_i}(\pi^e f) \geq 0$ is equivalent to $e \geq -\nu_{\Lambda_i}(f)$. Therefore we have the following expression for the exponent:

$$e = \max_{f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])} \{-\nu_{\Lambda_1}(f), -\nu_{\Lambda_2}(f), \ldots, -\nu_{\Lambda_n}(f)\} \tag{5}$$

which illustrates we can investigate $e$ by trying to understand the quantities $\nu_\Lambda(f)$.

## 3.3 A More Explicit Description of $e$

In this section, we will provide a more explicit description of the exponent by relating the quantities $\nu_\Lambda(f)$ to intersection numbers and the degree of a line bundle. We begin with a more general situation with $X$ an arithmetic surface over a DVR $R$ with residue field $k$ and $\mathcal{L}$ a line bundle on $X$ with connected special fiber.

**Definition 3.10.** Let $f \in H^0(X, \mathcal{L})$ be a non-zero global section and let $\{U_i\}$ be a trivializing open cover of $\mathcal{L}$ so that $\mathcal{L}(U_i) = \mathcal{O}_X(U_i)e_i$ for some generator $e_i \in \mathcal{L}(U_i)$. We can write $f|_{U_i} = f_i e_i$ for some $f_i \in \mathcal{O}_X(U_i)$. Then the system $\{(U_i, f_i)\}$ is an effective Cartier divisor of $X$, which we denote by $\mathrm{div}_{\mathcal{L}}(f)$ or simply $\mathrm{div}(f)$ if the line bundle is clear from context.

We can equivalently define $\mathrm{div}_{\mathcal{L}}(f)$ as a Weil divisor. Let $Z$ be a prime divisor of $X$ with generic point $\eta$. Then $\mathcal{O}_{X,\eta}$ is a DVR with valuation which we denote by $\nu_Z$. We can write the image of $f$ in the stalk of $x$ as $f = f_\eta e_\eta$ for some $f_\eta \in \mathcal{O}_{X,\eta}$ and generator $e_\eta \in \mathcal{L}_\eta$. Define $\nu_Z(f) := \nu_Z(f_\eta)$ which agrees with our valuation defined in Section 3.2 for $Z = \Lambda$ an irreducible component of $\bar{\mathfrak{X}}$ and is independent of choice of $e_\eta$. According to [Sta21, Tag02SE], the Weil divisor associated to $f$ is equal to

$$\mathrm{div}_{\mathcal{L}}(f) = \sum_Z \nu_Z(f)[Z]$$

where the sum is over prime divisors $Z$ of $X$. Decomposing the divisor $\mathrm{div}_{\mathcal{L}}(f)$ into its horizontal and vertical components, we can write

$$\mathrm{div}_{\mathcal{L}}(f) = H_f + \sum_\Lambda \nu_\Lambda(f)\,[\Lambda] \tag{6}$$

where $H_f$ is some effective horizontal divisor and the sum is over the irreducible components of $\bar{X}$.

**Proposition 3.11.** *Let $f \in H^0(X, \mathcal{L})$ be a non-zero global section. Then $\mathcal{L} \simeq \mathcal{O}_X(\mathrm{div}_{\mathcal{L}}(f))$.*

*Proof.* This is [Liu02, Exercise 7.1.13] (see also immediately before [BDP17, Proposition B.2.2.10]). □

Let $\Lambda_0, \ldots, \Lambda_n$ denote the irreducible components of $\bar{X}$. By Proposition 3.11 and Theorem 2.24d, we have

$$\begin{aligned} \deg_k(\mathcal{L}|_{\Lambda_j}) = &\quad \deg_k(\mathcal{O}_X(\mathrm{div}_{\mathcal{L}}(f))|_{\Lambda_j}) \\ = &\quad \mathrm{div}_{\mathcal{L}}(f).\Lambda_j. \end{aligned}$$

Decomposing $\mathrm{div}_{\mathcal{L}}(f)$ into its horizontal and vertical components, as in (6), we get

$$= \left( H_f + \sum_{i=0}^{n} \nu_{\Lambda_i}(f) \, [\Lambda_i] \right) .\Lambda_j$$

$$= H_f.\Lambda_j + \sum_{i=0}^{n} \nu_{\Lambda_i}(f) \Lambda_i.\Lambda_j.$$

Using the equation

$$\deg_k(\mathcal{L}|_{\Lambda_j}) = H_f.\Lambda_j + \sum_{i=0}^{n} \nu_{\Lambda_i}(f) \Lambda_i.\Lambda_j \tag{7}$$

we will provide an explicit expression for the quantities $\nu_{\Lambda_i}(f)/\mu_{\Lambda_i}$ for $i = 1, \ldots, n$ where $\mu_{\Lambda_i}$ is the multiplicity of $\Lambda_i$.

Let $M = (\Lambda_i.\Lambda_j)_{i,j=0,\ldots,n}$ denote the intersection matrix of $X$. Since $X$ is an arithmetic surface, Theorem 2.28 says $M$ is negative semi-definite and moreover (since the special fiber of $X$ is connected) that the kernel of $M$ is one-dimensional. Multiplication by $M$ induces an exact sequence

$$0 \to \ker M \to \mathbb{Q}^{n+1} \xrightarrow{M} \mathbb{Q}^{n+1}.$$

Let $\vec{\mu} = (\mu_{\Lambda_0}, \mu_{\Lambda_1}, \ldots, \mu_{\Lambda_n})$ which is a non-zero vector in $\mathbb{Q}^{n+1}$. Note that

$$(M\vec{\mu})_j = \sum_{i=0}^{n} \mu_{\Lambda_i}(\Lambda_i.\Lambda_j) = \left( \sum_{i=0}^{n} \mu_{\Lambda_i} \Lambda_i \right) .\Lambda_j = \bar{X}.\Lambda_j = 0$$

where the last equality is by Proposition 2.27(b). Therefore we can write $\ker(M) = \mathrm{span}_{\mathbb{Q}} \{\vec{\mu}\}$. The following lemma describes the image of $M$.

**Lemma 3.12.** *We have*

$$\mathrm{im}(M) = \left\{ \vec{a} = (a_0, \ldots, a_n) \in \mathbb{Q}^{n+1} : \sum_{j=0}^{n} \mu_{\Lambda_j} a_j = 0 \right\}.$$

*Proof.* Let $V$ denote the space of vectors $\vec{v} = (v_i) \in \mathbb{Q}^{n+1}$ such that $\sum_{j=0}^{n} \mu_{\Lambda_j} v_j = 0$. Let $\vec{a} = (a_j) \in \mathrm{im}(M)$ so $\vec{a} = M\vec{b}$ for some $\vec{b} = (b_j) \in \mathbb{Q}^{n+1}$. Then

$$a_j = (M\vec{b})_j = \sum_{i=0}^{n} b_i(\Lambda_i.\Lambda_j)$$

so

$$\begin{aligned} \sum_{j=0}^{n} \mu_{\Lambda_j} a_j &= \sum_{j=0}^{n} \mu_{\Lambda_j} \sum_{i=0}^{n} b_i(\Lambda_i.\Lambda_j) = \sum_{j=0}^{n} \mu_{\Lambda_j} \left( \sum_{i=0}^{n} b_i \Lambda_i \right).\Lambda_j \\ &= \left( \sum_{i=0}^{n} b_i \Lambda_i \right) . \left( \sum_{j=0}^{n} \mu_{\Lambda_j} \Lambda_j \right) = \left( \sum_{i=0}^{n} b_i \Lambda_i \right).\bar{X} = 0. \end{aligned}$$

Therefore $\mathrm{im}(M) \subseteq V$. Since $\mathrm{im}(M)$ and $V$ are both of dimension $n$, we get $\mathrm{im}(M) = V$. □

In order to isolate the terms $\nu_{\Lambda_i}(f)$ appearing in (7), we would ideally invert the matrix $M$. In light of $M$ being not invertible, we will instead invert an $n \times n$ submatrix of $M$ to obtain an expression for each $\nu_{\Lambda_i}(f)$, excluding $\nu_{\Lambda_0}(f)$, which suffices for our purpose.

Let $W = \left\{(x_0, \ldots, x_n) \in \mathbb{Q}^{n+1} : x_0 = 0\right\}$ and let $\mathrm{pr} : \mathbb{Q}^{n+1} \to W$ denote the projection map

$$(x_0, x_1, \ldots, x_n) \mapsto (0, x_1, \ldots, x_n).$$

Consider the restriction $\mathrm{pr}|_{\mathrm{im}(M)} : \mathrm{im}(M) \to W$. Given any $a_1, \ldots, a_n \in \mathbb{Q}$, we let

$$a_0 = -\frac{1}{\mu_{\Lambda_0}} \sum_{i=1}^{n} \mu_i a_i$$

which forces $\vec{a} := (a_0, a_1, \ldots, a_n) \in \mathrm{im}(M)$ by Lemma 3.12. Hence $\mathrm{pr}|_{\mathrm{im}(M)}$ is surjective. Since $\mathrm{im}(M)$ and $W$ are both of dimension $n$, the map $\mathrm{pr}|_{\mathrm{im}(M)}$ is an isomorphism of vector spaces.

Next we consider the restriction $M|_W : W \to \mathrm{im}(M)$. If $M\vec{w} = 0$ where $\vec{w} = (0, w_1, \ldots, w_n)$, then $\vec{w} = \alpha\vec{\mu}$ for some $\alpha \in \mathbb{Q}$. Coordinatewise, this means

$$(0, w_1, \ldots, w_n) = \vec{w} = \alpha\vec{\mu} = (\alpha\mu_{\Lambda_0}, \alpha\mu_{\Lambda_1}, \ldots, \alpha\mu_{\Lambda_n}).$$

Comparing the first coordinates and noting $\mu_{\Lambda_i} \neq 0$ for each $i$, we must have $\alpha = 0$. Hence $\vec{w} = 0$ so $M|_W$ is injective and therefore an isomorphism. Define $T : W \to W$ as the composition of our isomorphisms

$$T : W \xrightarrow{M|_W} \mathrm{im}(M) \xrightarrow{\mathrm{pr}|_{\mathrm{im}(M)}} W.$$

After identifying $W \simeq \mathbb{Q}^n$ by forgetting the first coordinate, we can identify $T$ as the lower right $n \times n$ submatrix of $M$.

The following is [BDP17, Proposition B.2.3.2].

**Proposition 3.13.** *For any $\vec{a} \in \mathrm{im}(M)$ and $\vec{b} = (b_0, b_1, \ldots, b_n) \in \mathbb{Q}^{n+1}$ such that $M\vec{b} = \vec{a}$, we have*

$$\vec{b} - \frac{b_0}{\mu_{\Lambda_0}}\vec{\mu} = T^{-1}(\mathrm{pr}(\vec{a})). \tag{8}$$

*Proof.* Note that $\vec{b} - \frac{b_0}{\mu_{\Lambda_0}}\vec{\mu} \in W$. Indeed, the first entry of this vector is $b_0 - \frac{b_0}{\mu_{\Lambda_0}}\mu_{\Lambda_0} = 0$. We now compute

$$T\left(\vec{b} - \frac{b_0}{\mu_{\Lambda_0}}\vec{\mu}\right) = \mathrm{pr}(M\vec{b}) - \frac{b_0}{\mu_{\Lambda_0}}\mathrm{pr}(M\vec{\mu}) = \mathrm{pr}(M\vec{b}) = \mathrm{pr}(\vec{a}).$$

Since $\vec{b} - \frac{b_0}{\mu_{\Lambda_0}}\vec{\mu} \in W$, we can invert $T$ to obtain our desired equality. □

We now apply Proposition 3.13 to our specific situation involving the quantities $\nu_{\Lambda_i}(f)$.

*Remark* 3.14. We make a quick remark about the term $\nu_{\Lambda_0}(f)$ that appears in the following theorem. Recall the equation for the exponent $e$ in (5). We claim that

$$e = \max_{\substack{1\le i\le n \\ f\in M_2(Np^r,\mathbb{Z}_p[\zeta_{Np^r}])}} \{-\nu_{\Lambda_i}(f)\} = \max_{\substack{1\le i\le n \\ f\in M_2(Np^r,\mathbb{Z}_p[\zeta_{Np^r}]) \\ \nu_{\Lambda_0}(f)=0}} \{\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)\}.$$

Indeed, the maximum on the left hand side must occur at some $f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$ with $\nu_{\Lambda_0}(f) = 0$. Otherwise if $f = \pi^a g$ with $g \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$, $\nu_{\Lambda_0}(g) = 0$, and $a > 0$, so

$$-\nu_{\Lambda_i}(g) = a - \nu_{\Lambda_i}(f).$$

Hence $-\nu_{\Lambda_i}(g) \ge -\nu_{\Lambda_i}(f)$ so the maximum must occur over such $g$.

Furthermore, the differences $\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)$ are visibly invariant under arbitrary scaling of $f$, so we have

$$\begin{aligned} e &= \max_{\substack{1\le i\le n \\ f\in H^0(\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})},\underline{\omega}^{\otimes 2}) \\ \nu_{\Lambda_0}(f)=0}} \{\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)\}. \\ &= \max_{\substack{1\le i\le n \\ f\in H^0(\mathfrak{X},\underline{\omega}^{\otimes 2})}} \{\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)\} \end{aligned}$$

In particular, this shows the exponent can be computed using Theorem 3.15 below, which provides a formula for the differences $\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)$ in terms of geometric data.

**Theorem 3.15.** *Let $f \in H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2})$ be non-zero and let $\Lambda_0, \ldots, \Lambda_n$ denote the irreducible components of $\bar{\mathfrak{X}}$. We have*

$$\nu_{\Lambda_i}(f) - \nu_{\Lambda_0}(f) = \sum_{j=1}^{n} (\deg_k(\underline{\omega}^{\otimes 2}|_{\Lambda_j}) - H_f.\Lambda_j)c^{i,j} \tag{9}$$

*where $c^{i,j}$ is the $(i,j)$ entry of $T^{-1}$ and $T$ is the matrix obtained by removing the first row and column of the intersection matrix of $\mathfrak{X}$.*

*Proof.* We will write out equation (8) entry-wise in the generality of Proposition 3.13, with $X$ an arithmetic surface, and $\mathcal{L}$ a line bundle on $X$. Then we apply our results to the situation of the modular curve.

Let $\vec{a} = (a_0, \ldots, a_n)$ and $\vec{b} = (b_0, \ldots, b_n)$ such that $M\vec{b} = \vec{a}$. Then equation 8 gives

$$\begin{pmatrix} 0 \\ b_1 - \frac{b_0}{\mu_{\Lambda_0}}\mu_{\Lambda_1} \\ \vdots \\ b_n - \frac{b_0}{\mu_{\Lambda_0}}\mu_{\Lambda_n} \end{pmatrix} = T^{-1} \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Recall $T = (\Lambda_i.\Lambda_j)_{i,j=1,\ldots,n}$ is the $n \times n$ lower right submatrix of the intersection matrix of $X$. Write $T^{-1} = (c^{i,j})_{i,j=1,\ldots,n}$ where $c^{i,j}$ is the $(i,j)$-entry of the inverse of $T$. Then we have

$$T^{-1} \begin{pmatrix} 0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1c^{1,1} + a_2c^{1,2} + \cdots + a_nc^{1,n} \\ a_1c^{2,1} + a_2c^{2,2} + \cdots + a_nc^{2,n} \\ \vdots \\ a_1c^{n,1} + a_2c^{n,2} + \cdots + a_nc^{n,n} \end{pmatrix}.$$

Therefore

$$b_i - \frac{b_0}{\mu_{\Lambda_0}}\mu_{\Lambda_i} = \sum_{j=1}^{n} a_j c^{i,j}.$$

Let $f \in H^0(X, \mathcal{L})$ be non-zero and let $\vec{b} = (\nu_{\Lambda_0}(f), \ldots, \nu_{\Lambda_n}(f))$. The $j$th coordinate of $\vec{a} = M\vec{b}$ is precisely

$$a_j = \sum_{i=0}^{n} \nu_{\Lambda_i}(f)\Lambda_i.\Lambda_j = \deg_k(\mathcal{L}|_{\Lambda_j}) - H_f.\Lambda_j$$

by equation (7). Thus we have

$$\nu_{\Lambda_i}(f) - \frac{\nu_{\Lambda_0}(f)}{\mu_{\Lambda_0}}\mu_{\Lambda_i} = \sum_{j=1}^{n} \left(\deg_k(\mathcal{L}|_{\Lambda_j}) - H_f.\Lambda_j\right) c^{i,j}. \tag{10}$$

Now we take $X = \mathfrak{X}$ to be our modular curve and $\mathcal{L} = \underline{\omega}^{\otimes 2}$ to be the modular sheaf. Recall $\bar{\mathfrak{X}}$ is reduced so $\mu_{\Lambda_i} = 1$ for each $i$. Then equation (10) becomes

$$\nu_{\Lambda_i}(f) - \nu_{\Lambda_0}(f) = \sum_{j=1}^{n} (\deg_k(\underline{\omega}^{\otimes 2}|_{\Lambda_j}) - H_f.\Lambda_j)c^{i,j}$$

as desired. □

Thus Theorem 3.15 expresses the quantities $\nu_{\Lambda_i}(f)$ in terms of $\deg_k(\underline{\omega}^{\otimes 2}|_{\Lambda_j})$, $H_f.\Lambda_j$, and the entries of the inverse of $T$.

# 4 Intersection Matrix

Throughout this chapter, we will use the following notation:

- $N \geq 3$ and $r \geq 1$ will be integers and $p \geq 2$ will be a prime such that $p \nmid N$
- $\mathbb{F}_q$ will denote the residue field $\mathbb{F}_p(\zeta_N)$ of $\mathbb{Z}_p[\zeta_{Np^r}]$ where $q = p^\ell$ where $\ell$ is the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^\times$.
- $\mathfrak{X}$ will denote the modular curve $\mathfrak{X}(Np^r)_{/\mathbb{Z}_p[\zeta_{Np^r}]}$.

## 4.1 Intersection matrix for $\mathfrak{X}$

In this section, we will obtain an explicit description of the intersection matrix of $\mathfrak{X}$. First we recall the description of the irreducible components of the special fiber $\bar{\mathfrak{X}}$ as in Theorem 2.47.

**Theorem 4.1.** *The special fiber of $\mathfrak{X}(Np^r)$ is the disjoint union, with crossings at the supersingular points of $\mathfrak{X}(N)_{/\mathbb{F}_q}$, of the exotic Igusa curves $\mathrm{ExIg}(p^r, r, N)$ over $\mathfrak{X}(N)_{/\mathbb{F}_q}$, indexed by*

$$(\mathbb{Z}/p^r\mathbb{Z})^\times/\mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z}).$$

*Furthermore, $\bar{\mathfrak{X}}(Np^r)$ is reduced.*

In the paragraph proceeding Corollary 2.49, a complete list of representatives in

$$(\mathbb{Z}/p^r\mathbb{Z})^\times/\mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z})$$

was given by

$$\begin{cases} \Lambda_{(1,-a)} & a \in \mathbb{Z}/p^r\mathbb{Z} \\ \Lambda_{(-pb,1)} & b \in \mathbb{Z}/p^{r-1}\mathbb{Z} \end{cases}.$$

We will identify an irreducible component of $\bar{\mathfrak{X}}$ by it's index $\Lambda$. By [KM85, 13.8.5], the local intersection number at a supersingular point $s$ between two distinct irreducible components $\Lambda_1$ and $\Lambda_2$ is precisely

$$i_s(\Lambda_1, \Lambda_2) = \left[\#\left((\mathbb{Z}/p^r\mathbb{Z})^2/(\ker\Lambda_1 + \ker\Lambda_2)\right)\right]^2. \tag{11}$$

*Remark* 4.2. In [KM85, 13.8.5], we require $s$ to be a $\mathbb{F}_q$-rational supersingular point. However, when $q = p$, the supersingular points may not be $\mathbb{F}_q$-rational. Indeed, [KM85, p. 96], shows that the supersingular points are $\mathbb{F}_{p^2}$-rational. However, we can instead compute the intersection numbers by etale base change. Consider the ring $R' = \mathbb{Z}_p[\zeta_{p^r}, \zeta_{p^2-1}]$. By [Ser79, IV, §4, Prop 16 & 17], $R'$ is the ring of integers of the local field $\mathbb{Q}_p(\zeta_{p^r}, \zeta_{p^2-1})$ so $R'$ is a DVR. Since $p$ has order 2 in $(\mathbb{Z}/(p^2-1)\mathbb{Z})^\times$, the residue field of $R'$ is $\mathbb{F}_{p^2}$. The map $\mathbb{Z}_p[\zeta_{p^r}] \to R'$ is unramified by [Ser79, IV, §4, Prop. 16], noting that $p \nmid (p^2-1)$, and flat because $R'$ is torsion-free over the DVR $\mathbb{Z}_p[\zeta_{p^r}]$ (see [Sta21, 0539]). Therefore $\mathrm{Spec}(R') \to \mathrm{Spec}(\mathbb{Z}_p[\zeta_{p^r}])$ is etale.

Let $\Lambda$ and $\Lambda'$ be two irreducible components of $\bar{\mathfrak{X}}$. Let $\mathfrak{X}' = \mathfrak{X}_{/R'}$ and let $s'$ be a supersingular point of $\mathfrak{X}'_{/\mathbb{F}_{p^2}}$ which maps to $s$. By [Liu02, 9.1.5, 9.1.6], we have

$$i_{s'}(f^*\Lambda, f^*\Lambda') = i_s(\Lambda, \Lambda').$$

Note that [Liu02, 9.2.15] uses a desingularization of $\mathfrak{X}'$ *i.e.* a proper birational morphism $\mathfrak{X}'' \to \mathfrak{X}'$ where $\mathfrak{X}''$ is regular. Since $R$ and $R'$ are both DVRs, they are both regular, excellent, and noetherian. Therefore $\mathfrak{X}'$ is an arithmetic surface (by Proposition 2.36), so we can take $\mathfrak{X}'' = \mathfrak{X}'$.

From equation (11), the local intersection number doesn't depend on the supersingular point. Let $\mathrm{S}(N)$ (resp. $S(Np^r)$) denote the supersingular locus of $\mathfrak{X}(N)_{/\mathbb{F}_q}$ (resp. $\bar{\mathfrak{X}}$). By [KM85, 12.7.2], the map $\mathrm{Ig}(p^r, N) \to \mathfrak{X}(N)_{/\mathbb{F}_q}$ is totally ramified at the supersingular points so $\deg \mathrm{S}(Np^r) = \deg \mathrm{S}(N)$. Therefore to obtain the global intersection number $\Lambda_1.\Lambda_2$ we multiply $i_s(\Lambda_1, \Lambda_2)$ by $\deg \mathrm{S}(N)$. We will now compute the local intersection number between each pair of irreducible components of $\bar{\mathfrak{X}}$, beginning with the case of distinct pairs.

**Proposition 4.3.** *Let $\nu_p$ denote the p-adic valuation normalized so $\nu_p(p) = 1$ and let $s$ be a supersingular point. We have*

$$\begin{aligned} i_s(\Lambda_{(1,-a)}, \Lambda_{(-pb,1)}) &= 1 \\ i_s(\Lambda_{(1,-a)}, \Lambda_{(1,-a')}) &= p^{2\nu_p(a'-a)} \\ i_s(\Lambda_{(-pb,1)}, \Lambda_{(-pb',1)}) &= p^{2\nu_p(b'-b)+2} \end{aligned}$$

*for $a, a' \in \mathbb{Z}/p^r\mathbb{Z}$ distinct and $b, b' \in \mathbb{Z}/p^{r-1}\mathbb{Z}$ distinct.*

*Proof.* Since each group homomorphism $\Lambda : (\mathbb{Z}/p^r\mathbb{Z})^2 \to \mathbb{Z}/p^r\mathbb{Z}$ corresponding to an irreducible component is a surjective, we know $\#\ker\Lambda = p^r$. By definition,

$$\Lambda_{(1,-a)}(a, 1) = a\Lambda_{(1,-a)}(1, 0) + \Lambda_{(1,-a)}(0, 1) = a - a = 0.$$

Therefore

$$\ker\Lambda_{(1,-a)} = \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}}\left\{\begin{pmatrix} a \\ 1 \end{pmatrix}\right\}.$$

Similarly, since

$$\Lambda_{(-pb,1)}(1, pb) = \Lambda_{(-pb,1)}(1, 0) + pb\Lambda_{(-pb,1)}(0, 1) = -pb + pb = 0$$

we have

$$\ker\Lambda_{(-pb,1)} = \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}}\left\{\begin{pmatrix} 1 \\ pb \end{pmatrix}\right\}.$$

By equation (11), we have

$$i_s(\Lambda_1, \Lambda_2) = (p^{2r}/\#(\ker\Lambda_1 + \ker\Lambda_2))^2.$$

We will now compute $\#(\ker\Lambda_1 + \ker\Lambda_2)$ by considering the following three cases.

**Case 1**: We have

$$\ker \Lambda_{(1,-a)} + \ker \Lambda_{(-pb,1)} = \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ pb \end{pmatrix} \right\}.$$

Note that

$$\det \begin{pmatrix} a & 1 \\ 1 & pb \end{pmatrix} = pab - 1$$

is invertible in $\mathbb{Z}/p^r\mathbb{Z}$. Therefore $\{(a,1),(1,pb)\}$ is a basis for $(\mathbb{Z}/p^r\mathbb{Z})^2$ hence

$$\ker \Lambda_{(1,-a)} + \ker \Lambda_{(-pb,1)} = (\mathbb{Z}/p^r\mathbb{Z})^2.$$

We conclude $i_s(\Lambda_{(1,-a)}.\Lambda_{(-pb,1)}) = 1$.

**Case 2**: We have

$$\begin{aligned} \ker \Lambda_{(1,-a)} + \ker \Lambda_{(1,-a')} &= \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} a' \\ 1 \end{pmatrix} \right\} \\ &= \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} a'-a \\ 0 \end{pmatrix} \right\} \end{aligned}$$

For any $0 \leq i \leq r$, consider the injective group homomorphism

$$m_i : \mathbb{Z}/p^{r-i}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z}$$

given by multiplication by $p^i$. This gives us a filtration

$$\mathbb{Z}/p^r\mathbb{Z} = \mathrm{im}(m_0) \supset \mathrm{im}(m_1) \supset \cdots \supset \mathrm{im}(m_{r-1}) \supset \mathrm{im}(m_r) = (0)$$

which is exhaustive and separated. Thus for any non-zero $c \in \mathbb{Z}/p^r\mathbb{Z}$, there exists a unique smallest $i$ such that $c \in \mathrm{im}(m_i)$. We let $\nu_p(c)$ denote the quantity $i$. We claim $c = p^{\nu_p(c)}u_c$ for some $u_c \in (\mathbb{Z}/p^r\mathbb{Z})^\times$. Indeed, if $u_c$ is not a unit of $(\mathbb{Z}/p^r\mathbb{Z})^\times$, then $u_c \in (p)$ so $u_c = pu_c'$ for some $u_c' \in \mathbb{Z}/p^r\mathbb{Z}$. Hence $c = p^{\nu_p(c)+1}u_c' \in \mathrm{im}(m_{\nu_p(c)+1})$, contradicting the minimality of $\nu_p(c)$.

We will show the sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}/p^{\nu_p(c)}\mathbb{Z} \xrightarrow{m_{r-\nu_p(c)}} \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\cdot c} \mathrm{span}_{\mathbb{Z}/p^r\mathbb{Z}}(c) \longrightarrow 0 \tag{12}$$

is exact. Exactness at the second and fourth term are clear. Let $x \in \mathbb{Z}/p^{\nu_p(c)}\mathbb{Z}$. Then

$$c \cdot m_{r-\nu_p(c)}(x) = c \cdot p^{r-\nu_p(c)}x = u_c p^{\nu_p(c)} p^{r-\nu_p(c)} x = u_c p^r x = 0$$

so $\mathrm{im}(m_{r-\nu_p(c)}) \subseteq \ker(\cdot c)$. On the other hand, let $d \in \ker(\cdot c)$ so $d \cdot c = 0$. Then

$$u_c u_d p^{\nu_p(c)} p^{\nu_p(d)} = 0.$$

Since $u_c, u_d \in (\mathbb{Z}/p^r\mathbb{Z})^\times$, we have $p^{\nu_p(c)+\nu_p(d)} = 0$. Hence $\nu_p(d) \geq r - \nu_p(c)$ or equivalently $\mathrm{im}(m_d) \subseteq \mathrm{im}(m_{r-\nu_p(c)})$. Therefore $\ker(\cdot c) \subseteq \mathrm{im}(m_{r-\nu_p(c)})$ allowing us to conclude exactness at the third term.

Let $i = \nu_p(a' - a)$. Consider the sequence of abelian groups

$$0 \to \mathbb{Z}/p^i\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{\psi} \text{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} a'-a \\ 0 \end{pmatrix} \right\} \to 0 \tag{13}$$

where $\phi(x) = (0, m_{r-i}(x))$ and

$$\psi(c,d) = c \begin{pmatrix} a \\ 1 \end{pmatrix} + d \begin{pmatrix} a'-a \\ 0 \end{pmatrix} = \begin{pmatrix} ca + d(a'-a) \\ c \end{pmatrix}.$$

One can show exactness similar to the above sequence (12). Indeed, showing $\text{im}(\phi) \subseteq \ker(\psi)$ is clear. Conversely, if $\psi(c,d) = (0,0)$, then $c = 0$ and consequently $d(a'-a) = 0$. Writing $p^{\nu_p(d)}p^i u_d u_{a'-a} = 0$, we similarly conclude $\nu_p(d) \geq r - i$ so $\ker(\psi) \subset \text{im}(\phi)$. Using exactness of (13), we get

$$\#\text{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} a'-a \\ 0 \end{pmatrix} \right\} = \frac{\#(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z})}{\#\mathbb{Z}/p^i\mathbb{Z}} = p^{2r-i} = p^{2r-\nu_p(a'-a)}.$$

We conclude $i_s(\Lambda_{(1,-a)}.\Lambda_{(1,-a')}) = p^{2\nu_p(a'-a)}$.

