A QUANTUM STOCHASTIC APPROACH TO POISSON MASTER EQUATION UNRAVELLINGS AND GHIRARDI-RIMINI-WEBER THEORY

by

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A Dissertation Submitted to the Faculty of the

GRADUATE INTERDISCIPLINARY PROGRAM IN APPLIED MATHEMATICS

In Partial Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College

THE UNIVERSITY OF ARIZONA

2022

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

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I'd like to thank the committee, Tom Kennedy and Sunder Sethuraman, and also Maciej Lewenstein for agreeing to review this thesis. I'd especially like to thank Jan Wehr, without whose patience and support this project would not have been possible. Lastly, but no less important, I'd like to thank my family for being an unending source of stability throughout my studies and throughout my entire life.

To Akasha, Sky, who masterfully watches over the entirety.

Contents

Abstract

The theory of quantum stochastic calculus is used to expand the traditional Ghirardi-Rimini-Weber theory to a fully quantum theory, where the noise can be then be interpreted as a new field interacting with quantum systems. A derivation of a stochastic unravellling of the GKSL master equation is first presented from the standpoint of a purely quantum theory, which is then specialized to the case of the GRW master equation. Reverse engineering this procedure gives rise to a new nonlinear quantum stochastic wave equation which preserves a generalized system norm and is an unravelling of the GKSL master equation. Comments on how this particular interpretation can be tested are given.

1. Introduction

The measurement problem has plagued physics since the inception of quantum mechanics, and yet a century's worth of effort has not led to its satisfactory resolution. While some physicists consider the problem resolved, there is not a unified position in the physics community on just what interpretation should be considered the conclusive one. The problem is of paramount importance from a philosophical perspective because its solution delineates the boundary between possible and actual, while from a physical standpoint this delineation is not as important as the predictions which the theory of quantum mechanics makes with resounding success. Quantum mechanics is the most accurate theory in any science we have, and so it is surprising that the exact process of actualization of possibilities is not well described. The prevailing interpretation for decades has been the Copenhagen interpretation which does not describe the process of actualization but leaves it somewhere at the boundary separating two incompatible dynamics, Schrödinger dynamics and projection onto an outcome of a measurement. This interpretation says that collapse happens when a classical system interacts with a quantum system, and that collapse following the Born rule, which says that probability, in the state $|\psi\rangle$, of obtaining an eigenvalue, λ, of an observable with eigenprojector P_λ, is $\| P_\lambda |\psi\rangle \|^2$, represents an irreducible indeterminism. In order to have a consistent theory of measurement, the state of the system changes from $|\psi\rangle$ to the *a posteriori* state

$$|\psi'\rangle = \frac{P_\lambda |\psi\rangle}{\| P_\lambda |\psi\rangle \|},$$

with this change occurring sometime during the process of measurement. This leaves the explanation of the mechanism of actualization incomplete, with nothing more being said on how or when the system collapses onto the *a posteriori* state, but due to the success of the theory at making accurate laboratory predictions the question was more or less dropped by the working physics community and left to theorists, while the rest of the community was left to a 'Shut up and Calculate!' culture. Nevertheless, there were some who continued to develop interpretations of quantum mechanics that were self consistent, such as H. Everett III, developing the Many Worlds Interpretation (MWI) in 1957, [12], and G. C. Ghirardi, A. Rimini and T. Weber, developing the GRW theory of spontaneous collapse in 1985, [14]. These theories solve the issue by eliminating collapse from the theory, replacing it with purely linear evolution as in MWI, or by adding collapse explicitly to the dynamics to unify the two opposing dynamics as in GRW and its cousin theory: Continuous Spontaneous Localization (CSL) [13].

We will approach the measurement problem in the framework of open quantum system theory, in which the universe is divided into two parts, system with Hilbert space $\mathscr{H}_S$, and environment with Hilbert space $\mathscr{H}_E$. An element of the tensor product $|\psi\rangle \in \widetilde{\mathscr{H}} = \mathscr{H}_S \otimes \mathscr{H}_E$ can be considered a universal state vector for the joint system. One of the results of this work is to put the GRW stochastic collapse models into the open quantum systems framework, creating a version with quantum noise. GRW and CSL are described by stochastic differential equations (SDEs), which are unravellings of the "master equation" of open quantum systems in the sense which we now describe.

