The Friedmann-Lemaître-Robertson-Walker Metric

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Abstract

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric used to describe the cosmic spacetime is based on the cosmological principle, which assumes homogeneity and isotropy throughout the Universe. It also adopts free-fall conditions via the selection of a constant lapse function, $g_{tt} = 1$, regardless of whether or not the chosen energy-momentum tensor $T^{\alpha\beta}$ produces an accelerated expansion. This is sometimes justified by arguing that one may shift the gauge, if necessary, transforming the time dt to a new coordinate $dt' \equiv \sqrt{g_{tt}}\, dt$, thereby re-establishing a unitary value for $g_{t't'}$. Previously, we have demonstrated that this approach is inconsistent with the Friedmann equations derived using comoving coordinates. In this paper, we advance this discussion significantly by using the Local Flatness Theorem in general relativity to *prove* that g_{tt} in FLRW is inextricably dependent on the expansion dynamics via the expansion factor $a(t)$, which itself depends on the equation-of-state in $T^{\alpha\beta}$. One is therefore not free to choose g_{tt} arbitrarily without ensuring its consistency with the energy-momentum tensor. We prove that the use of FLRW in cosmology is valid only for zero active mass, i.e., $\rho + 3p = 0$, where ρ and p are, respectively, the total energy density and pressure in the cosmic fluid.

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1 Introduction

The general relativistic collapse/expansion of a spherically symmetric distribution of matter and energy was first considered by Oppenheimer & Snyder [1], with important generalizations introduced later by McVittie [2], Misner & Sharp [3] and Thompson & Whitrow [4], among others. The spherically-symmetric spacetime metrics describing this process, of which the Friedmann-Lemaître-Robertson-Walker form (FLRW) is a special case, may be represented as

$$ds^2 = e^{2\Phi/c^2} c^2 dt^2 - e^{\lambda} dr^2 - R^2 d\Omega^2 \ , \tag{1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$, and Φ, λ, and R are each functions of the radius r and time t, and need to be determined by folding this ansatz through Einstein's equations. This exercise is often tedious, though straightforward, and allows one to find the dependence of the coefficients

$g_{tt} \equiv e^{2\Phi/c^2}$ and $g_{rr} \equiv e^{\lambda}$ on the energy-momentum tensor $T^{\alpha\beta}$, which is often assumed to have the perfect-fluid form,

$$T_{\alpha\beta} = \left(\rho_m + \frac{p}{c^2}\right) u_\alpha u_\beta - p g_{\alpha\beta} \ , \tag{2}$$

in terms of the co-moving energy density ρ and pressure p, and the four-velocity u_α.

But though FLRW is a member of this class, it has nevertheless been handled in a distinctly different manner than the rest. This is likely due to the fact that, whereas all other problems of expansion and collapse involve well-defined, localized mass distributions, the cosmic spacetime deals with a continuous source of gravity distributed throughout a presumably infinite medium. Today we understand how to handle this situation, using the now well-understood Birkhoff-Jebsen theorem [5, 6], but this was not yet fully appreciated during the FLRW metric's early years of development. The corollary to this theorem states that, even for such an infinite, isotropic medium, the spacetime within a spherical region is independent of what lies in the exterior (see also ref. [7]). The general relativistic description of the expansion of a spherical portion of the Universe, relative to an observer at the center of that region, should therefore closely follow the formalism used more commonly in the simulation of stellar collapse and explosion.

