GENERALIZING BONDAL–ORLOV CRITERIA FOR DELIGNE MUMFORD–STACKS

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ABSTRACT

If $F : D^b_{coh}(\mathcal{X}) \to T$ is a functor with left and right adjoints from a proper smooth Deligne–Mumford stack with projective coarse moduli space to a triangulated category, there is a Bondal–Orlov criterion determining the full-faithfulness of F. We develop techniques that allow for the proof of this criterion absent the assumption that the course moduli space of $\mathcal{X}$ is projective. Furthermore, if the functor F has a dg or infinity category enhancement, the assumption that F has left and right adjoints may also be relaxed.

CHAPTER 1

INTRODUCTION

Suppose that X is an object common to algebraic geometry (for example, a scheme, stack or algebraic space). Given a functor F from $D^b_{coh}(X)$ to a triangulated category $\mathcal{T}$, it is often of interest to know whether or not F is fully faithful. There has been extensive study of full faithfulness criteria for different categories of objects. For example, if X and Y are projective schemes over a field k of characteristic zero and $F : D^b_{coh}(Y) \to D^b_{coh}(X)$ is the Fourier-Mukai type functor with kernel P, there is the following theorem of Bondal and Orlov [5, Theorem 5.1]

Theorem 1.0.1 *The functor F is fully faithful if and only if for every point $y \in Y$,*

$$Hom_{D^b_{coh}(X)}(F(\mathcal{O}_y), F((O_y)) \simeq k$$

and for each pair of points $y_1, y_2 \in Y$ and each integer i,

$$Hom^i_{D^b_{coh}(X)}(F(\mathcal{O}_{y_1}), F(\mathcal{O}_{y_2})) = 0$$

unless $y_1 = y_2$ and $0 \leq i \leq dim(Y)$.

Due to their origin, such criteria are often referred to as "Bondal-Orlov full faithfulness criteria." Recently, there has been more interest in the study of algebraic stacks. As a result, there has been an effort to expand such criteria to the case of stacks. Throughout this paper, a Deligne–Mumford stack will be an algebraic stack whose atlas is etale as it is defined in [11]. In this paper, we intend to prove the following result for Deligne–Mumford stacks:

Theorem 1.0.2 *Let $\mathcal{X}$ be a proper, smooth Deligne Mumford stack over a field k of characteristic zero. Let $F : D^b_{coh}(\mathcal{X}) \to \mathcal{T}$ be a k-linear functor to a k-linear triangulated category with left and right adjoints. Suppose further that if G is the left adjoint of F, that $G \circ F$ is of Fourier-Mukai type. Then, F is fully faithful if and only if the following conditions hold:*

(1) For every generalized point (x,ξ) of $\mathcal{X}$,

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{x,\xi})) = k$$

(2) For every pair of generalized points (x,ξ) and (y,ν), we have

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), (F(\mathcal{O}_{y,\nu}[i])) = 0$$

unless $x \simeq y$ and $0 \leq i \leq \dim(\mathcal{X})$

and

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{y,\nu})) = 0$$

unless $(x,\xi) \simeq (y,\nu)$

Notably, there is a similar theorem of Poluschuk and Lim which asserts a similar criterion but with more assumptions (see [1, Theorem 1.2]). Namely, our result removes the assumption that $\mathcal{X}$ has a projective coarse moduli space. With some additional conditions, the assumption that F has left and right adjoints can also be dropped. If every contravariant cohomological functor on T is representable, then it is relatively simple to show that F will have left and right adjoints. The existence of the right adjoint is a direct consequence of representability. The left adjoint follows quickly from representability after an application of Serre Duality. The representability hypothesis can be weakened due to the following result:

Proposition 1.0.3 *Let $\mathcal{T}$ be a strongly generated k-linear triangulated category such that for any $A, B \in \mathcal{T}$, the sum $\sum_n dim(Hom(A, B[n]))$ is finite. Then, every contravariant cohomological functor on $\mathcal{T}$ is representable.*

Furthermore, the class of Fourier–Mukai type functors is very well understood. If F admits a dg or infinty category enhanced functor, it can be shown that the functor $G \circ F$ will be of Fourier–Mukai type absent the assumption. The final result is the following:

Theorem 1.0.4 *Let $\mathcal{X}$ be a proper, smooth Deligne–Mumford stack over a field k of characteristic zero. Let $F : D^b_{coh}(\mathcal{X}) \to \mathcal{T}$ be a k-linear functor to a k-linear triangulated category such that for any $A, B \in T$, the sum $\sum_n dim(Hom(A, B[n]))$ is finite. Then, F is fully faithful if and only if the following conditions hold:*

(1) For every generalized point (x, ξ) of $\mathcal{X}$,

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{x,\xi})) = k$$

(2) For every pair of generalized points (x, ξ) and (y, ν), we have

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), (F(\mathcal{O}_{y,\nu}[i])) = 0$$

unless $x \simeq y$ and $0 \le i \le \dim(\mathcal{X})$
and

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{y,\nu})) = 0$$

unless $(x, \xi) \simeq (y, \nu)$

We will begin with a brief discussion of the concept of generalized points. With this preparation completed, we proceed to the main argument. In [1, Section 4] it is shown that to show Theorem 1.0.2 holds, it is necessary and sufficient to show that the map on ext-groups $\mathrm{Ext}^1(\mathcal{O}_x,\mathcal{O}_x) \to \mathrm{Ext}^1(GF(\mathcal{O}_x),GF(\mathcal{O}_x))$ induced by $G \circ F$ is generically injective. In order to prove this we will take advantage of the fact that this is the Kodaira–Spencer map associated to $G \circ F$. We begin by constructing a moduli stack $Coh(\mathcal{X})$ which contains all generalized points of $\mathcal{X}$. Then, we can use the coarse moduli space and residual gerbe of $\mathcal{X}$ to construct a map $\mathcal{X}_{\mathcal{F}} : BG \times V \to Coh(\mathcal{X})$, where V is an open set of $\mathcal{X}$ and G is a finite group, such that $\mathcal{X}_{\mathcal{F}}$ is generically smooth. From f we can construct a functor $\chi : W \to Coh(\mathcal{X})$ where W is another open of $\mathcal{X}$ and χ is equivalent to $G \circ F$ on W. When $G \circ F$ is of Fourier–Mukai type, it can be shown that χ is faithful and universally injective on the level of k-valued points. It is then differential properties that allow us to show that χ is an injection on tangent vectors. Since $\mathrm{Ext}^1(\mathcal{O}_x,\mathcal{O}_x) \to \mathrm{Ext}^1(GF(\mathcal{O}_x),GF(\mathcal{O}_x))$ is the Kodaira–Spencer map of χ, this will complete the proof.

We proceed by providing some evidence that this method of proof will allow for more general results. In the penultimate section, we prove Theorem 7.0.1, a similar criterion for a Deligne–Mumford stack over a regular scheme. The next section discusses an example showing this particular criterion will not easily generalize to arbitrary characteristic.

The final three chapters address the further reduction of assumptions leading to the proof of Theorem 1.0.4. After the aforementioned discussion of the relaxation of the adjoint and Fourier–Mukai type assumptions, we discuss further the properties of the Fourier–Mukai kernel of $G \circ F$. Namely, we will show that the kernel must be psuedo-coherent.

CHAPTER 2

GENERALIZED POINTS

As discussed in the introduction, the premise of this paper will be to prove the following theorem.

Theorem 2.0.1 *Let $\mathcal{X}$ be a proper, smooth Deligne–Mumford stack over a field k of characteristic zero. Let $F : D^b_{coh}(\mathcal{X}) \to \mathcal{T}$ be a k-linear functor to a k-linear triangulated category such that for any $A, B \in T$, the sum $\sum_n dim(Hom(A, B[n]))$ is finite. Then, F is fully faithful if and only if the following conditions hold:*

(1) For every generalized point (x, ξ) of $\mathcal{X}$,

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{x,\xi})) = k$$

(2) For every pair of generalized points (x, ξ) and (y, ν), we have

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), (F(\mathcal{O}_{y,\nu}[i])) = 0$$

unless $x \simeq y$ and $0 \leq i \leq \dim(\mathcal{X})$
and

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{y,\nu})) = 0$$

unless $(x, \xi) \simeq (y, \nu)$

Remark The conditions (1) and (2) in the previous theorem will be referred to throughout this paper as the “Bondal–Orlov conditions”. Any time we reference the Bondal–Orlov conditions, we are referring to those conditions unless specified otherwise. There are many other results use conditions that are similar in structure. Should we need to reference such conditions, we will refer to them as the Bondal–Orlov conditions of the Theorem in question.

In order to prove this result, we will begin with a theorem of Polischuk and Lim [1, Theorem 1.2]. This result will be a starting point that we will build upon to reach the above result. More specifically, we will argue that the assumptions that $\mathcal{X}$ has projective coarse moduli space is unnecessary.