In the theory of open quantum systems, one studies the GKSL [16, 21] master equation, which is a differential equation

$$\frac{d}{dt}\rho(t) = \mathscr{L}[\rho],$$

where ρ is the system's density operator and $\mathscr{L}$ is an operator (or "superoperator") acting on density operators by

$$\mathscr{L}[\rho] = -i[H,\rho] + \frac{1}{2}\sum_i \left(2L_i\rho L_i^\dagger - L_i^\dagger L_i\rho - \rho L_i^\dagger L_i\right), \tag{1}$$

called the Lindbladian. An unravelling of the master equation is an SDE for a stochastic process $|\psi_t\rangle$ with values in the system's Hilbert space, $\mathscr{H}_0$, such that the expected value of the projection onto $|\psi_t\rangle$ solves the equation:

$$\mathbb{E}d|\psi_t\rangle\langle\psi_t| = \mathscr{L}[\mathbb{E}|\psi_t\rangle\langle\psi_t|]\,dt.$$

The unravelling may be thought of in the context of a Monte Carlo algorithm for integrating the GKSL equation, where instead of having to solve the full differential equation for the $n^2/2$ components of $\rho(t)$, we solve it for the n-dimensional $|\psi_t\rangle$ and take the average of $|\psi_t\rangle\langle\psi_t|$ over many realizations. For the GKSL equation, there are two well known unravellings. An unravelling driven by a Wiener process was derived by Belavkin and Staszewski [4, 5], but is more often known as the Gisin-Percival equation [15]

$$\begin{aligned} d|\psi_t\rangle = -iH|\psi_t\rangle dt + &\sum_\alpha \left(\left\langle L_\alpha^\dagger\right\rangle_{\psi_t} L_\alpha - \frac{1}{2}L_\alpha^\dagger L_\alpha - \frac{1}{2}|\langle L_\alpha\rangle_{\psi_t}|^2\right)|\psi_t\rangle dt \\ &+ \frac{1}{\sqrt{2}}\sum_\alpha \left(L_\alpha - \langle L_\alpha\rangle_{\psi_t}\right)|\psi_t\rangle dW_\alpha(t) \end{aligned} \tag{2}$$

where dW_α are complex valued Wiener processes satisfying the Itô rules (with dW_α^* denoting the complex conjugate of dW_α),

$$\begin{aligned} &dW_\alpha(t)dW_\beta(t) = dW_\alpha^*(t)dW_\beta^*(t) = 0, \\ &dW_\alpha^*(t)dW_\beta(t) = 2\delta_{\alpha\beta}dt, \end{aligned}$$

and $\langle L_\alpha\rangle_{\psi_t} = \langle\psi_t|L_\alpha|\psi_t\rangle$, making the Belavkin equation non-linear. Another such unravelling is driven by Poisson processes $N_\alpha(t)$,

$$\begin{aligned} d|\psi_t\rangle = -&\left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha - \left\langle L_\alpha^\dagger L_\alpha\right\rangle_{\psi_t}\right)|\psi_t\rangle dt \\ &+ \sum_\alpha \left(\frac{L_\alpha}{\langle L_\alpha^\dagger L_\alpha\rangle_{\psi_t}^{1/2}} - I\right)|\psi_t\rangle dN_\alpha(t). \end{aligned} \tag{3}$$