In spite of our better understanding of this phenomenon today, however, we still tend to follow the steps established in the early years. The principal difference in the use of FLRW compared with the other spherically-symmetric spacetimes is that, whereas the dynamical equations describing the gravitational collapse are obtained from Einstein's equations using the general form of the metric in Equation (1), the Friedmann equations (see Eqns. 21 and 23 below) are derived by first introducing all of the possible symmetries, such as homogeneity and isotropy, to greatly simplify the metric coefficients before folding Equation (1) through Einstein's equations. Quite inexplicably, this procedure adopts *free-fall* conditions by setting the lapse function, g_{tt}, equal to 1 (i.e., $\Phi = 0$), without considering the possible effects of time dilation due to an accelerated expansion of the spatial coordinates. The Hubble flow, for instance, is generally not inertial, so how can the FLRW metric, with $g_{tt} = 1$, adequately describe the cosmic expansion which, in the standard model, ΛCDM, has undergone various phases of deceleration and acceleration since the Big Bang?

This issue receives scant attention in the literature, mainly because the conventional view has it that g_{tt} in FLRW can at most be a function of t, not space, in order to comply with the adoption of homogeneity in the Cosmological principle. If necessary, the argument goes, one can carry out a gauge transformation $dt \to dt' \equiv \sqrt{g_{tt}}\, dt$ to ensure the disappearance of the lapse function in the FLRW metric using the new time coordinate t'. But it is not difficult to convince oneself that this attempted 'fix' is not necessarily consistent with general relativity, because a shift in gauge is necessarily a transformation from one distinct frame of reference to another. So if one begins with an assumed form of $T^{\alpha\beta}$ in Equation (2), using a four-velocity consistent with the comoving frame, a subsequent gauge transformation to eliminate the lapse function shifts the observer out of the comoving frame whence they started. By definition, this new frame must be in free-fall, since $g_{t't'} = 1$.

In previous work (see, e.g., refs. [8, 9]), we have argued that the imposition of free-fall conditions ($g_{tt} = 1$) on the FLRW metric necessarily constrains the stress-energy tensor $T^{\alpha\beta}$ to have an equation-of-state consistent with zero active mass, i.e., $\rho + 3p = 0$. In § 3 below we shall discuss how the cosmological observations support this constraint, given that ΛCDM with this condition accounts for the data better than the standard model without it.

Some have contested this view [10], however, arguing that the interpretation of a gauge transformation described above can be ignored, allowing one to use the FLRW metric with $g_{tt} = 1$ even in a non-inertial, comoving frame. The goal of this paper is to prove, via the Local Flatness Theorem in general relativity, that the lapse function is indeed inextricably linked to the cosmic expansion rate, and can therefore *not be chosen arbitrarily* without ensuring its consistency with the assumed energy-momentum tensor. We demonstrate that claims to the contrary are unsupportable, and we provide a complete physical justification for the adoption of the zero active mass condition in FLRW cosmology.

2 The Lapse Function

Instead of pre-assuming that the lapse function g_{tt} is 1 before folding the FLRW metric through Einstein's field equations, let us first find the constraints imposed on it by our choice of stress-energy tensor. Certainly, if the prescribed equation-of-state is consistent with a unitary value for this metric coefficient, the result $g_{tt} = 1$ should emerge automatically in the allowed solutions without us having to impose it by hand.

We write the metric coefficients, $g_{\mu\nu}$, and Christoffel symbols, $\Gamma^{\lambda}{}_{\mu\nu}$, in terms of the coordinates $x^{\mu} = (ct, x, y, z)$ in the co-moving frame. For simplicity, however, we assume a spatially flat cosmology since this is what the observations are telling us today [11]. And to keep the derivations as straightforward as possible, we shall write the FLRW metric as

$$ds^2 = g_{tt} c^2 dt^2 - a(t)^2 \left(dx^2 + dy^2 + dz^2\right) \ . \tag{3}$$

With these Cartesian coordinates, the metric coefficients are therefore

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & -a^2 \end{pmatrix} \ . \tag{4}$$