Theorem 2.0.2 *Let $\mathcal{X}$ be a proper, smooth Deligne–Mumford stack with projective coarse moduli space over an algebraically closed field k of characteristic zero. Let $F : D(\mathcal{X}) \to \mathcal{T}$ be an exact functor with right adjoint and left adjoint $G : T \to D(\mathcal{X})$ such that $G \circ F$ is a Fourier Mukai functor. Then, F is fully faithful if and only if the following two conditions hold:*

(1) For every generalized point (x, ξ) of $\mathcal{X}$,

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{x,\xi})) = k$$

(2) For every pair of generalized points (x, ξ) and (y, ν), we have

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), (\mathcal{O}_{y,\nu}[i]) = 0$$

unless $x \simeq y$ and $0 \le i \le \dim(\mathcal{X})$
and

$$Hom_{\mathcal{T}}(F(\mathcal{O}_{x,\xi}), F(\mathcal{O}_{y,\nu})) = 0$$

unless $(x, \xi) \simeq (y, \nu)$

We will not assume the truth of this result. Instead, we will develop a technique for its proof that does not make use of the projective coarse moduli space.

Before this, we will elaborate on certain concepts to be used in our argument. To begin, we will briefly elaborate on what is meant by a generalized point, following the narrative given by Polischuk and Lim.

Let k be an algebraically closed field and $\mathcal{X}$ a separated Deligne–Mumford stack of finite type over k. A k–point of a Deligne–Mumford stack $\mathcal{X}$ is a morphism $x : spec(k) \to \mathcal{X}$. There is a factorization of x over the inclusion of the residual gerbe

$$spec(k) \xrightarrow{p} B(Aut(x)) \xrightarrow{i_x} \mathcal{X} \ ,$$

where p is the canonical etale atlas for the classifying stack $B(Aut(x))$. For any finite group G, Maschke's Theorem gives an orthogonal decomposition of the derived category of G:

$$D(BG) = \oplus_{\xi \in Irr(G)} D(spec(k)) \otimes \xi.$$

Given the above decomposition and a representation ξ of $Aut(x)$, the object $\mathcal{O}_{spec(k)} \otimes \xi$ defines an object of $D(BAut(x))$. We define the generalized point sheaf $\mathcal{O}_{x,\xi}$ to be the pushforward of this object into $D(\mathcal{X})$. Namely, we have $\mathcal{O}_{x,\xi} = i_{x_*}(\mathcal{O}_{spec(k)} \otimes \xi)$.

Furthermore, under the canonical etale atlas $p_* \mathcal{O}_{spec(k)}$ is the regular representation of $B(Aut(x))$. As a result, we have a decomposition of $\mathcal{O}_x = x_* \mathcal{O}_{spec(k)}$ given by

$$\mathcal{O}_x = \oplus_{\xi \in Irr(Aut(x))} \xi^v \otimes \mathcal{O}_{x,\xi}$$

CHAPTER 3

THE STACK OF COHERENT SHEAVES

In this paper, there will be some discussion of the stack of coherent sheaves on a stack $\mathcal{X}$, denoted $Coh(\mathcal{X})$. An element of $Coh(\mathcal{X})$ is a flat family of coherent sheaves over $\mathcal{X}$. Such an element can be parameterized by a sheaf E on $\mathcal{X} \times S$ flat over S where S is a scheme. We make this more precise here

Definition Let $\mathcal{X} \to T$ be a proper morphism of finite presentation from a Noetherian Deligne-Mumford stack to a Noetherian scheme and let $S \to T$ be a morphism of schemes. The stack of coherent sheaves on $\mathcal{X}$, denoted $Coh(\mathcal{X}/T)$ is a category fibered in groupoids on the category Sch/T. An object of $Coh(\mathcal{X}/T)$ is a finitely presented quasi coherent sheaf E on $\mathcal{X} \times_T S$ that is flat over S. A morphism in $Coh(\mathcal{X}/T)$ is a morphism of sheaves on $\mathcal{X} \times_T S$ that is compatible with base change on the natural diagram arising from the fiber product given below

$$\begin{tikzcd} & \mathcal{X} \times_T S \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathcal{X} \arrow[dr] & & S \arrow[dl] \\ & T & \end{tikzcd}$$

By [7][Corollary 9.2], $Coh(\mathcal{X}/T)$ is an algebraic stack with affine diagonal.

Remark In the case of Theorem 1.0.2 our base T will be $spec(k)$. This construction defines a functor $(Sch/T) \to Coh(\mathcal{X}/T)$. In the case where T is $spec(k)$ and the functor is itself applied to $spec(k)$ we see that an object of $Coh(\mathcal{X}/spec(k))$ is a finitely presented quasi coherent sheaf on $\mathcal{X}\times_{spec(k)}spec(k) = \mathcal{X}$. In other words, we have that an object of $Coh(\mathcal{X}/spec(k))(spec(k))$ is a coherent sheaf on $\mathcal{X}$. Therefore, the field valued points of $Coh(\mathcal{X}/spec(k))$ are the coherent sheaves on $\mathcal{X}$. This is enough to conclude that all generalized points lie in this category. As a final note on notation, when $T = spec(k)$ we will omit $spec(k)$ from the notation in favor of the more compact $Coh(\mathcal{X})$.

We will continue this section by discussing the structure of $Coh(BG)$ where G is a finite group.

Example As discussed above, the k-points of $Coh(BG)$ are the coherent sheaves of BG. It is therefore essential to determine the coherent sheaves of BG. We claim that a coherent sheaf on BG is a representation of G. Note that BG is a quotient of $spec(k)$ so any coherent sheaf $\mathcal{F}$ on BG can be pulled back to a coherent sheaf on $spec(k)$. The coherent sheaves of $spec(k)$ are equivalent to finite dimensional vector spaces on k. Therefore, for every coherent sheaf $\mathcal{F}$ on BG there is an associated finite dimensional vector space over k, which we will denote as $V_{\mathcal{F}}$.

We now consider the fiber diagram

$$\begin{tikzcd} spec(k)\times_{BG} spec(k) \arrow[r, "p_1"] \arrow[d, "p_2"] & spec(k) \arrow[d] \\ spec(k) \arrow[r] & BG \end{tikzcd}$$

First, note that since BG is the quotient of $G \to spec(k)$ it follows that $spec(k)\times_{BG} spec(k) \simeq G$. It must also be that case that $p_1^* V_{\mathcal{F}} \simeq p_2^* V_{\mathcal{F}}$. There is then an isomorphism $\phi : k[G]\otimes V_{\mathcal{F}} \to V_{\mathcal{F}} \otimes k[G]$. This map is a descent datum for BG. Since G is a finite set of distinct k points, we know that on the level of stalks, ϕ restricts to a mapping of $V_{\mathcal{F}}$. In other words, for each $g \in G$, there is a map $\phi_g : V_{\mathcal{F}} \to V_{\mathcal{F}}$. These maps must obey the cocycle conditions of descent. To discern what this entails, we must consider the three projections of $spec(k) \times_{BG} spec(k) \times_{BG} spec(k) \to spec(k) \times_{BG} spec(k)$. We will denote these projections as pr_{12}, pr_{13} and pr_{23} with the subscript notation representing the obvious choices. The cocycle condition of descent asserts that $pr_{13}^*\phi = pr_{23}^*\phi \circ pr_{12}^*\phi$. Note that this map can be viewed as a map from $G \times_{BG} spec(k) \to G$, a map from $spec(k) \times_{BG} G$ or a map from $G \times G \to G$. In the latter case, there are three sensible maps. More specifically, we consider the two projections and the multiplication map which we will denote by q_1, q_2 and m respectively. The map q_1 corresponds to the first interpretation, so that $pr_{12} = q_1$. The second interpretation corresponds to q_2, so we have $pr_{23} = q_2$. This leaves us with $pr_{13} = m$. The cocycle condition then gives us $m^*\phi = q_1^*\phi \circ q_2^*\phi$. Again, because G is a disjoint union of a finite set of k-points we can evaluate this on stalks. On a point $(g,h) \in G \times G$, we see that the descent condition yields $\phi_{gh} = \phi_g\phi_h$. It follows that the map ϕ determines a representation of G. The result of this is that the coherent sheaves of BG are the representations of G.

For a more specific example, consider the case where $G = \mathbb{Z}/2$. As before, the k-points of $Coh(B(\mathbb{Z}/2))$ are the coherent sheaves of $B(\mathbb{Z}/2)$. These correspond to the representations of $\mathbb{Z}/2$. There are two such representations, namely the trivial representation and the sign representation.