**Case 3**: Lastly, we have

$$\begin{aligned} \ker \Lambda_{(-pb,1)} + \ker \Lambda_{(-pb',1)} &= \text{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} 1 \\ pb \end{pmatrix}, \begin{pmatrix} 1 \\ pb' \end{pmatrix} \right\} \\ &= \text{span}_{\mathbb{Z}/p^r\mathbb{Z}} \left\{ \begin{pmatrix} 1 \\ pb \end{pmatrix}, \begin{pmatrix} 0 \\ p(b'-b) \end{pmatrix} \right\} \end{aligned}$$

Arguing as above by creating a sequence similar to (13) and replacing $a$ (resp. $a'$) with $pb$ (resp. $pb'$), we get

$$\#\left(\ker \Lambda_{(-pb,1)} + \ker \Lambda_{(-pb',1)}\right) = p^r \cdot p^{r-(\nu_p(b'-b)-1)}.$$

We conclude

$$i_s(\Lambda_{(-pb,1)}.\Lambda_{(-pb',1)}) = (p^{2r}/p^{2r-\nu_p(b'-b)+1})^2 = p^{2\nu_p(b'-b)+2}. \qquad \square$$

To finish our calculation of intersection numbers, we will now compute each self-intersection. First we introduce the following lemma.

**Lemma 4.4.** *Let $r \geq 1$. We have*

$$\sum_{\substack{a' \in \mathbb{Z}/p^r\mathbb{Z} \\ a' \neq 0}} p^{2\nu_p(a')} = p^{2r-1} - p^{r-1}.$$

*Proof.* We will group the index $a' \in \mathbb{Z}/p^r\mathbb{Z}$ based on its $p$-adic valuation $m = \nu_p(a')$ and then sum over $m$. Observe that there are precisely $\varphi(p^{r-m})$-many elements in $\mathbb{Z}/p^r\mathbb{Z}$ with $p$-adic valuation equal to $m$. Thus

$$\begin{aligned}
\sum_{\substack{a'\in\mathbb{Z}/p^r\mathbb{Z}\\ a'\neq 0}} p^{2\nu_p(a')} &= \sum_{m=0}^{r-1}\varphi(p^{r-m})p^{2m}\\
&= \sum_{m=0}^{r-1} p^{r-m-1}(p-1)p^{2m}\\
&= \sum_{m=0}^{r-1} p^{r+m-1}(p-1)\\
&= (p-1)p^{r-1}\sum_{m=0}^{r-1} p^m\\
&= (p-1)p^{r-1}\frac{1-p^r}{1-p}\\
&= p^{2r-1}-p^{r-1}
\end{aligned}$$

as desired. □

**Proposition 4.5.** *For any* $\Lambda \in (\mathbb{Z}/p^r\mathbb{Z})^\times\backslash\mathrm{HomSurj}((\mathbb{Z}/p^r\mathbb{Z})^2, \mathbb{Z}/p^r\mathbb{Z})$*, the self-intersection number is*

$$\Lambda.\Lambda = -\deg \mathrm{S}(N)\cdot p^{2r-1}.$$

*Proof.* Recall Proposition 2.27b, which states

$$\Lambda.\Lambda = -\frac{1}{\mu_\Lambda}\sum_{\Lambda'\neq\Lambda}\mu_{\Lambda'}(\Lambda'.\Lambda)$$

where $\mu_{\Lambda'}$ is the multiplicity of $\Lambda'$ for any irreducible component $\Lambda'$. Since $\bar{\mathfrak{X}}$ is reduced, $\mu_\Lambda = 1$.

We compute the self-intersection for the two possible cases of $\Lambda$. First we have

$$\begin{aligned}
\Lambda_{(1,-a)}.\Lambda_{(1,-a)} &= -\sum_{\substack{a'\in\mathbb{Z}/p^r\mathbb{Z}\\ a'\neq a}}\Lambda_{(1,-a')}.\Lambda_{(1,-a)} - \sum_{b\in\mathbb{Z}/p^{r-1}\mathbb{Z}}\Lambda_{(-pb,1)}.\Lambda_{(1,-a)}\\
&= -\deg \mathrm{S}(N)\sum_{\substack{a'\in\mathbb{Z}/p^r\mathbb{Z}\\ a'\neq a}} p^{2\nu_p(a'-a)} - \deg \mathrm{S}(N)\sum_{b\in\mathbb{Z}/p^{r-1}\mathbb{Z}} 1\\
&= -\deg \mathrm{S}(N)\sum_{\substack{a'\in\mathbb{Z}/p^r\mathbb{Z}\\ a'\neq 0}} p^{2\nu_p(a')} - \deg \mathrm{S}(N)\cdot p^{r-1}
\end{aligned}$$

where we have used Proposition 4.3 to calculate the intersection numbers. By Lemma 4.4, we have

$$\sum_{\substack{a'\in\mathbb{Z}/p^r\mathbb{Z}\\ a'\neq 0}} p^{2\nu_p(a')} = p^{2r-1}-p^{r-1}.$$

Hence

$$\begin{aligned}\Lambda_{(1,-a)}.\Lambda_{(1,-a)} &= -\deg \mathrm{S}(N)(p^{2r-1}-p^{r-1}) - \deg \mathrm{S}(N)\cdot p^{r-1} \\ &= -\deg \mathrm{S}(N)\cdot p^{2r-1}.\end{aligned}$$

Next we consider the case $\Lambda = \Lambda_{(-pb,1)}$. We have

$$\begin{aligned}\Lambda_{(-pb,1)}.\Lambda_{(-pb,1)} &= -\sum_{\substack{b'\in\mathbb{Z}/p^{r-1}\mathbb{Z}\\ b'\neq b}} \Lambda_{(-pb',1)}.\Lambda_{(-pb,1)} - \sum_{a\in\mathbb{Z}/p^r\mathbb{Z}} \Lambda_{(1,-a)}.\Lambda_{(1,-pb)} \\ &= -\deg \mathrm{S}(N) \sum_{\substack{b'\in\mathbb{Z}/p^{r-1}\mathbb{Z}\\ b'\neq b}} p^{2\nu_p(b'-b)+2} - \deg \mathrm{S}(N) \sum_{a\in\mathbb{Z}/p^r\mathbb{Z}} 1 \\ &= -\deg \mathrm{S}(N) \sum_{\substack{b'\in\mathbb{Z}/p^{r-1}\mathbb{Z}\\ b'\neq 0}} p^{2\nu_p(b')+2} - \deg \mathrm{S}(N)\cdot p^r \\ &= -\deg \mathrm{S}(N)\cdot p^2 \sum_{\substack{b'\in\mathbb{Z}/p^{r-1}\mathbb{Z}\\ b'\neq 0}} p^{2\nu_p(b')} - \deg \mathrm{S}(N)\cdot p^r \\ &= -\deg \mathrm{S}(N)\cdot p^2\left(p^{2(r-1)-1}-p^{r-2}\right) - \deg \mathrm{S}(N)\cdot p^r \\ &= -\deg \mathrm{S}(N)\left(p^2(p^{2(r-1)-1}-p^{r-2})+p^r\right) \\ &= -\deg \mathrm{S}(N)(p^{2(r-1)+1}-p^r+p^r) = -\deg \mathrm{S}(N)\cdot p^{2r-1}.\end{aligned}$$

□

We will now describe the intersection matrix $M$ by specifying four blocks which make up $M$. We label the columns (and by symmetry the rows) of $M$ in the following order:

$$\Lambda_{(1,0)} \quad \Lambda_{(1,-1)} \quad \dots \quad \Lambda_{(1,-a)} \quad \dots \quad \Lambda_{(1,-(p^r-1))} \quad \Lambda_{(0,1)} \quad \Lambda_{(-p,1)} \quad \Lambda_{(-p2,1)} \quad \dots \quad \Lambda_{(-pb,1)} \quad \dots \quad \Lambda_{(-p(p^{r-1}-1),1)}$$

so that the $(i,j)$ entry of $M$ is equal to the intersection number between the $i$th row label and $j$th column label. Since $\deg \mathrm{S}(N)$ is a common factor among each entry of $M$, we will describe the matrix $\frac{1}{\deg \mathrm{S}(N)}M$ to simplify exposition.

Let $M_{11}$ (resp. $M_{22}$) denote the submatrix of $\frac{1}{\deg \mathrm{S}(N)}M$ corresponding to the column and row labels of the form $\Lambda_{(1,-a)}$ (resp. $\Lambda_{(-pb,1)}$). We let $M_{12}$ and $M_{21}$ denote the remaining two submatrices of $\frac{1}{\deg \mathrm{S}(N)}M$ so that

$$M = \deg \mathrm{S}(N)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}.$$

We also let $M(p^r)$ denote the matrix $M_{11}$ to highlight the dependence on $p^r$. By convention, we define $M(p^0)$ to be the $1\times 1$ matrix consisting of the entry $-\frac{1}{p}$.

The matrices $M_{11}$ and $M_{22}$ take on a special form as we will show shortly.

**Definition 4.6.** An $n \times n$ **circulant** matrix $C$ is of the form

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

where each column is equal to the previous column shifted downward by 1, looping around as appropriate.

Refer to Appendix A for a discussion on circulant matrices, including an explicit description of the eigenvalues, eigenvectors, and inverse in terms of the entries of the matrix and roots of unity.

**Proposition 4.7.** *a. The entries of $M_{12}$ and $M_{21}$ are all equal to 1.*

*b. The matrix $M_{11} = M(p^r)$ is a $p^r \times p^r$ circulant matrix whose first entry of the first column is $-p^{2r-1}$. For $2 \leq a \leq p^r$, the $a^{\text{th}}$ entry in the first column is equal to $p^{2\nu_p(a-1)}$.*

*c. The matrix $M_{22}$ is a $p^{r-1} \times p^{r-1}$ circulant matrix equal to $p^2 M(p^{r-1})$.*

*Proof.* Proposition 4.3 immediately tells us the entries of $M_{12}$ and $M_{21}$ are all 1. Next we describe the first column of $M_{11}$. The first entry is equal to the local self-intersection number of $\Lambda_{(1,0)}$, which is $-p^{2r-1}$ by Proposition 4.5. Using Proposition 4.3, for $2 \leq a \leq p^r$, the $a^{\text{th}}$ entry of the first column is equal to the local intersection number

$$\frac{1}{\deg \mathrm{S}(N)} \Lambda_{(1,-(a-1))} . \Lambda_{(1,0)} = p^{2\nu_p(a-1)}.$$

Now we show $M_{11}$ is circulant. Recall the $j$th column of $M_{11}$ corresponds to the label $\Lambda_{(1,-(j-1))}$ for $1 \leq j \leq p^r$. The $k$th entry in the $j$th column is equal to

$$\frac{1}{\deg \mathrm{S}(N)} \Lambda_{(1,-(k-1))} . \Lambda_{(1,-(j-1))} = \begin{cases} p^{2\nu_p((k-1)-(j-1))} & \text{if } j \neq k \\ -p^{2r-1} & \text{if } j = k \end{cases}.$$

Note for any nonzero $x, y \in \mathbb{Z}$, if $x \equiv y \pmod{p^r}$, then $\nu_p(x) = \nu_p(y)$. Therefore the quantity

$$\nu_p((k-1)-(j-1))$$

for $k \neq j$ remains unchanged if we take $(k-1)-(j-1)$ modulo $p^r$. We conclude

$$\Lambda_{(1,-(k-1))} . \Lambda_{(1,-(j-1))} = \Lambda_{(1,0)} . \Lambda_{(1,-(k-1)+(j-1))}$$

which says the $j$th column is equal to the first column with every entry shifted downward by $j-1$, looping around as appropriate. Hence $M_{11}$ is a circulant matrix.

Lastly, we show $M_{22} = p^2 M(p^{r-1})$. Suppose $r > 1$. By Proposition 4.3, the $(i,j)$ entry of $M_{22}$ for $i \neq j$ is equal to

$$p^{2\nu_p((j-1)-(i-1))+2} = p^2 p^{2\nu_p((j-1)-(i-1))}$$

which is equal to the $(i,j)$ entry of $M(p^{r-1})$ multiplied by $p^2$. When $i=j$, the $(i,j)$ entry of $M_{22}$ is $-p^{2r-1}$ while the $(i,j)$ entry of $p^2M(p^{r-1})$ is

$$-p^2p^{2(r-1)-1} = -p^2p^{2r-3} = -p^{2r-1}.$$

Thus $M_{22} = p^2M(p^{r-1})$.

When $r=1$, $M_{22}$ is a $1\times 1$ matrix consisting of the entry $(-p)$. By convention $M(p^0) = (-\frac{1}{p})$ so $M_{22} = p^2M(p^{r-1})$ in the case $r=1$. □

**Example 4.8.** The intersection matrix for $\mathfrak{X}(N\cdot 5)$ is the $6\times 6$ matrix

$$\deg \mathrm{S}(N)\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & -5 & 1 & 1 & 1 & 1 \\ 1 & 1 & -5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 1 & -5 & 1 \\ 1 & 1 & 1 & 1 & 1 & -5 \end{pmatrix}$$

where the entries in red comprise $M_{11} = M(p^r)$ and the entries in blue comprise $M_{22}$. The intersection matrix for $\mathfrak{X}(N3^2)$ is the $12\times 12$ matrix

$$\deg \mathrm{S}(N)\begin{pmatrix} -3^3 & 1 & 1 & 3^2 & 1 & 1 & 3^2 & 1 & 1 & 1 & 1 & 1 \\ 1 & -3^3 & 1 & 1 & 3^2 & 1 & 1 & 3^2 & 1 & 1 & 1 & 1 \\ 1 & 1 & -3^3 & 1 & 1 & 3^2 & 1 & 1 & 3^2 & 1 & 1 & 1 \\ 3^2 & 1 & 1 & -3^3 & 1 & 1 & 3^2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3^2 & 1 & 1 & -3^3 & 1 & 1 & 3^2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3^2 & 1 & 1 & -3^3 & 1 & 1 & 3^2 & 1 & 1 & 1 \\ 3^2 & 1 & 1 & 3^2 & 1 & 1 & -3^3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3^2 & 1 & 1 & 3^2 & 1 & 1 & -3^3 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3^2 & 1 & 1 & 3^2 & 1 & 1 & -3^3 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3^3 & 3^2 & 3^2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3^2 & -3^3 & 3^2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3^2 & 3^2 & -3^3 \end{pmatrix}$$

## 4.2 Inverting $M(p^r)$

Recall our goal is to invert the matrix $T$ obtained by removing the first row and column of $M$. For a general matrix $A$, let $A_{\hat{1},\hat{1}}$ denote the matrix obtained by removing the first row and column of $A$ and let $\mathbf{1}_{n\times m}$ denote the $n\times m$ matrix whose entries are all 1. Using Proposition 4.7, we have the following description of $T$:

$$T = \deg \mathrm{S}(N)\begin{pmatrix} M(p^r)_{\hat{1},\hat{1}} & \mathbf{1}_{p^{r-1}\times p^{r-1}} \\ \mathbf{1}_{p^{r-1}\times p^{r-1}} & p^2M(p^{r-1}) \end{pmatrix}.$$

We will use the following identity to invert $T$.

**Proposition 4.9.** *(Woodbury Matrix Identity) Let $A$ be an $n \times n$ invertible matrix, $C$ an invertible $k \times k$ matrix where $k \leq n$, $U$ an $n \times k$ matrix, and $V$ a $k \times n$ matrix. Then*

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}.$$

We write $T$ as a sum of two matrices $T = A + N$ where

$$A = \begin{pmatrix} M(p^r)_{\hat{1},\hat{1}} & 0 \\ 0 & p^2 M(p^{r-1}) \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & \mathbf{1}_{p^r-1 \times p^{r-1}} \\ \mathbf{1}_{p^{r-1} \times p^r-1} & 0 \end{pmatrix}.$$

Refer to Appendix B for a discussion on using the Woodbury Matrix Identity to compute the inverse of $A + UCV$ in the situation that $A$ is a block diagonal matrix with two blocks, $C$ is the $2 \times 2$ identity, and both $U$ and $V$ consists of 0's and 1's which we specify later. Note that the inverse of a block diagonal matrix is obtained by inverting each block. Thus computing $A^{-1}$ amounts to computing $M(p^r)^{-1}$ and $M(p^r)^{-1}_{\hat{1},\hat{1}}$ which we will do in this subsection.

Before we compute the eigenvalues of $M(p^r)$, we will need the following technical lemma.

**Lemma 4.10.** *Let $N$ and $J$ be integers $\geq 1$. We have*

$$\sum_{\substack{u=1 \\ p \nmid u}}^{p^N-1} \zeta_{p^N}^{-uJ} = \begin{cases} 0 & \text{if } p^{N-1} \nmid J \\ -p^{N-1} & \text{if } p^{N-1} \mid J \text{ and } p^N \nmid J \\ p^N - p^{N-1} & \text{if } p^N \mid J \end{cases}.$$

*Proof.* By the geometric partial sum formula, we have

$$\begin{aligned} \sum_{u=1}^{p^N-1} \zeta_{p^N}^{-uJ} &= \begin{cases} -1 + \frac{1-\zeta_{p^N}^{-Jp^N}}{1-\zeta_{p^N}^{-J}} & \text{if } p^N \nmid J \\ p^N - 1 & \text{if } p^N \mid J \end{cases} \\ &= \begin{cases} -1 & \text{if } p^N \nmid J \\ p^N - 1 & \text{if } p^N \mid J \end{cases}. \end{aligned}$$

Next we consider the sum

$$\sum_{\substack{u=1 \\ p \mid u}}^{p^N-1} \zeta_{p^N}^{-uJ} = \sum_{u=1}^{p^{N-1}-1} \zeta_{p^{N-1}}^{-uJ} = \begin{cases} -1 & \text{if } p^{N-1} \nmid J \\ p^{N-1} - 1 & \text{if } p^{N-1} \mid J \end{cases}.$$

Therefore

$$\sum_{\substack{u=1\\ p\nmid u}}^{p^N-1} \zeta_{p^N}^{-uJ} = \sum_{u=1}^{p^N-1} \zeta_{p^N}^{-uJ} - \sum_{\substack{u=1\\ p\mid u}}^{p^N-1} \zeta_{p^N}^{-uJ} = \begin{cases} 0 & \text{if } p^{N-1} \nmid J \\ -p^{N-1} & \text{if } p^{N-1} \mid J \text{ and } p^N \nmid J \\ p^N - p^{N-1} & \text{if } p^N \mid J \end{cases}. \qquad \square$$

**Lemma 4.11.** *The eigenvalues $\lambda_j$ of $M(p^r)$ are*

$$\lambda_1 = -p^{r-1}$$

*and*

$$\lambda_j = -p^{2r-2-\nu_p(j-1)}(p+1)$$

*for $2 \le j \le p^r$.*

*Proof.* By Lemma A.2, the eigenvalues of an $n \times n$ circulant matrix whose first column has entries $c_0, c_1, \ldots, c_{n-1}$ are given by

$$\lambda_j = \sum_{k=0}^{n-1} c_k \zeta_n^{j(n-k)}.$$

For $M(p^r)$, the eigenvalues are therefore given by

$$\lambda_j = -p^{2r-1} + \sum_{k=1}^{p^r-1} p^{2\nu_p(k)} \zeta_{p^r}^{(j-1)(p^r-k)} = -p^{2r-1} + \sum_{k=1}^{p^r-1} p^{2\nu_p(k)} \zeta_{p^r}^{-k(j-1)}$$

for $j = 1, \ldots, p^r$.

When $j = 1$, we have

$$\lambda_1 = -p^{2r-1} + \sum_{k=1}^{p^r-1} p^{2\nu_p(k)} = -p^{2r-1} + (p^{2r-1} - p^{r-1}) = -p^{r-1}$$

where we have used Lemma 4.4 to calculate the sum.Thm. 8.4

Assume $j > 1$. When $r = 1$, we have

$$\lambda_j = -p + \sum_{k=1}^{p-1} p^{2\nu_p(k)} \zeta_p^{-k(j-1)}.$$

Since $\nu_p(k) = 0$ for $1 \le k \le p-1$, we have

$$\lambda_j = -p + \sum_{k=1}^{p-1} \left(\zeta_p^{-(j-1)}\right)^k$$

As $2 \le j \le p$, we have $p \nmid (j-1)$ so $\zeta_p^{-(j-1)} \ne 1$. Using the geometric series partial sum formula, we get

$$\lambda_j = -p - 1 + \frac{1 - \zeta_p^{-(j-1)p}}{1 - \zeta_p^{-(j-1)}} = -p - 1.$$

Lastly we handle the $r > 1$ case. We compute the sum appearing in the expression for $\lambda_j$ by breaking it up according to the value of $\nu_p(k)$. We have

$$\sum_{k=1}^{p^r-1} p^{2\nu_p(k)}\zeta_{p^r}^{-k(j-1)} = \sum_{\substack{k=1\\ \nu_p(k)=0}}^{p^r-1} \zeta_{p^r}^{-k(j-1)} + p^2 \sum_{\substack{k=1\\ \nu_p(k)=1}}^{p^r-1} \zeta_{p^r}^{-k(j-1)} + p^4 \sum_{\substack{k=1\\ \nu_p(k)=2}}^{p^r-1} \zeta_{p^r}^{-k(j-1)} + \cdots + p^{2(r-1)} \sum_{\substack{k=1\\ \nu_p(k)=r-1}}^{p^r-1} \zeta_{p^r}^{-k(j-1)}.$$

We can rewrite the index in each sum as $k = up^{\nu_p(k)}$ where $p \nmid u$ and $1 \le u \le p^{r-\nu_p(k)} - 1$. Re-indexing, with $u \ge 1$, we get

$$= \sum_{\substack{u=1\\ p\nmid u}}^{p^r-1} \zeta_{p^r}^{-u(j-1)} + p^2 \sum_{\substack{u=1\\ p\nmid u}}^{p^{r-1}-1} \zeta_{p^r}^{-up(j-1)} + p^4 \sum_{\substack{u=1\\ p\nmid u}}^{p^{r-2}-1} \zeta_{p^r}^{up^2(j-1)} + \cdots + p^{2(r-1)} \sum_{\substack{u=1\\ p\nmid u}}^{p-1} \zeta_{p^r}^{-up^{r-1}(j-1)}.$$

Since $\zeta_{p^r}^{p^m} = \zeta_{p^{r-m}}$ for $0 \le m < r$, we have

$$= \sum_{\substack{u=1\\ p\nmid u}}^{p^r-1} \zeta_{p^r}^{-u(j-1)} + p^2 \sum_{\substack{u=1\\ p\nmid u}}^{p^{r-1}-1} \zeta_{p^{r-1}}^{-u(j-1)} + p^4 \sum_{\substack{u=1\\ p\nmid u}}^{p^{r-2}-1} \zeta_{p^{r-2}}^{-u(j-1)} + \cdots + p^{2(r-1)} \sum_{\substack{u=1\\ p\nmid u}}^{p-1} \zeta_{p}^{-u(j-1)} \tag{14}$$

Let $\ell = \nu_p(j-1)$. We will simplify (14) using Lemma 4.10 with $J = j-1$, and $N = 1, \ldots, r$. For ease of exposition, we split into two different cases depending on $\ell$ and will consequently obtain our desired expression for $\lambda_j$.

**Case 1**: Suppose $\ell = 0$. By Lemma 4.10, each sum in (14) is zero except for the last sum. We get

$$\lambda_j = -p^{2r-1} + p^{2(r-1)} \sum_{\substack{u=1\\ p\nmid u}}^{p-1} \zeta_p^{-u(j-1)} = -p^{2r-1} + p^{2(r-1)}(-1) = -p^{2r-1} - p^{2(r-1)}.$$

**Case 2**: Suppose $1 \le \ell \le r-1$. Since $p^{\ell+1} \nmid (j-1)$, we have

$$\sum_{\substack{u=1\\ p\nmid u}}^{p^N-1} \zeta_{p^N}^{-u(j-1)} = 0$$

for all $N \geq \ell + 2$. Expression (14) becomes

$$
\begin{aligned}
&= p^{2(r-(\ell+1))} \sum_{\substack{u=1 \\ p \nmid u}}^{p^{\ell+1}-1} \zeta_{p^{\ell+1}}^{-u(j-1)} + p^{2(r-\ell)} \sum_{\substack{u=1 \\ p \nmid u}}^{p^{\ell}-1} \zeta_{p^{\ell}}^{-u(j-1)} + p^{2(r-(\ell-1))} \sum_{\substack{u=1 \\ p \nmid u}}^{p^{\ell-1}-1} \zeta_{p^{\ell-1}}^{-u(j-1)} + \cdots + p^{2(r-1)} \sum_{\substack{u=1 \\ p \nmid u}}^{p-1} \zeta_{p}^{-u(j-1)} \\
&= p^{2(r-(\ell+1))}(-p^{\ell}) + p^{2(r-\ell)}(p^{\ell} - p^{\ell-1}) + p^{2(r-(\ell-1))}(p^{\ell-1} - p^{\ell-2}) + \cdots + p^{2(r-2)}(p^2 - p) + p^{2(r-1)}(p-1) \\
&= -p^{2(r-(\ell+1))+\ell} + p^{2(r-\ell)+\ell-1}(p-1) + p^{2(r-(\ell-1))+\ell-2}(p-1) + \cdots + p^{2(r-2)+1}(p-1) + p^{2(r-1)}(p-1) \\
&= -p^{2(r-1)-\ell} + p^{2r-(\ell+1)}(p-1) + p^{2r-\ell}(p-1) + \cdots + p^{2r-3}(p-1) + p^{2r-2}(p-1) \\
&= -p^{2(r-1)-\ell} + (p-1)p^{2r} \sum_{k=2}^{\ell+1} p^{-k} \\
&= -p^{2(r-1)-\ell} + (p-1)p^{2r} \left( -1 - p^{-1} + \frac{1 - p^{-(\ell+2)}}{1 - p^{-1}} \right) \\
&= -p^{2(r-1)-\ell} + (p-1)p^{2r} \cdot \frac{p^{-(\ell+1)}(p^{\ell} - 1)}{p-1} \\
&= -p^{2(r-1)-\ell} + p^{2r-(\ell+1)}(p^{\ell} - 1).
\end{aligned}
$$

We conclude

$$
\begin{aligned}
\lambda_j &= (-p^{2r-1}) - p^{2(r-1)-\ell} + p^{2r-(\ell+1)}(p^{\ell} - 1) \\
&= -p^{2r-1} - p^{2r-\ell-2} + p^{2r+1} - p^{2r-\ell-1} \\
&= -p^{2r-2-\ell}(p+1) \\
&= -p^{2r-2-\nu_p(j-1)}(p+1)
\end{aligned}
$$

as desired. □

**Corollary 4.12.** *$M(p^r)$ is invertible.*

*Proof.* By Lemma 4.11, all the eigenvalues of $M(p^r)$ are nonzero hence $M(p^r)$ is invertible. □

Now that we know the eigenvalues of $M(p^r)$, we can use Proposition A.4 to compute the inverse of $M(p^r)$.