Notice that, although g_{tt} is allowed to be an arbitrary function of time, we are explicitly imposing the symmetries associated with isotropy and homogeneity in the Cosmological principle, so it cannot depend on the spatial coordinates. Correspondingly, it is a straightforward exercise to find that the only non-zero Christoffel symbols are

$$
\begin{aligned}
\Gamma^0{}_{00} &= \frac{1}{2}\frac{1}{g_{tt}}\frac{\partial g_{tt}}{\partial\, ct} \\
\Gamma^0{}_{ii} &= \frac{1}{g_{tt}c}a\dot{a} \\
\Gamma^i{}_{i0} &= \Gamma^i{}_{0i} = \frac{1}{c}\frac{\dot{a}}{a}\ .
\end{aligned}
\tag{5}
$$

In general relativity, the Local Flatness Theorem (see, e.g., refs. [6, 12]) is a formal expression of the equivalence principle. It holds that in the neighborhood of any spacetime point X^μ in x^μ there must exist a *local* inertial frame, against which any acceleration in x^μ can be measured absolutely. That is, the local inertial frame, whose coordinates we shall label ξ^μ, is in free fall at X^μ, and therefore any non-inertial effects in x^μ, including time dilation, can be measured relative to ξ^μ. Non-inertial and time-dilation effects in x^μ, measured against ξ^μ, are absent only if our co-moving frame is itself in free fall.

The Cosmological principle tells us that the Universe looks identical, no matter where we place the observer. For convenience and without loss of generality, let us orient the ξ^μ frame so that its origin coincides with ours. Then, the mathematical consequence of the Local Flatness Theorem is that the coordinates ξ^μ must satisfy the equations [6]

$$
\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} = \Gamma^\lambda{}_{\mu\nu}\frac{\partial\xi^\alpha}{\partial x^\lambda}\ . \tag{6}
$$

These equations are exact. There is no approximation implied by them, nor any assumption other than the validity of the equivalence principle. They are the foundation upon which the principal result of this paper is based. As we shall see, we are not at all free to choose the lapse function arbitrarily, because Equation (6) couples g_{tt} to g_{ii}, i.e., $a(t)$ and, since the expansion factor depends critically on the equation-of-state in the cosmic fluid, the value of g_{tt} is critically dependent on our choice of stress-energy tensor.

Given the symmetry, ξ^0 clearly depends solely on t and the radius $r = \sqrt{x^2 + y^2 + z^2}$, and ξ^i can be a function only of t and x^i (with $i = 1, 2$ or 3). The $\alpha = \mu = \nu = 0$ component in Equation (6) thus yields the following relation:

$$
\xi^0 = A(r)\int^{ct}\sqrt{g_{tt}(t')}\, d(ct') + B(r)\ , \tag{7}
$$

where $A(r)$ and $B(r)$ are functions we shall determine below. The $\alpha = \mu = 0$, $\nu = i$ component gives

$$\frac{\partial}{\partial x^i}\left(\frac{\partial \xi^0}{\partial\, ct}\right) = \frac{1}{c}\frac{\dot{a}}{a}\left[\frac{x^i}{r}\frac{dA}{dr}\int^{ct}\sqrt{g_{tt}(t')}\,d(ct') + \frac{x^i}{r}\frac{dB}{dr}\right]\,. \tag{8}$$

From these two expressions, we thus conclude that $dB/dr = 0$, so we simply put $B(r) = 0$, because B represents a constant translation in time, which has no consequence on the cosmic dynamics. The most important result of this paper follows from Equations (7) and (8), from which we see that

$$\int^{ct}\sqrt{g_{tt}(t')}\,d(ct') = cg_{tt}(t)\frac{a}{\dot{a}}\,. \tag{9}$$

This is the critical expression that couples g_{tt} to $a(t)$ and, therefore, to the equation-of-state.