CHAPTER 4

TANGENT SPACES AND CORRESPONDENCE TO EXT GROUPS

4.1 Correspondence

Here, we will explain that there is a natural correspondence between the tangent space to $Coh(\mathcal{X})$ at a point and a certain ext group. An element of $Coh(\mathcal{X})$ is a flat family of coherent sheaves on $\mathcal{X}$. Such a family can be represented by a coherent sheaf E on $\mathcal{X} \times S$ flat over S where S is a scheme. An element of $T_E Coh(\mathcal{X})$ is a deformation of E. This can be represented by a sheaf E' on $\mathcal{X} \times S[I]$ flat over $S[I]$ where I is a square zero $\mathcal{O}_S$ module, together with an isomorphism $E'|_{\mathcal{X}\times S} \to E$. We first construct a diagram that will be used to define the relevant morphisms:

$$\begin{tikzcd}
\mathcal{X} \times S[I] \arrow[d, "f_{S[I]}"] & \mathcal{X} \times S \arrow[l, hook', "i"'] \arrow[r] \arrow[d, "f_S"] & \mathcal{X} \arrow[d, "f"] \\
S[I] & S \arrow[l, hook', "i_S"'] \arrow[r] & spec(k)
\end{tikzcd}$$

Due to the restriction property, there is a surjection of $\mathcal{O}_{\mathcal{X}\times S[I]}$ modules $E' \to i_*E$. Assigning K to be the kernel of this map, we have an exact sequence of $\mathcal{O}_{\mathcal{X}\times S[I]}$ modules

$$0 \longrightarrow K \longrightarrow E' \longrightarrow i_*E \longrightarrow 0$$

We claim that K is isomorphic to $i_*f_S^*I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} i_*E$. Note that there is a surjection $\mathcal{O}_{\mathcal{X}\times S[I]} \to i_*\mathcal{O}_{\mathcal{X}\times S}$ with kernel $i_*f_S^*I$ leading to an exact sequence

$$0 \longrightarrow i_*f_S^*I \longrightarrow \mathcal{O}_{\mathcal{X}\times S[I]} \longrightarrow i_*\mathcal{O}_{\mathcal{X}\times S} \longrightarrow 0$$

Due to flatness, this sequence can be tensored with E' to obtain a new exact sequence

$$0 \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow \mathcal{O}_{\mathcal{X}\times S[I]} \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow i_* \mathcal{O}_{\mathcal{X}\times S} \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow 0$$

Basic properties of tensor products indicate that the middle term in the above sequence is E' and the restriction property gives the last term as $i_* E$. Therefore, this sequence simplifies to

$$0 \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow E' \longrightarrow i_* E \longrightarrow 0$$

Given this, we have $K \simeq i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E'$. There is then an exact sequence

$$0 \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow E' \longrightarrow i_* E \longrightarrow 0$$

Tensoring with $i_* f_S^* I$ over $\mathcal{O}_{\mathcal{X}\times S[I]}$, we get the exact sequence:

$$i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} i_* E \longrightarrow 0$$

Noting that $i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \simeq (i_* f_S^* I) E'$, the first map of the above exact sequence then factors through $(i_* f_S^* I)^2 E'$. Recalling that I is square zero, we conclude that $i_* f_S^* I$ is as well since this property is preserved by pullback and pushforward. Since $(i_* f_S^* I)^2 = 0$, we conclude that the first map in the above series is zero. It follows that the second map is an isomorphism. Namely, our conclusion is that

$$K \simeq i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} E' \simeq i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} i_* E$$

This proves the initial claim, giving an exact sequence

$$0 \longrightarrow i_* f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}\times S}} i_* E \longrightarrow E' \longrightarrow i_* E \longrightarrow 0$$

From this exact sequence, we obtain a map of $\mathcal{O}_{\mathcal{X}\times S[I]}$ modules $i_*E \to i_*f_S^*I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} (i_*E)[1]$. We now observe that $i_*f_S^*I \otimes_{\mathcal{O}_{\mathcal{O}_{\mathcal{X}\times S[I]}}} (i_*E) \simeq i_*(f_S^*I \otimes_{\mathcal{O}_{\mathcal{X}\times S}} E)$. Therefore, $i_*f_S^*I \otimes_{\mathcal{O}_{\mathcal{X}\times S[I]}} (i_*E)[1]$ is a pushforward from $\mathcal{X} \times S$, and by adjunction we obtain a map of $\mathcal{O}_{\mathcal{X}\times S}$ modules $E \to f_S^*I \otimes_{\mathcal{O}_{\mathcal{X}\times S}} E[1]$. This determines an element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}\times S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}\times S}} f_S^*I)$. Note that this map is determined by the above exact sequence, so it is enough to give the exact sequence that the map is derived from when discussing elements of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}\times S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}\times S}} f_S^*I)$.

Remark This process is reversible, giving an equivalence between $T_E(Coh(\mathcal{X}))$ and $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}\times S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}\times S}} f_S^*I)$. The details are shown in [8, Chapter 4]. For our purposes, all that will be needed is the discussion above.

4.2 Isomorphism Calculations

Now, we will assume that F is a Fourier Mukai Type functor such that F induces a map $Y \to Coh(\mathcal{X})$ where Y is an open in $Coh(\mathcal{X})$. We denote F_S and $F_{S[I]}$ to be the pullbacks of S to $\mathcal{X} \times S$ and $\mathcal{X} \times S[I]$ respectively. Let p_1 and p_2 be the natural projections from $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and let $p_{1,S}$, $p_{2,S}$, $p_{1,S[I]}$ and $p_{2,S[I]}$ be the natural pullbacks. Let P be the Fourier-Mukai kernel associated to F. In this case F_S and $F_{S[I]}$ are Fourier-Mukai type as well, and we denote their kernels by P_S and $P_{S[I]}$ respectively. For the following discussion, we will need to show that $F_S(E \otimes_{\mathcal{O}_{\mathcal{X}\times S}} f_S^*I) \simeq F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}\times S}} f_S^*I$. For convenience, we will use the notation $\mathcal{X}_S$ and $\mathcal{X}_{S[I]}$ for $\mathcal{X} \times S$ and $\mathcal{X} \times S[I]$ respectively. Based on the notation defined above, we have

$$F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^*I) = p_{2,S,*}(P \otimes_{\mathcal{O}_{\mathcal{X}_S\times\mathcal{X}_S}} p_{1,S}^*(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^*I)).$$

Distributing $p_{1,S}^*$ through the tensor product, we see that this is equivalent to

$$F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) = p_{2,S,*}(P \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{1,S}^* E \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{1,S}^* f_S^* I).$$

Regrouping parentheses we have

$$F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) = p_{2,S,*}((P \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{1,S}^* E) \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{2,S}^* f_S^* I).$$

We then apply the projection formula to the term in the parentheses to obtain

$$F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) = p_{2,S,*}(P \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{1,S}^* E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I.$$

However, the first term in the tensor product above is $F_S(E)$. Therefore, we obtain the desired result.

As one final preliminary calculation, we will need to show that $F_{S[I]}(i_* E) \simeq i_* F_S(E)$. We begin with the standard definition of a Fourier-Mukai type functor giving:

$$F_{S[I]}(i_* E) = p_{1,S[I],*}(P_{S[I]} \otimes_{\mathcal{O}_{\mathcal{X}_{S[I]} \times \mathcal{X}_{S[I]}}} p_{2,S[I]}^* i_* E)$$

There is a commutative diagram of maps which we will utilize given below. Note that all maps which were previously defined are consistent with this diagram.

$$\begin{tikzcd}
\mathcal{X}_S \times \mathcal{X}_S \arrow[r, hook, "j"] \arrow[d, "p_{2,S}"] & \mathcal{X}_{S[I]} \times \mathcal{X}_{S[I]} \arrow[d, "p_{2,S[I]}"] \\
\mathcal{X}_S \arrow[r, "i"] & \mathcal{X}_{S[I]}
\end{tikzcd}$$

The map $p_{2,S[I]}$ is flat. By tor-independent base change we deduce that $p_{2,S[I]}^* i_* E \simeq j_* p_{2,S}^* E$ so that

$$F_{S[I]}(i_* E) \simeq p_{1,S[I],*}(P_{S[I]} \otimes_{\mathcal{O}_{\mathcal{X}_{S[I]} \times \mathcal{X}_{S[I]}}} j_* p_{2,S}^* E).$$

By the projection formula, we obtain

$$F_{S[I]}(i_*E) \simeq p_{1,S[I],*}j_*(j^*P_{S[I]} \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{2,S}^* E).$$

Since $p_{1,S[I]}j = p_{1,S}i$, by functoriality we see that

$$F_{S[I]}(i_*E) \simeq i_*p_{1,S,*}(P_S \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{2,S}^* E).$$

Since $F_S(E) = p_{1,S,*}(P_S \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{2,S}^* E)$, we now have the desired result:

$$F_{S[I]}(i_*E) \simeq i_*p_{1,S,*}(P_S \otimes_{\mathcal{O}_{\mathcal{X}_S \times \mathcal{X}_S}} p_{2,S}^* E) \simeq i_*F_S(E).$$

4.3 Induced Maps on Tangent Spaces and Ext Groups

Given the map $\underline{F} : Y \subseteq Coh(\mathcal{X}) \to Coh(\mathcal{X})$ induced by F, there is an induced map on tangent spaces $\underline{F_{tan}} : T_E Coh(\mathcal{X}) \to T_{F(E)} Coh(X)$ as well as an induced map on ext groups $\underline{F_{ext}} : \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(F_S(E), F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I)$. This leads to a diagram:

$$\begin{tikzcd}
T_E Coh(\mathcal{X}) \arrow[r, "\underline{F_{tan}}"] \arrow[d] & T_{F(E)} Coh(\mathcal{X}) \arrow[d] \\
\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) \arrow[r, "\underline{F_{ext}}"] & \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(F_S(E), F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I)
\end{tikzcd}$$

where the vertical arrows are given by the construction discussed in section 4.1. We will show that the diagram commutes.

