Modeling Orbital Motion in a Circular Conic Reference Frame

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Two-body orbital trajectories conform to conic sections. However, typically in the literature their motion is analyzed in a plane. Kepler modeled elliptic orbital motion in a plane, stated his second law, and derived the geometric position-time relationship as the uniform change in area with respect to time. Kepler's equation has been applied extensively and proven to give time as a function of position for exact solutions to orbital problems. An identical equation has been derived without reliance on geometry alone by applying basic principle of classical mechanics and the calculus. When the elliptic orbit is analyzed as a section of a circular cone and represented in three dimensions, additional variables relate position and time. In a conical reference frame, the planar and conic representations merge. This paper combines the conic section knowledge and characteristics of the cone to introduce a third dimension to modeling orbital motion. Force and potential energy have indeterminate limits because potential energy approaches zero as the radius approaches infinity and the gravitational force approaches infinity as the radius vector approaches zero, a singularity. The conic frame includes the apex where the singularity of a radius of zero is an established point in the reference system.

I. Nomenclature

F = force field center position
S = satellite position
a = semi-major axis
e = eccentricity
h = specific angular momentum
p = semi-latus rectum, semi-parameter
r = radius, radial distance
t = time
ε = derived elliptic orbit eccentric anomaly
ξ = specific mechanical energy
μ = gravitational constant
ρ = cone slant height
ϕ = spherical and conic frame reference angle
θ = true anomaly, spherical and conic frame reference angle

Subscripts

K = kinetic
$\perp$ = perpendicular

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II. Background

Prior work derived time versus position equations by an energy approach that reproduced Kepler's equation for elliptic orbits and an equation for hyperbolic orbits in trigonometric terms that equals that in the literature. Figure 1 shows the forces on a satellite in a two-body orbit, and the opposing centrifugal and gravitational forces are continually aligned with the radius vector. Their vector difference is equal to radial acceleration. A point mass of unity is assumed.[1]

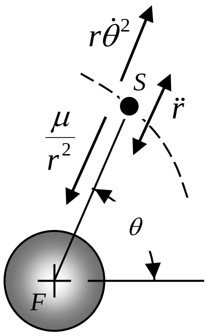

Fig. 1 Forces on a Satellite

The integration approach presented in Ref. [1] is summarized to give the derivation steps consistent with classical mechanics. Figure 1 shows that the gravity force vector and the centrifugal force vector are aligned with the radius vector, and the vector sum equals acceleration. Through a sequence of steps adhering to classical mechanics and the calculus, Kepler's equation is written for elliptic orbits and other orbital trajectories.

1) Sum forces $\ddot{\boldsymbol{r}} = \overline{r\dot{\theta}^2} - \overline{\mu/r^2}$

2) Dot-multiply by $\dot{\boldsymbol{r}}$ $\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = \overline{\left(\frac{\mu p}{r^3}\right)}\cdot\dot{\boldsymbol{r}} - \overline{\left(\frac{\mu p}{r^2}\right)}\cdot\dot{\boldsymbol{r}}$

3) Integrate with respect to time to get $\frac{\dot{r}^2}{2} = \frac{\mu}{r} - \frac{\mu p}{2r^2} - \frac{\mu}{2a}$, the energy equation. (1)

4) Rearrange $\dot{r}^2 = \frac{\mu}{a}\left(\frac{2ar - ap - r^2}{r^2}\right)$

5) Separate variables $\sqrt{\frac{\mu}{a}}\int dt = \int \frac{r\,dr}{\sqrt{2ar - ap - r^2}}$

6) Integrate and adjust angle $\sqrt{\frac{\mu}{a^3}}\,t = \varepsilon - e\sin\varepsilon$

Where ε is equal to E in Kepler's equation. The symbol ε identifies the method of derivation. When the eccentricity is equal to zero, the orbit is circular. When eccentricity is one, the trajectory is rectilinear, and if the total energy is unchanged, the linear elliptic trajectory represents a limiting case for orbits with this value of total specific energy.[1]

7) When $e = 1$ and $\varepsilon = \pi/2$ $\sqrt{\frac{\mu}{a^3}}\,t = \frac{\pi}{2} - 1$

Transcendental equations of motion are not simplified by this method. However:

- The integration approach is verified by compliance with classical mechanics and with Kepler's equation.
- Following the same approach with $e = 1$ produces the special linear case for elliptic and hyperbolic orbits.

III. Reference Frame

Simplification of a derivation of equations for solving the Kepler problem is the objective of this paper. Principles of orbital mechanics, nomenclature, and orbital position as a function of time are referenced in Curtis [2]. Ref. [3] is the source for mathematical formulas.