**Proposition 4.13.** *Let $b_{i,j}$ denote the $(i,j)$-entry of $M(p^r)^{-1}$. We have*

$$
b_{i,j} = \begin{cases} -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} & \text{if } i = j \\ -p^{1-2r} - \frac{p^{-3r+2}}{p+1} \cdot (-p^{r-1} + \nu_p(i-j)p^{r-1}(p-1)) & \text{otherwise} \end{cases}.
$$

*Proof.* By Lemma A.4, the $(i,j)$-entry of $M(p^r)^{-1}$ is equal to

$$\frac{1}{p^r}\sum_{k=1}^{p^r}\lambda_k^{-1}\zeta_{p^r}^{(k-1)(i-j)}$$

where $\lambda_k$ are the eigenvalues of $M(p^r)$ as in Lemma 4.11. Continuing,

$$\begin{aligned}
&= \frac{1}{p^r}\left(-p^{1-r}+\sum_{k=2}^{p^r}\frac{-1}{p+1}p^{-2r+2+\nu_p(k-1)}\zeta_{p^r}^{(k-1)(i-j)}\right)\\
&= -p^{1-2r}-\frac{p^{-3r+2}}{p+1}\sum_{k=2}^{p^r}p^{\nu_p(k-1)}\zeta_{p^r}^{(k-1)(i-j)}\\
&= -p^{1-2r}-\frac{p^{-3r+2}}{p+1}\sum_{k=1}^{p^r-1}p^{\nu_p(k)}\zeta_{p^r}^{k(i-j)}.
\end{aligned}$$

We split into two cases, breaking down the sum in a similar manner as in the proof of Lemma 4.11. We have

$$\sum_{k=1}^{p^r-1}p^{\nu_p(k)}\zeta_{p^r}^{k(i-j)}=\sum_{\substack{u=1\\ p\nmid u}}^{p^r-1}\zeta_{p^r}^{u(i-j)}+p\sum_{\substack{u=1\\ p\nmid u}}^{p^{r-1}-1}\zeta_{p^{r-1}}^{u(i-j)}+p^2\sum_{\substack{u=1\\ p\nmid u}}^{p^{r-2}-1}\zeta_{p^{r-2}}^{u(i-j)}+\cdots+p^{r-1}\sum_{\substack{u=1\\ p\nmid u}}^{p-1}\zeta_p^{u(i-j)}$$

**Case 1**: Suppose $i=j$. Then

$$\begin{aligned}
\sum_{k=1}^{p^r-1}p^{\nu_p(k)}\zeta_{p^r}^{k(i-j)} &= \sum_{\substack{u=1\\ p\nmid u}}^{p^r-1}1+p\sum_{\substack{u=1\\ p\nmid u}}^{p^{r-1}-1}1+p^2\sum_{\substack{u=1\\ p\nmid u}}^{p^{r-2}-1}1+\cdots+p^{r-1}\sum_{\substack{u=1\\ p\nmid u}}^{p-1}1\\
&= p^{r-1}(p-1)+pp^{r-2}(p-1)+p^2p^{r-3}(p-1)+\cdots+p^{r-1}(p-1)\\
&= p^{r-1}(p-1)+p^{r-1}(p-1)+p^{r-1}(p-1)+\cdots+p^{r-1}(p-1)=rp^{r-1}(p-1).
\end{aligned}$$

Thus the $(i,i)$-entry of $M(p^r)^{-1}$ is

$$-p^{1-2r}-\frac{p^{-3r+2}}{p+1}rp^{r-1}(p-1)=-p^{1-2r}-\frac{p-1}{p+1}\cdot r\cdot p^{1-2r}.$$

**Case 2**: Suppose $i\neq j$ and let $\ell=\nu_p(i-j)$. Then this situation resembles that of Equation (14) which we have already computed. We have

$$\begin{aligned}
&\sum_{k=1}^{p^r-1}p^{\nu_p(k)}\zeta_{p^r}^{k(i-j)}\\
&= p^{r-(\ell+1)}\sum_{p\nmid u}^{p^{\ell+1}-1}\zeta_{p^{\ell+1}}^{u(i-j)}+p^{r-\ell}\sum_{p\nmid u}^{p^{\ell}-1}\zeta_{p^{\ell}}^{u(i-j)}+p^{r-(\ell-1)}\sum_{p\nmid u}^{p^{\ell-1}-1}\zeta_{p^{\ell-1}}^{u(i-j)}+\cdots+p^{r-1}\sum_{p\nmid u}^{p-1}\zeta_{p}^{u(i-j)}\\
&= -p^{r-(\ell+1)+\ell}+p^{r-\ell+\ell-1}(p-1)+p^{r-(\ell-1)+\ell-2}(p-1)+\cdots+p^{r-2+1}(p-1)+p^{r-1}(p-1)\\
&= -p^{r-1}+\ell p^{r-1}(p-1).
\end{aligned}$$

Thus the $(i,j)$-entry of $M(p^r)^{-1}$ is

$$-p^{1-2r} - \frac{p^{-3r+2}}{p+1}\left(-p^{r-1} + \ell p^{r-1}(p-1)\right) = -p^{1-2r} - \frac{p^{-3r+2}}{p+1}\left(-p^{r-1} + \nu_p(i-j)p^{r-1}(p-1)\right)$$

as desired. □

## 4.3 Inverting $M(p^r)_{\hat{1},\hat{1}}$

Having calculated the entries of $M(p^r)^{-1}$, we can calculate the entries of $M(p^r)^{-1}_{\hat{1},\hat{1}}$ using the following result. We will provide a sketch of the proof. A full proof can be found in [JCP16, Theorem 2.2].

**Proposition 4.14.** *Let $A$ be an invertible $n \times n$ matrix and let $A^{-1} = (m_{ij})$. Let $s,t \in \{1,\dots,n\}$ and let $A_{\hat{s},\hat{t}}$ denote the matrix obtain from $A$ by removing the $s^{\text{th}}$ row and $t^{\text{th}}$ column. Then the $(i,j)$-entry of $A^{-1}_{\hat{s},\hat{t}} = (a_{ij})$ is given by*

$$a_{ij} = m_{ij} - \frac{m_{is}m_{tj}}{m_{ts}}$$

*for $i,j = 1,\dots,n$ with $i \neq t$ and $j \neq s$.*

*Proof.* Write $A = (w_{ij})$. Let $u$ denote the $s^{\text{th}}$ column of $A^{-1}$ after removing the $t^{\text{th}}$ component and let $v$ denote the $s^{\text{th}}$ row of $A$ after removing the $t^{\text{th}}$ component. Then one can verify

$$(A_{\hat{s},\hat{t}})^{-1} = \left(I_{n-1} - uv^T\right)^{-1}(A^{-1})_{\hat{t},\hat{s}}$$

where $I_{n-1}$ is the $(n-1)\times(n-1)$ identity matrix. Using the Sherman-Morrison formula, which is a special case of Proposition 4.9, to calculate $(I_{n-1} - uv^T)^{-1}$, we get

$$\begin{aligned}(A_{\hat{s},\hat{t}})^{-1} &= \left(I_{n-1} + \frac{uv^T}{1 - v^Tu}\right)(A^{-1})_{\hat{t},\hat{s}} \\ &= \left(I_{n-1} + \frac{uv^T}{w_{st}m_{ts}}\right)(A^{-1})_{\hat{t},\hat{s}.}\end{aligned}$$

Therefore the $(i,j)$ entry of $(A_{\hat{s},\hat{t}})^{-1}$ is

$$\begin{aligned}a_{ij} &= m_{ij} + \frac{m_{is}}{w_{st}m_{ts}}\sum_{k\neq q} w_{sk}m_{kj} \\ &= m_{ij} - \frac{m_{is}m_{tj}}{m_{ts}}\end{aligned}$$

□

**Proposition 4.15.** *Let* $(a_{i,j}) = M(p^r)^{-1}_{\hat{1},\hat{1}}$ *and let* $\ell_i = \nu_p(i)$. *We have*

$$a_{i,j} = \begin{cases} -p^{1-2r} - \dfrac{p-1}{p+1}rp^{1-2r} + \dfrac{p^{1-2r}(\ell_i p - \ell_i + p)^2}{(p+1)(pr+p-r+1)} & \text{if } i = j \\ -\dfrac{p^{1-2r}(\ell_{i-j}p + p - \ell_{i-j})}{p+1} + \dfrac{p^{1-2r}(\ell_i p + p - \ell_i)(\ell_j p + p - \ell_j)}{(p+1)(pr+p-r+1)} & \text{otherwise} \end{cases}$$

*Proof.* We will apply Proposition 4.13 and Proposition 4.14 to compute $a_{i,j}$. When $i = j$, we have

$$\begin{aligned} a_{i,i} &= m_{i,i} - \frac{m_{i,1}m_{1,i}}{m_{1,1}} \\ &= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{\left(-p^{1-2r} - \frac{p^{-3r+2}}{p+1}\cdot(-p^{r-1} + \ell_{i+1}p^{r-1}(p-1))\right)^2}{-p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r}} \\ &= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} + \frac{p^{1-2r}(\ell_{i+1}p - \ell_{i+1} + p)^2}{(p+1)(pr+p-r+1)}. \end{aligned}$$

When $i \neq j$, we have

$$\begin{aligned} a_{i,j} &= m_{i,j} - \frac{m_{i,1}m_{1,j}}{m_{1,1}} \\ &= -p^{1-2r} - \frac{p^{-3r+2}}{p+1}\cdot(-p^{r-1} + \ell_{i-j}p^{r-1}(p-1)) \\ &\quad - \frac{\left(-p^{1-2r} - \frac{p^{-3r+2}}{p+1}\cdot(-p^{r-1} + \ell_{i+1}p^{r-1}(p-1))\right)\left(-p^{1-2r} - \frac{p^{-3r+2}}{p+1}\cdot(-p^{r-1} + \ell_{j+1}p^{r-1}(p-1))\right)}{\left(-p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r}\right)} \\ &= -\frac{p^{1-2r}(\ell_{i-j}p - \ell_{i-j} + p)}{p+1} + \frac{\frac{p^{1-2r}(\ell_{i+1}p - \ell_{i+1} + p)}{p+1}\cdot\frac{p^{1-2r}(\ell_{j+1}p - \ell_{j+1} + p)}{p+1}}{\frac{p^{1-2r}(pr+p-r+1)}{p+1}} \\ &= -\frac{p^{1-2r}(\ell_{i-j}p + p - \ell_{i-j})}{p+1} + \frac{p^{1-2r}(\ell_{i+1}p + p - \ell_{i+1})(\ell_{j+1}p + p - \ell_{j+1})}{(p+1)(pr+p-r+1)}. \end{aligned}$$

Note that the indices $i, j$ in Proposition 4.14 range $2 \leq i, j \leq p^r$ in the situation $s = t = 1$; we will shift our index down by 1 so that $1 \leq i, j \leq p^r - 1$, giving our desired expression for $a_{i,j}$ in the statement of the proposition. □

## 4.4 Inverting $T$

Recall in Section 4.2 we wrote $T = A + N$ where

$$A = \begin{pmatrix} M(p^r)_{\hat{1},\hat{1}} & 0 \\ 0 & p^2M(p^{r-1}) \end{pmatrix} \text{ and } N = \begin{pmatrix} 0 & \mathbf{1}_{p^{r-1}\times p^{r-1}} \\ \mathbf{1}_{p^{r-1}\times p^{r-1}} & 0 \end{pmatrix}.$$

The matrix $N$ is rank 2 and can be written as $N = UI_2V$ where $I_2$ is the $2 \times 2$ identity matrix, $U$ is the $(p^r - 1 + p^{r-1}) \times 2$ matrix whose first and last column are the same as those of $N$, and $V$ is the $2 \times (p^r - 1 + p^{r-1})$

matrix

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

where the first $p^r - 1$ entries of the first row of $V$ are all 1 with the remaining $p^{r-1}$ entries are all 0 and the first $p^r - 1$ entries of the second row are 0 while the remaining $p^{r-1}$ entries are all 1. Note that $V = U^T$.

In the more general situation where $A$ is an invertible block diagonal matrix with

$$A^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & & & \\ \vdots & \ddots & \vdots & & \mathbf{0} & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & b_{11} & \cdots & b_{1m} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & b_{m1} & \cdots & b_{mm} \end{pmatrix}$$

an explicit formula for the entries of $(A+UV)^{-1}$ is provided in the appendix, Proposition B.1. We state it here for convenience:

**Proposition 4.16.** *Let $T = A + UV$ where $A, U$, and $V$ are the given matrices above. Let $c^{i,j}$ denote the $(i,j)$-entry of $T^{-1}$. We have*

$$c^{i,j} = \begin{cases} a_{i,j} + \dfrac{\beta}{1-\alpha\beta}\left(\displaystyle\sum_{k=1}^{n} a_{i,k}\right)\left(\displaystyle\sum_{k=1}^{n} a_{k,j}\right) & \text{if } 1 \le i,j \le n \\ \dfrac{-1}{1-\alpha\beta}\left(\displaystyle\sum_{k=1}^{n} a_{i,k}\right)\left(\displaystyle\sum_{k=1}^{m} b_{k,j}\right) & \text{if } 1 \le i \le n \text{ and } n < j \le n+m \\ \dfrac{-1}{1-\alpha\beta}\left(\displaystyle\sum_{k=1}^{m} b_{i,k}\right)\left(\displaystyle\sum_{k=1}^{n} a_{k,j}\right) & \text{if } 1 \le j \le n \text{ and } n < i \le n+m \\ b_{i,j} + \dfrac{\alpha}{1-\alpha\beta}\left(\displaystyle\sum_{k=1}^{m} b_{i,k}\right)\left(\displaystyle\sum_{k=1}^{m} b_{k,j}\right) & \text{if } n < i,j \le n+m \end{cases}$$

*where $\alpha = \sum a_{i,j}$ is the sum of all entries in the first block in $A^{-1}$ and $\beta = \sum b_{i,j}$ is the sum of all entries in the second block in $A^{-1}$.*

To obtain a closed formula for the entries $c^{i,j}$, we will therefore need to calculate the row and column sums of $M(p^r)^{-1}_{\hat{1},\hat{1}}$ and $p^{-2}M(p^{r-1})^{-1}$. Note that both these matrices are symmetric so it suffices to compute, say, the row sums.

The following lemma will be used when computing these sums.

**Lemma 4.17.** *We have*

$$\sum_{m=1}^{p^r-1} \nu_p(m) = \frac{p^r - pr + r - 1}{p-1}.$$

*Proof.* The number of positive integers $< p^r$ with valuation $\nu_p(m) = \ell$ is precisely $\varphi(p^{r-\ell}) = p^{r-\ell-1}(p-1)$.

Hence

$$\sum_{m=1}^{p^r-1} \nu_p(m) = \sum_{\ell=0}^{r-1} \ell p^{r-\ell-1}(p-1) = p^{r-1}(p-1)\sum_{\ell=0}^{r-1} \ell p^{-\ell}.$$

Using the identity

$$\sum_{\ell=0}^{n} \ell X^\ell = \frac{X(nX^{n+1} - (n+1)X^n + 1)}{(X-1)^2}$$

with $X = p^{-1}$ and $n = r-1$, we obtain

$$\begin{aligned} p^{r-1}(p-1)\sum_{\ell=0}^{r-1} \ell p^{-\ell} &= p^{r-1}(p-1)\cdot \frac{p^{-1}((r-1)p^{-r} - rp^{-(r-1)} + 1)}{(p^{-1}-1)^2} \\ &= \frac{p^r - pr + r - 1}{p-1}. \end{aligned}$$

□

**Corollary 4.18.** *Let $1 \le i \le p^r - 1$. We have*

$$\sum_{\substack{j=1 \\ j\neq i}}^{p^r-1} \nu_p(j-i) = -\nu_p(i) + \frac{p^r - pr + r - 1}{p-1}.$$

*Proof.* We have

$$\sum_{\substack{j=1 \\ j\neq i}}^{p^r-1} \nu_p(j-i) = \sum_{\substack{j=1-i \\ j\neq 0}}^{p^r-1-i} \nu_p(j) = \sum_{j=1-i}^{-1} \nu_p(j) + \sum_{j=1}^{p^{r-1}-1-i} \nu_p(j).$$

Since $\nu_p(j) < r$, we have $\nu_p(j) = \nu_p(j+p^r)$. Continuing,

$$\begin{aligned} &= \sum_{j=1-i}^{-1} \nu_p(j+p^r) + \sum_{j=1}^{p^{r-1}-i-1} \nu_p(j) \\ &= \sum_{j=p^r-i+1}^{p^r-1} \nu_p(j) + \sum_{j=1}^{p^r-i-1} \nu_p(j) \\ &= -\nu_p(p^r - i) + \sum_{j=1}^{p^r-1} \nu_p(j) \\ &= -\nu_p(i) + \sum_{j=1}^{p^r-1} \nu_p(j). \end{aligned}$$

By Lemma 4.17, we finally get

$$= -\nu_p(i) + \frac{p^r - pr + r - 1}{p-1}.$$

□

Write $M(p^r)^{-1}_{\hat{1},\hat{1}} = (a_{ij})$ and $p^{-2}M(p^{r-1})^{-1} = (b_{ij})$. We first compute the row and column sums of $p^{-2}M(p^{r-1})^{-1}$ and the quantity $\beta$, the sum of all the entries of $(b_{ij})$.

**Lemma 4.19.** *Consider the matrix $p^{-2}M(p^{r-1})^{-1} = (b_{ij})$. We have*

$$\sum_{m=1}^{p^{r-1}} b_{i,m} = -p^{-r} \text{ and } \beta = -p^{-1}$$

*for all $1 \leq i \leq p^{r-1}$.*

*Proof.* Since $M(p^{r-1})$ is circulant, the inverse $M(p^{r-1})^{-1}$ is circulant by Corollary A.5. Therefore all the row sums are the same. Furthermore, by Lemma 4.11, $\lambda_1 = -p^{r-2}$ is an eigenvalue of $M(p^{r-1})$ with corresponding eigenvector $v_1 = (1, 1, \ldots, 1)$ (see Lemma A.2).

Note that the entries of $M(p^{r-1})v_1 = -p^{r-2}v_1$ are precisely the row sums of $M(p^{r-1})$. Hence the row sums of $M(p^{r-1})$ are all $-p^{r-2}$. Observe that

$$M(p^{r-1})^{-1}v_1 = -p^{2-r}v_1$$

so the row sums of $M(p^{r-1})^{-1}$ are all $-p^{2-r}$. Consequently the row sums of $p^{-2}M(p^{r-1})^{-1}$ are all $-p^{-r}$.

The matrix $p^{-2}M(p^{r-1})^{-1}$ has $p^{r-1}$ rows so

$$\beta = (p^{r-1})(-p^{-r}) = -p^{-1}$$

as desired. □

Next we compute the row sums for $M(p^r)^{-1}_{\hat{1},\hat{1}}$ which is substantially more tedious than Lemma 4.19, noting that $M(p^r)_{\hat{1},\hat{1}}$ fails to be circulant in general.

**Lemma 4.20.** *Consider the matrix $M(p^r)^{-1}_{\hat{1},\hat{1}} = (a_{ij})$ and let $\ell_i = \nu_p(i)$. Fix a row $i \geq 1$. We have*

$$\sum_{j=1}^{p^r-1} a_{ij} = \frac{p^{1-r}(-pr + r - 1)}{(p-1)r + p + 1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r + p + 1}\right)\ell_i$$

*and*

$$\alpha = -\frac{(p-1)pr}{(p-1)r + p + 1}.$$

*Proof.* Using Proposition 4.15, we compute

$$\begin{aligned}
\sum_{j=1}^{p^r-1} a_{ij} &= a_{ii} + \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} a_{ij} \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \left( -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} + \frac{p^{1-2r}(\nu_p(i)p+p-\nu_p(i))(\nu_p(j)p+p-\nu_p(j))}{(p+1)(pr+p-r+1)} \right) \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} \qquad (15) \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \left( -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} \right) \qquad (16) \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \left( \frac{p^{1-2r}(\nu_p(i)p+p-\nu_p(i))(\nu_p(j)p+p-\nu_p(j))}{(p+1)(pr+p-r+1)} \right). \qquad (17)
\end{aligned}$$

We will now compute the sums above, starting with (16).

$$\begin{aligned}
\sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \left( -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} \right) &= -\frac{p^{1-2r}}{p+1} \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} (\nu_p(i-j)p + p - \nu_p(i-j)) \\
&= -\frac{p^{1-2r}}{p+1} \left[ \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} p + (p-1) \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \nu_p(j-i) \right] \\
&= -\frac{p^{1-2r}}{p+1} \left[ p(p^r-2) + (p-1) \sum_{\substack{j=1\\ j\neq i}}^{p^r-1} \nu_p(j-i) \right].
\end{aligned}$$

Using Corollary 4.18, we have

$$\begin{aligned}
&= -\frac{p^{1-2r}}{p+1} \left[ p(p^r-2) + (p-1) \left( -\nu_p(i) + \frac{p^r - pr + r - 1}{p-1} \right) \right] \\
&= -\frac{p^{1-2r}}{p+1} \left[ p(p^r-2) - (p-1)\nu_p(i) + p^r - pr + r - 1 \right].
\end{aligned}$$

Now we compute (17). Note that the third expression in (15) is what would be the $j = i$ term in the

sum. We can combine them and compute instead

$$\begin{aligned}
&\sum_{j=1}^{p^r-1} \frac{p^{1-2r}(\nu_p(i)p+p-\nu_p(i))(\nu_p(j)p+p-\nu_p(j))}{(p+1)(pr+p-r+1)} \\
&= \frac{p^{1-2r}(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)} \sum_{j=1}^{p^r-1} (\nu_p(j)p+p-\nu_p(j)) \\
&= \frac{p^{1-2r}(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)} \sum_{j=1}^{p^r-1} (p+\nu_p(j)(p-1)) \\
&= \frac{p^{2-2r}(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)}(p^r-1) + \frac{p^{1-2r}(p-1)(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)} \sum_{j=1}^{p^r-1} \nu_p(j) \\
&= \frac{p^{2-2r}(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)}(p^r-1) + \frac{p^{1-2r}(p-1)(\nu_p(i)p+p-\nu_p(i))}{(p+1)(pr+p-r+1)} \left(\frac{p^r-pr+r-1}{p-1}\right) \\
&= (\nu_p(i)p+p-\nu_p(i)) \left(\frac{p^{2-2r}(p^r-1)}{(p+1)(pr+p-r+1)} + \frac{p^{1-2r}(p^r-pr+r-1)}{(p+1)(pr+p-r+1)}\right) \\
&= (\nu_p(i)p+p-\nu_p(i))p^{1-2r}\left(\frac{p^r}{(p-1)r+p+1} - \frac{1}{p+1}\right).
\end{aligned}$$

Combining everything together, the $i$th row sum is

$$\begin{aligned}
\sum_{j=1}^{p^r-1} a_{ij} &= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{p^{1-2r}}{p+1}\left[p(p^r-2)-(p-1)\nu_p(i)+p^r-pr+r-1\right] \\
&+ (\nu_p(i)p+p-\nu_p(i))p^{1-2r}\left(\frac{p^r}{(p-1)r+p+1} - \frac{1}{p+1}\right) \\
&= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{p^{1-2r}}{p+1}p(p^r-2) + \frac{p^{1-2r}}{p+1}(p-1)\nu_p(i) - \frac{p^{1-2r}}{p+1}(p^r-pr+r-1) \\
&+ ((p-1)\nu_p(i)+p)\left(\frac{p^rp^{1-2r}}{(p-1)r+p+1} - \frac{p^{1-2r}}{p+1}\right) \\
&= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{p^{1-2r}}{p+1}p(p^r-2) + \frac{p^{1-2r}}{p+1}(p-1)\nu_p(i) - \frac{p^{1-2r}}{p+1}(p^r-pr+r-1) \\
&+ (p-1)\left(\frac{p^{1-r}}{(p-1)r+p+1} - \frac{p^{1-2r}}{p+1}\right)\nu_p(i) + \left(\frac{p^{1-r}}{(p-1)r+p+1} - \frac{p^{1-2r}}{p+1}\right)p \\
&= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{p^{1-2r}}{p+1}p(p^r-2) + \frac{p^{1-2r}}{p+1}(p-1)\nu_p(i) - \frac{p^{1-2r}}{p+1}(p^r-pr+r-1) \\
&+ \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1} - \frac{p^{1-2r}(p-1)}{p+1}\right)\nu_p(i) + \left(\frac{p^{2-r}}{(p-1)r+p+1} - \frac{p^{2-2r}}{p+1}\right) \\
&= -p^{1-2r} - \frac{p-1}{p+1}rp^{1-2r} - \frac{p^{1-2r}}{p+1}p(p^r-2) - \frac{p^{1-2r}}{p+1}(p^r-pr+r-1) + \left(\frac{p^{2-r}}{(p-1)r+p+1} - \frac{p^{2-2r}}{p+1}\right) \\
&+ \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1} - \frac{p^{1-2r}(p-1)}{p+1} + \frac{p^{1-2r}}{p+1}(p-1)\right)\nu_p(i) \\
&= \frac{p^{1-2r}(p-(p+1)p^r)}{p+1} + \left(\frac{p^{2-r}}{(p-1)r+p+1} - \frac{p^{2-2r}}{p+1}\right) + \left(\frac{(p-1)p^{1-r}}{(p-1)r+p+1}\right)\nu_p(i) \\
&= \frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i).
\end{aligned}$$

Lastly, we compute $\alpha$:

$$\begin{aligned}
\alpha &= \sum_{i=1}^{p^r-1} \left( \frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left( \frac{p^{1-r}(p-1)}{(p-1)r+p+1} \right) \nu_p(i) \right) \\
&= \sum_{i=1}^{p^r-1} \frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \frac{p^{1-r}(p-1)}{(p-1)r+p+1} \sum_{i=1}^{p^r-1} \nu_p(i) \\
&= \frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1}(p^r-1) + \frac{p^{1-r}(p-1)}{(p-1)r+p+1} \left( \frac{p^r-pr+r-1}{p-1} \right) \\
&= -\frac{(p-1)pr}{(p-1)r+p+1}.
\end{aligned}$$

□

We will now provide an explicit description of the entries of $T^{-1}$.

**Theorem 4.21.** *Let $c^{i,j}$ denote the $(i,j)$-entry of $T^{-1}$. We have*

$$\deg \mathrm{S}(N) c^{ij} = \begin{cases}
-\frac{2p^{1-2r}(pr-r+1)}{p+1} + \frac{2p^{1-2r}(p-1)}{p+1}\nu_p(i) & \text{if } 1 \le i,j \le p^r-1 \text{ and } i=j \\
-p^{1-2r}\frac{(pr-r+1)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p\left(\frac{1}{j}-\frac{1}{i}\right) & \text{if } 1 \le i,j \le p^r-1 \text{ and } i \ne j \\
-\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(i) & \text{if } 1 \le i \le p^r-1 \text{ and } p^r \le j \le p^r-1+p^{r-1} \\
-\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(j) & \text{if } 1 \le j \le p^r-1 \text{ and } p^r \le i \le p^r-1+p^{r-1} \\
-\frac{2p^{1-2r}(pr-r+1)}{p+1} & \text{if } p^r \le i,j \le p^r-1+p^{r-1} \text{ and } i=j \\
-\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p(i-j) & \text{if } p^r \le i,j \le p^r-1+p^{r-1} \text{ and } i \ne j
\end{cases}.$$

*Proof.* We will break into four cases, using Proposition 4.16 to calculate the $c^{i,j}$ along with our results in Lemma 4.19 and Lemma 4.20.

**Case 1**. Suppose $1 \le i,j \le p^r-1$. Then

$$c^{ij} = a_{i,j} + \frac{\beta}{1-\alpha\beta} \left( \sum_{k=1}^{p^r-1} a_{i,k} \right) \left( \sum_{k=1}^{p^r-1} a_{k,j} \right).$$

If $i = j$, then $c^{ii}$ is equal to

$$
\begin{aligned}
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} + \frac{\beta}{1-\alpha\beta}\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i)\right)^2 \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} - \frac{pr+p-r+1}{p(p+1)}\left(\frac{p^{2-2r}((p-1)\nu_p(i) + (1-p)r - 1)^2}{(pr+p-r+1)^2}\right) \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} - \frac{p^{1-2r}(pr+p-r+1)}{(p+1)}\left(\frac{((p-1)\nu_p(i) + (1-p)r - 1)^2}{(pr+p-r+1)^2}\right) \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2}{(p+1)(pr+p-r+1)} - \frac{p^{1-2r}((p-1)\nu_p(i) + (1-p)r - 1)^2}{(p+1)(pr+p-r+1)} \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{p^{1-2r}(\nu_p(i)p - \nu_p(i) + p)^2 - p^{1-2r}((p-1)\nu_p(i) + (1-p)r - 1)^2}{(p+1)(pr+p-r+1)} \\
&= -p^{1-2r} - \frac{p-1}{p+1} r p^{1-2r} + \frac{(p-1)(2\nu_p(i) - r + 1)p^{1-2r}}{p+1} \\
&= \frac{2p^{1-2r}((p-1)\nu_p(i) - pr + r - 1)}{p+1} \\
&= -\frac{2p^{1-2r}(pr-r+1)}{p+1} + \frac{2p^{1-2r}(p-1)}{p+1}\nu_p(i).
\end{aligned}
$$

If $i \neq j$, then $c^{ij}$ is equal to

$$
\begin{aligned}
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} + \frac{p^{1-2r}(\nu_p(i)p + p - \nu_p(i))(\nu_p(j)p + p - \nu_p(j))}{(p+1)(pr+p-r+1)} + \\
&+ \frac{\beta}{1-\alpha\beta}\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i)\right)\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \frac{p^{1-r}(p-1)}{(p-1)r+p+1}\nu_p(j)\right) \\
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} + \frac{p^{1-2r}(\nu_p(i)p + p - \nu_p(i))(\nu_p(j)p + p - \nu_p(j))}{(p+1)(pr+p-r+1)} \\
&- \frac{pr+p-r+1}{p(p+1)}\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i)\right)\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \frac{p^{1-r}(p-1)}{(p-1)r+p+1}\nu_p(j)\right) \\
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} + \frac{p^{1-2r}(\nu_p(i)p + p - \nu_p(i))(\nu_p(j)p + p - \nu_p(j))}{(p+1)(pr+p-r+1)} \\
&- \frac{pr+p-r+1}{p(p+1)}\left(\frac{p^{2-2r}(pr - p\nu_p(i) - r + \nu_p(i) + 1)(pr - p\nu_p(j) - r + \nu_p(j) + 1)}{(pr+p-r+1)^2}\right) \\
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} + \frac{p^{1-2r}(\nu_p(i)p + p - \nu_p(i))(\nu_p(j)p + p - \nu_p(j))}{(p+1)(pr+p-r+1)} \\
&- \frac{p^{1-2r}(pr - p\nu_p(i) - r + \nu_p(i) + 1)(pr - p\nu_p(j) - r + \nu_p(j) + 1)}{(p+1)(pr+p-r+1)} \\
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} - \frac{(p-1)p^{1-2r}(pr+p-r+1)(r - \nu_p(i) - \nu_p(j) - 1)}{(p+1)(pr+p-r+1)} \\
&= -\frac{p^{1-2r}(\nu_p(i-j)p + p - \nu_p(i-j))}{p+1} - \frac{(p-1)p^{1-2r}(r - \nu_p(i) - \nu_p(j) - 1)}{p+1} \\
&= -p^{1-2r}\frac{\nu_p(i-j)p + p - \nu_p(i-j) + (p-1)(r - \nu_p(i) - \nu_p(j) - 1)}{p+1} \\
&= -p^{1-2r}\frac{(pr-r+1)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\left(\nu_p(i-j) - \nu_p(i) - \nu_p(j)\right) \\
&= -p^{1-2r}\frac{(pr-r+1)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p\left(\frac{1}{j} - \frac{1}{i}\right).
\end{aligned}
$$

**Case 2**. Suppose $1 \leq i \leq p^r - 1$ and $p^r \leq j \leq p^r - 1 + p^{r-1}$. Then

$$\begin{aligned} c^{ij} &= \frac{-1}{1-\alpha\beta}\left(\sum_{k=1}^{n} a_{i,k}\right)\left(\sum_{k=1}^{m} b_{k,j}\right) \\ &= \frac{-1}{1-\alpha\beta}\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i)\right)\left(-p^{-r}\right) \\ &= -\left(\frac{pr+p-r+1}{p+1}\right)\left(-\frac{p^{1-2r}(-pr+r-1)}{(p-1)r+p+1} - \left(\frac{p^{1-2r}(p-1)}{(p-1)r+p+1}\right)\nu_p(i)\right) \\ &= -\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(i). \end{aligned}$$

**Case 3**. Suppose $p^r \leq i \leq p^r - 1 + p^{r-1}$ and $1 \leq j \leq p^r - 1$. Then

$$\begin{aligned} c^{ij} &= \frac{-1}{1-\alpha\beta}\left(\sum_{k=1}^{m} b_{i,k}\right)\left(\sum_{k=1}^{n} a_{k,j}\right) \\ &= -\left(\frac{pr+p-r+1}{p+1}\right)\left(-p^{-r}\right)\left(\frac{p^{1-r}(-pr+r-1)}{(p-1)r+p+1} + \left(\frac{p^{1-r}(p-1)}{(p-1)r+p+1}\right)\nu_p(j)\right) \\ &= -\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(j). \end{aligned}$$

**Case 4**. Suppose $p^r \leq i, j \leq p^r - 1 + p^{r-1}$. If $i = j$, then

$$\begin{aligned} c^{ii} &= b_{i,i} + \frac{\alpha}{1-\alpha\beta}\left(\sum_{k=1}^{m} b_{i,k}\right)\left(\sum_{k=1}^{m} b_{k,i}\right) \\ &= \frac{1}{p^2}\left(-p^{3-2r} - \frac{p-1}{p+1}(r-1)p^{3-2r}\right) + \frac{\alpha}{1-\alpha\beta}(-p^{-r})^2 \\ &= \frac{2p^{1-2r}(-pr+r-1)}{p+1} = -\frac{2p^{1-2r}(pr-r+1)}{p+1} \end{aligned}$$

If $i \neq j$, then

$$\begin{aligned} c^{ij} &= \frac{1}{p^2}\left(-p^{3-2r} - \frac{p^{5-3r}}{p+1}(-p^{r-2} + \nu_p(i-j)p^{r-2}(p-1))\right) + \frac{\alpha}{1-\alpha\beta}(-p^{-r})^2 \\ &= -\frac{((p-1)\nu_p(i-j)+p)p^{1-2r}}{p+1} - \frac{(p-1)pr}{p+1}(-p^{-r})^2 \\ &= -\frac{((p-1)\nu_p(i-j)+p)p^{1-2r} + (p-1)p^{1-2r}r}{p+1} \\ &= -\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p(i-j). \end{aligned}$$

□

The following corollary will be useful when we find an upper bound for the exponent later on.