This equation was derived by Belavkin [2] and was written in this form by Barchielli and Belavkin [1]. The Itô rule for Poisson processes is

$$dN_\alpha(t)dN_\beta(t) = \delta_{\alpha\beta}dN_\alpha(t).$$

In both cases, $|\psi_t\rangle$ can be shown to be an unravelling of the GKSL master equation by using the Itô rules, together with the product rule

$$d(|\psi_t\rangle\langle\psi_t|) = (d|\psi_t\rangle)\langle\psi_t| + |\psi_t\rangle d\langle\psi_t| + d|\psi_t\rangle d\langle\psi_t|,$$

and the expectations $\mathbb{E}dW(t) = 0$, $\mathbb{E}dN(t) = \mathbb{E}\left\langle L_\alpha^\dagger L_\alpha \right\rangle_{\psi_t} dt$. Note, that while the standard Poisson process has expected value $\mathbb{E}dN(t) = dt$, here a change of distribution was used making the Poisson process inhomogeneous with stochastic rate parameter. We will elaborate more on this later. With appropriate choices of the operators L_α, the Belavkin equations become the equations of GRW or CSL models. Belavkin equations are thus of central importance for stochastic collapse models of quantum mechanics. They form the subject of this dissertation, where they are rederived and put into the context of the measurement problem.

2. The Measurement Problem

Attempts to solve the measurement problem fall into three general categories:

I. Non-collapse models: de Broglie–Bohm [6, 7, 8], and Many Worlds Interpretation, [12, 11].

II. Schrödinger with Collapse models: Copenhagen Interpretation [9], Consistent Histories [17], Conscious collapse [31]

III. Stochastic Collapse models: GRW [14] and CSL [13]

All these interpretations account for the process of decoherence which occurs when a system is open to environmental influence. In particular, for the Many Worlds Interpretation, decoherence is integral to the interpretation itself, although the original idea, due to H. Everett III, precedes the formulation of the idea of decoherence by some 30 years.

2.1 Decoherence

Decoherence (see e.g. [32, 28]) is essential to the de Broglie–Bohm and Many Worlds Interpretations because it is responsible for emergence of the classical world and individuation of histories from the whole state vector. The environment, which may be considered a fermionic or bosonic bath of particles, is responsible for decoherence. To make predictions, we trace over the environment's degrees of freedom. We describe the procedure, originally due to von Neumann [30], known as the von Neumann measurement scheme. Consider a Hilbert space $\widetilde{\mathscr{H}} = \mathscr{H}_S \otimes \mathscr{H}_E$—with a system Hilbert space $\mathscr{H}_S$ and an environment Hilbert space $\mathscr{H}_E$. A joint state, ρ, is defined on $\widetilde{\mathscr{H}}$, with reduced states, ρ^S, and ρ^E, which give marginal states defined on either subspace, $\mathscr{H}_S$ or $\mathscr{H}_E$, and are are defined as the partial trace over the opposite space, e.g.

$$\rho^S = \mathrm{Tr}_E[\rho] \text{ and } \rho^E = \mathrm{Tr}_S[\rho].$$

For convenience, we will study the example in which the states of the system Hilbert space describe the spin of a spin-$\frac{1}{2}$ particle. Let $|s_1\rangle, |s_2\rangle$ be eigenstates of the spin operator σ_z and consider an interaction Hamiltonian of the form

$$H_I = |s_1\rangle\langle s_1| \otimes \hat{E}_1 + |s_2\rangle\langle s_2| \otimes \hat{E}_2.$$

This Hamiltonian commutes with the system spin observable σ_z. We consider a ready state of the environment $|e_0\rangle$ and an initial system state $|\psi_0\rangle = a_1|s_1\rangle + a_2|s_2\rangle$, and denote the resulting state after time t by $|\psi_t\rangle$:

$$|\psi_0\rangle|e_0\rangle \quad \mapsto \quad |\psi_t\rangle = a_1|s_1\rangle|e_1(t)\rangle + a_2|s_2\rangle|e_2(t)\rangle. \tag{4}$$