Polar coordinates are typically applied to analyze two-body orbital motion. The motion is planar, and representation by the radial distance from the force field center and the angle from an identified point in an orbit is ideal. All trajectories trace one of the conic sections depending on the eccentricity and total specific mechanical energy of the orbit. In this paper r represents the radial distance and θ represents angular displacement from periapsis, the true anomaly.[1]

Equations of motion are fully described in polar coordinates. A conic reference frame selected as a special case of a spherical reference frame provides another representation of orbits and includes the mathematical relationships for conic sections. Figure 2 shows a spherical frame with its origin coinciding with the origin of a cartesian reference frame. The common nomenclature has θ representing angular rotation in the x-y plane measured from the x-axis and ϕ representing angular deflection from the z-axis. Distance from the origin is denoted by ρ. A cone having its apex at the origin is shown with a fixed value of ϕ and the other two parameters varying. The plane shown has ϕ fixed, and the sphere has ρ fixed. The nomenclature is as given in Ref. [4] apart from the primes on θ and ϕ. These symbols are applied without primes in later applications of those angles.

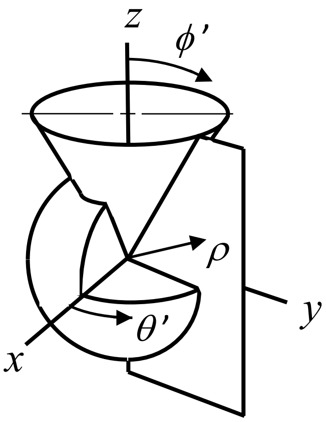

Fig. 2 Spherical Coordinate System

Figure 3 shows a general plane in a spherical reference frame. At a given value of rho, ρ_0, the plane described varies θ and ϕ such that the variable r remains perpendicular to ρ_0. Then construct a cone as shown in Figure 1 with the plane in Figure 2 intersecting it to form the conic section. Change nomenclature to match the polar coordinates for orbits with θ being angular deflection from a reference point. The Greek letter ϕ is the angular rotation from ρ_0.

Let the radial vector $\boldsymbol{\rho}_0$ define the center of a central force field. Then generate a plane by extending a perpendicular vector $\boldsymbol{r}$ with θ varying. The construction defines an elliptic orbit described by $\boldsymbol{r}$ and θ when they are equated by the conic section equation:

$$r = \frac{p}{1 + e\cos\theta}$$

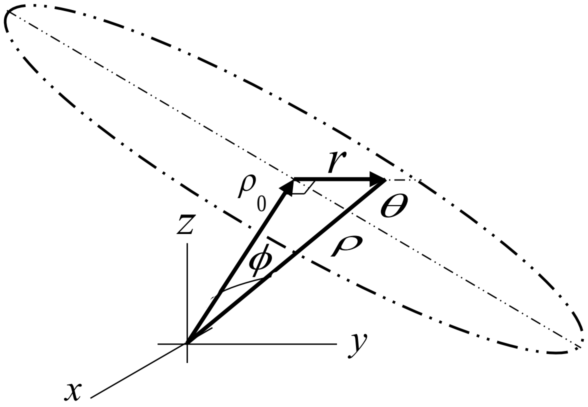

Fig. 3 Construction of a Plane in Polar Coordinates

IV. Elliptic Orbits

Consider a right circular cone and the elliptic orbit formed by a plane that cuts that cone an oblique angle. Then extend the angular momentum vector to pass through the apex of the cone and the primary focus of the ellipse perpendicular to the plane. The orbit's angular momentum vector coincides with this line. In Figure 4 the orbital area swept by the radius vector is shown as the shaded area. This area progresses uniformly with time as stated in Kepler's second law.

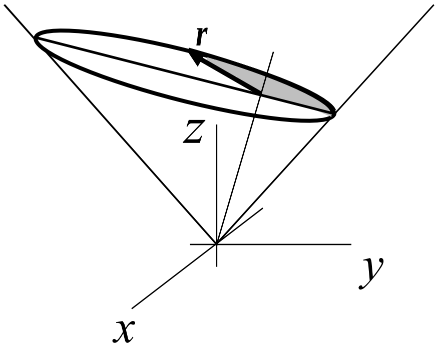

Fig. 4 Elliptic Orbit Shown as a Conic Section

An edge view shows the geometric construction of the cone and plane in Figure 5 with *F* as the center of the force field.