**Corollary 4.22.** *Let $c^{i,j}$ denote the $(i,j)$-entry of $T^{-1}$. Then each $c^{i,j}$ is negative.*

*Proof.* Based on our result in Theorem 4.21, we will show the third case is negative *i.e.* we will show

$$-\frac{p^{1-2r}(pr-r+1)}{p+1}+\frac{p^{1-2r}(p-1)}{p+1}\nu_p(i) \tag{18}$$

is negative for $1 \leq i \leq p^r - 1$. The other cases are either clearly negative or are essentially the same as this case.

Since the largest value $\nu_p(i)$ attains is $r-1$, the largest value expression (18) attains is

$$-\frac{p^{1-2r}(pr-r+1)}{p+1}+\frac{p^{1-2r}(p-1)}{p+1}(r-1)=-\frac{p^{2-2r}}{p+1}$$

which is always negative. □

# 5 Computing $\deg(\underline{\omega}^{\otimes 2}|_\Lambda)$

Throughout this chapter, we will use the following notation:

- $N \geq 3$ and $r \geq 1$ will be integers and $p \geq 2$ will be a prime such that $p \nmid N$
- $\mathbb{F}_q$ will denote the residue field $\mathbb{F}_p(\zeta_N)$ of $\mathbb{Z}_p[\zeta_{Np^r}]$ where $q = p^\ell$ where $\ell$ is the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^\times$.
- $\mathfrak{X}$ will denote the modular curve $\mathfrak{X}(Np^r)_{/\mathbb{Z}_p[\zeta_{Np^r}]}$.
- $R = \mathbb{Z}_p[\zeta_{Np^r}]$

## 5.1 Decomposing the Modular Sheaf

Recall Theorem 2.34, the Kodaira-Spencer isomorphism, which we restate here for convenience.

**Theorem 5.1.** *Let $R$ be a noetherian, regular, excellent $\mathbb{Z}[\zeta_N]$-algebra containing $1/N$. The Kodaira-Spencer isomorphism*

$$\underline{\omega}^{\otimes 2}_{\mathfrak{Y}(N)_{/R}} \simeq \Omega^1_{\mathfrak{Y}(N)_{/R}/R}$$

*on $\mathfrak{Y}(N)_{/R}$ extends to an isomorphism*

$$\underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)_{/R}} \simeq \Omega^1_{\mathfrak{X}(N)_{/R}/R}(\mathfrak{C}(N)_{/R})$$

*on $\mathfrak{X}(N)_{/R}$.*

The following definition is from [Liu02, 6.4.7].

**Definition 5.2.** Let $Y$ be a locally noetherian scheme and let $f : X \to Y$ be a quasi-projective l.c.i. Let $i : X \to Z$ be an immersion into a scheme $Z$ that is smooth over $Y$. We define the **canonical sheaf** of $X \to Y$ to be the invertible sheaf

$$\Omega_{X/Y} := \det(\mathcal{C}_{X/Z})^\vee \otimes_{\mathcal{O}_X} i^*(\det \Omega^1_{Z/Y})$$

where $\mathcal{C}_{X/Z}$ is the conormal sheaf of $X$ in $Z$ (see [Liu02, 6.3.7]), det indicates the determinant of a locally free sheaf (see [Liu02, §6.4.1]), and $\Omega^1_{Z/Y}$ is the sheaf of Kahler differentials of $Z$ over $Y$. According to [Liu02, 6.4.7], $\Omega_{X/Y}$ is independent of $Z$, up to isomorphism, and is an invertible sheaf. When $f$ is smooth, the canonical sheaf $\Omega_{X/Y}$ coincides with the sheaf of Kahler differentials $\Omega^1_{X/Y}$.

More generally, suppose $f : X \to Y$ is a proper morphism of relative dimension $\leq r$. The following definition is from [Liu02, 6.4.18].

**Definition 5.3.** A **relative ($r$-)dualizing sheaf** for $f : X \to Y$ is a quasi-coherent sheaf $\Omega_f$ on $X$, endowed with a homomorphism of $\mathcal{O}_X$-modules

$$\mathrm{tr}_f : R^r f_* \Omega_f \to \mathcal{O}_Y$$

such that for any quasi-coherent sheaf $\mathcal{F}$ on $X$, the natural bilinear map

$$f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_f) \times R^r f_* \mathcal{F} \to R^r f_* \Omega_f \xrightarrow{\mathrm{tr}_f} \mathcal{O}_Y$$

induces an isomorphism

$$f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \Omega_f) \simeq \mathcal{H}om_{\mathcal{O}_Y}(R^r f_* \mathcal{F}, \mathcal{O}_Y).$$

By [Liu02, 6.4.19], uniqueness of $\Omega_f$ is automatic once we have existence. If $Y$ is locally noetherian and $f : X \to Y$ is a projective morphism with fibers of dimension $\leq r$, then as remarked in [Liu02, 6.4.30], the relative $r$-dualizing sheaf exists. The following, which is [Liu02, Theorem 6.4.32], says the relative dualizing sheaf generalizes the canonical sheaf.

**Theorem 5.4.** *Let $Y$ be a locally noetherian scheme, and let $f : X \to Y$ be a flat projective l.c.i. of relative dimension $r$. Then the $r$-dualizing sheaf $\Omega_f$ is isomorphic to $\Omega_{X/Y}$.*

We will use the following result, which is in [Liu02, Theorem 6.4.9] known as the adjunction formula, to eventually relate $\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}$ with the relative dualizing sheaf of $\mathfrak{X}(Np^r)$ and of $\mathfrak{X}(N)$.

**Theorem 5.5.** *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-projective l.c.i.s. We have a canonical isomorphism of canonical sheaves*

$$\Omega_{X/Z} \simeq \Omega_{X/Y} \otimes_{\mathcal{O}_X} f^* \Omega_{Y/Z}.$$

We cannot directly apply the Kodaira-Spencer isomorphism to our modular curve $\mathfrak{X}$ since $p$ and consequently the level $Np^r$, is not invertible in $R$. Instead, we will apply it to the modular curve $\mathfrak{X}(N)$ over $R$ since the level $N$ is invertible in $\mathbb{Z}_p \subset R$. Consider

$$\mathfrak{X} \xrightarrow{\mathrm{pr}} \mathfrak{X}(N)_{/R} \xrightarrow{g} \mathrm{Spec}(R) \tag{19}$$

where pr is the projection map and $g$ is the structural morphism. For convenience, we let $\mathfrak{X}(N)$ denote the base change $\mathfrak{X}(N)_{/R}$.

According to [Liu02, 6.3.18], if $X \to Y$ is a morphism of finite type of regular locally noetherian schemes, then $X \to Y$ is an l.c.i. Therefore the maps pr and $g$ are l.c.i.s. By Theorem 2.17, $g$ and $g \circ \mathrm{pr}$ are projective. Hence by [KM85, 3.3.32(e)], pr is projective. Applying the adjunction formula to (19), we have

$$\Omega_{\mathfrak{X}/R} \simeq \Omega_{\mathfrak{X}/\mathfrak{X}(N)} \otimes_{\mathcal{O}_\mathfrak{X}} \mathrm{pr}^* \Omega_{\mathfrak{X}(N)/R}.$$

Combining this with the Kodaira-Spencer isomorphism applied to $\mathfrak{X}(N)$, we get

$$\Omega_{\mathfrak{X}/R} \simeq \Omega_{\mathfrak{X}/\mathfrak{X}(N)} \otimes_{\mathcal{O}_{\mathfrak{X}(Np^r)}} \mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)). \tag{20}$$

We will later show in Lemma 5.21

$$\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)) \simeq \underline{\omega}^{\otimes 2}_{\mathfrak{X}}(-p^r \mathfrak{C}(Np^r))$$

which is where the sheaf $\underline{\omega}^{\otimes 2}_{\mathfrak{X}}$ appears in (20). This will then allow us to identify $\underline{\omega}^{\otimes 2}_{\mathfrak{X}}$ with $\Omega_{\mathfrak{X}/R}(\mathfrak{C}(Np^r))$, the relative dualizing sheaf twisted by the cuspidal divisor. Thus, computing $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}}|_\Lambda)$ amounts to computing $\deg(\Omega_{\mathfrak{X}}(\mathfrak{C}(Np^r))|_\Lambda)$. Our first step will be to investigate $\Omega_{\mathfrak{X}/\mathfrak{X}(N)}$.

## 5.2 The Relative Dualizing Sheaf $\Omega_{\mathfrak{X}/\mathfrak{X}(N)}$

In this section, our goal will be to better understand the relative dualizing sheaf $\Omega_{\mathfrak{X}/\mathfrak{X}(N)}$. By Theorem 5.4, $\Omega_{\mathfrak{X}/\mathfrak{X}(N)}$ is isomorphic to the canonical sheaf of pr, which is an invertible $\mathcal{O}_{\mathfrak{X}}$-module (see Definition 5.2), so we have $\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}(\mathcal{R})$ for some divisor $\mathcal{R}$ of $\mathfrak{X}$. As we will see, $\mathcal{R}$ will be the divisor associated to the *different* of the morphism

$$\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}(N)_{/R}.$$

We begin by discussing the trace map which generalizes the usual notion over a finite extension of fields. Let $A \to B$ be a finite, flat map of noetherian rings. According to [Sta21, Tag0BSY], $B$ is a finite locally free $A$-module and so we can consider the trace $\mathrm{Tr}_{B/A}(b)$ of the $A$-linear map $B \to B$ given by multiplication by $b$. This gives us an $A$-linear map $\mathrm{Tr}_{B/A} : B \to A$. The following definition is from [Sta21, Tag0BW0].

**Definition 5.6.** Let $A \to B$ be a ring map and let $K = \mathrm{Frac}(A)$, the total ring of fractions of $A$ (see [Sta21, 02C5], note when $A$ is a domain, $\mathrm{Frac}(A)$ coincides with the field of fractions), and $L = B \otimes_A K$. We say the **Dedekind different is defined** if $A$ is noetherian, $A \to B$ is finite and maps any non-zerodivisor of $A$ to a non-zerodivisor of $B$, and $K \to L$ is étale. In this situation, $K \to L$ is finite flat. Let

$$\mathcal{L}_{B/A} = \left\{ x \in L : \mathrm{Tr}_{L/K}(bx) \in A \text{ for all } b \in B \right\}.$$

We define the **Dedekind different** of $A \to B$ to be the inverse of $\mathcal{L}_{B/A}$:

$$\mathfrak{D}_{B/A} = \mathcal{L}_{B/A}^{-1} = \left\{ x \in L : x\mathcal{L}_{B/A} \subset B \right\}$$

viewed as a sub $B$-module of $L$.

*Remark* 5.7. Let $A$ be a Dedekind domain, $K = \mathrm{Frac}(A)$, $L$ a finite separable extension of $K$, and $B$ the integral closure of $A$ in $L$. In this situation, [Ser79, §4.3] defines the different $\mathfrak{D}_{B/A}$ in the same manner as we have done. Since $A$ is normal and noetherian, by [Sta21, Tag032L], $A \to B$ is finite. Furthermore, $L = \mathrm{Frac}(B)$ and $L = B \otimes_A K$ so indeed, the Dedekind different is defined for $A \to B$. We record a few useful facts for calculating the different in this situation.

**Proposition 5.8.** *Let $A$ be a Dedekind domain, $K = \mathrm{Frac}(A)$, $L$ a finite separable extension of $K$, and $B$ the integral closure of $A$ in $L$.*

*a. Let $\mathfrak{P}$ be a non-zero prime of $B$ such that the corresponding residue extension is separable and let $e_{\mathfrak{P}}$ denote the ramification index of $\mathfrak{P}$. Then the exponent of $\mathfrak{P}$ in the different $\mathfrak{D}_{B/A}$ is greater than or equal to $e_{\mathfrak{P}} - 1$ with equality precisely when $\mathfrak{P}$ is tamely ramified.*

*b. Suppose for each prime $\mathfrak{P}$ of $B$, the corresponding residue extension is separable. The annihilator of the $B$-module $\Omega^1_{B/A}$ of Kahler differentials is equal to $\mathfrak{D}_{B/A}$.*

*Proof.* (a) is [Ser79, III, §6, Prop 13] while (b) is [Ser79, III, §7, Prop. 14]. □

**Lemma 5.9.** *Suppose the Dedekind different is defined for $A \to B$. Let $S \subset A$ be a multiplicatively closed subset such that the Dedekind different is defined for $S^{-1}A \to S^{-1}B$. Then $S^{-1}\mathfrak{D}_{B/A} = \mathfrak{D}_{S^{-1}B/S^{-1}A}$ as $S^{-1}B$-modules.*

*Proof.* First we show $S^{-1}\mathcal{L}_{B/A} = \mathcal{L}_{S^{-1}B/S^{-1}A}$. By definition,

$$\mathcal{L}_{S^{-1}B/S^{-1}A} = \left\{ \frac{x}{s} \in S^{-1}L : \mathrm{Tr}_{S^{-1}L/K}\left(\frac{b}{s'}\frac{x}{s}\right) \in S^{-1}A \text{ for all } \frac{b}{s'} \in S^{-1}B \right\}.$$

Now

$$\begin{aligned} &\mathrm{Tr}_{S^{-1}L/K}\left(\frac{bx}{ss'}\right) \in S^{-1}A \text{ for all } \frac{b}{s'} \in S^{-1}B \\ \iff &\frac{1}{ss'}\mathrm{Tr}_{S^{-1}L/K}(bx) \in S^{-1}A \text{ for all } \frac{b}{s'} \in S^{-1}B \\ \iff &\mathrm{Tr}_{L/K}(bx) \in A \text{ for all } b \in B. \end{aligned}$$

Thus $\mathcal{L}_{S^{-1}B/S^{-1}A}$ can be identified with

$$\begin{aligned} &= \left\{ \frac{x}{s} \in S^{-1}L : \mathrm{Tr}_{L/K}(bx) \in A \text{ for all } b \in B \right\} \\ &= S^{-1}\mathcal{L}_{B/A}. \end{aligned}$$

Lastly, we show $S^{-1}(\mathcal{L}^{-1}_{B/A}) = (S^{-1}\mathcal{L}_{B/A})^{-1}$. Let $\frac{x}{s} \in S^{-1}(\mathcal{L}^{-1}_{B/A})$ so $x\mathcal{L}_{B/A} \subseteq B$. Then $\frac{x}{s}S^{-1}\mathcal{L}_{B/A} \subseteq S^{-1}B$ hence $S^{-1}(\mathcal{L}^{-1}_{B/A}) \subseteq (S^{-1}\mathcal{L}_{B/A})^{-1}$.

For the other inclusion, we first note that $\mathcal{L}_{B/A}$ is finitely generated since $B$ is noetherian. Let $x_1, \ldots, x_n \in \mathcal{L}_{B/A}$ denote the generators of $\mathcal{L}_{B/A}$ over $B$. Let $x \in (S^{-1}\mathcal{L}_{B/A})^{-1}$ so $xS^{-1}\mathcal{L}_{B/A} \subseteq S^{-1}B$. Then for each $i = 1, \ldots, n$ there exists $s_i \in S$ such that $s_i x x_i \in B$. Let $s = \prod_{i=1}^n s_i$. Then $s x x_i \in B$ so $sx\mathcal{L}_{B/A} \subseteq B$. Therefore $x\mathcal{L}_{B/A} \subset S^{-1}B$ so $x \in S^{-1}(\mathcal{L}^{-1}_{B/A})$.

In conclusion,

$$S^{-1}\mathfrak{D}_{B/A} = S^{-1}(\mathcal{L}^{-1}_{B/A}) = (S^{-1}\mathcal{L}_{B/A})^{-1} = (\mathcal{L}_{S^{-1}B/S^{-1}A})^{-1} = \mathfrak{D}_{S^{-1}B/S^{-1}A}.$$ □

Let $f : Y \to X$ be a proper[3] morphism of locally noetherian schemes. According to [Sta21, Tag0BVG], the relative dualizing sheaf $\Omega_f$ is the unique coherent $\mathcal{O}_Y$-module such that for every pair of affine opens $\mathrm{Spec}(B) \subset Y$ and $\mathrm{Spec}(A) \subset X$ with $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$, we have a canonical isomorphism

$$H^0(\mathrm{Spec}(B), \Omega_f) \simeq \mathrm{Hom}_A(B, A).$$

If we further assume $f$ is flat, then by [Sta21, Tag0BVJ], there exists a global section $\tau_{Y/X} \in H^0(Y, \Omega_f)$ such that whenever $A \to B$ is finite, $\tau_{Y/X}|_{\mathrm{Spec}(B)}$ is identified with $\mathrm{Tr}_{B/A}$ under the isomorphism.

**Definition 5.10.** Let $f : Y \to X$ be a flat, proper morphism of noetherian schemes. The **different** $\mathfrak{D}_f$ is the annihilator of the cokernel

$$\mathrm{Coker}(\mathcal{O}_Y \xrightarrow{\tau_{Y/X}} \Omega_f)$$

which is a coherent ideal sheaf $\mathfrak{D}_f \subset \mathcal{O}_Y$.

**Lemma 5.11.** *Let $f : Y \to X$ be a proper morphism of noetherian schemes. Let $\mathrm{Spec}(B) \subset Y$ and $\mathrm{Spec}(A) \subset X$ be affine open subschemes such that $f(\mathrm{Spec}(B)) \subset \mathrm{Spec}(A)$. If the Dedekind different of $A \to B$ is defined, then*

$$\mathfrak{D}_f|_{\mathrm{Spec}(B)} = (\mathfrak{D}_{B/A})^\sim$$

*where $(\mathfrak{D}_{B/A})^\sim$ is the quasi-coherent sheaf induced by the $B$-module $\mathfrak{D}_{B/A}$.*

*Proof.* This is [Sta21, Tag0BW5]. □

**Corollary 5.12.** *Let $f : Y \to X$ and let $\mathrm{Spec}(B) \subset Y$ and $\mathrm{Spec}(A) \subset X$ satisfy the hypothesis of Lemma 5.11. Let $x \in \mathrm{Spec}(B)$ and suppose the Dedekind different is defined for $\mathcal{O}_{X,f(x)} \to \mathcal{O}_{Y,x}$. Then $\mathfrak{D}_{f,x} \simeq \mathfrak{D}_{\mathcal{O}_{Y,x}/\mathcal{O}_{X,f(x)}}$.*

*Proof.* Let $\mathfrak{p}$ be the prime of $B$ corresponding to $x$ and $\mathfrak{q}$ be the prime of $A$ corresponding to $f(x)$. By Lemma 5.11 and Lemma 5.9, we have

$$\mathfrak{D}_{f,x} \simeq (\mathfrak{D}_{B/A})_\mathfrak{p} \simeq \mathfrak{D}_{B_\mathfrak{p}/A_\mathfrak{q}} = \mathfrak{D}_{\mathcal{O}_{Y,x}/\mathcal{O}_{X,f(x)}}.$$ □

Let $\mathcal{R} \subseteq X$ denote the closed subscheme associated to $\mathfrak{D}_f$ and let $[\mathcal{R}]$ denote the Weil divisor associated to $\mathcal{R}$. The following is [Sta21, Tag0BWA].

**Proposition 5.13.** *Let $f : Y \to X$ be a proper morphism of noetherian schemes. If $\Omega_f$ is invertible and $f$ is étale at the associated points of $Y$, then $\mathcal{R}$ is an effective Cartier divisor and $\Omega_f \simeq \mathcal{O}_Y(\mathcal{R})$.*

[3] More generally, one can define the different of a locally quasi-finite morphism of locally noetherian schemes, as in [Sta21, Tag0BTC].

Explicitly, the Weil divisor associated to $\mathcal{R}$ is

$$[\mathcal{R}] = \sum_{\substack{x \in Y \\ \dim \mathcal{O}_{Y,x}=1}} \mathrm{mult}_x(\mathcal{R}) \cdot \overline{\{x\}}.$$

By definition, $\mathrm{mult}_x(\mathcal{R}) = \mathrm{length}_{\mathcal{O}_{Y,x}}(\mathcal{O}_{Y,x}/\mathcal{R}_x\mathcal{O}_{Y,x})$ and $\mathcal{R}_x = \mathfrak{D}_{f,x}$.

We will now show that the usual projection morphism $\mathrm{pr} : \mathfrak{X} \to \mathfrak{X}(N)_{/R}$ satisfies the necessary properties to use the above results on the different. By [KM85, 5.5.1(1)], the map pr is finite and flat so the different $\mathfrak{D}_{\mathrm{pr}}$ is defined.

Suppose $x \in \mathfrak{X}$ is a codimension 1 point. Let

$$\mathrm{pr}_x : \mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}_{\mathfrak{X},x}$$

denote the induced map on stalks. Since $x$ is of codimension 1 and pr is flat, we have $\mathrm{pr}(x)$ is also of codimension 1. Consequently, both $\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ are DVRs.

**Proposition 5.14.** *Let $x \in \mathfrak{X}$ be a codimension 1 point. The induced map on stalks $\mathrm{pr}_x$ is a finite ring map. Furthermore, the induced map on fraction fields*

$$K = \mathrm{Frac}(\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}) \to L = \mathrm{Frac}(\mathcal{O}_{\mathfrak{X}(Np^r),x})$$

*is finite separable and $\mathcal{O}_{\mathfrak{X},x}$ is the integral closure of $\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ in $L$.*

*Proof.* Let $\mathrm{Spec}(B) \subset \mathfrak{X}(N)$ be an affine open containing $\mathrm{pr}(x)$ and $\mathrm{Spec}(A) \subset \mathfrak{X}$ be an affine open containing $x$ such that $\mathrm{pr}(\mathrm{Spec}(A)) \subset \mathrm{Spec}(B)$. Since pr is finite, it is also integral so the induced map $A \to B$ is integral. By [Sta21, Tag034K], the induced map on localizations remains integral so $\mathrm{pr}_x$ is integral. Since $\mathfrak{X}(N)$ is an integral scheme, we have inclusions

$$A \hookrightarrow \mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)} \hookrightarrow \mathrm{Frac}(A).$$

Therefore $K = \mathrm{Frac}(A)$. Similarly, we conclude $L = \mathrm{Frac}(B)$. Both $\mathfrak{X}(N)$ and $\mathfrak{X}$ are normal schemes so $\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ (resp. $\mathcal{O}_{\mathfrak{X},x}$) is integrally closed in $K$ (resp. $L$). Having established

$$\mathrm{pr}_x : \mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}_{\mathfrak{X},x}$$

is integral, the integral closure of $\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ in $L$ is precisely $\mathcal{O}_{\mathfrak{X},x}$.

Since pr is a finite map between two integral schemes, the extension of function fields

$$K(\mathfrak{X}(N)) \to K(\mathfrak{X})$$

is a finite extension of characteristic zero fields, and hence separable. Note that $K(\mathfrak{X}(N)) = K$ and $K(\mathfrak{X}) = L$ so $L/K$ is a finite separable extension. By [Ser79, I, §IV, Prop. 8], we can conclude $\mathrm{pr}_x$ is finite. □

Since $\mathfrak{X}$ is integral, it's only associated point is its unique generic point. In Proposition 5.14, we deduced

the map on function fields $K(\mathfrak{X}) \to K(\mathfrak{X}(N))$ is finite separable so pr is étale at the generic point of $\mathfrak{X}$. By Proposition 5.13, we have

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}([\mathcal{R}]) = \mathcal{O}_{\mathfrak{X}}\left(\sum_{\substack{x\in\mathfrak{X}\\ \dim \mathcal{O}_{\mathfrak{X},x}=1}} \mathrm{mult}_x(\mathcal{R})\cdot\overline{\{x\}}\right) \tag{21}$$

Since $\mathcal{O}_{\mathfrak{X},x}$ is a DVR in this case, $\mathrm{mult}_x(\mathcal{R})$ is equal to the valuation of $\mathcal{R}_x = \mathfrak{D}_{\mathrm{pr},x}$ in $\mathcal{O}_{\mathfrak{X},x}$. By Corollary 5.12 and Proposition 5.14, $\mathfrak{D}_{\mathrm{pr},x} = \mathfrak{D}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}}$. Also, by Proposition 5.8(b), we can identify $\mathfrak{D}_{\mathrm{pr},x}$ with

$$\mathrm{Ann}_{\mathcal{O}_{\mathfrak{X},x}}(\Omega^1_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}}).$$

Let $d_x$ denote the valuation of the different $\mathfrak{D}_{\mathrm{pr},x}$ in the DVR $\mathcal{O}_{\mathfrak{X},x}$. Recall from Proposition 2.19 that the codimension 1 points of $\mathfrak{X}$ are precisely the closed points of the generic fiber, and the generic points of the special fiber. We will compute $d_x$ when $x$ is a closed point of the generic fiber. If $x$ is a generic point of the special fiber, then we show all $d_x$ contributions are the same which taken together contribute nothing to the different of pr. In this case, however, we provide a method of computing $d_x$ in Appendix C using strict Henselizations.

## 5.3 Computing $d_x$ for the closed points of the generic fiber

The generic fiber of $\mathfrak{X}$ is open so it suffices to compute $d_x$ over $\mathbb{Q}_p(\zeta_{Np^r})$ i.e. for the map

$$\mathrm{pr}_x : \mathcal{O}_{\mathfrak{X}(N)_{/\mathbb{Q}_p(\zeta_{Np^r})},\mathrm{pr}(x)} \to \mathcal{O}_{\mathfrak{X},x}.$$

First we show the value of $d_x$ does not change after base changing $\mathbb{Q}_p(\zeta_{Np^r})$. In particular, we can compute the value of $d_x$ over $\mathbb{C}$ and use the classical theory of modular curves as compact Riemann surfaces.