We begin with an element of $T_E Coh(\mathcal{X})$, a sheaf E' on $\mathcal{X} \times S$, flat over S, together with an isomorphism $\phi : E'|_{\mathcal{X} \times S} \to E$. If we follow the upper portion of the diagram, we begin by applying $\underline{F_{tan}}$. Note that as a map of stacks F is a functor, and is hence defined when applied to maps as well as objects. The image of this element under $\underline{F_{tan}}$ will be the sheaf $F_{S[I]}(E')$ on $\mathcal{X} \times S[I]$ together with the isomorphism $F_S(\phi) : F_{S[I]}(E')|_{\mathcal{X}\times S} \to F_S(E)$. The corresponding element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(F_S(E), F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I)) = \mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(F_S(E), F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I)$ is then determined by the exact sequence

$$0 \longrightarrow i_*(F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) \longrightarrow F_{S[I]}(E') \longrightarrow i_* F_S(E) \longrightarrow 0 .$$

If we instead begin with the downward arrow, we first obtain the element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}_S}}(E, E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I)$ determined by the exact sequence

$$0 \longrightarrow i_*(f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}_S}} i_* E) \longrightarrow E' \longrightarrow i_* E \longrightarrow 0$$

as discussed above. Applying $\underline{F_{ext}}$ will then simply apply the functor $F_{S[I]}$ to this map resulting in an exact sequence:

$$0 \longrightarrow F_{S[I]}(i_*(f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}_S}} E)) \longrightarrow F_{S[I]}(E') \longrightarrow F_{S[I]}(i_* E) \longrightarrow 0 .$$

In section 4.2, we showed that the pushforward i_* can be factored outside of $F_{S[I]}$ so this exact sequence reduces to

$$0 \longrightarrow i_* F_S(f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}_S}} E) \longrightarrow F_{S[I]}(E') \longrightarrow i_* F_S(E) \longrightarrow 0 .$$

We also showed in the previous section that $F_S(E \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I) \simeq F_S(E) \otimes_{\mathcal{O}_{\mathcal{X}_S}} f_S^* I$ so that this further reduces to

$$0 \longrightarrow i_*(f_S^* I \otimes_{\mathcal{O}_{\mathcal{X}_S}} F_S(E)) \longrightarrow F_{S[I]}(E') \longrightarrow i_* F_S(E) \longrightarrow 0 .$$

Notice that this is the same exact sequence as before and recall that the corresponding Ext group elements are determined by these exact sequences. Therefore, the diagram commutes as claimed. The commutativity of this diagram is enough to justify considering $\underline{F_{tan}}$ and $\underline{F_{ext}}$ as interchangable when discussing their properties. However, it is worth noting that because the vertical arrows are equivalences, $\underline{F_{tan}}$ and $\underline{F_{ext}}$ are actually the same map.

CHAPTER 5

THE KODAIRA SPENCER MAP

In the discussion to follow, there will be some mention of the "Kodaira–Spencer map" of a map of stacks. The Kodaira–Spencer is in fact the induced map on tangent spaces, but viewed in a different way. The purpose of this remark will be to elaborate on this as it relates to the current argument. The tangent space to $Coh(\mathcal{X})$ at the point $\mathcal{O}_x$, denoted $T_{\mathcal{O}_x}Coh(\mathcal{X})$ is defined as the set of 2-commutative diagrams

$$\begin{tikzcd} spec(k) \arrow[d, hook] \arrow[dr, "\mathcal{O}_x"] & \\ spec(k[\epsilon]) \arrow[r, "f"] & Coh(\mathcal{X}) \end{tikzcd}$$

up to equivalence. In other words, this is the set of maps f which take the closed point of $spec(k[\epsilon])$ to $\mathcal{O}_x$ in $Coh(\mathcal{X})$.

Remark This definition of an element of the tangent space to $Coh(\mathcal{X})$ is formulated differently than the one given in the previous section. However, they are equivalent in that the two definitions are simply two ways of expressing what is meant by a deformation of the point in question. We choose this formulation here because it is convenient for the following discussion.

Since GF is Fourier Mukai type, it is given by a kernel $\mathcal{Q}$ which is a sheaf in $Coh(\mathcal{X}) \times Coh(\mathcal{X})$. Given an element of the tangent space defined by the map f as in the diagram above, we can pull $\mathcal{Q}$ back to obtain a deformation of the sheaf

$\mathcal{Q}_x$ on $Coh(\mathcal{X})$ with base $spec(k[\epsilon])$. This defines an element of $\mathrm{Ext}^1(\mathcal{Q}_x, \mathcal{Q}_x)$. The implied map

$$T_{\mathcal{O}_x} Coh(\mathcal{X}) \to \mathrm{Ext}^1(\mathcal{Q}_x, \mathcal{Q}_x)$$

sending f to the aforementioned deformation is the Kodaira–Spencer map of GF. However, because there is a natural identification of $\mathrm{Ext}^1(\mathcal{Q}_x, \mathcal{Q}_x)$ with the tangent space $T_{\mathcal{Q}_x} Coh(\mathcal{X})$ one can view this as the induced map on tangent spaces

$$T_{\mathcal{O}_x} Coh(\mathcal{X}) \to T_{\mathcal{Q}_x} Coh(\mathcal{X}).$$

Noting that under this notation $\mathcal{Q}_x = GF(\mathcal{O}_x)$, one can view the same map as the map of Ext groups induced by GF:

$$\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(\mathcal{Q}_x, \mathcal{Q}_x)$$

CHAPTER 6

ARGUMENT FOR THEOREM 1.0.2

We begin with some preparatory remarks. First, we define the coarse moduli space of a stack.

Definition Let $\mathcal{X}$ be an algebraic stack. A course moduli space for $\mathcal{X}$ is a map $\pi : \mathcal{X} \to X$ where X is an algebraic space such that π is initial among maps from $\mathcal{X}$ to algebraic spaces and π is a bijection on k-points.

While not all stacks have coarse moduli spaces, it is the case that any separated Deligne–Mumford stack does. This is due to the Keel-Mori theorem. Below we give a refinement of said theorem as given by [6][Theorem 1.1]

Theorem 6.0.1 *Let S be a scheme and $\mathcal{X}$ an algebraic stack locally of finite presentation over S and with finite inertia stack. Then, $\mathcal{X}$ admits a coarse moduli space $\pi : \mathcal{X} \to X$.*

A proper Deligne–Mumford stack will then also admit a coarse moduli space, so we may invoke its existence in our argument going forward. We will denote the coarse moduli space of $\mathcal{X}$ by X and the map $\mathcal{X} \to X$ by π. There is a dense open substack $\mathcal{U}$ of $\mathcal{X}$ which is a gerbe [13, Tag 06RB]. There will be a dense open U in X which is then the coarse moduli space of $\mathcal{U}$. The space U is an algebraic space, so it has an etale covering by a scheme V. Since $\mathcal{U}$ is a gerbe lying over U, it then follows that there is an etale covering $q : BG \times V \to \mathcal{U}$ where BG is the classifying stack of some constant finite group. The result of the preceding discussion is the following diagram.

$$\begin{tikzcd}
V \arrow[r, "p"] & BG \times V \arrow[r, "q"] \arrow[d] & \mathcal{U} \arrow[r, hook] \arrow[d, "\pi_U"] & \mathcal{X} \arrow[d, "\pi"] \\
 & V \arrow[r] & U \arrow[r, hook] & X
\end{tikzcd}$$

Note that the injection maps in the above diagrams are open inclusions. Now, we will use the maps within this diagram to define an object $\mathcal{F}$. Let j be the map from $BG \times V \to \mathcal{X}$ obtained by following the above diagram. Now, consider the following diagram

$$\begin{tikzcd}
BG \times V \arrow[r, "\Delta"] \arrow[ddr, "id"] & (BG \times V) \times (BG \times V) \arrow[d, "id \times j"] \\
 & (BG \times V) \times \mathcal{X} \arrow[d, "proj"] \\
 & BG \times V
\end{tikzcd}$$

We then define $\mathcal{F}$ to be the object

$$\mathcal{F} = (id \times j)_* \Delta_* p_* \mathcal{O}_V$$

First, we claim that $\mathcal{F}$ is $BG \times V$ flat. Indeed, j is separated and of finite diagonal. Furthermore, $(id \times j)\Delta$ is a section to the projection as displayed in the above diagram. Therefore, $(id \times j)\Delta$ is finite. Since $p_*\mathcal{O}_V$ is a vector bundle and $(id \times j)\Delta$ is finite, it follows that $\mathcal{F}$ is a vector bundle as well and is therefore $BG \times V$ flat.