Were we to choose a lapse function $g_{tt} = 1$, we would simultaneously restrict the possible kinds of expansion factor $a(t)$ permitted by the Local Flatness Theorem. Clearly, putting $g_{tt}(t) = 1$ would permit only two functional forms of $a(t)$. The first is

$$a(t) \equiv \left(\frac{t}{t_0}\right)\,, \tag{10}$$

written conventionally for a spatially flat metric, scaled so that $a(t_0) = 1$ at the current age, t_0, of the Universe. The second is $a =$ constant, when we realize that Equation (6) is also trivially satisfied with $\xi^0 = ct$ and $g_{tt}(t) = 1$. In this case, the spacetime curvature is identically zero, and $\Gamma^\alpha{}_{\mu\nu} = 0$ (i.e., Minkowski space).

A related conclusion from the result shown in Equation (9) is that a choice of $a(t)$ different from that shown in Equation (10) (or the Minkowski space solution) would then require the identification of g_{tt} as a function of t. As we shall point out § III below, the real Universe appears to have 'chosen' an expansion factor consistent with $g_{tt} = 1$, so the need for finding the correct functional form of the lapse function for a range of expansion scenarios different from this is not critical. Nevertheless, one can see directly from Equation (9) that for each specific choice of $a(t)$, the procedure for finding the corresponding g_{tt} amounts to 'guessing' the ansatz satisfying this integral equation.

In summary, the lapse function g_{tt} in FLRW may be set equal to 1 only for two very special cases of the cosmic spacetime: (1) Minkowski space, for which $a =$ constant; and (2) $a(t) = (t/t_0)$. Any other expansion factor would represent a net acceleration or deceleration of the x^μ frame, rendering it non-inertial, and requiring a necessary time dilation relative to the time ξ^0 in the local free-falling frame, ξ^μ.

But Equation (6) contains other terms. Is it possible to circumvent this restriction somehow, based on what its other components are telling us to do? This is very easy to check. If $g_{tt}(t) = 1$, the time coordinate in Equation (7) may be written $\xi^0 = A(r)ct$. Then, from the $\alpha = 0$, $\mu = \nu = i$ component of Equation (6), we find that

$$\frac{(x^i)^2}{r^2}\frac{d^2A}{dr^2} + \frac{(x^j)^2 + (x^k)^2}{r^3}\frac{dA}{dr} = \frac{1}{(ct_0)^2}A\ , \tag{11}$$

where $i \neq j \neq k$. To second order in r, the solution to this equation may be written

$$A(r) = 1 + \frac{1}{2}\left(\frac{r}{ct_0}\right)^2\ . \tag{12}$$

Thus, since the coordinates ξ^μ belong to a *locally* inertial frame in the vicinity of $\vec{x} = 0$, we find that $\xi^0 \to ct$, as expected.

We may extract additional constraints from the spatial components of Equation (6), including the expression

$$\xi^i = C(x^i)\int^{ct}\sqrt{g_{tt}(t')}\,d(ct')\ , \tag{13}$$

in which the function $C(x^i)$ will be derived shortly. If we now choose $g_{tt}(t) = 1$,

$$\xi^i = C(x^i)ct\ . \tag{14}$$

But these coordinates must also satisfy the equation

$$\frac{\partial^2\xi^i}{\partial ct\,\partial x^i} = \frac{1}{c}\frac{\dot{a}}{a}\frac{\partial\xi^i}{\partial x^i} \tag{15}$$

for each individual value of the index 'i'. Thus,

$$\frac{dC(x^i)}{dx^i} = \frac{1}{c}\frac{\dot{a}}{a}\left(\frac{dC(x^i)}{dx^i}ct\right)\ , \tag{16}$$

so clearly the expansion factor $a(t)$ must again satisfy the equation

$$\frac{1}{c}\frac{\dot{a}}{a}ct = 1\ , \tag{17}$$

confirming the result in Equation (10).