**Lemma 5.15.** *Let $\pi : X \to Y$ be a finite type morphism of normal curves over a field $K$ and let $L$ be a field extension of $K$. Let $p : X_L \to X$ denote the usual projection morphism from base change and let $x \in X_L$. Then we have $d_x = d_{p(x)}$.*

*Proof.* We can equate $d_x$ with the valuation of the annihilator ideal of $\Omega^1_{X_L/Y_L,x}$ in $\mathcal{O}_{X_L,x}$. Since Kahler differentials are compatible with base change, we have

$$\Omega^1_{X_L/Y_L,x} \simeq (p^*\Omega^1_{X/Y})_x \simeq \Omega^1_{X/Y,p(x)} \otimes_{\mathcal{O}_{X,p(x)}} \mathcal{O}_{X_L,x}.$$

Since $K \to L$ is flat, the map $\mathcal{O}_{X,p(x)} \to \mathcal{O}_{X_L,x}$ is flat. Furthermore, $\Omega^1_{X/Y,p(x)}$ is finite over $\mathcal{O}_{X,p(x)}$ since $\pi$ is finite type. Thus, by Lemma 3.5 we have

$$\begin{aligned}\mathrm{Ann}_{\mathcal{O}_{X_L},x}(\Omega^1_{X_L/Y_L,x}) &= \mathrm{Ann}_{\mathcal{O}_{X_L},x}(\Omega^1_{X/Y,p(x)} \otimes_{\mathcal{O}_{X,p(x)}} \mathcal{O}_{X_L,x})\\ &= \mathrm{Ann}_{\mathcal{O}_{X,p(x)}}(\Omega^1_{X/Y,x})\mathcal{O}_{X_L,x}.\end{aligned}$$

Hence $d_x = d_{p(x)}$. □

Thus we can compute $d_x$ in the situation that our modular curves are over $\mathbb{C}$. Let $X(M)$ denote the modular curve over $\mathbb{C}$ of level $\Gamma(M)$. In this situation, $\mathcal{O}_{X(Np^r),x}$ is tamely ramified over $\mathcal{O}_{X(N),\mathrm{pr}(x)}$ so by Proposition 5.8(a), we have $d_x = e_x - 1$ where $e_x$ is the ramification index of $x$. We will investigate the ramification index of all points in $X(Np^r)$ under the usual projection map $\mathrm{pr} : X(Np^r) \to X(N)$.

Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. We also let $\mathcal{H}$ denote the (complex) upper half plane and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. As we will see shortly, investigating the ramification of a point $z \in \mathcal{H}^*$ amounts to understanding the stabilizer group

$$\Gamma_z = \{\gamma \in \Gamma : \gamma \cdot z = z\}.$$

The points of $z \in \mathcal{H}^*$ can be classified depending on which elements of $\Gamma$ fix $z$. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = z \iff cz^2 + (d-a)z - b = 0$$

for matrices not equal to $\pm I$. Hence the fixed points of a given $\gamma \in \Gamma$ are either conjugate complex numbers, a single real number, or two distinct real numbers. Based on this fact, we make the following definition.

**Definition 5.16.** We say $z \in \mathcal{H}^*$ is a/an

- **ellipitic point of** $\Gamma$ if there exists $\gamma \in \Gamma_z$ such that $|\mathrm{tr}(\gamma)| < 2$, or equivalently $\gamma$ has two distinct fixed points $z$ and $\bar{z}$.
- **cusp of** $\Gamma$ if there exists $\gamma \in \Gamma_z$ such that $|\mathrm{tr}(\gamma)| = 2$, or equivalently $\gamma$ has a unique real fixed point $z$.
- **hyperbolic point of** $\Gamma$ if there exists $\gamma \in \Gamma_z$ such that $|\mathrm{tr}(\gamma)| > 2$, or equivalently $\gamma$ has two distinct real fixed points.
- **ordinary point of** $\Gamma$ if $\Gamma$ does not fix $z$, excluding $\pm I$.

Since the matrices we consider are in $\mathrm{SL}_2(\mathbb{Z})$, we will not have any hyperbolic points appearing. We denote $\bar{\Gamma} = \Gamma/\Gamma \cap \{\pm I\}$. The following is [Miy89, Theorem 1.5.4] which describes the stabilizer groups $\Gamma_z$.

**Theorem 5.17.** *a. If $z \in \mathcal{H}$ is an elliptic point of $\Gamma$, then $\Gamma_z$ is a finite cyclic group.*
*b. If $z \in \mathbb{Q} \cup \{\infty\}$ is a cusp of $\Gamma$, then $\bar{\Gamma}_z \simeq \mathbb{Z}$.*

The following proposition, which is [Shi94, Prop. 1.37], relates the ramification index with the index of stabilizer groups.

**Proposition 5.18.** *Let $\Gamma' \leq \Gamma$ be a finite index subgroup and consider the projection $\pi : \Gamma'\backslash\mathcal{H}^* \to \Gamma\backslash\mathcal{H}^*$. The ramification index $e_z$ of a point $z \in \Gamma'\backslash\mathcal{H}^*$ under $\pi$ is equal to*

$$e_z = \left[\bar{\Gamma}_z : \bar{\Gamma}'_z\right].$$

We are now ready to compute the ramification index under our map pr.

**Proposition 5.19.** *Let* $\mathrm{pr} : X(Np^r) \to X(N)$ *denote the usual projection map and let* $z \in X(Np^r)$. *We have*

$$e_z = \begin{cases} p^r & \text{if } z \text{ is a cusp} \\ 1 & \text{otherwise} \end{cases}.$$

*Proof.* By [Shi94, Prop. 1.39], the congruence subgroup $\Gamma(M)$ has no elliptic elements for any $M > 1$. Consequently, the ordinary points are precisely the points of $\Gamma(M)\backslash\mathcal{H}$. We will now apply Proposition 5.18 in the case $\Gamma' = \Gamma(Np^r)$ and $\Gamma = \Gamma(N)$.

Note that the image of a cusp under pr is again a cusp and similarly for ordinary points. If $z \in X(Np^r)$ is ordinary, then its stabilizer group is trivial so $e_z = 1$. If $z$ is a cusp, then there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot z = \infty$. Therefore

$$\overline{\Gamma(N)}_z/\overline{\Gamma(Np^r)}_z \simeq \gamma\overline{\Gamma(N)}_z\gamma^{-1}/\gamma\overline{\Gamma(Np^r)}_z\gamma^{-1} = \overline{\Gamma(N)}_\infty/\overline{\Gamma(Np^r)}_\infty$$

so it suffices to compute $e_\infty$.

Let $A(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$. According to [Shi94, bottom p. 22], we have for $M > 1$ that $\overline{\Gamma(M)}_\infty = \langle A(d) \rangle$. Furthermore, $-I \in \Gamma(M)$ if and only if $-1 \equiv 1$ modulo $M$, or equivalently $M = 2$. Since $M \geq 3$ in our situation, we always have $\overline{\Gamma(M)}_\infty = \Gamma(M)_\infty$. Note that for any $m \geq 0$, we have $A(d)^m = A(md)$. Therefore the order of $A(d)$ in $\overline{\Gamma(N)}_\infty/\overline{\Gamma(Np^r)}_\infty$ is equal to $p^r$ so

$$e_\infty = [\overline{\Gamma(N)}_\infty : \overline{\Gamma(Np^r)}_\infty] = p^r. \qquad \square$$

## 5.4 Relating $\underline{\omega}_{\mathfrak{X}}^{\otimes 2}$ with $\Omega_{\mathfrak{X}/R}$

Let $x \in \mathfrak{X}$ be a generic point of the special fiber. Recall the value of $d_x$ is equal to the valuation of the different ideal corresponding to the induced map on stalks

$$\mathrm{pr}_x : \mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}_{\mathfrak{X},x}.$$

We will use our earlier discussion in Section 2.7 to provide an explicit description of $\mathrm{pr}_x$. Recall in Theorem 2.43 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ig}(p^r, N) & \xrightarrow{\simeq} & \mathrm{ExIg}(p^r, i, N) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ \mathfrak{X}(N)^{(\sigma-i)}_{\mathbb{F}_q} & \xrightarrow{F^i} & \mathfrak{X}(N)_{\mathbb{F}_q}. \end{array}$$

where $\rho$ and $\rho'$ are the usual projection maps. By Theorem 2.47, the restriction of pr to any irreducible component of $\bar{\mathfrak{X}}$ is the map $\rho'$. We get a commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}(N)_{/\mathbb{F}_q},\mathrm{pr}(x)} & \xrightarrow{F^r} & \mathcal{O}_{\mathfrak{X}(N)^{(p^{-r})}_{/\mathbb{F}_q},\mathrm{pr}(x)} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_x} & & \Big\downarrow{\scriptstyle \rho_x} \\ \mathcal{O}_{\mathrm{ExIg}(p^r,r,N),x} & \xrightarrow{\simeq} & \mathcal{O}_{\mathrm{Ig}(p^r,N),x} \end{array} \tag{22}$$

In particular, the map $\mathrm{pr}_x$ is the same for each generic point $x$ of the special fiber. Hence, the value of $d_x$ is independent of $x$ in this situation.

**Lemma 5.20.** *We have*

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}((p^r-1)\mathfrak{C}(Np^r)).$$

*Proof.* Recall from equation (21) and the paragraphs proceeding it, we have

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}\left(\sum_{x\in\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})}} d_x\cdot\overline{\{x\}} + \sum_{x\in\mathfrak{X}_{/\mathbb{F}_q}} d_x\cdot\overline{\{x\}}\right)$$

where the first sum is over closed points of the generic fiber and the second sum is over the generic points of the irreducible components of the special fiber. By Proposition 5.19, we have

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}\left(\sum_{x\in\mathfrak{C}(Np^r)} (p^r-1)\cdot\overline{\{x\}} + \sum_{x\in\mathfrak{X}_{/\mathbb{F}_q}} d_x\cdot\overline{\{x\}}\right)$$

where the first sum is over all the cusps in the generic fiber. As discussed above, the $d_x$ values appearing in the second sum are independent of $x$. Using the fact that the special fiber is reduced, we have

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}((p^r-1)\mathfrak{C}(Np^r) + d_x\mathfrak{X}_{/\mathbb{F}_q}.$$

Note that $\mathfrak{X}_{/\mathbb{F}_q}$, when viewed as a divisor, is principal (see Proposition 2.26). Thus

$$\Omega_{\mathfrak{X}/\mathfrak{X}(N)} \simeq \mathcal{O}_{\mathfrak{X}}((p^r-1)\mathfrak{C}(Np^r))$$

as desired. □

Recall the map from (19):

$$\mathfrak{X} \xrightarrow{\mathrm{pr}} \mathfrak{X}(N)_{/R} \xrightarrow{g} \mathrm{Spec}(R)$$

Again, for convenience, we let $\mathfrak{X}(N)$ denote the base change $\mathfrak{X}(N)_{/R}$. To compute $\deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}|_\Lambda)$, our strategy is to first prove that

$$\underline{\omega}_{\mathfrak{X}}^{\otimes 2} \simeq \Omega_{\mathfrak{X}/R}(\mathfrak{C}(Np^r))$$

using the isomorphism in (20):

$$\Omega_{\mathfrak{X}/R} \simeq \Omega_{\mathfrak{X}/\mathfrak{X}(N)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)).$$

Our next step is to provide a better description of $\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N))$.

**Lemma 5.21.** *We have an isomorphism of sheaves on $\mathfrak{X}(Np^r)$*

$$\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)) \simeq \underline{\omega}^{\otimes 2}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(-p^r \mathfrak{C}(Np^r)).$$

*Proof.* We have

$$\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)) \simeq \mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)} \otimes_{\mathcal{O}_{\mathfrak{X}(Np^r)}} \mathrm{pr}^* \mathcal{O}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)).$$

By Proposition 2.33 and Proposition 2.35, we have

$$\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)} = \underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}.$$

Since pr maps $\mathfrak{C}(Np^r)$ onto $\mathfrak{C}(N)$ and is ramified at the cusps of degree $p^r$ by Proposition 5.19, we have

$$\mathrm{pr}^* \mathcal{O}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)) \simeq \mathcal{O}_{\mathfrak{X}(Np^r)}(-p^r \mathfrak{C}(Np^r)).$$

Thus

$$\mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)) \simeq \underline{\omega}^{\otimes 2}_{\mathfrak{X}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}(-p^r \mathfrak{C}(Np^r))$$

as desired. □

**Theorem 5.22.** *We have*

$$\underline{\omega}^{\otimes 2}_{\mathfrak{X}} \simeq \Omega_{\mathfrak{X}/R}(\mathfrak{C}(Np^r)).$$

*Proof.* Recall the isomorphism in (20) states

$$\Omega_{\mathfrak{X}/R} \simeq \Omega_{\mathfrak{X}/\mathfrak{X}(N)} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathrm{pr}^* \underline{\omega}^{\otimes 2}_{\mathfrak{X}(N)}(-\mathfrak{C}(N)).$$

By Lemma 5.20 and Lemma 5.21, we have

$$\Omega_{\mathfrak{X}/R} \simeq \mathcal{O}_{\mathfrak{X}}((p^r - 1)\mathfrak{C}(Np^r)) \otimes \underline{\omega}^{\otimes 2}_{\mathfrak{X}} \otimes \mathcal{O}_{\mathfrak{X}}(-p^r \mathfrak{C}(Np^r))$$

$$\simeq \underline{\omega}^{\otimes 2}_{\mathfrak{X}}(-\mathfrak{C}(Np^r)).$$

□

**Corollary 5.23.** *Let $\Lambda$ be an irreducible component of the special fiber of $\mathfrak{X}$. We have*

$$\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}}|_\Lambda) = \deg(\Omega_{\mathfrak{X}/R}|_\Lambda) + \deg(\mathfrak{C}(Np^r)|_\Lambda).$$

The following will allow us to calculate $\deg(\mathfrak{C}(Np^r)|_\Lambda)$.

**Lemma 5.24.** $\deg(\mathfrak{C}(Np^r)|_\Lambda)$ *is equal to the number of cusps* $\overline{\{x\}}$ *which intersect* $\Lambda$.

*Proof.* Viewed as a divisor, $\mathfrak{C}(Np^r)$ is equal to the closure of all the cusps of the generic fiber of $\mathfrak{X}$ by Proposition 2.37. By Theorem 2.24(d),

$$\deg(\mathfrak{C}(Np^r)|_\Lambda) = \mathfrak{C}(Np^r).\Lambda = \sum_x \overline{\{x\}}.\Lambda$$

where the sum is indexed over $x \in \mathfrak{C}(\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})})$ *i.e.* over the cusps of the generic fiber.

All the cusps are rational, so by Corollary 2.30, $\overline{\{x\}}$ intersects at a single irreducible component of the special fiber. Therefore

$$\overline{\{x\}}.\Lambda = \begin{cases} 1 & \text{if } \overline{\{x\}} \cap \Lambda \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

so $\deg(\mathfrak{C}(Np^r)|_\Lambda)$ is equal to the number of cusps which intersect $\Lambda$ as desired. □

For convenience, we let $C(Np^r) = \mathfrak{C}(\mathfrak{X}_{/\mathbb{Q}_p(\zeta_{Np^r})})$, the set of all cusps of the generic fiber. Using Lemma 5.24 and Proposition 2.53, we obtain the following:

**Proposition 5.25.** *We have*

$$\deg(\mathfrak{C}(Np^r)|_\Lambda) = \varphi(p^r)\#C(N).$$

To compute $\deg(\Omega_{\mathfrak{X}(Np^r)/R}|_\Lambda)$, we will use [Liu02, Theorem 9.1.37]:

**Theorem 5.26.** *Let* $X \to S$ *be a regular fibered surface,* $s \in S$ *a closed point, and* $E \in \mathrm{Div}_s(X)$ *such that* $0 < E \leq X_s$. *Then we have*

$$\Omega_{E/k(s)} \simeq (\mathcal{O}_X(E) \otimes \Omega_{X/S})|_E.$$

**Corollary 5.27.** *We have*

$$\deg(\Omega_{\mathfrak{X}/R}|_\Lambda) = p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24} - \varphi(p^r)\#C(N) + \deg \mathrm{S}(N) \cdot p^{2r-1}$$

*Proof.* Applying Theorem 5.26 to $\mathfrak{X} \to \mathrm{Spec}(R)$ and $E = \Lambda$, we get

$$\Omega_{\Lambda/\mathbb{F}_q} \simeq (\mathcal{O}_{\mathfrak{X}}(\Lambda) \otimes \Omega_{\mathfrak{X}/R})|_\Lambda.$$

Therefore

$$\begin{aligned}\deg(\Omega_{\Lambda/\mathbb{F}_q}) &= \deg((\mathcal{O}_{\mathfrak{X}}(\Lambda)\otimes\Omega_{\mathfrak{X}/R})|_\Lambda)\\ &= \deg(\mathcal{O}_{\mathfrak{X}}(\Lambda)|_\Lambda)+\deg(\Omega_{\mathfrak{X}/R}|_\Lambda)\\ &= \Lambda.\Lambda+\deg(\Omega_{\mathfrak{X}/R}|_\Lambda).\end{aligned}$$

Thus

$$\deg(\Omega_{\mathfrak{X}/R}|_\Lambda)=\deg(\Omega_{\Lambda/\mathbb{F}_q})-\Lambda.\Lambda.$$

By [Liu02, Corollary 7.3.31(a)],

$$\deg(\Omega_{\Lambda/\mathbb{F}_q})=2\rho_a(\Lambda)-2$$

where $\rho_a$ is the arithmetic genus. By [KM85, 12.9.4] along with [KM85, Corollary 10.13.12], we have

$$\begin{aligned}2\rho_a(\Lambda) &= 2\rho_a(\mathrm{Ig}(p^r,N))\\ &= p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}+2-\varphi(p^r)\#C(N)\end{aligned}$$

We have already shown $\Lambda.\Lambda=-\deg \mathrm{S}(N)\cdot p^{2r-1}$ in Proposition 4.5. Thus

$$\begin{aligned}\deg(\Omega_{\mathfrak{X}/R}|_\Lambda) &= (2\rho_a(\Lambda)-2)-\Lambda.\Lambda\\ &= p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}-\varphi(p^r)\#C(N)+\deg \mathrm{S}(N)\cdot p^{2r-1}.\end{aligned}$$

as desired. □

Recall that the space of cusp forms, by definition, are the global sections of $\underline{\omega}_{\mathfrak{X}}^{\otimes 2}(-\mathfrak{C}(Np^r))$.

**Theorem 5.28.** *We have*

$$\deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}|_\Lambda)=\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\left[\frac{1}{12}(p-1)p^{2r-1}\right]$$

*and*

$$\deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}(-\mathfrak{C}(Np^r))|_\Lambda)=\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})(p-1)\left[\frac{p^{2r-1}}{12}-\frac{p^{r-1}}{2N}\right].$$

*Proof.* By Corollary 5.23, Proposition 5.25, and Corollary 5.27, we have

$$\begin{aligned}\deg(\underline{\omega}_{\mathfrak{X})}^{\otimes 2}|_\Lambda) &= p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}-\varphi(p^r)\#C(N)+\deg \mathrm{S}(N)\cdot p^{2r-1}+\varphi(p^r)\#C(N)\\ &= p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}+\deg \mathrm{S}(N)\cdot p^{2r-1}\end{aligned}$$

By [KM85, 12.4.5], $\deg \mathrm{S}(N) = \frac{(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}$. Continuing, we have

$$\begin{aligned}
\deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}|_\Lambda) &= p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24} + \frac{(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24}\cdot p^{2r-1} \\
&= \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\left[\frac{p^r\varphi(p^r)}{24} + \frac{p^{2r-1}(p-1)}{24}\right] \\
&= \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\left[\frac{p^r p^{r-1}(p-1)}{24} + \frac{p^{2r-1}(p-1)}{24}\right] \\
&= \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\left[\frac{1}{12}(p-1)p^{2r-1}\right].
\end{aligned}$$

By Proposition 5.25 along with Lemma 2.51, we have

$$\begin{aligned}
\deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}(-\mathfrak{C}(Np^r))|_\Lambda) &= \deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}) - \deg(\mathfrak{C}(Np^r)|_\Lambda) \\
&= \deg(\underline{\omega}_{\mathfrak{X}}^{\otimes 2}) - \varphi(p^r)\#C(N) \\
&= \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})\left[\frac{1}{12}(p-1)p^{2r-1}\right] - \left[\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{2N}\right] \\
&= \#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})(p-1)\left[\frac{p^{2r-1}}{12} - \frac{p^{r-1}}{2N}\right].
\end{aligned}$$

□

## 5.5 An Upper Bound

Recall at the end of Remark 3.14 we have the following expression for the exponent

$$e = \max_{\substack{1\le i\le n \\ f\in H^0(\mathfrak{X},\underline{\omega}^{\otimes 2})}} \left\{\nu_{\Lambda_0}(f) - \nu_{\Lambda_i}(f)\right\}.$$

At the end of Section 3.3 we established

$$\nu_\Lambda(f) - \nu_{\Lambda_{(1,0)}}(f) = \sum_{\Lambda'\neq\Lambda_{(1,0)}} \left(\deg(\underline{\omega}^{\otimes k}|_{\Lambda'}) - H_f.\Lambda'\right) c^{\Lambda,\Lambda'}$$

where the sum is over all irreducible components of the special fiber excluding $\Lambda_{(1,0)}$. We will now provide an upper bound for $e$. First we need to compute the sums $\sum c^{\Lambda,\Lambda'}$, where $c^{\Lambda,\Lambda'}$ is the entry of $T^{-1}$ corresponding to row label $\Lambda$ and column label $\Lambda'$ (see Section 4.2).

**Lemma 5.29.** *For $1 \le a' \le p^r - 1$, we have*

$$\sum_{\substack{a=1 \\ a\neq a'}}^{p^r-1} \nu_p\left(\frac{1}{a} - \frac{1}{a'}\right) = -(p^r-2)\nu_p(a').$$

*Proof.* We have

$$\sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p\left(\frac{1}{a}-\frac{1}{a'}\right) = \sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p\left(\frac{a'-a}{aa'}\right) = \sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p(a'-a) - \sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p(a) - \sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p(a')$$

$$= \sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p(a-a') - \left(\sum_{a=1}^{p^r-1} \nu_p(a) - \nu_p(a')\right) - (p^r-2)\nu_p(a') \tag{23}$$

Note that

$$\begin{aligned}
\sum_{\substack{a=1\\a\neq a'}}^{p^r-1} \nu_p(a-a') &= \sum_{a=1}^{a'-1} \nu_p(a-a') + \sum_{a=a'+1}^{p^r-1} \nu_p(a-a') \\
&= \sum_{a=1}^{a'-1} \nu_p(a) + \sum_{a=1}^{p^r-1-a'} \nu_p(a) \\
&= \sum_{a=-a'+1}^{-1} \nu_p(a) + \sum_{a=1}^{p^r-1-a'} \nu_p(a) \\
&= \sum_{a=p^r-a'+1}^{p^r-1} \nu_p(a) + \sum_{a=1}^{p^r-a'-1} \nu_p(a) \\
&= -\nu_p(a') + \sum_{a=1}^{p^r-1} \nu_p(a).
\end{aligned}$$

Continuing equation (23), we have

$$\begin{aligned}
&= -\nu_p(a') + \sum_{a=1}^{p^r-1} \nu_p(a) - \left(\sum_{a=1}^{p^r-1} \nu_p(a) - \nu_p(a')\right) - (p^r-2)\nu_p(a') \\
&= -(p^r-2)\nu_p(a').
\end{aligned}$$

□

**Proposition 5.30.** *We have*

$$\deg \mathrm{S}(N) \sum_{\Lambda' \neq \Lambda_{(1,0)}} c^{\Lambda,\Lambda'} = \begin{cases} -p^{-r}(pr-r+1) + p^{-r}(p-1)\nu_p(a) & \text{if } \Lambda = \Lambda_{(1,-a)} \\ -p^{-r}(pr-r+1) & \text{if } \Lambda = \Lambda_{(-pb,1)} \end{cases}$$

*Proof.* We split into two cases, depending on $\Lambda$.

Case 1. Suppose $\Lambda = \Lambda_{(1,-a')}$. Using Theorem 4.21, we compute

$$\begin{aligned}
\sum_{\Lambda \neq \Lambda_{(1,0)}} c^{\Lambda_{(1,a')},\Lambda} &= \sum_{a=1}^{p^r-1} c^{\Lambda_{(1,-a')},\Lambda_{(1,-a)},} + \sum_{b=0}^{p^{r-1}-1} c^{\Lambda_{(1,-a')},\Lambda_{(-pb,1)}} \\
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + \sum_{\substack{a=1 \\ a\neq a'}}^{p^r-1} \left( -p^{1-2r}\frac{(pr-r+1)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p\left(\frac{1}{a}-\frac{1}{a'}\right)\right) \\
&\quad + \sum_{b=0}^{p^{r-1}-1} \left( -\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(a')\right) \\
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + (p^r-2)\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) - \frac{p^{1-2r}(p-1)}{p+1}\sum_{\substack{a=1 \\ a\neq a'}}^{p^r-1}\nu_p\left(\frac{1}{a}-\frac{1}{a'}\right) \\
&\quad -(p^{r-1})\frac{p^{1-2r}(pr-r+1)}{p+1} + p^{r-1}\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a')
\end{aligned}$$

Using Lemma 5.29 on the sum $\sum v_p(1/a - 1/a')$, we have

$$\begin{aligned}
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + (p^r-2)\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) - \frac{p^{1-2r}(p-1)}{p+1}\left(-(p^r-2)\nu_p(a')\right) \\
&- (p^{r-1})\frac{p^{1-2r}(pr-r+1)}{p+1} + p^{r-1}\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a') \\
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + (p^r-2+p^{r-1})\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) - \frac{p^{1-2r}(p-1)}{p+1}\left(-(p^r-2)\nu_p(a')\right) + p^{r-1}\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a') \\
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + (p^r-2+p^{r-1})\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) + \frac{p^{1-2r}(p-1)}{p+1}\left((p^r-2)\nu_p(a')\right) + p^{r-1}\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a') \\
&= c^{\Lambda_{(1,a')},\Lambda_{(1,a')}} + (p^r-2+p^{r-1})\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) + (p^r+p^{r-1}-2)\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a') \\
&= -\frac{2p^{1-2r}(pr-r+1)}{p+1} + \frac{2p^{1-2r}(p-1)}{p+1}\nu_p(a') + (p^r-2+p^{r-1})\left(-p^{1-2r}\frac{(pr-r+1)}{p+1}\right) \\
&+ (p^r+p^{r-1}-2)\frac{p^{1-2r}(p-1)}{p+1}\nu_p(a') \\
&= -p^{-r}(pr-r+1) + p^{-r}(p-1)\nu_p(a').
\end{aligned}$$

Case 2: Suppose $\Lambda' = \Lambda_{(-pb',1)}$. Then

$$\begin{aligned}
\sum_{\Lambda \neq \Lambda_{(1,0)}} c^{\Lambda_{(-pb',1)},\Lambda} &= \sum_{a=1}^{p^r-1} c^{\Lambda_{(-pb',1)},\Lambda_{(1,-a)},} + \sum_{b=0}^{p^{r-1}-1} c^{\Lambda_{(-pb',1)},\Lambda_{(-pb,1)}} \\
&= \sum_{a=1}^{p^r-1} \left( -\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\nu_p(a) \right) \\
&\quad -\frac{2p^{1-2r}(pr-r+1)}{p+1} + \sum_{\substack{b=0 \\ b\neq b'}}^{p^{r-1}-1} \left( -\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\nu_p(b'-b) \right) \\
&= -(p^r-1)\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\sum_{a=1}^{p^r-1}\nu_p(a) \\
&\quad -\frac{2p^{1-2r}(pr-r+1)}{p+1} - (p^{r-1}-1)\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\sum_{\substack{b=0 \\ b\neq b'}}^{p^{r-1}-1}\nu_p(b'-b)
\end{aligned}$$

Using Lemma 4.17, we get

$$\begin{aligned}
&= -(p^r-1)\frac{p^{1-2r}(pr-r+1)}{p+1} + \frac{p^{1-2r}(p-1)}{p+1}\left(\frac{p^r-pr+r-1}{p-1}\right) \\
&- \frac{2p^{1-2r}(pr-r+1)}{p+1} - (p^{r-1}-1)\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\left(-\nu_p(b') + \frac{p^{r-1}-p(r-1)+r-2}{p-1} + \nu_p(b')\right) \\
&= -\frac{(p-1)rp^{1-r}}{p+1} - \frac{2p^{1-2r}(pr-r+1)}{p+1} - (p^{r-1}-1)\frac{p^{1-2r}(pr+p-r)}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\left(\frac{p^{r-1}-p(r-1)+r-2}{p-1}\right) \\
&= \frac{p^{-2r}(-p^{r+1}-rp^{r+2}+rp^r-p^2(r-1)+p(r-2))}{p+1} - \frac{p^{1-2r}(p-1)}{p+1}\left(\frac{p^{r-1}-p(r-1)+r-2}{p-1}\right) \\
&= -p^{-r}(pr-r+1). \qquad \square
\end{aligned}$$

**Theorem 5.31.** *Let $N \geq 3$, and $r \geq 1$ be integers and $p \geq 2$ a prime such that $p \nmid N$. Let $e$ be the exponent of $\pi$ in the annihilator of $M_2(\Gamma(Np^r), \mathbb{Z}_p[\zeta_{Np^r}])/H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2})$. Then*

$$e \leq 2p^{r-1}(pr-r+1).$$

*Proof.* For any $f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$, we always have $\nu_{\Lambda_{(1,0)}}(f) \geq 0$. Thus

$$\begin{aligned}
-\nu_\Lambda(f) &\leq \nu_{\Lambda_{(1,0)}}(f) - \nu_\Lambda(f) \\
&= \sum_{\Lambda' \neq \Lambda_{(1,0)}} \left( H_f.\Lambda' - \deg(\underline{\omega}^{\otimes 2}|_{\Lambda'}) \right) c^{\Lambda,\Lambda'} \\
&= \sum_{\Lambda' \neq \Lambda_{(1,0)}} (H_f.\Lambda')c^{\Lambda,\Lambda'} - \sum_{\Lambda' \neq \Lambda_{(1,0)}} \deg(\underline{\omega}^{\otimes 2}|_{\Lambda'})c^{\Lambda,\Lambda'}.
\end{aligned}$$

By Corollary 4.22, $c^{\Lambda,\Lambda'} < 0$. Also note that $H_f$ is an effective horizontal divisor since $f$ has no poles while $\Lambda'$ is an effective vertical divisor. Since $H_f$ and $\Lambda'$ do not have any common components, the intersection

number $H_f.\Lambda'$ is positive. Thus $\sum_{\Lambda' \neq \Lambda_{(1,0)}} (H_f.\Lambda') c^{\Lambda,\Lambda'}$ will always be negative. Continuing, we have

$$\leq - \sum_{\Lambda' \neq \Lambda_{(1,0)}} \deg(\underline{\omega}^{\otimes 2}|_{\Lambda'}) c^{\Lambda,\Lambda'}$$

Using Theorem 5.28, the above expression becomes

$$-\frac{(p-1)p^{2r-1}\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{12} \frac{24}{(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} \deg \mathrm{S}(N) \sum_{\Lambda' \neq \Lambda_{(1,0)}} c^{\Lambda,\Lambda'}$$

$$= -2p^{2r-1} \deg \mathrm{S}(N) \sum_{\Lambda' \neq \Lambda_{(1,0)}} c^{\Lambda,\Lambda'}.$$

Using Proposition 5.30, we have

$$= \begin{cases} 2p^{r-1}(pr-r+1) - 2p^{r-1}(p-1)\nu_p(a) & \text{if } \Lambda = \Lambda_{(1,-a)} \\ \\ 2p^{r-1}(pr-r+1) & \text{if } \Lambda = \Lambda_{(-pb,1)} \end{cases}$$

which is maximized whenever $a$ is coprime to $p$ and attains a value of $2p^{r-1}(pr-r+1)$. □

By replacing $\underline{\omega}^{\otimes 2}$ with $\underline{\omega}^{\otimes 2}(-\mathfrak{C}(Np^r))$ in the proof of Theorem 5.31, we obtain an upper bound for the exponent in the situation of cusp forms.