The object $\mathcal{F}$ defines a map $\chi_{\mathcal{F}} : BG \times V \to Coh(\mathcal{X})$. We will prove that this map has desirable properties.

Lemma 6.0.2 *The map $\chi_{\mathcal{F}}$ is smooth.*

Proof This result can be shown via the lifting criterion for smoothness [7, Lemma 4.2]. More specifically, if A is a local Artinian ring with maximal ideal m and I is an ideal of A such that $m_I = 0$, the map $\chi_{\mathcal{F}}$ is smooth if the diagram below fills in as indicated.

$$\begin{tikzcd} Spec(A/I) \arrow[r] \arrow[d, hook] & BG \times V \arrow[d, "\mathcal{F}"] \\ SpecA \arrow[r] \arrow[ur, dotted, "\exists"] & Coh(\mathcal{X}) \end{tikzcd}$$

Standard techniques in deformation theory show that the obstruction to this lifting lies in $\mathrm{Ext}^1(\mathcal{F}|_{\mathcal{X}_{A/I}}, \mathcal{F}|_{\mathcal{X}_{A/I}} \otimes \mathcal{I})$ where $\mathcal{I}$ is the ideal sheaf corresponding to I [7, Theorem 4.4]. Therefore, if we can show this ext group to be zero, the claim will be proven. To prove this, we will make use of base change from the following diagram

$$\begin{tikzcd} T_2 \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ T_1 \arrow[r] \arrow[d] & BG \times V \arrow[d, "\Delta"] \\ Spec(A/I) \times (BG \times V) \arrow[r] \arrow[d] & (BG \times V) \times (BG \times V) \arrow[d, "id \times j"] \\ Spec(A/I) \times \mathcal{X} \arrow[r] \arrow[d] & (BG \times V) \times \mathcal{X} \arrow[d, "proj"] \\ Spec(A/I) \arrow[r] & BG \times V \end{tikzcd}$$

Note that T_2 and T_1 are defined to be the necessary spaces to complete the diagram such that all boxes are Cartesian. Furthermore, all horizontal maps are affine. Also, p is finite etale and as discussed earlier, $(id \times j)\Delta$ is finite. It is also etale because j is an open immersion. It follows that the map from V to $(BG \times V) \times \mathcal{X}$ in the above diagram is finite etale. We now label the map from T_2 to T_1 in the above diagram as g. It now follows that g is also finite etale. By affine base change, we discern that

$$\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}|_{\mathcal{X}_{A/I}}, \mathcal{F}|_{\mathcal{X}_{A/I}} \otimes \mathcal{I}) = \mathrm{Ext}^1_{\mathcal{O}_{T_1}}(g_*\mathcal{O}_{T_2}, g_*\mathcal{O}_{T_2})$$

Then, by adjunction we see that

$$\mathrm{Ext}^1_{\mathcal{O}_{T_2}}(g_*\mathcal{O}_{T_2}, g_*\mathcal{O}_{T_2}) = \mathrm{Ext}^1_{\mathcal{O}_{T_2}}(g^*g_*\mathcal{O}_{T_2}, \mathcal{O}_{T_2})$$

Finally, we notice that T_2 is affine. Therefore, the last Ext group above is zero, since the first Ext group of vector bundles on an affine is always zero. By earlier observations, the claim of smoothness is proven.

We now know that the map $\chi_{\mathcal{F}}$ is smooth. Since $\chi_{\mathcal{F}}$ is smooth, its image is an open substack $W \subseteq Coh(\mathcal{X})$. Now, consider the universal family of coherent sheaves on $\mathcal{X} \times Coh(\mathcal{X})$. Restricting this to $\mathcal{X} \times W$ and applying GF we get a bounded complex of coherent sheaves Q on $\mathcal{X} \times W$. Notice that if we pull Q back to $(BG \times V) \times \mathcal{X}$, we retrieve $\mathcal{F}$. It then follows that there is a morphism χ filling in the diagram below.

$$\begin{tikzcd}
BG \times V \arrow[d, two heads] \arrow[dr, "\chi_{\mathcal{F}}"] & \\
W \arrow[r, dashed, "\chi"] & Coh(\mathcal{X})
\end{tikzcd}$$

Note also that χ agrees with GF on W. We now show that χ has the properties we require to make use of the differential properties of the Kodaira–Spencer map.

Lemma 6.0.3 *The morphism χ is representable and universally injective.*

Proof Both of the properties asserted in the statement may be checked on k-points. We begin with representability, which is the same as faithfulness of the underlying functor. Note that on the level of k-points, χ is simply equivalent to GF. Therefore, we need only check that the natural map

$$\mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_w) \to \mathrm{Hom}(GF(\mathcal{O}_w), GF(\mathcal{O}_w))$$

is injective for all $w \in W$. Notice first by adjunction that

$$\mathrm{Hom}(GF(\mathcal{O}_w), GF(\mathcal{O}_w)) \simeq \mathrm{Hom}(F(\mathcal{O}_w), FGF(\mathcal{O}_w)).$$

Adjunction gives a splitting of the map $F(\mathcal{O}_w) \to FGF(\mathcal{O}_w)$, so that the map

$$\mathrm{Hom}(F(\mathcal{O}_w), F(\mathcal{O}_w)) \to \mathrm{Hom}(F(\mathcal{O}_w), FGF(\mathcal{O}_w))$$

is injective. The first Bondal–Orlov condition implies that $\mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_w) \simeq \mathrm{Hom}(F(\mathcal{O}_w), F(\mathcal{O}_w))$. We then have a sequence of maps

$$\mathrm{Hom}(\mathcal{O}_w, \mathcal{O}_w) \xrightarrow{\simeq} \mathrm{Hom}(F(\mathcal{O}_w), F(\mathcal{O}_w)) \hookrightarrow \mathrm{Hom}(F(\mathcal{O}_w), FGF(\mathcal{O}_w)).$$

The composition of the two maps above is clearly injective as it is formed from an isomorphism and an injection. By adjunction, we have

$$\mathrm{Hom}(F(\mathcal{O}_w), FGF(\mathcal{O}_w)) \simeq \mathrm{Hom}(GF(\mathcal{O}_w), GF(\mathcal{O}_w))$$

Therefore, the natural map given at the outset of this argument is injective.

We now move on to universal injectivity. Again, this can be checked on the level of k-points. The second Bondal–Orlov condition implies that GF must separate points. Since χ and GF are the same on the level of k-points, the result is trivial.

We now proceed to the conclusion of the argument. Below we present the final proof of Theorem 1.0.2.

Proof A reduction process exists in the stacky case we are interested in. In [1, Section 4] it is shown that the generic injectivity of the map

$$\mathrm{Ext}^1(\mathcal{O}_x, \mathcal{O}_x) \to \mathrm{Ext}^1(GF(\mathcal{O}_x), GF(\mathcal{O}_x))$$

will suffice to conclude the result. Recall that this is the Kodaira–Spencer map associated to the morphism χ for $x \in W$ as discussed in section 5.

Now, there is a factorization of χ

$$V \xrightarrow{\widehat{\chi}} Z \hookrightarrow Coh(\mathcal{X})$$

where Z is closed in $Coh(\mathcal{X})$ and $\widehat{\chi}$ is schematically dominant and representable. Thus, the map from Z into $Coh(\mathcal{X})$ is a closed immersion. Furthermore, χ is representable and universally injective. The map $\widehat{\chi}$ is then also universally injective. By generic smoothness, it follows that there exists a dense open W' of W for which the map $W' \to Z$ is an open immersion and compatible with $Z \subseteq Coh(\mathcal{X})$. Then, the resulting map $W' \to Coh(\mathcal{X})$ is unramified. It follows that the map $W' \to Coh(\mathcal{X})$ is an injection on tangent vectors. Equivalently, this map has an injective Kodaira– Spencer map. Since W' is open in W, the respective maps from W' and W to $Coh(\mathcal{X})$ have the same Kodaira–Spencer map. Thus, they are both injective, and in turn the Kodaira–Spencer map of GF is generically injective, finishing the proof.

We now proceed to a brief discussion of the case of relative stacks.

CHAPTER 7

THE RELATIVE CASE

In this section we will outline how the techniques in the preceding sections can be used to demonstrate a similar result in the relative case, namely when $\mathcal{X}$ is a stack with a base S that is non-trivial. First, we define a new set of conditions for this case.

Definition Let $\mathcal{X} \to S$ be a proper, smooth Deligne-Mumford stack with proper and smooth structure map over a regular scheme S. Let $F : \mathcal{X} \to T$ be an exact functor to a triangulated category T. We say that F obeys the relative Bondal-Orlov conditions if for every $s \in S$, F obeys the standard Bondal-Orlov conditions on $\mathcal{X}_s$.