The last component in Equation (6) demonstrates that the local free-falling frame is actually the Hubble flow. To see this, consider the $\mu = \nu = 1$ component:

$$C(x^i) = C_0 e^{x^i/ct_0}\ , \tag{18}$$

in terms of an integration constant C_0. A simple redefinition of the spatial scale,

$$\chi^i \equiv C_0 c t_0 e^{x^i / c t_0} \ , \tag{19}$$

then allows us to formally write

$$\xi^i = a(t) \chi^i \ . \tag{20}$$

The outcome of this application of the Local Flatness Theorem to the FLRW metric is therefore quite clear. The transformation relations from the local free-falling frame to our comoving coordinates necessarily couple the lapse function g_{tt} to the expansion factor $a(t)$. Choosing a unitary value $g_{tt} = 1$ in the comoving frame restricts the expansion factor to be uniquely $a(t) = (t/t_0)$, or $a =$ constant in Minkowski space. And the local inertial frame is then simply the Hubble flow with $\xi^\mu = (ct, a\chi^1, a\chi^2, a\chi^3)$.

Clearly, Minkowski space is not relevant to a Universe with total energy density $\rho \neq 0$. So what equation-of-state can be accommodated by $a(t) = (t/t_0)$? This is most easily and directly answered via the second Friedmann equation, often called the Raychaudhuri equation [13], which gives the acceleration of cosmic expansion:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \left(\rho + 3p \right) \ , \tag{21}$$

in terms of ρ and the total pressure p. A Universe with $a(t) = (t/t_0)$ thus requires the so-called *zero active mass* condition in general relativity:

$$\rho + 3p = 0 \ . \tag{22}$$

And in the Discussion section below we shall demonstrate how the observations have been strongly pointing to this equation-of-state for over a decade now.

3 Discussion

We first consider the physical meaning of the Raychaudhuri equation, analogously to the approach sometimes taken to interpret the Friedmann equation [14],

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \rho - \frac{kc^2}{a^2} \ , \tag{23}$$

where H is the Hubble parameter. This expression emerges from Einstein's field equations, but can also be derived with a Newtonian approach [15], justified via the use of the Birkhoff-Jebsen theorem [5]. The Friedmann equation represents the conservation of local kinetic energy (the term on the left-hand side) and gravitational potential energy (first term on the right), via the conserved net energy proportional to the negative of the spatial curvature constant k.

Following the Newtonian approach, one can also show from the First Law of Thermodynamics ($dE = dQ - p\,dV$) that the energy density ρ changes as the Universe expands according to the expression

$$\dot{\rho} = -3H(\rho + p) \ . \tag{24}$$

The appearance of the pressure p in this equation simply represents the loss of energy due to '$p\,dV$' work. It does not, as is sometimes incorrectly described, represent an 'effective' increase in density, ρ, due to some relativistic effect missing in classical mechanics. Equations (23) and (24) are two linearly independent expressions for the evolution of the variables $a(t)$ and ρ as the Universe expands. They each represent a conservation of energy (kinetic plus gravitational in the former, and ρ in the latter). The rate at which the expansion takes place, however, is given by the acceleration (or Raychaudhuri) equation, which has nothing to do with the energy. Nevertheless, the pressure affects the rate at which ρ changes, which ultimately impacts the acceleration, and p therefore appears in Equation (21) because it is derived from the other two. Again, the $3p$ in this equation is not meant to be an addition to ρ, creating an 'effective' density different from ρ. So the term 'active mass' should simply be interpreted as an indication of the overall source of curvature, which includes not only ρ itself, but also the agent (i.e., pressure) that affects the rate at which ρ changes.

The argument we have made in § 2 above unequivocally establishes the requirement that $\rho + 3p$ be zero for the FLRW metric. Any assumed stress-energy tensor with an equation-of-state different from $p = -\rho/3$ cannot be described by this metric with a lapse function $g_{tt} = 1$. As we have noted on several occasions, the comoving frame in such alternative cosmologies is not inertial, and one must therefore be able to measure a time dilation in such frames relative to the local free-falling observer.