**Corollary 5.32.** *Let $N \geq 3$, and $r \geq 1$ be integers and $p \geq 2$ a prime such that $p \nmid N$. The exponent $e$ of $\pi$ in the annihilator of $S_2(\Gamma(Np^r), \mathbb{Z}_p[\zeta_{Np^r}])/H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2}(-\mathfrak{C}(Np^r)))$ is bounded above by*

$$2p^{r-1}(pr-r+1) - \frac{12}{Np}(pr-r+1).$$

*Proof.* Using Theorem 5.28 to compute $\deg(\underline{\omega}^{\otimes 2}(-\mathfrak{C}(Np^r))|_\Lambda)$, we get an upper bound of

$$-\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})(p-1)\left[\frac{p^{2r-1}}{12} - \frac{p^{r-1}}{2N}\right] \frac{24}{(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} \deg \mathrm{S}(N) \sum_{\Lambda' \neq \Lambda_{(1,0)}} c^{\Lambda,\Lambda'}$$

$$\begin{aligned} &= \left[\frac{12p^{r-1}}{N} - 2p^{2r-1}\right] \deg \mathrm{S}(N) \sum_{\Lambda' \neq \Lambda_{(1,0)}} c^{\Lambda,\Lambda'} \\ &\leq -\left[\frac{12p^{r-1}}{N} - 2p^{2r-1}\right] p^{-r}(pr-r+1) \\ &= 2p^{r-1}(pr-r+1) - \frac{12}{Np}(pr-r+1). \end{aligned}$$

□

*Remark* 5.33. We compare our result in Corollary 5.32 with Edixhoven's method in [Edi06]. He considers the situation of weight $k = 2$ and level $\Gamma_0(N)$ cusp forms where $\mathrm{ord}_p(N) = 1$, and bounds $e$ via the inequality

$$e < \frac{\deg(\Omega_{\mathfrak{X}_0(N)}|_\Lambda)}{\deg \mathrm{S}(N)}.$$

We will show a similar inequality holds our situation. Let $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ denote the irreducible components of $\bar{\mathfrak{X}}$ where $\Lambda_0$ contains the cusp $\infty$. Let $f \in M_2(Np^r, \mathbb{Z}_p[\zeta_{Np^r}])$ be a non-zero cusp form such that $\nu_{\Lambda_0}(f) = 0$ and, without loss of generality, let $-m := \nu_{\Lambda_1}(f) < 0$ be the minimum among the values in $\{\nu_{\Lambda_i}(f)\}_{i=1}^n$. By scaling, we can assume $\nu_{\Lambda_0}(f) = m$, $\nu_{\Lambda_1}(f) = 0$, and $\nu_{\Lambda_i}(f) \geq 0$ for $i = 2, \ldots, n$ so that $f$ has non-negative valuation along every irreducible component *i.e.* $f \in H^0(\mathfrak{X}, \underline{\omega}^{\otimes 2})$. We can write the divisor associated to $f$ as

$$\mathrm{div}(f) = H_f + m\Lambda_0 + \sum_{i=2}^{n} \nu_{\Lambda_i}(f)\Lambda_i$$

where $H_f$ is the horizontal part of the divisor $\mathrm{div}(f)$. By Theorem 5.22, the sheaf of cusp forms $\underline{\omega}^{\otimes 2}(-\mathfrak{C}(Np^r))$ is isomorphic to the relative dualizing sheaf $\Omega_{\mathfrak{X}/R}$. Since $\mathfrak{X}$ is an arithmetic surface, we can use intersection theory (see Theorem 2.24), along with Proposition 3.11, to compute:

$$\begin{aligned}
\deg \Omega_{\mathfrak{X}/R}|_{\Lambda_1}(-m\mathrm{S}(N)) &= \deg(\Omega_{\mathfrak{X}/R}|_{\Lambda_1}) - m \deg \mathrm{S}(N) \\
&= \deg(\mathrm{div}(f)|_{\Lambda_1}) - m \deg \mathrm{S}(N) \\
&= \mathrm{div}(f).\Lambda_1 - m \deg \mathrm{S}(N) \\
&= H_f.\Lambda_1 + m\Lambda_0.\Lambda_1 + \sum_{i=2}^{n} \nu_{\Lambda_i}(f)\Lambda_i.\Lambda_1 - m \deg \mathrm{S}(N) \\
&= H_f.\Lambda_1 + m(\Lambda_0.\Lambda_1 - \deg \mathrm{S}(N)) + \sum_{i=2}^{n} \nu_{\Lambda_i}(f)\Lambda_i.\Lambda_1.
\end{aligned}$$

Since $f$ has no poles, $H_f$ is effective so $H_f.\Lambda_1 \geq 0$. By assumption, $\nu_{\Lambda_i}(f) \geq 0$ for $i = 2, \ldots, n$; we also have $\Lambda_i.\Lambda_1 \geq 0$ since $\Lambda_i$ and $\Lambda_1$ have no common components. Lastly, $\Lambda_0$ and $\Lambda_1$ intersect precisely at the supersingular points, so $\Lambda_0.\Lambda_1 \geq \deg \mathrm{S}(N)$. Therefore we conclude

$$0 \leq \deg \Omega_{\mathfrak{X}/R}|_{\Lambda_1}(-m\mathrm{S}(N)) = \deg(\Omega_{\mathfrak{X}/R}|_{\Lambda_1}) - m \deg \mathrm{S}(N).$$

Using Corollary 5.27, we have:

$$\begin{aligned}
m &< \frac{\deg(\Omega_{\mathfrak{X}/R}|_{\Lambda_1})}{\deg \mathrm{S}(N)} \\
&= \frac{p^r\varphi(p^r)\frac{\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24} - \varphi(p^r)\#C(N) + \deg \mathrm{S}(N) \cdot p^{2r-1}}{\deg \mathrm{S}(N)} \\
&= \frac{24p^r\varphi(p^r)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{24(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} - \frac{24\varphi(p^r)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}{2N(p-1)\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})} + p^{2r-1} \\
&= \frac{p^r p^{r-1}(p-1)}{p-1} - \frac{12p^{r-1}(p-1)}{N(p-1)} + p^{2r-1} \\
&= p^{2r-1} - \frac{12p^{r-1}}{N} + p^{2r-1} \\
&= 2p^{2r-1} - \frac{12p^{r-1}}{N}
\end{aligned}$$

which provides an upper bound for $e$. When $r = 1$, this bound agrees with the bound in Corollary 5.32, but in general is larger by a factor of $p^r/(pr - r + 1)$.

# 6 A Lower Bound

In this section, we will use Klein forms to build an explicit modular form in $M_2(\Gamma_1(p), \mathbb{Z}_p[\zeta_p])$ for each prime level $p$. By viewing these modular forms at level $\Gamma(Np^r)$ with coefficients in $\mathbb{Z}_p[\zeta_{Np^r}]$, we will obtain a lower bound for the exponent.

**Definition 6.1.** A **nearly holomorphic modular form** is a modular form which is allowed to be meromorphic at the cusps.

Fix $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$. Let $\tau \in \mathcal{H}$, $q = e^{2\pi i \tau}$, and $q_z = e^{2\pi i z}$ where $z = r_1\tau + r_2$. Define the **Klein form**

$$\kappa_{(r_1,r_2)}(\tau) = e^{\pi i r_2(r_1-1)} q^{\frac{1}{2}r_1(r_1-1)}(1-q_z)\prod_{n=1}^{\infty}(1-q^n q_z)(1-q^n q_z^{-1})(1-q^n)^{-2}. \tag{24}$$

The following result, which is [EKS11, Theorem. 2.6], builds upon results in [KL81, §2.1, §3.4] which establish a criterion for when a product of Klein forms is a nearly holomorphic modular form.

For $r \in \mathbb{R}$, we let $\langle r \rangle$ denote the fractional part of $r$. Note $\langle r \rangle = 0$ precisely when $r \in \mathbb{Z}$.

**Theorem 6.2.** *For an integer $N \geq 2$, let $\{m(t)\}_{t=1}^{N-1}$ be a family of integers. Then the product*

$$\kappa(\tau) = \prod_{t=1}^{N-1} \kappa_{(t/N,0)}(N\tau)^{m(t)}$$

*is a nearly holomorphic modular form for $\Gamma_1(N)$ of weight $k = -\sum_{t=1}^{N-1} m(t)$ if*

$$\sum_{t=1}^{N-1} m(t)t^2 \equiv 0 \pmod{\gcd(2,N)\cdot N}.$$

*Furthermore, for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$\mathrm{ord}_q\left(\kappa(\tau)|[\alpha]_k\right) = \frac{\gcd(c,N)^2}{2N}\sum_{t=1}^{N-1} m(t)\left\langle \frac{at}{\gcd(c,N)}\right\rangle\left(\left\langle \frac{at}{\gcd(c,N)}\right\rangle - 1\right).$$

Using this result when $N = p$ is an odd prime, we will choose a family of integers $\{m(t)\}_{t=1}^{N-1}$ such that $k = 2$ and $\mathrm{ord}_q\left(\kappa(\tau)|[\alpha]_2\right) \geq 0$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. This guarantees that $\kappa(\tau)$ is a weight 2 (holomorphic) modular form of level $\Gamma_1(p)$. As $\Gamma(Np^r) \leq \Gamma(p) \leq \Gamma_1(p)$, this also gives us a modular form of level $\Gamma(Np^r)$ as originally desired.

It may be natural to choose $m(t)$ to be zero for most values of $t$ in order to simplify the quadratic condition in Theorem 6.2 and the expression for the order. Consider the situation when $m(t_1), m(t_2)$, and $m(t_3)$ are the only non-zero values for some distinct $t_1, t_2, t_3 \in \{1, 2, \ldots, p-1\}$. By Theorem 6.2, we seek $m(t_1), m(t_2)$, and $m(t_3)$ such that

$$m(t_1) + m(t_2) + m(t_3) = -2$$

$$m(t_1)t_1^2 + m(t_2)t_2^2 + m(t_3)t_3^2 \equiv 0 \pmod{2p}$$

and

$$\frac{\gcd(c,p)^2}{2p}\left(\sum_{i=1}^{3} m(t_i)\left\langle\frac{at_i}{\gcd(c,p)}\right\rangle\left(\left\langle\frac{at_i}{\gcd(c,p)}\right\rangle - 1\right)\right) \geq 0$$

for all $a, c \in \mathbb{Z}$.

To satisfy the first condition, we can take $m(t_1) = m(t_2) = -2$ and $m(t_3) = 2$. The second condition becomes

$$-t_1^2 - t_2^2 + t_3^2 \equiv 0 \pmod{p} \text{ or equivalently } t_1^2 + t_2^2 \equiv t_3^2 \pmod{p}$$

which can be satisfied if we take $(t_1, t_2, t_3)$ to be a Pythagorean triple. As we now show, taking $t_1 = 3, t_2 = 4$, and $t_3 = 5$ suffices if $p > 5$. For convenience, we let $(x, y)$ denote $\gcd(x, y)$ for $x, y \in \mathbb{Z}$.

**Proposition 6.3.** *The product of Klein forms*

$$\kappa(\tau) = \kappa_{(3/p,0)}(p\tau)^{-2}\kappa_{(4/p,0)}(p\tau)^{-2}\kappa_{(5/p,0)}(p\tau)^2$$

*is a weight 2 modular form of level* $\Gamma_1(p)$ *for* $p > 5$.

*Proof.* Define $m : \{1, \ldots, p-1\} \to \mathbb{Z}$ by

$$m(t) = \begin{cases} -2 & \text{if } t = 3, 4 \\ 2 & \text{if } t = 5 \\ 0 & \text{otherwise} \end{cases}$$

By Theorem 6.2, the product $\kappa(\tau)$ defined by $m(t)$ is a weight 2 nearly holomorphic modular form of level $\Gamma_1(p)$. It remains to show $\kappa(\tau)$ is holomorphic at the cusps. The order of $\kappa(\tau)$ is given by

$$\operatorname{ord}_q(\kappa(\tau)|[\alpha]_2)$$

$$= \frac{(c,p)^2}{2p}\left(-2\left\langle\frac{3a}{(c,p)}\right\rangle\left(\left\langle\frac{3a}{(c,p)}\right\rangle - 1\right) - 2\left\langle\frac{4a}{(c,p)}\right\rangle\left(\left\langle\frac{4a}{(c,p)}\right\rangle - 1\right) + 2\left\langle\frac{5a}{(c,p)}\right\rangle\left(\left\langle\frac{5a}{(c,p)}\right\rangle - 1\right)\right)$$

where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. We will show this expression is always non-negative.

Since $(c,p)^2/2p > 0$, we can ignore this factor. Dividing through by 2, our goal is to show

$$-\left\langle\frac{3a}{(c,p)}\right\rangle\left(\left\langle\frac{3a}{(c,p)}\right\rangle - 1\right) - \left\langle\frac{4a}{(c,p)}\right\rangle\left(\left\langle\frac{4a}{(c,p)}\right\rangle - 1\right) + \left\langle\frac{5a}{(c,p)}\right\rangle\left(\left\langle\frac{5a}{(c,p)}\right\rangle - 1\right)$$

or equivalently

$$\left\langle\frac{3a}{(c,p)}\right\rangle\left(1 - \left\langle\frac{3a}{(c,p)}\right\rangle\right) + \left\langle\frac{4a}{(c,p)}\right\rangle\left(1 - \left\langle\frac{4a}{(c,p)}\right\rangle\right) - \left\langle\frac{5a}{(c,p)}\right\rangle\left(1 - \left\langle\frac{5a}{(c,p)}\right\rangle\right)$$

is non-negative for all $a, c \in \mathbb{Z}$.

Let

$$f(x) = \langle 3x\rangle\,(1-\langle 3x\rangle) + \langle 4x\rangle\,(1-\langle 4x\rangle) - \langle 5x\rangle\,(1-\langle 5x\rangle)\,,$$

noting that $f(a/(c,p))$ is the expression we are showing is non-negative. Since $\langle x\rangle$ is periodic with period $0 \le x < 1$, it suffices to show $f(x) \ge 0$ for all $0 \le x < 1$. Noting that

$$\langle -x\rangle\,(1-\langle -x\rangle) = (1-\langle x\rangle)(1-(1-\langle x\rangle)) = \langle x\rangle\,(1-\langle x\rangle)$$

we can conclude $f(-x) = f(x)$. Therefore it suffices to show $f(x) \ge 0$ for all $0 \le x \le 1/2$.

We accomplish this by finding an explicit piecewise defined expression for $f(x)$ and then computing its derivative. Note that

$$\langle 3x\rangle\,(1-\langle 3x\rangle) = \begin{cases} 3x(1-3x) & \text{if } 0 \le x < 1/3 \\ 3(x-1/3)(1-3(x-1/3)) & \text{if } 1/3 \le x < 2/3 \\ 3(x-2/3)(1-3(x-2/3)) & \text{if } 2/3 \le x < 1 \end{cases}$$

$$= \begin{cases} 3x(1-3x) & \text{if } 0 \le x < 1/3 \\ (3x-1)(2-3x) & \text{if } 1/3 \le x < 2/3\,. \\ 3(3x-2)(1-x) & \text{if } 2/3 \le x < 1 \end{cases}$$

Similar expressions can be obtained for $\langle 4x\rangle\,(1-\langle 4x\rangle)$ and $\langle 5x\rangle\,(1-\langle 5x\rangle)$. Putting these together, we have

$$f(x) = \begin{cases} 3x(1-3x) + 4x(1-4x) - 5x(1-5x) & \text{if } 0 \le x < 1/5 \\ 3x(1-3x) + 4x(1-4x) - (5x-1)(2-5x) & \text{if } 1/5 \le x < 1/4 \\ 3x(1-3x) + 2(4x-1)(1-2x) - (5x-1)(2-5x) & \text{if } 1/4 \le x < 1/3\,. \\ (3x-1)(2-3x) + 2(4x-1)(1-2x) - (5x-1)(2-5x) & \text{if } 1/3 \le x < 2/5 \\ (3x-1)(2-3x) + 2(4x-1)(1-2x) - (5x-2)(3-5x) & \text{if } 2/5 \le x < 2/4 \end{cases}$$

Thus we can compute the derivative directly:

$$f'(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/5 \\ 2-8x & \text{if } 1/5 \le x < 1/4 \\ 0 & \text{if } 1/4 \le x < 1/3 \\ 6x-2 & \text{if } 1/3 \le x < 2/5 \\ 2-4x & \text{if } 2/5 \le x < 2/4 \end{cases}$$

This shows that the pieces of $f(x)$ are either strictly increasing, strictly decreasing, or constant in their appropriate interval. For each interval, the following table shows if $f(x)$ is increasing, decreasing, or constant based on $f'(x)$. The value of $f$ at the rightmost endpoint is also calculated for each interval.

| $a \le x < b$ | $0 \le x < 1/5$ | $1/5 \le x < 1/4$ | $1/4 \le x < 1/3$ | $1/3 \le x < 2/5$ | $2/5 \le x < 1/2$ |
|---|---|---|---|---|---|
| Inc, Dec, Con | increasing | decreasing | constant | increasing | decreasing |
| $f(b)$ | $2/5$ | 0 | 0 | $2/5$ | 0 |

Thus we conclude $f(x) \ge 0$ for all $x$. Consequently, $\kappa(\tau)$ is holomorphic at each cusp so is indeed a modular form. □

For our purposes, we view $\kappa(\tau)$ as a modular form of weight 2 and level $\Gamma(Np^r)$ over $\mathbb{Z}_p[\zeta_{Np^r}]$. We compute the valuation of its $q$-expansion at each cusp. The $q$-expansion at $\infty$ for $\kappa_{(r_1,r_2)}(\tau)$ was given earlier

in equation (6.2). In particular, the $q$-expansion of $\kappa_{(a/p,0)}(p\tau)$ is

$$\kappa_{(a/p,0)}(p\tau) = q^{p\frac{1}{2}\frac{a}{p}(\frac{a}{p}-1)}(1-q^a)\prod_{n=1}^{\infty}(1-q^{pn}q^a)(1-q^{pn}q^{-a})(1-q^{pn})^{-2}.$$

Thus the $q$-expansion of $\kappa(\tau)$ is:

$$\kappa(\tau) = \kappa_{(3/p,0)}(p\tau)^{-2}\kappa_{(4/p,0)}(p\tau)^{-2}\kappa_{(5/p,0)}(p\tau)^2$$

$$= \frac{q^{p\frac{1}{2}\frac{5}{p}(\frac{5}{p}-1)2}(1-q^5)^2\prod_{n=1}^{\infty}(1-q^{pn}q^5)^2(1-q^{pn}q^{-5})^2(1-q^{pn})^{-4}}{q^{p\frac{1}{2}\frac{3}{p}(\frac{3}{p}-1)2}(1-q^3)^2\prod_{n=1}^{\infty}(1-q^{pn}q^3)^2(1-q^{pn}q^{-3})^2(1-q^{pn})^{-4}q^{p\frac{1}{2}\frac{4}{p}(\frac{4}{p}-1)2}(1-q^4)^2\prod_{n=1}^{\infty}(1-q^{pn}q^4)^2(1-q^{pn}q^{-4})^2(1-q^{pn})^{-4}}$$

$$= \frac{q^{p\frac{5}{p}(\frac{5}{p}-1)}(1-q^5)^2}{q^{p\frac{3}{p}(\frac{3}{p}-1)}(1-q^3)^2q^{p\frac{4}{p}(\frac{4}{p}-1)}(1-q^4)^2}\prod_{n=1}^{\infty}\frac{(1-q^{pn}q^5)^2(1-q^{pn}q^{-5})^2(1-q^{pn})^{-4}}{(1-q^{pn}q^3)^2(1-q^{pn}q^{-3})^2(1-q^{pn})^{-4}(1-q^{pn}q^4)^2(1-q^{pn}q^{-4})^2(1-q^{pn})^{-4}}$$

$$= q^{p\frac{2}{p}}\frac{(1-q^5)^2}{(1-q^3)^2(1-q^4)^2}\prod_{n=1}^{\infty}\frac{(1-q^{pn}q^5)^2(1-q^{pn}q^{-5})^2(1-q^{pn})^{-4}}{(1-q^{pn}q^3)^2(1-q^{pn}q^{-3})^2(1-q^{pn})^{-4}(1-q^{pn}q^4)^2(1-q^{pn}q^{-4})^2(1-q^{pn})^{-4}}$$

$$= q^2\frac{(1-q^5)^2}{(1-q^3)^2(1-q^4)^2}\prod_{n=1}^{\infty}\frac{(1-q^{pn}q^5)^2(1-q^{pn}q^{-5})^2(1-q^{pn})^4}{(1-q^{pn}q^3)^2(1-q^{pn}q^{-3})^2(1-q^{pn}q^4)^2(1-q^{pn}q^{-4})^2}$$

which has all integral coefficients.

To compute the $q$-expansion of $\kappa(\tau)$ at the other cusps, we use the following identity for Klein forms, which can be found in [EKS11, Prop. 2.1]. For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\kappa_{(r_1,r_2)}(\alpha\tau) = \kappa\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-1}\kappa_{\alpha(r_1,r_2)}(\tau).$$

We let $\pi_r = 1-\zeta_{Np^r}$ which is a uniformizer of $\mathbb{Z}_p[\zeta_{Np^r}]$.

**Proposition 6.4.** *The $\pi_r$-adic valuation of $\kappa(\tau)$ at its $q$-expansions around the cusp 0 is $-2p^r$.*

*Proof.* Let $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$; note $\sigma\cdot\infty = 0$. The $q$-expansion of $\kappa(\tau)$ at the cusp 0 is given by

$$\kappa(\tau)|[\sigma]_2 = (-\tau)^{-2}\kappa(\sigma\cdot\tau) = \frac{\kappa_{(5/p,0)}(p\cdot\sigma\tau)^2}{\tau^2\kappa_{(3/p,0)}(p\cdot\sigma\tau)^2\kappa_{(4/p,0)}(p\cdot\sigma\tau)^2}.$$

Observe that

$$\kappa_{(r_1,r_2)}(p\cdot\sigma\tau) = \kappa_{(r_1,r_2)}(-p/\tau) = \kappa_{(r_1,r_2)}(\sigma\cdot(\tau/p))$$
$$= (-\tau/p)^{-1}\kappa_{\sigma(r_1,r_2)}(\tau/p) = -(\tau/p)^{-1}\kappa_{(r_2,-r_1)}(\tau/p).$$

Thus

$$\begin{aligned}\kappa(\tau)|[\sigma]_2 &= \frac{(\tau/p)^{-2}\kappa_{(0,-5/p)}(\tau/p)^2}{\tau^2(\tau/p)^{-2}\kappa_{(0,-3/p)}(\tau/p)^2(\tau/p)^{-2}\kappa_{(0,-4/p)}(\tau/p)^2} \\ &= \frac{\kappa_{(0,-5/p)}(\tau/p)^2}{p^2\kappa_{(0,-3/p)}(\tau/p)^2\kappa_{(0,-4/p)}(\tau/p)^2}.\end{aligned}$$

Using equation (24), the $q$-expansion of $\kappa_{(0,-a/p)}(\tau/p)$ is

$$\kappa_{(0,-a/p)}(\tau/p) = e^{\pi i \frac{a}{p}}(1-\zeta_p^{-a})\prod_{n=1}^{\infty}(1-q^{n/p}\zeta_p^{-a})(1-q^{n/p}\zeta_p^{a})(1-q^{n/p})^{-2}$$

where $\zeta_p = e^{2\pi i/p}$. Therefore

$$\kappa(\tau)|[\sigma]_2$$

$$= \frac{e^{\pi i\frac{5}{p}}(1-\zeta_p^{-5})^2\prod_{n=1}^{\infty}(1-q^{n/p}\zeta_p^{-5})^2(1-q^{n/p}\zeta_p^{5})^2(1-q^{n/p})^{-4}}{p^2e^{\pi i\frac{3}{p}}(1-\zeta_p^{-3})^2\prod_{n=1}^{\infty}(1-q^{n/p}\zeta_p^{-3})^2(1-q^{n/p}\zeta_p^{3})^2(1-q^{n/p})^{-4}e^{\pi i\frac{4}{p}}(1-\zeta_p^{-4})\prod_{n=1}^{\infty}(1-q^{n/p}\zeta_p^{-4})^2(1-q^{n/p}\zeta_p^{4})^2(1-q^{n/p})^{-4}}$$

$$= \frac{e^{-\pi i\frac{2}{p}}(1-\zeta_p^{-5})^2}{p^2(1-\zeta_p^{-3})^2(1-\zeta_p^{-4})^2}\prod_{n=1}^{\infty}\frac{(1-q^{n/p}\zeta_p^{-5})^2(1-q^{n/p}\zeta_p^{5})^2(1-q^{n/p})^4}{(1-q^{n/p}\zeta_p^{-3})^2(1-q^{n/p}\zeta_p^{3})^2(1-q^{n/p}\zeta_p^{-4})^2(1-q^{n/p}\zeta_p^{4})^2}$$

$$= \frac{\zeta_p^{-1}(1-\zeta_p^{-5})^2}{p^2(1-\zeta_p^{-3})^2(1-\zeta_p^{-4})^2}\prod_{n=1}^{\infty}\frac{(1-q^{n/p}\zeta_p^{-5})^2(1-q^{n/p}\zeta_p^{5})^2(1-q^{n/p})^4}{(1-q^{n/p}\zeta_p^{-3})^2(1-q^{n/p}\zeta_p^{3})^2(1-q^{n/p}\zeta_p^{-4})^2(1-q^{n/p}\zeta_p^{4})^2}. \tag{25}$$

Observe that the $q$-product appearing above in (25) has $q$-expansion of the form $1+O(q)$. Note that $1-\zeta_{p^r}^a$ is a uniformizer for $\mathbb{Z}_p[\zeta_{Np^r}]$ for $a$ coprime to $p$. Furthermore,

$$(1-\zeta_{p^r}^a)^{p^{r-1}(p-1)}\mathbb{Z}_p[\zeta_{Np^r}] = p\mathbb{Z}_p[\zeta_{Np^r}] = (1-\zeta_p^a)^{p-1}\mathbb{Z}_p[\zeta_{Np^r}].$$

Thus the minimal valuation among the coefficients of the $q$-expansion of $\kappa(\tau)|[\sigma]_2$ is

$$\nu_{\pi_r}\left(\frac{\zeta_p^{-1}(1-\zeta_p^{-5})^2}{p^2(1-\zeta_p^{-3})^2(1-\zeta_p^{-4})^2}\right) = \nu_{\pi_r}\left(\frac{1}{p^2(1-\zeta_p^{-4})^2}\right) = -2\left(\nu_{\pi_r}(p)+\nu_{\pi_r}(1-\zeta_p^{-4})\right)$$

$$= -2\left(p^{r-1}(p-1)+p^{r-1}\right) = -2p^r. \qquad \square$$

Next we will handle the case of $p=5$.

**Proposition 6.5.** *The product of Klein forms*

$$\kappa(\tau) = \kappa_{(1/5,0)}(5\tau)^4\kappa_{(2/5,0)}(5\tau)^{-2}\kappa_{(3/5,0)}(5\tau)^{-4}$$

*is a weight 2 modular form of level* $\Gamma_1(5)$.