With these conditions, there is a similar full-faithfullness criterion:

Theorem 7.0.1 *Let $\mathcal{X} \to S$ be a proper, smooth Deligne-Mumford stack with proper and smooth structure map over a regular scheme S. Let $F : \mathcal{X} \to \mathcal{T}$ be an exact functor to a triangulated category $\mathcal{T}$. Then, F is fully faithful if and only if it obeys the relative Bondal-Orlov conditions.*

There are known results which will allow us to restrict our attention to the fibers over points in s. For example, the following result is a generalization of one from [12, Proposition 2.15]:

Proposition 7.0.2 *Suppose that $\mathcal{X} \to S$ and $\mathcal{Y}$ are proper and smooth Deligne–Mumford stacks over a regular scheme and that K is an object of $D^b_{coh}(\mathcal{X} \times_S \mathcal{Y})$ of finite homological dimension over both $\mathcal{X}$ and $\mathcal{Y}$. The relative integral functor Φ with kernel K is fully faithful if and only if the functor Φ_s is fully faithful for every closed point $s \in S$.*

This result is essential to the argument to come, so we will provide a proof following the techniques used in the source paper. Note that we only discuss the reverse direction here.

Proof First, we note that the function in question has a right adjoint as any integral functor does. The right adjoint to Φ will be called H. We assume that Φ_s is fully faithful for all $s \in S$. As is standard, to check that Φ is fully faithful, we must show that the unit of adjunction $\eta : Id \to H \circ \Phi$ is an isomorphism. For every complex $\mathcal{F} \in D^b_{coh}(\mathcal{X})$ there is a distinguished triangle

$$\mathcal{F} \xrightarrow{\eta(\mathcal{F})} (H \circ \Phi)(\mathcal{F}) \longrightarrow Cone(\eta(\mathcal{F})) \longrightarrow \mathcal{F}[1]$$

Let $j_s : \mathcal{X}_s \times \mathcal{Y}_s \to \mathcal{X} \times_S \mathcal{Y}$ be the natural embedding. Note that there is a base change isomorphism $Lj_s^* \Phi(\mathcal{F}) \simeq \Phi_s(Lj_{s_*}(\mathcal{F}))$. Applying Lj_s^* to the triangle and applying base change, we get a new triangle

$$Lj_s^*(\mathcal{F}) \longrightarrow (H_s \circ \Phi_s)(Lj_s^*(\mathcal{F})) \longrightarrow Lj_s^*(Cone(\eta(\mathcal{F}))) \longrightarrow Lj_s^*(\mathcal{F}[1])$$

By [12, Lemma 2.14], for every i there is an integer r small enough so that $\mathcal{H}^i(Lj_s^*(\mathcal{F})) \simeq \mathcal{H}^i(\tau_{\geq r} Lj_s^*(\mathcal{F}))$ and $\mathcal{H}^i((H_s \circ \Phi_s)(Lj_s^*(\mathcal{F})) \simeq \mathcal{H}^i(\tau_{\geq r}(H_s \circ \Phi_s)(Lj_s^*(\mathcal{F}))$. By definition $\tau_{\geq r} Lj_s^*(\mathcal{F})$ is a bounded complex and by assumption Φ_s is fully faithful. As a result, we have that $\eta_s : Id \to (H_s \circ \Phi_s)$ is an isomorphism. It follows that $\mathcal{H}^i(\tau_{\geq r} Lj_s^*(\mathcal{F})) \simeq \mathcal{H}^i((H_s \circ \Phi_s)(\tau_{\geq r} Lj_s^*(\mathcal{F}))$. Therefore, the map $Lj_s^*(\mathcal{F}) \to$

$(H_s \circ \Phi_s)(Lj_s^*(\mathcal{F}))$ in the above distinguished triangle induces isomorphisms on all of the cohomology sheaves. It then follows that $Lj_s^*(Cone(\eta(\mathcal{F}))$ is zero. This holds for all closed points s in S, so [12, Lemma 2.3] finishes the proof.

Using the previous proposition, we now need only check that our result holds on the fibers $\mathcal{X}_s$ for each $s \in S$. We can use a similar argument as before in order to do this by defining a moduli functor for each point in S.

Definition Given a point $s \in S$ we define a psuedo-functor $D_s : Sch/s \to Coh(\mathcal{X}_s/s)$. The image under D_s of an s-scheme T is a finitely presented quasi coherent sheaf E_T on $\mathcal{X}_s \times sT$ that is flat over T. The image under D_s of a morphism of s-schemes $f : T \to T'$ is the map of sheaves arising from the morphism lying over f in the following diagram:

$$\begin{tikzcd} \mathcal{X}_s \times_s T \arrow[r] \arrow[d] & \mathcal{X}_s \times_s T' \arrow[d] \\ T \arrow[r, "f"] & T' \end{tikzcd}$$

Note that algebricity is closed under fiber products, so each $\mathcal{X}_s$ will be algebraic.

Remark The above definition is essentially the same as the stack of coherent sheaves defined in section 3. The image of a scheme T under D_s will be $Coh(\mathcal{X}_s/s)(T)$.

As before, the s-valued points of the $Coh(\mathcal{X}_s/s)$ are coherent sheaves on $\mathcal{X}_s$. However, there is one potential issue in that the pullbacks of points from the base may not be perfect, or even coherent depending on the base. We see an example below.

Example Consider the case where $X = S = spec(k[\epsilon])$ with the identity as a structure map. In the dual numbers $k[\epsilon] = k[x]/(x^2)$, there is a resolution of k:

$$\cdots \longrightarrow k[x]/(x^2) \xrightarrow{*x} k[x]/(x^2) \xrightarrow{*x} k[x]/(x^2) \longrightarrow k \longrightarrow 0$$

We now consider the Tor dimension of the point corresponding to k in $k(\epsilon)$. We will compute $\mathrm{Tor}_i^{k(\epsilon)}(k,k)$. Tensoring with k and dropping the base term of the previous resolution, we get the complex

$$\cdots \longrightarrow k[\epsilon] \otimes_{k[\epsilon]} k \xrightarrow{*x \otimes_{k[\epsilon])} 1} k[\epsilon] \otimes_{k[\epsilon]} k \xrightarrow{*x \otimes_{k[\epsilon])} 1} k[\epsilon] \otimes_{k[\epsilon]} k \longrightarrow 0$$

We claim that all of the maps in this complex are actually zero maps. Indeed, the element $1 \otimes 1$ is mapped to $x \otimes 1$. This is equivalent to $1 \otimes x * 1$. Viewing k as $k[x]/(x)$, we have $1 \otimes x * 1 = 1 \otimes 0 = 0$. This suffices to show that these maps are zero. The group $\mathrm{Tor}_i^{k[\epsilon]}(k,k)$ is the homology of the above complex at the ith position. Since the kernel of each map is $k[\epsilon] \otimes_{k[\epsilon]} k = k$ and the images are all zero, we conclude that $\mathrm{Tor}_i^{k[\epsilon]}(k,k) = k$ for all positive integers i.

Since k has infinite Tor dimension in $k[\epsilon]$ the corresponding sheaf will as well. That sheaf will therefore not be perfect, or reside in $D^b_{coh}(X)$, the bounded derived category.

Examples such as the above are the reason we assume the scheme S is regular. In a regular scheme, all local rings are regular, disallowing such infinite resolutions. In the regular case, the pullbacks of all points from S will be of finite tor dimension, and in turn perfect.

Therefore, all generalized points of $\mathcal{X}_s$ are contained in $Coh(\mathcal{X}_s/s)(s)$. Furthermore, since $\mathcal{X}$ is smooth, so to are all of the fibers over the points of S. Therefore, each $\mathcal{X}_s$ is smooth. With this in mind, we can apply the argument laid out in section 6 to each $\mathcal{X}_s$. Then, proposition 7.0.2 will complete the proof of the Theorem 7.0.1.

If we wish to have a result where the base is not required to be regular, we will need to examine a different category of objects.