Some reach the opposite conclusion [10] for the simple reason that they believe they can carry out a gauge transformation without actually changing their frame of reference. They incorrectly claim that an FLRW metric with $g_{tt} = f(t)$ may be transformed into its standard form (with $g_{t't'} = 1$) by simply changing the time coordinate to $dt' \equiv \sqrt{f(t)}dt$, while still remaining in the comoving frame. In other words, they argue that one may choose the stress-energy tensor arbitrarily, evaluate the metric coefficients in the comoving frame (presumably with $g_{tt} \neq 1$) and—in spite of the fact that this frame is non-inertial—then transform the time coordinate to impose the free-falling condition $g_{t't'} = 1$, *while remaining in the non-inertial comoving frame*. This procedure is clearly unphysical; it violates the tenants of general relativity at a very fundamental level. The proof we have presented in § 2 above quite conclusively exposes the fallacy with this line of reasoning.

Given this highly restricted validity of the FLRW metric, one should then ask why ΛCDM, absent the zero active mass condition, appears to account quite well for the observations. It is well known that the pressure in this model is typically given as $p = p_{\rm m} + p_{\rm r} + p_{\rm de}$, with each of the contributions coming from matter ($\rho_{\rm m}$), radiation ($\rho_{\rm r}$) and dark energy ($\rho_{\rm de}$), the latter often assumed to be simply a cosmological constant Λ. Clearly, with $p_{\rm m} \approx 0$, $p_{\rm r} = \rho_{\rm r}/3$ and $p_{\rm de} \sim -\rho_{\rm de}$, one cannot claim that $p = -\rho/3$ is maintained in this model. Yet the data are telling us that the ΛCDM Universe has been expanding at a rate consistent with $p = -\rho/3$ averaged over a Hubble time [7, 15]. Indeed, the observations indicate that the Hubble radius today, $R_{\rm h} \equiv c/H_0$, equals ct_0 to within the measurement error. As it turns out, this equality is a direct consequence of the expansion rate $a(t) = (t/t_0)$ [16, 7, 17], and cannot be a coincidence because it could only have occurred once in the entire history of the Universe, and it is happening right now, just when we happen to be looking at it. The most likely explanation for this remarkable and highly unlikely result is that $R_{\rm h}$ equals ct all the time, not just now, giving rise to the eponymously named cosmology 'The $R_{\rm h} = ct$ universe.'

But the observational support for the proof discussed in § 2 is far stronger than this initial constraint. The $R_{\rm h} = ct$ universe is basically ΛCDM together with the requirement that $p_{\rm r} + p_{\rm de} = -(\rho_{\rm m} + \rho_{\rm r} + \rho_{\rm de})/3$. It is thus quite straightforward to carry out comparative tests between these two versions of ΛCDM, one with $\rho + 3p = 0$ and the other without. And this has now been done for over 27 different kinds of observation, from low to high redshifts, using integrated measures of distance and differential rates of expansion. A relatively recent survey of these extensive studies may be found in Table 2 of ref. [18], and a thorough discussion of this work appears in the recently published monograph 'The Cosmic Spacetime' [15].

The great improvement in the agreement between the observations and standard model predictions with the adoption of the zero active mass condition is only the beginning. Probing the internal structure of ΛCDM in greater detail we uncover the existence of over ten paradoxes and inconsistencies that have plagued this model for several decades. Quite impressively, however, the imposition of zero active mass on the equation of state in the standard model alleviates—if not completely removes—these problems. As such, there is not only a large body of experimental/observational evidence supporting the theoretical conclusion we are making in this paper, but there are fundamental reasons based on the internal self-consistency of ΛCDM that require it as well. Let us examine the most prominent among these.