*Proof.* Define $m:\{1,2,3,4\}\to\mathbb{Z}$ by

$$m(t) = \begin{cases} 4 & \text{if } t=1 \\ -2 & \text{if } t=2 \\ -4 & \text{if } t=3 \\ 0 & \text{if } t=4 \end{cases}$$

Note that

$$m(1)+m(2)+m(3) = -2$$

and

$$m(1)\cdot 1^2 + m(2)\cdot 2^2 + m(3)\cdot 3^2 = 4\cdot 1^2 - 2\cdot 2^2 - 4\cdot 3^2$$

$$= -40 \equiv 0 \pmod{10}.$$

By Theorem 6.2, the product $\kappa(\tau)$ defined by $m(t)$ is a weight 2 nearly holomorphic modular form of level $\Gamma_1(5)$. It remains to show $\kappa(\tau)$ is holomorphic at the cusps. Similar to the proof in Proposition 6.3, it suffices to show

$$4\left\langle \frac{a}{(c,5)}\right\rangle\left(\left\langle \frac{a}{(c,5)}\right\rangle - 1\right) - 2\left\langle \frac{2a}{(c,5)}\right\rangle\left(\left\langle \frac{2a}{(c,5)}\right\rangle - 1\right) - 4\left\langle \frac{3a}{(c,5)}\right\rangle\left(\left\langle \frac{3a}{(c,5)}\right\rangle - 1\right) \geq 0$$

for all $a, c \in \mathbb{Z}$. Let

$$f(x) = 4\langle x\rangle(\langle x\rangle - 1) - 2\langle 2x\rangle(\langle 2x\rangle - 1) - 4\langle 3x\rangle(\langle 3x\rangle - 1).$$

As before, it suffices to show $f(x) \geq 0$ for $0 \leq x \leq 1/2$. Since we consider $x$ of the form $a/(c,5)$, it suffices to show $f(x) \geq 0$ when $x = 0, 1/5, 2/5$ or $1/2$. We compute directly:

$$f(0) = f(1/2) = 0$$

$$f(1/5) = -4\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) + 6\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) = \frac{4}{5}$$

$$f(2/5) = -4\left(\frac{2}{5}\right)\left(\frac{3}{5}\right) + 6\left(\frac{4}{5}\right)\left(\frac{1}{5}\right) = 0.$$

Consequently, $\kappa(\tau)$ is holomorphic at each cusp so is indeed a modular form. □

The same proof holds from Proposition 6.4 to show the $q$-expansion around the cusp 0 of our modular form $\kappa(\tau)$ from Proposition 6.5 has $\pi_r$-adic valuation equal to $2(5^r)$.

**Theorem 6.6.** *For $p \geq 5$, a lower bound for $e$ is given by $2p^r$.*

**Corollary 6.7.** *Let $p \geq 5$ be prime. The exponent $e$ of $\pi$ in the annihilator of $M_2(\Gamma(Np), \mathbb{Z}_p[\zeta_{Np}])/H^0(\mathfrak{X}(Np), \underline{\omega}^{\otimes 2})$ is equal to $2p$.*

*Proof.* When $r = 1$, a lower bound for $e$ is given by $2p$. By Theorem 5.31, an upper bound for $e$ is given by

$$2p^{r-1}(pr - r + 1) = 2p^0(p - 1 + 1) = 2p$$

which agrees with the lower bound. Hence $e = 2p$. □

# A Appendix – Circulant Matrices

**Definition A.1.** An $n \times n$ **circulant** matrix $C$ over a field is any matrix of the form

$$C = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}$$

where each column is equal to the previous column shifted downward by 1, looping around as appropriate.

The entries $c_{i,j}$ of an $n \times n$ circulant matrix can be characterized by the equation

$$c_{i,j} = c_{i-j+1,1}$$

for all $1 \leq i, j \leq n$ where the index $i-j+1$ is taken modulo $n$ among the residue classes in $\{0, 1, \ldots, n-1\}$.

Let $\zeta_n = e^{2\pi i/n}$ denote a primitive $n$th root of unity. The following provides an explicit description of the eigenvalues and corresponding eigenvectors of a circulant matrix.

**Lemma A.2.** *The eigenvalues of a circulant matrix $C = (c_{ij})$ are precisely*

$$\lambda_j = \sum_{k=0}^{n-1} c_k \zeta_n^{(j-1)(n-k)}$$

*for $j = 1, \ldots, n$. A corresponding eigenvector of $\lambda_j$ is given by*

$$\vec{v}_j = \frac{1}{\sqrt{n}} \left(1, \zeta_n^{j-1}, \zeta_n^{2(j-1)}, \ldots, \zeta_n^{(n-1)(j-1)}\right).$$

*Proof.* We will verify that $C\vec{v}_j = \lambda_j \vec{v}_j$ for all $j$. For $1 \leq i \leq n$, the $i^{\text{th}}$ entry of $C\vec{v}_j$ is

$$\frac{1}{\sqrt{n}} \left(c_{i-1} + c_{i-2}\zeta_n^{j-1} + c_{i-3}\zeta_n^{2(j-1)} + \cdots + c_0\zeta^{(i-1)(j-1)} + c_{n-1}\zeta_n^{i(j-1)} \cdots + c_i\zeta_n^{(n-1)(j-1)}\right)$$

$$\begin{aligned} &= \frac{1}{\sqrt{n}} \sum_{k=0}^{i-1} c_k \zeta_n^{(i-1-k)(j-1)} + \frac{1}{\sqrt{n}} \sum_{\ell=i}^{n-1} c_\ell \zeta_n^{(n-1+i-\ell)(j-1)} \\ &= \frac{1}{\sqrt{n}} \sum_{k=0}^{i-1} c_k \zeta_n^{(n+i-1-k)(j-1)} + \frac{1}{\sqrt{n}} \sum_{\ell=i}^{n-1} c_\ell \zeta_n^{(n-1+i-\ell)(j-1)} \\ &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k \zeta^{(n+i-1-k)(j-1)} \end{aligned}$$

while the $i^{\text{th}}$ entry of $\lambda_j \vec{v}_j$ is

$$\begin{aligned}\frac{1}{\sqrt{n}}\zeta_n^{(i-1)(j-1)} \sum_{k=0}^{n-1} c_k \zeta_n^{(j-1)(n-k)} &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k \zeta_n^{(j-1)(n-k)+(i-1)(j-1)} \\ &= \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} c_k \zeta_n^{(j-1)(n+i-1-k)}\end{aligned}$$

which agrees with the $i^{\text{th}}$ entry of $C\vec{v}_j$ for all $i$. □

Circulant matrices are always diagonalizable (see [KW01, §2]) so we can write $C = PDP^{-1}$ where the columns of $P$ are the eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ and $D$ is a diagonal matrix whose diagonal entries are the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Explicitly,

$$P = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \zeta_n & \zeta_n^2 & \cdots & \zeta_n^{n-2} & \zeta_n^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \zeta_n^{n-2} & \zeta_n^{(n-2)2} & \cdots & \zeta_n^{(n-2)(n-2)} & \zeta_n^{(n-2)(n-1)} \\ 1 & \zeta_n^{n-1} & \zeta_n^{(n-1)2} & \cdots & \zeta_n^{(n-1)(n-2)} & \zeta_n^{(n-1)(n-1)} \end{pmatrix}.$$

We see the $(i,j)$-entry of $P$ is equal to $\zeta_n^{(i-1)(j-1)}/\sqrt{n}$ for $1 \le i, j \le n$.

**Lemma A.3.** *The matrix $P$ is unitary i.e. the inverse of $P$ is equal to the conjugate transpose $P^*$.*

*Proof.* Note that $P$ is symmetric and the conjugate of $\zeta_n$ is $\bar{\zeta}_n = \zeta_n^{-1}$. The $(i,j)$-entry of $PP^*$ is equal to

$$\begin{aligned}\frac{1}{n} \sum_{k=0}^{n-1} \zeta_n^{k(i-1)} \bar{\zeta}_n^{k(j-1)} &= \frac{1}{n} \sum_{k=0}^{n-1} \zeta_n^{k(i-1)} \zeta_n^{-k(j-1)} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \zeta_n^{k(i-j)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}.\end{aligned}$$

Hence $PP^* = I_n$ so $P^* = P^{-1}$. □

Now that we know $C = PDP^*$, we can describe the $(i,j)$ entry of $C^{-1}$, if it exists.

**Proposition A.4.** *Let $C$ be an $n \times n$ invertible circulant matrix whose entries along the first column are $c_0, c_1, \ldots, c_{n-1}$ in that order. Then the $(i,j)$-entry of $C^{-1}$ is equal to*

$$\frac{1}{n} \sum_{k=1}^{n} \lambda_{k-1}^{-1} \zeta_n^{(k-1)(i-j)}$$

*where $\lambda_j = \sum_{m=0}^{n-1} c_m \zeta_n^{(j-1)(n-m)}$ are the eigenvalues of $C$ for $1 \le j \le n$.*

*Proof.* We compute out the product $C^{-1} = PD^{-1}P^*$. The matrix $PD^{-1}$ will be of the form

$$PD^{-1} = \frac{1}{\sqrt{n}} \begin{pmatrix} \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} & \cdots & \lambda_{n-1}^{-1} & \lambda_n^{-1} \\ \lambda_1^{-1} & \lambda_2^{-1}\zeta_n & \lambda_3^{-1}\zeta_n^2 & \cdots & \lambda_{n-1}^{-1}\zeta_n^{n-2} & \lambda_n^{-1}\zeta_n^{n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_1^{-1} & \lambda_2^{-1}\zeta_n^{n-2} & \lambda_3^{-1}\zeta_n^{(n-2)2} & \cdots & \lambda_{n-1}^{-1}\zeta_n^{(n-2)(n-2)} & \lambda_n^{-1}\zeta_n^{(n-2)(n-1)} \\ \lambda_1^{-1} & \lambda_2^{-1}\zeta_n^{n-1} & \lambda_3^{-1}\zeta_n^{(n-1)2} & \cdots & \lambda_{n-1}^{-1}\zeta_n^{(n-1)(n-2)} & \lambda_n^{-1}\zeta_n^{(n-1)(n-1)} \end{pmatrix}.$$

Thus the $(i,j)$-entry of $PD^{-1}$ is equal to

$$\lambda_j^{-1}\zeta_n^{(i-1)(j-1)}/\sqrt{n}$$

and so the $(i,j)$-entry of $PD^{-1}P^*$ is equal to

$$\frac{1}{n}\sum_{k=1}^{n} \lambda_k^{-1}\zeta_n^{(i-1)(k-1)}\zeta_n^{-(k-1)(j-1)} = \frac{1}{n}\sum_{k=1}^{n} \lambda_k^{-1}\zeta_n^{(k-1)(i-j)}.$$

□

Using our explicit description of the entries of $C^{-1}$, we can establish the following.

**Corollary A.5.** *The inverse of an invertible circulant matrix is circulant.*

*Proof.* Let $C = (c_{i,j})$ be an $n \times n$ circulant matrix whose entries along the first column are ordered $c_0, c_1, \ldots, c_{n-1}$ so that $c_{m+1,1} = c_m$ for $0 \leq m \leq n-1$. By Lemma A.4, the $(i,j)$-entry of $C^{-1}$ is equal to

$$c^{i,j} = \frac{1}{n}\sum_{k=1}^{n}\left(\sum_{m=0}^{n-1} c_m\zeta_n^{(k-1)(n-m)}\right)\zeta_n^{(k-1)(i-j)}.$$

To show $C^{-1}$ is circulant, we will establish the relationship $c^{i,j} = c^{i-j+1,1}$ where the index $i-j+1$ is taken modulo $n$ in the residue class $\{1,2,\ldots,n\}$. In the above expression for $c^{i,j}$, the index $(i,j)$ only appears in the term $\zeta_n^{(k-1)(i-j)}$. Observe that

$$\zeta_n^{(k-1)((i-j+1)-1)} = \zeta_n^{(k-1)(i-j)}.$$

Hence $c^{i,j} = c^{i-j+1,1}$ so $C^{-1}$ is circulant. □

# B Appendix – Inverse via the Woodbury Matrix Identity

Let $A$ be an $n \times n$ invertible matrix, $C$ an invertible $k \times k$ matrix where $k \leq n$, $U$ an $n \times k$ matrix, and $V$ a $k \times n$ matrix. The Woodbury matrix identity states

$$(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}.$$

A proof of this identity can be found in [HS81, 1.3]. This identity allows us to compute the inverse of $A+UCV$ provided we can easily compute the inverses of $A$ and $C^{-1}+VA^{-1}U$.

In general, we can use the Woodbury Matrix Identity to compute the inverse of $A+N$ where $A$ is an $n \times n$ invertible matrix and $N$ is an $n \times n$ matrix of rank $k$, provided $A+N$ is invertible. Let $U$ be the $n \times k$ matrix whose columns $\vec{v}_1, \ldots, \vec{v}_k$ are the $k$ linearly independent columns of $N$. Let $\vec{u}_i$ denote the vector in the $i$th column of $N$ which can be expressed as

$$\vec{u}_i = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

We define the $i$th column of the $k \times n$ matrix $V$ to consist of entries $c_1, \ldots, c_k$ in that order. Consequently, $N = UCV$ where $C = I_k$ is the $k \times k$ identity matrix.

Using the Woodbury Matrix Identity, we will provide a formula for the inverse in the following situation as encountered in Section 4.3. Let $A$ be a block diagonal matrix with two invertible blocks of sizes $n \times n$ and $m \times m$. The inverse $A^{-1}$ is also block diagonal with blocks of the same size. We write

$$A^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & & & \\ \vdots & \ddots & \vdots & & \mathbf{0} & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & b_{11} & \cdots & b_{1m} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & b_{m1} & \cdots & b_{mm} \end{pmatrix}.$$

We also consider the case

$$U = \begin{pmatrix} 0_1 & 1_1 \\ 0_2 & 1_2 \\ \vdots & \vdots \\ 0_n & 1_n \\ 1_1 & 0_1 \\ \vdots & \vdots \\ 1_m & 0_m \end{pmatrix} \text{ and } V = \begin{pmatrix} 1_1 & 1_2 & \cdots & 1_n & 0_1 & \cdots & 0_m \\ 0_1 & 0_2 & \cdots & 0_n & 1_1 & \cdots & 1_m \end{pmatrix}$$

where the subscripts on the 0 and 1 entries are there to help keep track of their position.

**Proposition B.1.** *Let $T = A + UV$ where $A, U$, and $V$ are the given matrices above. Let $c^{i,j}$ denote the $(i,j)$-entry of $T^{-1}$. We have*

$$c^{ij} = \begin{cases} a_{i,j} + \dfrac{\beta}{1-\alpha\beta}\left(\sum_{k=1}^{n} a_{i,k}\right)\left(\sum_{k=1}^{n} a_{k,j}\right) & \text{if } 1 \le i,j \le n \\ \dfrac{-1}{1-\alpha\beta}\left(\sum_{k=1}^{n} a_{i,k}\right)\left(\sum_{k=1}^{m} b_{k,j}\right) & \text{if } 1 \le i \le n \text{ and } n < j \le n+m \\ \dfrac{-1}{1-\alpha\beta}\left(\sum_{k=1}^{m} b_{i,k}\right)\left(\sum_{k=1}^{n} a_{k,j}\right) & \text{if } 1 \le j \le n \text{ and } n < i \le n+m \\ b_{i,j} + \dfrac{\alpha}{1-\alpha\beta}\left(\sum_{k=1}^{m} b_{i,k}\right)\left(\sum_{k=1}^{m} b_{k,j}\right) & \text{if } n < i,j \le n+m \end{cases}$$

*where $\alpha = \sum a_{i,j}$ is the sum of all entries in the first block in $A^{-1}$ and $\beta = \sum b_{i,j}$ is the sum of all entries in the second block in $A^{-1}$.*

*Proof.* By the Woodbury Matrix Identity,

$$T^{-1} = A^{-1} - A^{-1}U(I_2 + VA^{-1}U)^{-1}VA^{-1}.$$

We first compute

$$\begin{aligned} VA^{-1}U &= V \begin{pmatrix} a_{11} & \cdots & a_{1n} & & & \\ \vdots & \ddots & \vdots & & \mathbf{0} & \\ a_{n1} & \cdots & a_{nn} & & & \\ & & & b_{11} & \cdots & b_{1m} \\ & \mathbf{0} & & \vdots & \ddots & \vdots \\ & & & b_{m1} & \cdots & b_{mm} \end{pmatrix} \begin{pmatrix} 0_1 & 1_1 \\ 0_2 & 1_2 \\ \vdots & \vdots \\ 0_n & 1_n \\ 1_1 & 0_1 \\ \vdots & \vdots \\ 1_m & 0_m \end{pmatrix} \\ &= \begin{pmatrix} 1_1 & 1_2 & \cdots & 1_n & 0_1 & \cdots & 0_m \\ 0_1 & 0_2 & \cdots & 0_n & 1_1 & \cdots & 1_m \end{pmatrix} \begin{pmatrix} 0 & \sum_{i=1}^{n} a_{1i} \\ 0 & \sum_{i=1}^{n} a_{2i} \\ \vdots & \vdots \\ 0 & \sum_{i=1}^{n} a_{n,i} \\ \sum_{i=1}^{m} b_{1,i} & 0 \\ \vdots & \vdots \\ \sum_{i=1}^{m} b_{m,i} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sum_{i,j=1}^{n} a_{ij} \\ \sum_{i,j=1}^{m} b_{ij} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}. \end{aligned}$$

Hence

$$(I_2 + VA^{-1}U)^{-1} = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}^{-1} = \frac{1}{1-\alpha\beta}\begin{pmatrix} 1 & -\alpha \\ -\beta & 1 \end{pmatrix}.$$

Next we will compute

$$\begin{aligned} U(I_2^{-1} + VA^{-1}U)^{-1}V &= \frac{1}{1-\alpha\beta}U\begin{pmatrix} 1 & -\alpha \\ -\beta & 1 \end{pmatrix}\begin{pmatrix} 1_1 & 1_2 & \cdots & 1_n & 0_1 & \cdots & 0_m \\ 0_1 & 0_2 & \cdots & 0_n & 1_1 & \cdots & 1_m \end{pmatrix} \\ &= \frac{1}{1-\alpha\beta}\begin{pmatrix} 0_1 & 1_1 \\ 0_2 & 1_2 \\ \vdots & \vdots \\ 0_n & 1_n \\ 1_1 & 0_1 \\ \vdots & \vdots \\ 1_m & 0_m \end{pmatrix}\begin{pmatrix} 1_1 & 1_2 & \cdots & 1_n & -\alpha_1 & \cdots & -\alpha_m \\ -\beta_1 & -\beta_2 & \cdots & -\beta_n & 1_1 & \cdots & 1_m \end{pmatrix} \\ &= \frac{1}{1-\alpha\beta}\begin{pmatrix} -\beta & -\beta & \cdots & -\beta & 1 & \cdots & 1 \\ -\beta & -\beta & \cdots & -\beta & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta & -\beta & \cdots & -\beta & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 & -\alpha & \cdots & -\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & -\alpha & \cdots & -\alpha \end{pmatrix}. \end{aligned}$$

Lastly, we compute

$$\begin{aligned} &A^{-1}U(I_2^{-1} + VA^{-1}U)^{-1}VA^{-1} \\ &= \frac{1}{1-\alpha\beta}\begin{pmatrix} a_{11} & \cdots & a_{1,n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{m,1} & \cdots & b_{m,m} \end{pmatrix}\begin{pmatrix} -\beta & \cdots & -\beta & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta & \cdots & -\beta & 1 & \cdots & 1 \\ 1 & \cdots & 1 & -\alpha & \cdots & -\alpha \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & -\alpha & \cdots & -\alpha \end{pmatrix}A^{-1} \\ &= \frac{1}{1-\alpha\beta}\begin{pmatrix} -\beta\sum a_{1,\ell} & \cdots & -\beta\sum a_{1,\ell} & \sum a_{1,\ell} & \cdots & \sum a_{1,\ell} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\beta\sum a_{n,\ell} & \cdots & -\beta\sum a_{n,\ell} & \sum a_{n,\ell} & \cdots & \sum a_{n,\ell} \\ \sum b_{1,\ell} & \cdots & \sum b_{1,\ell} & -\alpha\sum b_{1,\ell} & \cdots & -\alpha\sum b_{1,\ell} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum b_{m,\ell} & \cdots & \sum b_{m,\ell} & -\alpha\sum b_{m,\ell} & \cdots & -\alpha\sum b_{m,\ell} \end{pmatrix}A^{-1}. \end{aligned}$$

We now describe the $(i,j)$-entry of the above product in four different cases.

**Case 1** ($1 \leq i, j \leq n$): The $(i,j)$-entry is equal to

$$\frac{1}{1-\alpha\beta}\sum_{k=1}^{n}\left(-\beta\sum_{m=1}^{n}a_{i,m}\right)a_{k,j} = -\frac{\beta}{1-\alpha\beta}\left(\sum_{m=1}^{n}a_{i,m}\right)\left(\sum_{k=1}^{n}a_{k,j}\right) = -\frac{\beta}{1-\alpha\beta}\left(\sum_{k=1}^{n}a_{i,k}\right)\left(\sum_{k=1}^{n}a_{k,j}\right).$$

**Case 2** ($1 \leq i \leq n$ and $n < j \leq n+m$): The $(i,j)$-entry is equal to

$$\frac{1}{1-\alpha\beta}\sum_{k=1}^{m}\left(\sum_{m=1}^{n}a_{i,m}\right)b_{k,j} = \frac{1}{1-\alpha\beta}\left(\sum_{k=1}^{n}a_{i,k}\right)\left(\sum_{k=1}^{m}b_{k,j}\right).$$

**Case 3** ($1 \leq j \leq n$ and $n < i \leq n+m$): The $(i,j)$-entry is equal to

$$\frac{1}{1-\alpha\beta}\sum_{k=1}^{n}\left(\sum_{m=1}^{m}b_{i,m}\right)a_{k,j} = \frac{1}{1-\alpha\beta}\left(\sum_{k=1}^{m}b_{i,k}\right)\left(\sum_{k=1}^{n}a_{k,j}\right).$$

**Case 4** ($n < i, j \leq n+m$): The $(i,j)$-entry is equal to

$$\frac{1}{1-\alpha\beta}\sum_{k=1}^{p^{r-1}}\left(-\alpha\sum_{m=1}^{m}b_{i,m}\right)b_{k,j} = -\frac{\alpha}{1-\alpha\beta}\left(\sum_{k=1}^{m}b_{i,k}\right)\left(\sum_{k=1}^{m}b_{k,j}\right).$$

Subtracting these quantities from corresponding $(i,j)$-entry of $A^{-1}$ gives our desired result. □

# C Appendix – Computation of $d_x$ for the generic points of the special fiber

Throughout this chapter, we will use the following notation:

- $N \geq 3$ and $r \geq 1$ will be integers and $p \geq 2$ will be a prime such that $p \nmid N$
- $\mathfrak{X}(Np^r)$ and $\mathfrak{X}(N)$ denote the modular curves of level $[\Gamma(Np^r)]^{\text{can}}$ (resp. $[\Gamma(N)]^{\text{can}}$) over $R = \mathbb{Z}_p[\zeta_{Np^r}]$.
- $\mathbb{F}_q$ will denote the residue field $\mathbb{F}_p(\zeta_N)$ of $\mathbb{Z}_p[\zeta_{Np^r}]$ where $q = p^\ell$ where $\ell$ is the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^\times$.

Let $x \in \mathfrak{X}(Np^r)$ be a generic point of the special fiber and let

$$\text{pr} : \mathfrak{X}(Np^r) \to \mathfrak{X}(N)_{/R}$$

denote the usual projection map as shown in (19). At the end of Section 5.2, we considered the quantity $d_x$, which was the valuation of the different ideal in $\mathcal{O}_{\mathfrak{X}(Np^r),x}$ from the induced map on stalks

$$\text{pr}_x : \mathcal{O}_{\mathfrak{X}(N),\text{pr}(x)} \to \mathcal{O}_{\mathfrak{X}(Np^r),x}.$$

Equivalently, $d_x$ is the valuation of

$$\text{Ann}_{\mathcal{O}_{\mathfrak{X}(Np^r),x}}(\Omega^1_{\mathcal{O}_{\mathfrak{X}(Np^r),x}/\mathcal{O}_{\mathfrak{X}(N),\text{pr}(x)}})$$

in $\mathcal{O}_{\mathfrak{X}(Np^r),x}$. At the beginning of Section 5.4, we deduced that the map $\text{pr}_x$ fits in a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}(N)_{/\mathbb{F}_q},\text{pr}(x)} & \xrightarrow{F^r} & \mathcal{O}_{\mathfrak{X}(N)^{(p^{-r})}_{/\mathbb{F}_q},\text{pr}(x)} \\ \Big\downarrow{\scriptstyle \text{pr}_x} & & \Big\downarrow{\scriptstyle \rho_x} \\ \mathcal{O}_{\text{ExIg}(p^r,r,N),x} & \xrightarrow{\simeq} & \mathcal{O}_{\text{Ig}(p^r,N),x} \end{array} \tag{26}$$

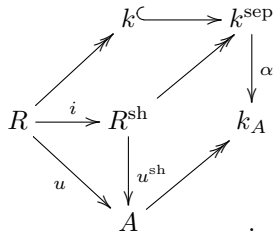
where $\rho : \text{Ig}(p^r, N) \to \mathfrak{X}(N)_{/\mathbb{F}_q}$ is the usual projection map.

Since $x$ is not a supersingular point, by Proposition 2.41(b), $\rho_x$ is finite étale. Furthermore, $x$ is the generic point of $\text{Ig}(p^r, N)$ and $\text{pr}(x)$ is the generic point of $\mathfrak{X}(N)_{/\mathbb{F}_q}$ so $\rho_x$ is a finite separable extension of fields. By passing to the strict Henselization, $\rho_x$ becomes an isomorphism so $\rho'_x$ becomes identified with the $r$th-fold Frobenius on strict Henselizations. In other words, after taking the strict Henselization of $\text{pr}_x$, the induced map on residue fields $\overline{\text{pr}}_x = \rho_x$ can be identified with the $r$th-fold Frobenius. We will develop the necessary results on strict Henselizations to allow us to compute the valuation of $\text{Ann}_{\mathcal{O}_{\mathfrak{X}(Np^r),x}}(\Omega^1_{\text{pr}_x})$, where $\Omega^1_{\text{pr}_x}$ is the module of relative Kahler differentials of $\text{pr}_x$, given this information.

In section 5.4, the quantities $d_x$ show up in the computation of $\deg(\underline{\omega}^{\otimes 2}_{\mathfrak{X}(Np^r)}|_\Lambda)$ where $\Lambda$ is an irreducible component of the special fiber. Since the value of $d_x$ is the same for every generic point $x$ of the special fiber, we did not need to know the exact value of $d_x$. However, if we consider the modular curve of balanced $\Gamma_1(Np^r)$-level structure, as defined in [KM85, §3.3], then the values of $d_x$ are not all the same. Indeed, [Cai18, B.17] provides an explicit description of the projection map restricted to the various irreducible components of the special fiber, which are all not necessarily the same.

**Definition C.1.** Let $R$ be a local ring with residue field $k$. We say $R$ is **Henselian** if it satisfies Hensel's lemma *i.e.* for each monic polynomial $P \in R[T]$, every simple $k$-rational root of the reduction $\bar{P} \in k[T]$ lifts to an $R$-rational root of $P$. Additionally, if $k$ is separably closed, then we say $R$ is **strictly Henselian**. For example, all complete DVRs are Henselian.

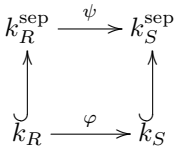
Choose a separable algebraic closure $k^{\text{sep}}$ of $k$. A **strict Henselization** of a local ring $R$ is a strictly Henselian ring $R^{\text{sh}}$ with residue field $k^{\text{sep}}$ along with a local map $i : R \to R^{\text{sh}}$ satisfying the following universal property: for any local map $u : R \to A$, where $A$ is strictly Henselian, and any $k$-map $\alpha : k^{\text{sep}} \to k_A$, where $k_A$ is the residue field of $A$, there exists a unique local map $u^{\text{sh}} : R^{\text{sh}} \to A$ such that the diagram



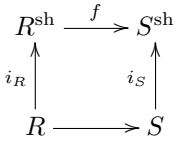

commutes. Note that $u^{\text{sh}}$ induces the map $\alpha$ on residue fields. According to [Sta21, Tag04GP], the strict Henselization of $R$ always exists.

The following lemma, which is [Sta21, Tag04GU], shows the strict Henselization is determined by a map on the separable algebraic closure on residue fields.

**Lemma C.2.** *Let $R \to S$ be a local map of local rings and $\varphi : k_R \to k_S$ the induced map on residue fields. Given a commutative diagram*




*there exists a unique local ring map $f : R^{\text{sh}} \to S^{\text{sh}}$ such that the diagram*

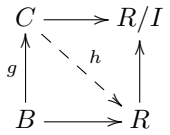



*commutes and $f$ induces $\psi$ on the residue fields.*

While in general the map $R \to R^{\text{sh}}$ fails to be finite type so is not étale, it is formally étale.

**Definition C.3.** A map of $A$-algebras $g : B \to C$ is **formally smooth** if for every commutative diagram

of rings




where $I \subset R$ is an ideal such that $I^2 = 0$, there exists a map of $A$-algebras $h : C \to R$ such that the above diagram remains commutative.

If the map $h : C \to R$ is unique (without necessarily existing), we say $g : B \to C$ is **formally unramified**. If $h$ exists and is unique, then we say $g$ is **formally étale**.

**Proposition C.4.** *Consider ring maps $A \xrightarrow{f} B \xrightarrow{g} C$. We have an exact sequence*

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \to 0$$

*of $C$-modules where $\alpha$ is given by $\alpha(db \otimes c) = c \cdot db$. Moreover, if $g$ is formally smooth, then $\alpha$ is injective.*

*Proof.* Exactness is proven in [Liu02, Prop. 6.1.8(b)] which we will briefly summarize. We will then adapt the proof of [Mat86, Thm. 25.1(2)] to prove the injectivity of $\alpha$ when $g$ is formally smooth.