CHAPTER 8

A COUNTEREXAMPLE IN ARBITRARY CHARACTERISTIC

In this section, we will recount a counterexample given in [12] showing that Theorem 2.0.2 is not true in arbitrary characteristic. Suppose that X is a smooth projective scheme of dimension m over a field k of nonzero characteristic p. Consider the Frobenius morphism $F : X \to X^{(p)}$. Take Φ to be the Fourier-Mukai type functor with kernel equal to the graph of F. It is well known that Φ corresponds to the pushforward of F. For any point x, we have $F_*(\mathcal{O}_x) = \mathcal{O}_{F(x)}$. Therefore, we have

$$\mathrm{Hom}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) = \mathrm{Hom}(F_*(\mathcal{O}_x), F_*(\mathcal{O}_x)) = \mathrm{Hom}(\mathcal{O}_{F(x)}, \mathcal{O}_{F(x)}) \simeq k$$

for any $x \in X$. Furthermore, if we have two distinct points $x, y \in X$, we have

$$\mathrm{Hom}^i(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_y)) = \mathrm{Hom}^i(\mathcal{O}_{F(x)}, \mathcal{O}_{F(y)}) = 0$$

for $i \leq m$. These two conditions are the Bondal Orlov conditions for schemes, so we might expect that Φ should be fully faithful. However, it can be shown that $F_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_{X^{(p)}}$ module of rank p^m. As a result, we have

$$\mathrm{Hom}^0(\Phi(\mathcal{O}_X), \Phi(\mathcal{O}_x)) = \mathrm{Hom}^0(F_*(\mathcal{O}_X), \mathcal{O}_{F(x)}) \simeq k^{p^m}$$

However, it is also the case that

$$\mathrm{Hom}^0(\mathcal{O}_X, \mathcal{O}_x) = k$$

so that Φ is not fully faithful. This counterexample demonstrates that theorems of this sort will not hold even in the case of projective schemes. In the scheme case, one can alter the conditions to exclude this type of example to obtain a result, as shown in the previously cited paper. However, regardless of the proof method, such results will require alteration in the case of nonzero characteristic. In the sections to follow, we will begin our discussion of relaxing the hypotheses of theorem 1.0.2.

CHAPTER 9

COMPACT GENERATION

In this section, we will discuss some preliminary results that will be necessary in proofs that will follow. In the interest of proving theorem 1.0.4, it will be essential to show that the category $D^b_{coh}(\mathcal{X})$ is saturated as defined below.

Definition Assume $\mathcal{D}$ is Ext-finite. Then $\mathcal{D}$ is saturated if every contravariant cohomological functor of finite type $H : \mathcal{D} \to Vect(k)$ is representable.

We wish to apply the following theorem of Bondal and Van Den Bergh [2, Thm 1.3]

Theorem 9.0.1 *Assume that $\mathcal{D}$ is Ext finite and has a strong generator. Assume in addition that $\mathcal{D}$ is Karoubian. Then $\mathcal{D}$ is saturated.*

Given that Ext-finiteness is one of our hypotheses, the main difficulty in showing that this is applicable is the strong generation hypothesis. To show that this will be satisfied, we aim to generalize another result of the same authorship [2, Thm 3.1.4]

Theorem 9.0.2 *Assume that X is a smooth scheme over a field. Then $D^b_{coh}(X)$ is strongly finitely generated.*

In the above result X is a scheme, so it clearly does not apply directly. However, the proof given in [2] would hold in our situation so long as the results it relies upon are still legitimate. All but one of these results are categorical, and apply in any situation involving category theory. The last is the following [2, Thm 3.1.1].

Theorem 9.0.3 *Assume that X is a quasi-compact, quasi-separated scheme. Then*

1. *The compact objects in $D_{qcoh}(X)$ are precisely the perfect complexes.*
2. *$D_{qcoh}(X)$ is generated by a single perfect complex.*

The second conclusion is the result that is useful in the proof of the previous theorem. There is an analogous result that applies in our case given by [9, Thm A]. From this, we can conclude that the strong generation hypothesis in theorem 9.0.1 is satisfied.

We will also need to invoke the existence of a Serre functor on $D^b_{coh}(\mathcal{X})$. This is shown in [10, Thm 2.22, Cor 2.30].

CHAPTER 10

ADJOINTS

In this section, we will show that under the conditions of Theorem 1.0.2, the functor F will have left and right adjoints. To show that F has a right adjoint, we use the fact that $D^b_{coh}(\mathcal{X})$ is saturated. This means that any contravariant cohomological functor on this category is representable.

Then, we consider the functor $\mathrm{Hom}_{\mathcal{T}}(F(-), N)$ for some $N \in \mathcal{T}$. This is a contravariant functor on $D^b_{coh}(\mathcal{X})$, so it is representable by some $H(N) \in D^b_{coh}(\mathcal{X})$ where H is a functor from $T \to D^b_{coh}(\mathcal{X})$. Therefore, we have that $\mathrm{Hom}_{\mathcal{T}}(F(-), N) \simeq \mathrm{Hom}_{\mathcal{X}}(-, H(N))$. It follows that H is a right adjoint to F.

To see that F has a left adjoint, we consider the functor $\mathrm{Hom}_{\mathcal{T}}(M, F(-))^v$. Given any $N \in D^b_{coh}(\mathcal{X})$ we apply representability to find that

$$\mathrm{Hom}_{\mathcal{T}}(M, F(N))^v \simeq \mathrm{Hom}_{\mathcal{X}}(N, G(M))$$

where G is a functor from $T \to D^b_{coh}(\mathcal{X})$. Dualizing, we see that

$$\mathrm{Hom}_{\mathcal{T}}(M, F(N)) \simeq \mathrm{Hom}_{\mathcal{X}}(N, G(M))^v.$$

We then observe that the category $D^b_{coh}(\mathcal{X})$ has a Serre functor, which we will denote S_X. This functor also has an inverse, as it is an equivalence. Applying the definition of a Serre functor to the right side of the above, we find

$$\mathrm{Hom}_{\mathcal{T}}(M, F(N)) \simeq \mathrm{Hom}_{\mathcal{X}}(S_X^{-1} G(M), N).$$

Therefore, $S_X^{-1}G$ is left adjoint to F.

CHAPTER 11

INTEGRALITY AND FURTHER OBSERVATIONS

Let G represent the left adjoint to the functor F described in Theorem 1.0.4. We will now demonstrate that in the situation of Theorem 1.0.4, the functor $G \circ F$ must be of Fourier Mukai type. Indeed, observe the following result given in [3, Thm 1.2] regarding the classification of functors on stacks. Note that we only include the relevant portions of the aforementioned result.

Theorem 11.0.1 *For $\mathcal{X} \to \mathcal{Y}$ a perfect morphism to a derived stack $\mathcal{Y}$ with affine diagonal with $\mathcal{X}'$ arbitrary. Then, there is a canonical equivalence*

$$QC(\mathcal{X} \times_Y \mathcal{X}') = Fun_{QC(\mathcal{Y})}(QC(\mathcal{X}), QC(\mathcal{Y}))$$

between the infinity category of sheaves on the derived fiber product and the infinity category of colimit preserving $QC(\mathcal{Y})$ linear functors.

This result is significantly more general than what we require. The application of this that we intend to use for our purposes is given in section 1 of [4] by equation 4. The statement in question is that if $\mathcal{X} \to S$ and $\mathcal{Y} \to S$ are perfect stacks, then all functors between quasi-coherent sheaves on $\mathcal{X}$ and $\mathcal{Y}$ are given by Fourier Mukai (integral) transforms with kernel in $QC(\mathcal{X} \times_S \mathcal{Y})$.

For our purposes, we will use the above statement but $\mathcal{X}$ and $\mathcal{Y}$ are the same stack and the base S is $spec(k)$ where k is a field of characteristic zero. In order to use this result, we will need to show that under the conditions of Theorem 1.0.4, $\mathcal{X}$ is a perfect stack. A stack is perfect if it has affine diagonal, its quasi coherent sheaves are compactly generated, and compact objects and dualizable objects coincide. The first condition is trivial under these assumptions. The second is proven by the Theorem of Bondal and Van den Berg discussed in a previous section. The third is proven in [9] section 4. Namely, Lemma 4.3 of the aforementioned paper proves that the notions of dualizability and perfection are identical for algebraic stacks. Lemma 4.4 (1) shows that all compact objects are perfect and thus dualizable. Lastly, Lemma 4.5 gives conditions for a perfect object to be compact. Condition (2) must be satisfied in our case due to the ext-finiteness condition. Therefore, under our conditions the stack $\mathcal{X}$ must be perfect.

The functor $G \circ F : D^b_{coh}(\mathcal{X}) \to D^b_{coh}(\mathcal{X})$ does fall inside the category of functors between quasi coherents on $\mathcal{X}$. Therefore, by the consequences of Theorem 11.0.1, $G \circ F$ is a Fourier Mukai type functor. However, it is not necessarily the case that its kernel is coherent. Unfortunately, it does not seem to be the case that functors between coherent complexes are in correspondence with integral transforms with coherent kernels. It is worth mentioning that if $\mathcal{X}$ is smooth and proper there is a correspondence between functors between perfect complexes and integral transforms with perfect kernels ([4] equation 4). However, $G \circ F$ is a functor between coherent complexes, not necessarily perfect complexes.

Remark It is worth mentioning that at this point, Theorem 1.0.4 has been demonstrated. Since the adjoint condition and the Fourier Mukai type condition of Theorem 1.0.2 have been shown to be superfluous, we can simply apply Theorem 1.0.2 to obtain the desired result.