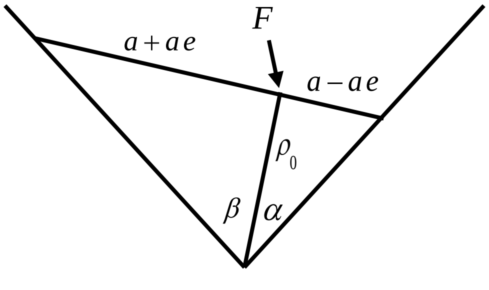

Fig. 5 A Linear Elliptic Trajectory is on the Slant Height of the Cone

Relating the right triangles in Figure 5 gives a preferred value for the apex angle.

Describe the angles α and β with the cotangent formula: $$\cot\alpha = \frac{\rho_0}{a(1-e)}$$

and $$\cot\beta = \frac{\rho_0}{a\left(1+e\right)}$$

The cotangent formula for the sum of α and b is[3] $$\cot\left(\alpha+\beta\right) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

Substitute: $$\cot\left(\alpha+\beta\right) = \frac{\left(\rho_0 \big/ a\left(1-e\right)\right)\left(\rho_0 \big/ a\left(1+e\right)\right) - 1}{\rho_0 \big/ a\left(1-e\right) + \rho_0 \big/ a\left(1+e\right)}$$

Simplify: $$\cot\left(\alpha+\beta\right) = \frac{{\rho_0}^2 - a^2\left(1-e^2\right)}{2a\rho_0}$$

Rearrange: $${\rho_0}^2 - 2a\cot\left(\alpha+\beta\right)\rho_0 - a^2\left(1-e^2\right) = 0 \qquad (2)$$

A simplified solution to Eq. (2) is found when $\cot\left(\alpha+\beta\right) = \pi/2$, and $\rho_0 = \sqrt{ap}$. The right triangle on the right-hand side of Figure 6 shows the relationship between r, ρ , and ϕ for a general point in the orbit.

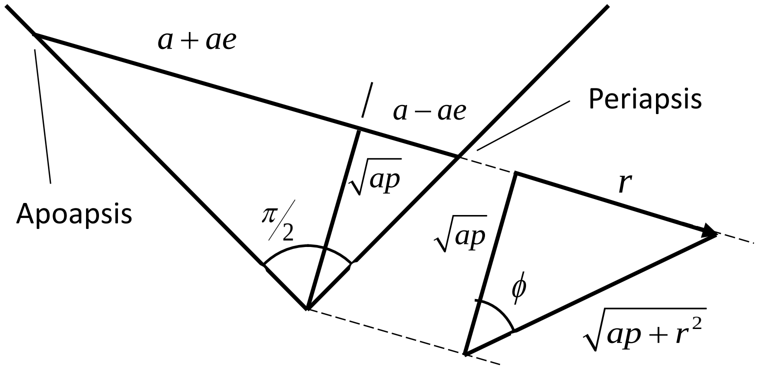

Fig. 6 Elliptic Orbit with Varying ϕ

The magnitude of angular momentum is known from the literature as $h = r^2\dot{\theta}$, and the circular component of velocity normal to r is $r\dot{\theta}$. Also known from the literature is the magnitude of the angular momentum as $h^2 = \mu p$.[2] Combining these equations gives the kinetic energy component normal to r :

$$\xi_{\mathrm{K}\perp} = \frac{\mu p}{2r^2}$$

The value and geometry of $\sqrt{ap}$ in Figure 6 have the significance that when $r = \sqrt{ap}$, $\xi_{\mathrm{K}\perp} = \mu/2a$. This corresponds to a velocity of $\sqrt{\mu/a}$, and area swept by this uniform circular motion with a radius of $\sqrt{ap}$ equals the area swept stated by Kepler. Trigonometric functions apply in the model, for example $r = \sqrt{\mu/a}\,\tan\phi$. Trigonometric functions provide additional useful relationships.

A family of elliptic orbits with the same total energy of $-\mu/2a$ but varying values of eccentricity can be described in the conic reference frame between the limiting cases of a circle and a linear trajectory. Figure 7 depicts several of these orbits in edge view. All these orbits have a rotational component except for the linear trajectory. The term with e in the energy equation has a value of eccentricity between zero (circle) and one (linear).

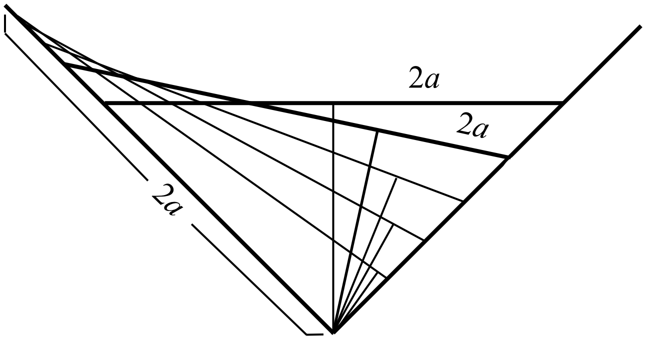

Fig. 7 Elliptic Orbits in Edge View with Equal Total Energy

Ref. [1] showed that a linear elliptic trajectory is represented by Kepler's equation when $e = 1$ and velocity is zero at $r = 2a$. In a conic reference frame, this is a slant height modeled as shown in Figure 8.