Starting with Big Bang nucleosynthesis (BBN), we note that ΛCDM does not really predict the He abundance, as is often incorrectly assumed. In the standard model, one fine-tunes the Boltzmann factor (for protons and neutrons) to ensure the correct He abundance. That means protons and neutrons need to be in thermal equilibrium, but if the radiation is also in equilibrium with them, the baryon-to-photon ratio is of order 10^{-18}, removing any possibility of getting the light elements to work correctly (see, e.g., ref. [15]). So BBN in ΛCDM instead argues that protons and neutrons are in equilibrium, while radiation is not, and the baryon-to-photon ratio is then introduced by hand at the well-known value $\sim 5 \times 10^{-10}$. With it, the He abundance is adjusted to its empirical value, and once these conditions are selected, one then runs the rest of the nuclear burning network to produce the remaining light elements. Contrary to popular belief, however, the only additional element that may be 'predicted' in this model is ^{7}Li and, as is well know, ΛCDM gets it wrong by a factor of over 400%. ^{6}Li is less reliable (and measurable), but ΛCDM gets this one wrong by as much as ten orders of magnitude.

With the introduction of the zero active mass condition, on the other hand (and actually most linearly expanding cosmologies), the BBN proceeds quite differently, as demonstrated more quantitatively in ref [19]. While all of the essential burning in ΛCDM must take place within the neutron's lifetime (i.e., at 3 min $< t <$ 18 min), the neutron population with the zero active mass condition is sustained by weak interactions for over 100 Myr. During this time, BBN proceeds steadily and quiescently, avoiding the extreme physical conditions in the standard model. The He abundance is again $\sim$ 25%, but the ^{7}Li anomaly is almost completely gone (due to the quiescent conditions). And no ad hoc assumptions need to be made in order to prevent the radiation from being in equilibrium with the other elements. BBN thus works better in ΛCDM when the zero active mass equation of state is introduced, consistently with the conclusion in this paper.

Probing even more deeply, ΛCDM currently has no explanation for how quantum fluctuations in the inflaton field classicalized to form the large-scale structure we see today. It therefore has no physically self-consistent picture for the formation of stars, galaxies and clusters. In the standard model, the classicalization of quantum fluctuations into macroscopic perturbations growing under their own self-gravity is simply 'put in by hand.' With the zero active mass condition, on the other hand, there is a natural mechanism for this transition to occur at the Planck scale itself [20].

But even before this transition, ΛCDM cannot explain how quantum fluctuations formed and evolved within the context of quantum mechanics below the Compton wavelength, i.e., below the Planck scale. Known as the 'trans-Planckian' anomaly [21], this major inconsistency has never been resolved, and remains a significant flaw in standard inflationary cosmology [22, 23]. The introduction of the zero active mass condition completely avoids this problem, since the quantum fluctuations with this equation of state emerged into the semi-classical universe at the Planck scale [24].

Once the quantum fluctuations classicalized, they would evolved into the anisotropies we see in the microwave background (CMB) temperature. The latest analysis of these features, however, has demonstrated that the inferred primordial fluctuation spectrum must have had a cutoff $k_{\rm min}$, significantly different from zero [25, 26]. This cutoff is a direct measure of the time at which inflation could have started, and the measured value of $k_{\rm min}$ makes it impossible for inflation to simultaneously have solved the temperature horizon problem and accounted for the formation of the primordial fluctuation spectrum [27]. Yet without inflation, ΛCDM cannot survive. The standard model with the inclusion of the zero active mass condition, however, has no horizon problems at all [28, 29], so it does not need inflation.

Even if inflation could miraculously solve the temperature horizon problem, though, ΛCDM has no explanation for how to solve the even worse electroweak horizon problem, which is now firmly established following the discovery of the Higgs boson [30]. But with the inclusion of the zero active mass condition, the standard model has no horizon problems at all, solving the inconsistency associated with the electroweak transition simply and elegantly [29].

The standard model also suffers due to an inconsistency with the first and second laws of thermodynamics. It has no explanation for how cosmic entropy could have started so low and grown enormously by redshift ~ 1090 to produce the observed CMB. Known as the cosmic entropy anomaly, this problem has never been solved. By comparison, cosmic entropy in the bulk is constant when ΛCDM adopts the zero active mass condition, so this problem never appears [31].