Proceeding from here, we ask if the kernel of $G \circ F$ can be shown to have any properties aside from quasi coherence. Indeed, we have the following result:

Theorem 11.0.2 *Let G be the left adjoint to F where F is the functor described in Theorem 1.0.4. Under the hypotheses of Theorem 1.0.4, the functor $G \circ F$ is of Fourier Mukai type with pseudo coherent kernel.*

Remark There are many equivalent descriptions of the property of psuedo coherence. In this paper, pseudo coherence will be taken to mean a cohomologically bounded above complex with coherent cohomology sheaves. Motivated by this description, we will denote the category of pseudo coherent complexes on X as $D^{-}_{coh}(X)$.

In order to prove this, we will first require a more technical result.

Theorem 11.0.3 *Let X be a proper Deligne Mumford stack over Spec(A) where A is a Noetherian ring. Then, the following are equivalent:*

(1) M is pseudo coherent over X

(2) $RHom(P,M)$ is pseudo coherent for all perfect P on X.

(3) $R\Gamma(X, E \otimes M)$ is pseudo coherent for all pseudo coherent E on X.

Proof First, we will show that (1) implies (3). Let M and E be pseudo coherent complexes on X. It follows that $M \otimes E$ is a pseudo coherent complex. Direct images preserve pseudo coherence, so $R\Gamma(X, E \otimes M)$ is pseudo coherent.

Next, we will show that (2) and (3) are equivalent. To show that (3) implies (2) observe that

$$R\mathrm{Hom}(E,M) = R\Gamma(X, \mathcal{RH}om(E,M))$$

By [13, Tag 08DQ]), we know that

$$R\mathrm{Hom}(E,M) = M \otimes^L E^v$$

Substituting this in to the previous result, we conclude that

$$R\mathrm{Hom}(E,M) = R\Gamma(X, M \otimes E^v)$$

Noting that E and E^v are perfect or not perfect simultaneously and that any perfect complex is also pseudo coherent, this suffices to prove the claim that (3) implies (2).

To show that (2) implies (3), we require an additional assumption, namely, that any pseudo coherent complex admits a perfect approximation. Assume that (2) holds. By the previous argument, this clearly implies that $R\Gamma(X, P \otimes M)$ is pseudo coherent for any perfect P. Any pseudo coherent complex can be represented by a complex which is bounded above, so choose a representative for M which has the property that for some $b \in \mathbb{Z}$, $H^i(M) = 0$ for all $i \geq b$. Further, note that the functor $R\Gamma(X,-)$ has finite cohomological dimension N. Let E be a pseudo coherent complex and choose a perfect approximation $P \to E$ on the $(X, E, m-N-1-b)$ where m is an arbitrary integer. We then complete this to a distinguished triangle

$$P \longrightarrow E \longrightarrow C \longrightarrow P[1]$$

The cohomology sheaves of C are zero above $m-N-1-b$. Tensoring with M can increase this bound by at most b. Therefore, the cohomology sheaves of $C \otimes M$ are zero above $m-N-1$. The cohomological dimension of $R\Gamma(X,-)$ being N

implies that the cohomology sheaves of $R\Gamma(X, C \otimes M)$ are zero above m-1. From our distinguished triangle, there is an induced exact sequence

$$0 \longrightarrow R\Gamma(X, P \otimes M) \longrightarrow R\Gamma(X, E \otimes M) \longrightarrow R\Gamma(X, C \otimes M)$$

Due to the properties discussed above, the first map is an isomorphism above degree m. By assumption $R\Gamma(X, P \otimes M)$ is pseudo coherent. Therefore, its cohomology sheaves are coherent above m and are eventually zero. Since there is an isomorphism between the cohomology sheaves of $R\Gamma(X, P \otimes M)$ and those of $R\Gamma(X, E \otimes M)$ above degree m, it follows that the latter cohomology sheaves will have the same properties. Thus, $R\Gamma(X, E \otimes M)$ is m pseudo coherent. Since m was an arbitrary integer, the result follows.

We will now show that (2) implies (1) to complete the proof. To do so, we will use the coarse moduli space of X. While we have not assumed it's existence in this case, it is the case that all proper Deligne–Mumford stacks admit a coarse moduli space. Let X_{cms} be the coarse moduli space for X with structure map $\pi : X \to X_{cms}$. Assume that (2) holds, namely, that $R\mathrm{Hom}_{\mathcal{O}_X}(P, M)$ is pseudo coherent for all perfect P on X. Since X_{cms} is a proper algebraic space, it follows that $R\pi_* R\mathrm{Hom}_{\mathcal{O}_X}(P, M)$ is pseudo coherent on X_{cms}. The goal is to show that $M \in D^-_{coh}(X)$. This question is local on X_{cms} under the etale topology. Therefore, we can replace X_{cms} by an affine scheme $SpecB$ admitting a finite etale cover $p : Spec(C) \to X$. This gives rise to the following diagram.

$$\begin{tikzcd}
Spec(C) \arrow[d, "p_B"] \arrow[dr, "p"] & \\
X_B \arrow[r] \arrow[d, "\pi_B"] & X \arrow[d, "\pi"] \\
Spec(B) \arrow[r] & X_{cms}
\end{tikzcd}$$

Because the question is etale local and p is a finite etale cover, we need only show that $p^*M \in D^-_{coh}(Spec(C))$. We now consider the object $R\mathrm{Hom}_{\mathcal{O}_X}(p_*\mathcal{O}_{Spec(C)}, M)$. Note that this object is pseudo coherent because $p_*\mathcal{O}_{Spec(C)}$ is a vector bundle and is therefore perfect. By adjunction, we have that

$$R\mathrm{Hom}_{\mathcal{O}_X}(p_*\mathcal{O}_{Spec(C)}, M) \simeq R\mathrm{Hom}_{\mathcal{O}_{Spec(C)}}(\mathcal{O}_{Spec(C)}, p^*M)$$

Then, we observe that

$$R\mathrm{Hom}_{\mathcal{O}_{Spec(C)}}(\mathcal{O}_{Spec(C)}, p^*M) \simeq p^*M.$$

It follows that $p^*M \in D^-_{coh}(\mathcal{O}_{Spec(C)})$, completing the proof.

To finish our argument for Theorem 11.0.2, we require one more result. Together with Theorem 11.0.3, the following result will trivialize the argument in question.

Theorem 11.0.4 *Let $X \to S$ and $Y \to S$ be perfect stacks with X proper and Deligne Mumford. Let F a coproduct preserving, Fourier Mukai type functor with kernel M on $X \times_S Y$. Then $M \in D^+_{coh}(X \times_S Y)$.*

Proof We will use Theorem 11.0.2 to prove this result. The relevant implication will be that (2) implies (1). Again, the question is etale local, so we may assume that the etale cover of Y is affine. Let $p : Spec(R) \to Y$ be an etale cover of Y. We now consider the following diagram:

$$\begin{tikzcd}
X_R \arrow[r, "p_l"] \arrow[d, "q"] & Spec(R) \arrow[d, "p"] \\
X \times_S Y \arrow[r, "p_Y"] \arrow[d, "p_X"] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

In the above diagram we will denote the composed map $X_R \to X$ as l. Furthermore, as an additional notation, the object $M_{(-)}$ will represent the kernel of a Fourier Mukai type functor. In this notation, we have $M_F = M$. As a final note, the map l and the map from $Spec(R) \to S$ in the above diagram are quasi affine. Using the above diagram, we will now examine the functor $p^*F : D^-_{coh}(X) \to D^-_{coh}(Spec(R))$. Let N be a member of $D^-_{coh}(X)$. For compactness, we will use underived notation. Since F is a Fourier Mukai type functor with kernel M_F, we have the formula

$$p^*F(N) = p^*p_{y,*}(p_x^*N \otimes M_F)$$

By base change, we find that

$$p^*F(N) = p_{l,*}q^*(p_x^*N \otimes M_F)$$

It follows that $q^*M_F \simeq M_{p^*F}$.

Now, let P be a perfect on X and consider the object $R\mathrm{Hom}_{\mathcal{O}_{X_R}}(l^*P, M_{p^*F})$. By well-known formulas, we find that

$$R\mathrm{Hom}_{\mathcal{O}_{X_R}}(l^*P, M_{p^*F}) \simeq R\Gamma(X_R, (l^*P)^v \otimes M_{p^*F})$$

Note that perfects are dualizable in this setting and that the dual to any perfect will also be perfect. For this reason, taking the dual of l^*P will cause no problems going forward. Applying our previous conclusion, we observe that

$$R\mathrm{Hom}_{\mathcal{O}_{X_R}}(l^*P, M_{p^*F}) \simeq R\Gamma(X_R, (l^*P)^v \otimes q^*M_F)$$

However, the right hand side of this equation is $p^*F(P^v)$. This object is pseudo coherent, thus, so to is $R\mathrm{Hom}_{\mathcal{O}_{X_R}}(l^*P, M_{p^*F})$. By Theorem 11.0.3, it follows that $M_{p^*F} \simeq q^*M_F$ is pseudo coherent. By the proper pushforward Theorem, M_F is pseudo coherent, finishing the proof.

As discussed above, the conditions of Theorem 1.0.4 imply that the hypotheses of Theorem 11.0.4 are satisfied, so the argument is completed.

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