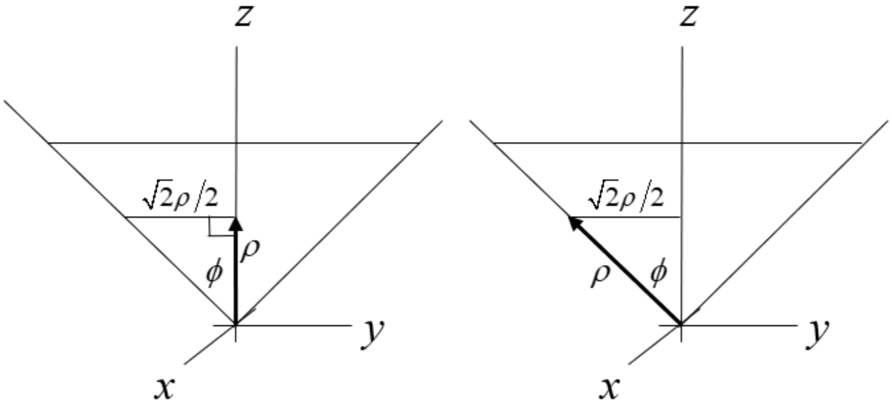

Fig. 8 Slant Height Linear Trajectories in Two Views

Consider the special case of a body in a gravitational force field with no rotational component with total specific energy of zero. Velocity approaches zero as r approaches infinity. By the integration step for conservation of energy:

$$\dot{r}^2/2 = \mu/r + C \tag{3}$$

If this constant of integration is zero, the trajectory is linear parabolic with limits of $\dot{r} \to 0$ as $r \to \infty$ and $\dot{r} \to \infty$ as $r \to 0$. Assume that ρ is defined as having linear parabolic motion as the slant height of the right circular cone and that the constant of integration is defined at the next step. In this case, $\rho = r$. Then Equations (3) becomes $\dot{r} = \sqrt{2\mu/r}$,

and $$\sqrt{r}\,\dot{r} = \sqrt{2\mu}$$

Separate variables and integrate $$\sqrt{2}/3\, r^{3/2} = \sqrt{\mu}\, t + C$$

The apex of the cone is a singularity for time-position calculations not limited by the dimensions of a real primary body. The question arises, is it mathematically acceptable to model orbits from this point? Consider the force, velocity, and radius vectors in the model. Translating these such that the center of the force field coincides with the apex is proper as shown in Figure 9.

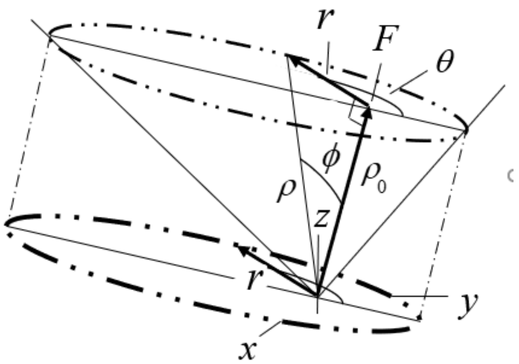

Fig. 9 Orbit Vectors Translated to the Apex of the Cone

V. Conclusions and Recommendations

Classical representation of two-body orbits as conic sections and Kepler's exact equation for time as a function of orbital position are proven. Combining them such that a cone is the frame for representing orbits provides another reference system for orbital motion. With the apex of the cone at the center of the spherical frame and a right angle as the apex angle, new equations are written. The model is constructed to extend the angular momentum vector on a line through the apex. An equation relating this constant line, the radius, and the slant height of the cone at any point results. It is recommended as an extension to write equations with trigonometric functions corresponding to the equations developed in this paper.

Additional analyses will examine a transposition of the parameters in this work to orbits with simultaneous equations represented by a linear trajectory along a slant height of the cone and circular motion. These meet at a right angle on the cone's surface as shown in Figure 8. Translating forces, velocities, and distances from the cone described herein to radial and circular components will yield new position-time equations.

References

References

[1] May, D. H., Orbital Motion Equations with Dynamic Models. Paper AAS 05-354 presented at the AAS/AIAA Astrodynamics Specialist Conference, Lake Tahoe, CA, August 7-11, 2005.

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