ΛCDM also has no explanation for how supermassive black holes could have formed by $z \sim 6$ via standard, astrophysical accretion processes, other than to invoke exotic mechanisms that have never been seen anywhere in the Universe. Instead, with the zero active mass condition, the timeline exactly matches that required to form these objects seen at high redshifts [32].

In a closely related time compression problem, the standard model has no explanation for how $\sim 10^8\ M_\odot$ galaxies could have formed by $z \sim 10-12$ via standard star-formation processes [33, 34, 35, 36, 37, 38]. The required high star-formation rates have never been seen anywhere in the Universe. Again, the timeline in ΛCDM with the zero active mass condition is an exact match to the observational requirements [39].

Continuing with large-scale structure formation, ΛCDM predicts a halo mass and redshift distribution at $z \sim 4-10$ different from what is observed by over 4 orders of magnitude [40]. This is not merely some tension—it represents a major problem with the theory of large-scale structure formation in the standard model. This problem completely disappears, however, once the zero active mass condition is introduced into its equation of state [41].

And finally, but not least, ΛCDM relies on a cosmological constant to get its expansion history right, but this constant Λ differs from what is expected from standard quantum field theory by over 120 orders of magnitude. Nowhere else in science does one see such enormous disparity between theory and experiment. But with the introduction of the zero active mass condition, dark energy becomes dynamic, and is likely to be a new field in extensions to the standard model of particle physics [15].

4 Conclusion

The Local Flatness Theorem, a formal expression of the equivalence principle, is indispensable to general relativity. According to Equations (6) and (9), this theorem unavoidably couples the lapse function g_{tt} in the FLRW metric to the expansion factor $a(t)$, and thereby to the equation-of-state in the stress-energy tensor. One may therefore not impose the free-fall condition $g_{tt} = 1$ arbitrarily without ensuring consistency with the expansion dynamics. The comoving frame is inertial only for the cosmology with $a(t) = (t/t_0)$ (and the much less relevant Minkowski space).

Any other equation-of-state yields an accelerating/decelerating Universe that necessarily requires time dilation relative to the local free-falling frame, specified by a time-dependent lapse function $g_{tt} = f(t)$.

Our Universe is apparently expanding at a rate consistent with the zero active mass condition, a necessary requirement of the $R_{\rm h} = ct$ model with $a(t) = (t/t_0)$. As such, the preponderance of available cosmological data today support our conclusion in § 2 that an application of the FLRW metric to the cosmic spacetime is valid only when the expansion proceeds at a constant rate.

The prospects for additional testing of the zero active mass condition in cosmology are quite promising, particularly in terms of measuring the real-time redshift drift of distant quasars [42]. More than any of the other upcoming observational campaigns, this one in particular aims to probe the underlying cosmology with a truly model-independent approach. Insofar as the zero active mass condition is concerned, this test should be the cleanest by far, since it relies on a simple yes/no response. Future surveys using ELT-HIRES [43] and the SKA Phase 2 array [44], will be able to measure the first and second order redshift derivatives of distant sources, such as quasars. The unambiguous prediction of $R_{\rm h} = ct$, with the zero active mass condition, is *zero* drift at all redshifts, contrasting sharply with all other cosmologies, such as ΛCDM, in which the expansion rate is variable. These campaigns are expected to differentiate between zero and non-zero redshift drift at $\sim 3\sigma$ after 5 years of observation, improving this to $\sim 5\sigma$ over a baseline of 20 years.

For the moment, though, we rely on the rigorous demonstration in this paper that the use of FLRW in cosmology is valid only for a stress-energy tensor that avoids producing an accelerated expansion of the Universe. We affirm our claim that the FLRW spacetime, with a lapse function $g_{tt} = 1$, is valid only for an equation-of-state derived from the zero active mass condition, $\rho + 3p = 0$.

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