Let $N$ be a $B$-module. By the universal property of Kahler differentials, we have an isomorphism of $A$-modules

$$\mathrm{Hom}_B(\Omega^1_{B/A}, N) \to \mathrm{Der}_A(B, N)$$

between $B$-module homomorphisms and $A$-derivations, given by $\phi \mapsto \phi \circ d$ where $d : B \to \Omega^1_{B/A}$ is the universal derivation.

Proving exactness amounts to proving exactness of the dual sequence

$$0 \to \mathrm{Hom}_C(\Omega^1_{C/B}, N) \to \mathrm{Hom}_C(\Omega^1_{C/A}, N) \to \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, N).$$

Using the fact that $\mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, N) = \mathrm{Hom}_B(\Omega^1_{B/A}, N)$ and the universal property of Kahler differentials, it suffices to show

$$0 \to \mathrm{Der}_B(C, N) \to \mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N)$$

is exact. The first map in the above sequence does not change the derivation while the second map is given by $D \mapsto D \circ g$ which can be readily seen to be exact.

Suppose now $g$ is formally smooth. We will show $\alpha$ is injective by showing $\mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N)$ is surjective. Define the ring $C * N$ to be the abelian group $C \oplus N$ endowed with multiplication given by

$$(c, n) \cdot (c', n') = (cc', cn' + c'n).$$

Note that the subring $\{(0, n) : n \in N\}$ of $C*N$ is isomorphic to $N$, and $N^2 = 0$. Furthermore, $(C*N)/N \simeq C$.

Let $D \in \mathrm{Der}_A(B, N)$. Consider the following commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{id}_C} & C \\ {\scriptstyle g}\uparrow & & \uparrow \\ B & \xrightarrow{\varphi} & C * N \end{array}$$

where $\varphi(b) = (g(b), D(b))$. Since $g$ is formally smooth, there exists a map $h : C \to C * N$ such that the above diagram remains commutative. For any $c \in C$, we write $h(c) = (c, D'(c))$. Since $h \circ g = \varphi$, we have $D = D' \circ g$. It remains to show $D' \in \mathrm{Der}_A(C, N)$; in particular we must show that $D'$ is $A$-linear map satisfying the Leibniz rule.

Since $h$ is already $A$-linear, we have $D'$ is $A$-linear. Let $c_1, c_2 \in C$. We compute

$$(c_1c_2, D'(c_1c_2)) = h(c_1c_2) = h(c_1)h(c_2)$$

$$= (c_1, D'(c_1)) \cdot (c_2, D'(c_2)) = (c_1c_2, c_1D'(c_2) + c_2D'(c_1)).$$

Hence $D'(c_1c_2) = c_1D'(c_2) + c_2D'(c_1)$ as desired. □

Now we will list some properties of strict Henselizations.

**Proposition C.5.** *Let $A$ be a local ring.*

*a. The map $A \to A^{\mathrm{sh}}$ is formally étale and faithfully flat.*

*b. $A$ is a DVR if and only if $A^{\mathrm{sh}}$ is a DVR.*

*Proof.* Statement (a) is [Sta21, Tag07QM, Tag07QN] and (b) is [Sta21, Tag0AP3]. □

We will now show that taking strict Henselizations, as well as completions, does not change the valuation of the different.

**Lemma C.6.** *Let $A \to B$ be a finite local map of DVRs. The induced maps $\hat{A} \to \hat{B}$ on completions and $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$ on strict Henselizations are finite.*

*Proof.* Since $A$ is noetherian and $A \to B$ is finite, by [Sta21, Tag00MA], we have $\hat{B} \simeq \hat{A} \otimes_A B$. Thus tensoring $A \to B$ by $\hat{A}$ , yields a finite map

$$\hat{A} = A \otimes_A \hat{A} \to B \otimes_A \hat{A} \simeq \hat{B}$$

so $\hat{B}$ is finite over $\hat{A}$. Since $A \to B$ is finite, the map on residue fields $k_A \to k_B$ is also finite, hence $k_A^{\mathrm{sep}} \simeq k_B^{\mathrm{sep}}$. Tensoring $A \to B$ by $A^{\mathrm{sh}}$ yields a finite map $A^{\mathrm{sh}} \to B \otimes_A A^{\mathrm{sh}}$. By [FK88, Corollary 3.2], $B \otimes_A A^{\mathrm{sh}} \simeq B^{\mathrm{sh}}$ hence $B^{\mathrm{sh}}$ is finite over $A^{\mathrm{sh}}$. □

**Proposition C.7.** *Let $A \to B$ be a finite local map of DVRs. The valuation of the different ideals $\mathfrak{D}_{B/A}$, $\mathfrak{D}_{B^{\mathrm{sh}}/A^{\mathrm{sh}}}$, and $\mathfrak{D}_{\hat{B}/\hat{A}}$ in their respective rings are equal.*

*Proof.* By Proposition 5.8(b), we have $\mathfrak{D}_{B/A} = \mathrm{Ann}_B(\Omega^1_{B/A})$ and by Lemma C.6, we can make the same identification for $\hat{A} \to \hat{B}$ and $A^{\mathrm{sh}} \to B^{\mathrm{sh}}$. Using Proposition C.4, the commutative diagram

$$\begin{array}{ccc} A^{\mathrm{sh}} & \longrightarrow & B^{\mathrm{sh}} \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

gives us two exact sequences

$$\Omega^1_{A^{\mathrm{sh}}/A} \otimes_{A^{\mathrm{sh}}} B^{\mathrm{sh}} \to \Omega^1_{B^{\mathrm{sh}}/A} \to \Omega^1_{B^{\mathrm{sh}}/A^{\mathrm{sh}}} \to 0$$

and

$$0 \to \Omega^1_{B/A} \otimes_B B^{\mathrm{sh}} \to \Omega^1_{B^{\mathrm{sh}}/A} \to \Omega^1_{B^{\mathrm{sh}}/B} \to 0.$$

Note that $B \to B^{\mathrm{sh}}$ is formally étale. By [[Sta21], Tag00UO], the Kahler differentials of any formally unramified map of rings is trivial hence $\Omega^1_{A^{\mathrm{sh}}/A} = \Omega^1_{B^{\mathrm{sh}}/B} = 0$. Thus from our two exact sequences we conclude

$$\Omega^1_{B/A} \otimes_B B^{\mathrm{sh}} \simeq \Omega^1_{B^{\mathrm{sh}}/A} \simeq \Omega^1_{B^{\mathrm{sh}}/A^{\mathrm{sh}}}.$$

As $B \to B^{\mathrm{sh}}$ is flat and $\Omega^1_{B/A}$ is a finitely generated $B$-module, we can use Lemma 3.5 to conclude

$$\mathfrak{D}_{B^{\mathrm{sh}}/A^{\mathrm{sh}}} = \mathrm{Ann}_{B^{\mathrm{sh}}}(\Omega^1_{B^{\mathrm{sh}}/A^{\mathrm{sh}}}) = \mathrm{Ann}_{B^{\mathrm{sh}}}(\Omega^1_{B/A} \otimes_B B^{\mathrm{sh}})$$

$$= \mathrm{Ann}_B(\Omega^1_{B/A}) B^{\mathrm{sh}} = \mathfrak{D}_{B/A} B^{\mathrm{sh}}$$

which shows $\mathfrak{D}_{B/A}$ and $\mathfrak{D}_{B^{\mathrm{sh}}/A^{\mathrm{sh}}}$ have the same valuation.

For completions, since $\hat{B} \simeq B \otimes_A \hat{A}$ and Kahler differentials are compatible with base change, we have

$$\Omega^1_{\hat{B}/\hat{A}} \simeq \Omega^1_{(B \otimes_A \hat{A})/\hat{A}} \simeq \Omega^1_{B/A} \otimes_B (B \otimes_A \hat{A}) \simeq \Omega^1_{B/A} \otimes_B \hat{B}.$$

As $B \to \hat{B}$ is flat, we can again apply Lemma 3.5 to conclude $\mathfrak{D}_{\hat{B}/\hat{A}} = \mathfrak{D}_{B/A}\hat{B}$. Hence $\mathfrak{D}_{B/A}$ and $\mathfrak{D}_{\hat{B}/\hat{A}}$ have the same valuation. □

Thus taking strict Henselizations and completions does not affect the valuation of the different ideal. Taking the strict Henselization of the map $\mathrm{pr}_x$, we get a map $\mathrm{pr}_x^{\mathrm{sh}} : \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(Np^r),x}$ whose induced map on residue fields $\overline{\mathrm{pr}}_x^{\mathrm{sh}}$ fits in the diagram

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(N)_{/\mathbb{F}_q},\mathrm{pr}(x)} & \xrightarrow{F^r} & \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(N)^{(p^{-r})}_{/\mathbb{F}_q},\mathrm{pr}(x)} \\ \downarrow{\scriptstyle \overline{\mathrm{pr}}_x^{\mathrm{sh}}} & & \downarrow{\scriptstyle \simeq} \\ \mathcal{O}^{\mathrm{sh}}_{\mathrm{ExIg}(p^r,r,N),x} & \xrightarrow{\simeq} & \mathcal{O}^{\mathrm{sh}}_{\mathrm{Ig}(p^r,N),x} \end{array}$$

coming from (26). However, we still do not know explicitly what the map $\mathrm{pr}_x^{\mathrm{sh}}$ does; we have only established that the map on residue fields is the $r$th-fold Frobenius. To remedy this, we will instead define an explicit map on completions $g : \hat{\mathcal{O}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \hat{\mathcal{O}}_{\mathfrak{X}(Np^r),x}$ such that the valuation of $\mathfrak{D}_g$ agrees with $d_x$. First we justify our ability to do so.

**Lemma C.8.** *Let $R$ be a noetherian local ring. Consider the diagram*

$$\begin{array}{ccccc} \hat{R} & \xrightarrow{\hat{\iota}} & \hat{R}^{\mathrm{sh}} & \xrightarrow{\ell} & (\hat{R}^{\mathrm{sh}})^{\wedge} \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j^{\mathrm{sh}}} & & \uparrow{\scriptstyle j^{\mathrm{sh},\wedge}} \\ R & \xrightarrow{\iota} & R^{\mathrm{sh}} & \xrightarrow{k} & (R^{\mathrm{sh}})^{\wedge} \end{array}$$

*where $\iota$ is the strict Henselization map, and $j, k, \ell$ are the usual embeddings into the completion. Then the map $j^{\mathrm{sh},\wedge} : (R^{\mathrm{sh}})^{\wedge} \to (\hat{R}^{\mathrm{sh}})^{\wedge}$ is an isomorphism.*

*Proof.* This is [Sta21, Tag06LJ]. □

**Theorem C.9.** *Let $f : A \to B$ and $g : \hat{A} \to \hat{B}$ be local ring maps of DVRs. Suppose the following:*

1. *$f$ and $g$ are finite, and*
2. *the induced maps $\bar{f}^{\mathrm{sh}}, \bar{g}^{\mathrm{sh}} : k_A^{\mathrm{sep}} \to k_B^{\mathrm{sep}}$ on the separable closures of residue fields agree.*

*Then the valuation of $\mathfrak{D}_f$ in $B$ agrees with the valuation of $\mathfrak{D}_g$ in $\hat{B}$.*

*Proof.* Consider the canonical maps

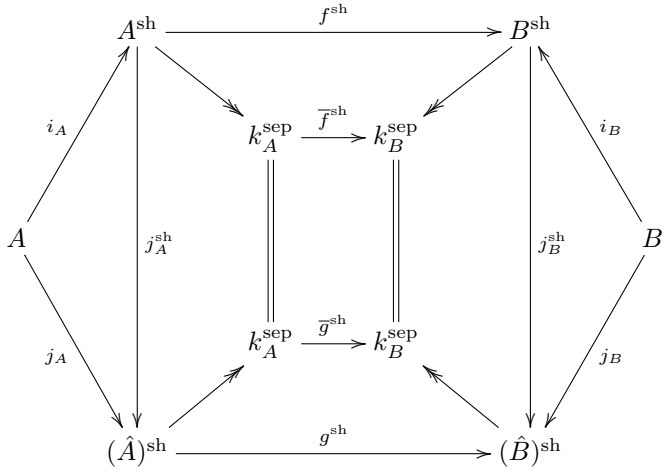
$$j_A : A \to \hat{A} \to (\hat{A})^{\mathrm{sh}} \text{ and } j_B : B \to \hat{B} \to (\hat{B})^{\mathrm{sh}}.$$

Let $k_A$ (resp. $k_B$) denote the residue field of $A$ (resp. $B$). By the universal property of strict Henselization, we have a commutative diagram

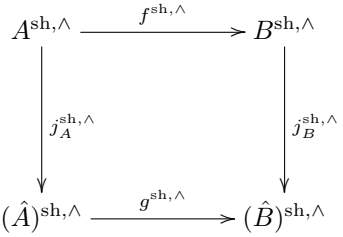
$$\begin{array}{ccccc} A & \xrightarrow{i_A} & A^{\mathrm{sh}} & \twoheadrightarrow & k_A^{\mathrm{sep}} \\ & \searrow^{j_A} & \downarrow{\scriptstyle j_A^{\mathrm{sh}}} & & \| \\ & & (\hat{A})^{\mathrm{sh}} & \twoheadrightarrow & k_A^{\mathrm{sep}} \end{array}$$

and a similar one for $B$. Combining these diagrams along with those coming from $f^{\mathrm{sh}}$ and $g^{\mathrm{sh}}$, we obtain

the following commutative diagram




Consider the compositions $j_B^{\mathrm{sh}} \circ f^{\mathrm{sh}}, g^{\mathrm{sh}} \circ j_A^{\mathrm{sh}} : A^{\mathrm{sh}} \to (\hat{B})^{\mathrm{sh}}$. Since these induce the same map on residue fields, we must have $j_B^{\mathrm{sh}} \circ f^{\mathrm{sh}} = g^{\mathrm{sh}} \circ j_A^{\mathrm{sh}}$ by Lemma C.2. Taking completions, we obtain a commutative diagram




where $j_A^{\mathrm{sh},\wedge}$ and $j_B^{\mathrm{sh},\wedge}$ are isomorphisms by Lemma C.8. Hence the valuation of $\mathfrak{D}_{f^{\mathrm{sh},\wedge}}$ and $\mathfrak{D}_{g^{\mathrm{sh},\wedge}}$ agree in their respective rings. Consequently, by Proposition C.7, the valuations of $\mathfrak{D}_f$ and $\mathfrak{D}_g$ agree in their respective rings. □

## C.1 Constructing $g$

In light of Theorem C.9, our goal is to construct an explicit map

$$g : \hat{\mathcal{O}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \hat{\mathcal{O}}_{\mathfrak{X}(Np^r),x}$$

which agrees with $\overline{\mathrm{pr}}_x^{\mathrm{sh}}$ on residue fields and whose different is computable. We will be able to explicitly describe the completed DVRs via Cohen's structure theorem.

**Definition C.10.** Let $A$ be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field $k$. Suppose $A$ has mixed characteristic with $\mathrm{char}(k) = p > 0$. A subring $R \subset A$ is a **coefficient ring** of $A$ if $R$ is a complete Noetherian local ring with maximal ideal $pR$ and residue field $k$.

The following, which is Theorem 29.3 and the Corollary proceeding Theorem 29.2 in [Mat86], guarantees

the existence and uniqueness of a coefficient ring.

**Proposition C.11.** *Any complete local ring $(A, \mathfrak{m}, k)$ with* $\mathrm{char} k > 0$ *has a coefficient ring $R$. If $A$ has characteristic 0, then $R$ is a complete DVR and is unique up to isomorphism.*

**Example C.12.** Let $q = p^\ell$ where $\ell$ is the order of $p$ in $(\mathbb{Z}/N\mathbb{Z})^\times$. Consider the field $\mathbb{F}_q(t)$ which is a transcendental extension of $\mathbb{F}_q$. Consider the complete local ring

$$R = \left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge$$

where the completion is taken with respect to the unique maximal ideal of the localization $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}$ generated by $p$. Then the residue field of $R$ is

$$\begin{aligned} R/pR &\simeq \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}/p\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \\ &\simeq (\mathbb{Z}_p[\zeta_N, t]/p\mathbb{Z}_p[\zeta_N, t])_{p\mathbb{Z}_p[\zeta_N,t]} \\ &\simeq \mathbb{F}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \\ &\simeq \mathbb{F}_q(t). \end{aligned}$$

Note that $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}$ is a DVR since it is a Noetherian local ring whose maximal ideal is principal. Hence $\left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge$ is the coefficient ring of any complete local ring $A$ of mixed characteristic $(0, p)$ and residue field $\mathbb{F}_q(t)$.

The residue field of $\mathcal{O}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ (resp. $\mathcal{O}_{\mathfrak{X}(Np^r),x}$) is the function field of $\mathfrak{X}(N)_{/\mathbb{F}_q}$ (resp. $\mathrm{ExIg}(p^r, r, N)$) where $q = p^\ell$. Both $\mathfrak{X}(N)_{/\mathbb{F}_q}$ and $\mathrm{ExIg}(p^r, r, N)$ are smooth curves over $\mathbb{F}_q$ hence are geometrically reduced. By [Liu02, Prop. 3.2.15] the function fields of $\mathfrak{X}(N)_{/\mathbb{F}_q}$ and $\mathrm{ExIg}(p^r, r, N)$ are finite separable extensions of $\mathbb{F}_q(t)$. The next lemma establishes the coefficient ring in this situation. Note that any finite separable extension $L/\mathbb{F}_q(t)$ is of the form $L = \mathbb{F}_q(t)(\alpha)$ for some $\alpha \in L$ by the primitive element theorem.

**Lemma C.13.** *Let $\mathbb{F}_q(t)(\alpha)$ be a finite separable extension of $\mathbb{F}_q(t)$ and let $h(X) \in \mathbb{F}_q(t)[X]$ be the minimal polynomial of $\alpha$. Consider the ring*

$$R = \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}[X]/(H(X))$$

*where $H(X) \in \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}[X]$ is a lift of $h(X)$. Then $R$ is a DVR whose maximal ideal is generated by $p$ with residue field $\mathbb{F}_q(t)(\alpha)$.*

*Proof.* First we show $H(X)$ is irreducible over $S := \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}$. Suppose $H(X)$ is reducible over $\mathrm{Frac}(S)$. By Gauss' Lemma, $H(X)$ is reducible over $S$ hence $h(X)$ is reducible over $S/pS = \mathbb{F}_q(t)$, a contradiction. Thus $H(X)$ must be irreducible over $\mathrm{Frac}(S)$ so is irreducible over $S$. Consequently, $R = S[X]/(H(X))$ is an integral domain.

Let $\mathfrak{m}$ be a maximal ideal of $R$. Note that $S$ is a DVR with maximal ideal $pS$ so we necessarily have

$\mathfrak{m} \cap S = pS$. The kernel of the surjective map

$$R \to \mathbb{F}_q(t)[X]/(h(x))$$

obtained by reduction mod $p$ is equal to $pR$. Hence

$$R/pR \simeq \mathbb{F}_q(t)[X]/(h(x)) \simeq \mathbb{F}_q(t)(\alpha)$$

is a field so $pR$ is maximal and $\mathfrak{m} = pR$. Hence $R$ has a unique maximal ideal generated by $p$ and consequently is a DVR. □

We will now show that upon taking strict Henselizations, the map $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \to R$ becomes an isomorphism. Hence we do not need to worry about knowing what $H(X)$ is.

**Corollary C.14.** *Let $R = \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}[X]/(H(X))$ be as in Lemma C.13. The map*

$$\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \to \mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}[X]/(H(X)) = R$$

*is an étale map of rings.*

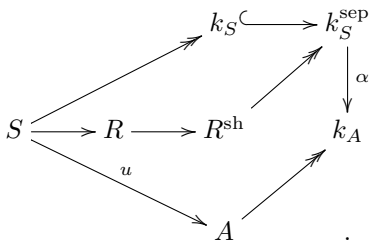
*Proof.* Both $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}$ and $R$ have $p$ as a uniformizer hence our map is unramified. It remains to show our map is flat. Both our rings are DVRs so in particular, $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}$ is regular and $R$ is Cohen-Macaulay. By miracle flatness *i.e.* [Sta21, Tag00R4], to show $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \to R$ is flat it suffices to show

$$\dim(R) = \dim(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}) + \dim(R/pR). \tag{27}$$

Indeed, the map $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \to R$ is integral so both rings have the same dimension. Since $R/pR$ is a field, $\dim(R/pR) = 0$ so (27) holds. Thus $\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]} \to R$ is flat. □

**Lemma C.15.** *Let $S \to R$ be an étale extension of DVRs. Then the induced map $S^{\mathrm{sh}} \to R^{\mathrm{sh}}$ is an isomorphism.*

*Proof.* We will show $R^{\mathrm{sh}}$ satisfies the same universal property for the strict Henselization of $S$. We already have a map $S \to R \to R^{\mathrm{sh}}$ and $k_S^{\mathrm{sep}} = k_R^{\mathrm{sep}}$. Consider the diagram

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where $A$ is a strictly Henselian ring. Since $S \to R$ is étale, by [Sta21, Tag04GT], we have a unique map $R \to S^{\mathrm{sh}}$ fitting in the commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & S^{\mathrm{sh}} \\ \uparrow & & \uparrow \\ S & \xrightarrow{\mathrm{id}} & S \end{array}$$

By the universal property of strict Henselizations, we have a unique map $S^{\mathrm{sh}} \to A$ hence we obtain a map $R \to S^{\mathrm{sh}} \to A$. Again, the universal property of $R^{\mathrm{sh}}$ gives us a map $R^{\mathrm{sh}} \to A$ extending $R \to A$ which induces $\alpha$ on residue fields. Hence $R^{\mathrm{sh}}$ satisfies the universal property for $S^{\mathrm{sh}}$. □

The next proposition will provide an explicit description of the completed stalks $\hat{\mathcal{O}}_{\mathfrak{X}(N),\mathrm{pr}(x)}$ and $\hat{\mathcal{O}}_{\mathfrak{X}(Np^r),x}$.

**Proposition C.16.** *Let $A$ be a complete DVR whose residue field is a finite separable extension of $\mathbb{F}_q(t)$. Let $\mathfrak{m}$ be the maximal ideal of $A$ and suppose $p \in \mathfrak{m}^2$. Then $A = \hat{R}[\pi_A]$ where $R$ is as in Lemma C.13 and $\pi_A$ is a uniformizer of $A$.*

*Proof.* This is essentially [Mat86, Thm. 29.8(ii)]; we provide the proof so that it matches our situation. By Lemma C.13, $A$ has $R$ as a coefficient ring; taking completions we get $\hat{R} \subset A$. We will show $A$ is finite over $\hat{R}$.

First note

$$A/pA = \bigoplus_{i=0}^{v} \pi_A^i \cdot A/\mathfrak{m}$$

where $v$ is the $\pi_A$-adic valuation of $p$ and $A/\mathfrak{m} \simeq \hat{R}/p\hat{R}$. Hence $A/pA$ is finite over $\hat{R}/p\hat{R}$. Since $pA \subset \mathfrak{m}$ and $A$ is $\mathfrak{m}$-adically separated, we have $A$ is also $p$-adically separated. Combined with the fact that $\hat{R}$ is $p$-adically complete, we can use [Mat86, Thm. 8.4] to conclude $A$ is finite over $\hat{R}$.

Since

$$A/pA = \bigoplus_{i=0}^{v} \pi_A^i \cdot \hat{R}/p\hat{R},$$

we have $A = pA + \hat{R}[\pi_A]$. By Nakayama's lemma, we conclude $A = \hat{R}[\pi_A]$.

□

We will now construct an explicit map

$$\hat{\mathcal{O}}_{\mathfrak{X}(N),x} \to \hat{\mathcal{O}}_{\mathfrak{X}(Np^r),\mathrm{pr}(x)}$$

which is a lift of the $r$th-fold Frobenius on the residue fields.

We will apply Theorem C.9 to compute the valuation of the different for the map

$$\mathrm{pr}_x^{\mathrm{sh}} : \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}^{\mathrm{sh}}_{\mathfrak{X}(Np^r),x}$$

by constructing a map on the completions

$$g : \mathcal{O}^{\mathrm{sh},\wedge}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \mathcal{O}^{\mathrm{sh},\wedge}_{\mathfrak{X}(Np^r),x}.$$

Write $\mathbb{F}_q(t)(\alpha)$ (resp. $\mathbb{F}_q(t)(\beta)$) for the residue field of $\mathcal{O}_{\mathfrak{X}(N),x}$ (resp. $\mathcal{O}_{\mathfrak{X}(Np^r),\mathrm{pr}(x)}$) where $\alpha$ and $\beta$ are the appropriate generators. By Proposition C.16, we can identify

$$\hat{\mathcal{O}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \simeq \left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}[\alpha]\right)^\wedge[\pi_1] \text{ and } \hat{\mathcal{O}}_{\mathfrak{X}(Np^r),x} \simeq \left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}[\beta]\right)^\wedge[\pi_2]$$

where $\pi_1$ (resp. $\pi_2$) is a uniformizer of $\mathcal{O}_{\mathfrak{X}(N),x}$ (resp. $\mathcal{O}_{\mathfrak{X}(Np^r),\mathrm{pr}(x)}$).

By Corollary C.14, the map

$$\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]} \to \mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}[\alpha]$$

is étale, so the induced map

$$\left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_1] \to \left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}[\alpha]\right)^\wedge[\pi_1] \simeq \hat{\mathcal{O}}_{\mathfrak{X}(N),\mathrm{pr}(x)}$$

is étale. By Lemma C.15, this map becomes an isomorphism upon taking strict Henselizations. We will consider an explicit map

$$h : \left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_1] \to \left(\mathbb{Z}_p[\zeta_N,t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_2].$$

Note that the induced map on strict Henselizations takes on the form

$$h^{\mathrm{sh}} : \hat{\mathcal{O}}^{\mathrm{sh}}_{\mathfrak{X}(N),\mathrm{pr}(x)} \to \hat{\mathcal{O}}^{\mathrm{sh}}_{\mathfrak{X}(Np^r),x}.$$

The desired map $g$ is then obtained from the induced map on completions $h^{\mathrm{sh},\wedge}$ after applying Lemma C.8:

$$\begin{array}{ccc} \hat{\mathcal{O}}^{\mathrm{sh},\wedge}_{\mathfrak{X}(N),\mathrm{pr}(x)} & \xrightarrow{h^{\mathrm{sh},\wedge}} & \hat{\mathcal{O}}^{\mathrm{sh},\wedge}_{\mathfrak{X}(Np^r),x} \\ \simeq \uparrow & & \uparrow \simeq \\ \mathcal{O}^{\mathrm{sh},\wedge}_{\mathfrak{X}(N),\mathrm{pr}(x)} & \xrightarrow{g} & \mathcal{O}^{\mathrm{sh},\wedge}_{\mathfrak{X}(Np^r),x} \end{array}$$

By Proposition C.7, the valuation of $\mathfrak{D}_g$, which is the same as the valuation of $\mathfrak{D}_{h^{\mathrm{sh},\wedge}}$, agrees with the valuation of $\mathfrak{D}_h$.

We will now define $h$. Note for any $u \in \mathbb{Z}_p$, we have $u - u^{p^r} \in p\mathbb{Z}_p$. Hence the identity $\mathrm{id} : \mathbb{Z}_p \to \mathbb{Z}_p$ induces the $r$th-fold Frobenius on the residue fields $\mathbb{F}_p \to \mathbb{F}_p$. We extend this to a map $\mathbb{Z}_p[\zeta_N,t] \to \mathbb{Z}_p[\zeta_N,t]$ by sending $\zeta_N \mapsto \zeta_N^{p^r}$ and $t \mapsto t^{p^r}$ which induces a map on localizations $\mathbb{Z}_p[\zeta_N,t]_p \to \mathbb{Z}_p[\zeta_N,t]_p$ and then on completions. Finally, we send $\pi_1 \mapsto \pi_2$.

**Theorem C.17.** *Let $d_x$ denote the valuation of the different $\mathfrak{D}_{\mathrm{pr}_x}$ when $x$ is a generic point of the special fiber of $\mathfrak{X}(Np^r)$. We have*

$$d_x = r \cdot \nu_{\mathcal{O}_{\mathfrak{X}(Np^r),x}}(p) = r\varphi(p^r).$$

*Proof.* Recall

$$\mathrm{pr} : \mathfrak{X}(Np^r) \to \mathfrak{X}(N)_{/\mathbb{Z}_p[\zeta_{Np^r}]}$$

is a map of $\mathbb{Z}_p[\zeta_{Np^r}]$-schemes. By Theorem C.9, it suffices to compute the valuation of the different ideal $\mathfrak{D}_g$, or equivalently of $\mathfrak{D}_h$, where $g$ and $h$ are defined above. Since $\mathfrak{D}_h = \mathrm{Ann}(\Omega^1_h)$, we will find a generator for $\Omega^1_h$.

Note that $\left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_2]$ is generated by $1, t, \ldots, t^{p^r-1}$ over $\left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_1]$ hence $\Omega^1_h$ is generated by $dt$ as an $\left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_2]$-module. Note that

$$p^r t^{p^r-1} dt = d(t^{p^r}) = 0.$$

Since $t$ is invertible in $\left(\mathbb{Z}_p[\zeta_N, t]_{p\mathbb{Z}_p[\zeta_N,t]}\right)^\wedge[\pi_2]$, we conclude $p^r$ generates $\mathrm{Ann}(\Omega^1_h) = p^r$. Hence $d_x$ is equal to the valuation of $p^r$ inside $\mathcal{O}_{\mathfrak{X}(Np^r),x}$ which is precisely $r\varphi(p^r)$. □

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