Individually the states $|s_i\rangle|e_i(t)\rangle$ can be seen as the conditions when the system is in state $|s_i\rangle$ and the environment unambiguously determines that the system is in state $|s_i\rangle$ with the corresponding 'pointer' state $|e_i(t)\rangle$. We may then calculate the total density matrix to be

$$\begin{aligned} |\psi_t\rangle\langle\psi_t| = |a_1|^2|s_1\rangle|e_1(t)\rangle\langle s_1|\langle e_1(t)| + |a_2|^2|s_2\rangle|e_2(t)\rangle\langle s_2|\langle e_2(t)| \\ + \left(a_1^* a_2|s_2\rangle|e_2(t)\rangle\langle s_1|\langle e_1(t)| + h.c.\right), \end{aligned}$$

where *h.c.* stands for Hermitian conjugation. Decoherence is used to explain the non-observability of macroscopic interferences. Macrosopic quantum interferences are manifested by the presence of off-diagonal terms of the reduced density matrix. We might imagine our spin-$\frac{1}{2}$ particle to be put through a Stern-Gerlach apparatus and the corresponding environmental states to be observing *which-path* information on the particle as it passes through the apparatus. Then the interference terms, which contain the factors $\langle e_i(t)|e_j(t)\rangle$, and represent macroscopic superpositions, are shown to rapidly decrease in time. This leads to a reduced density matrix which is a classical mixture of the two eigenstates with probabilities satisfying the Born rule.

We make the following important remarks: (i) It is possible to design an interaction Hamiltonian which correlates a system eigenstate of an observable unambiguously with an environmental pointer state. The Hamiltonian carries out a so-called quantum *non-demolition* measurement where the Hamiltonian commutes with the system observable. (ii) The rapid decrease of the non-diagonal elements of the density matrix converts the reduced system density operator from a quantum state, which may represent a superposition of eigenstates, into a classical mixture of eigenstates of the system observable, which may be thought of as a *quantum-to-classical* transition. (iii) Although the dynamics has produced a classical mixture, it has not provided any mechanism for selection of a single outcome among the ensemble. Thus, nothing like a quantum collapse has occurred. (iv) The structure of the interaction Hamiltonian is specific to the eigenbasis of the observable being resolved—a different observable need not be resolvable by the environment. This is known as the *preferred basis* problem. (v) The environmental pointer vectors must be robust under action of the interaction Hamiltonian and other vectors representing nonclassical states must decohere extremely fast. This imposes a superselection rule of the environment for pointer bases, a process termed *einselection* by Zurek [32].

Decoherence is observed in any measured physical system, as coupling to the environment is essential to the procedure of observation by macroscopic observers. It is also quite ubiquitous and autonomous due to the presence of particles, be they thermal photons or cosmic rays, throughout the Universe. Experimental apparatuses must be designed to single out a type of decoherence which allows for the system observable to be measured effectively, isolating it from other parts of the universe which might affect the outcomes of the measurement. For this reason, all quantum mechanical interpretations of the measurement problem must be consistent with the process of decoherence and in many cases are partially helped by it. Remark (ii) above solves part of the measurement problem by describing why macroscopic interferences are not observed, while remarks (iv) and (v) delineate what must be accounted for by the interaction Hamiltonian on the quantum side of the quantum-to-classical transition. In the following we will give a brief survey of the interpretations.

2.2 de Broglie–Bohm Interpretation

Originally, the de Broglie–Bohm interpretation (see [8] for a comprehensive treatment) had a dual ontology of particle and wavefunction. Particles were guided along physically real trajectories by the wavefunction, or 'pilot wave'. When the wavefunction is written in polar form

$$\psi = Re^{iS}$$

and substituted into the Schrödinger equation, one obtains a modified Hamilton–Jacobi equation for the phase

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar}{2m}\frac{\nabla^2 R}{R} = 0,$$

and continuity equation

$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right).$$

for the modulus. The term

$$Q := -\frac{\hbar}{2m}\frac{\nabla^2 R}{R}$$

can be considered a kind of potential, called the quantum potential. One may actually interpret S as action, corresponding to a Hamiltonian and then define the momentum as the gradient of action which allows to generalize this theory to the relativistic case, where particle number is not conserved, by considering an algebraic interpretation of the action [10]. However, as originally proposed, particles were intrinsic to the theory and traveled along integral curves of the momentum field. Collapse never happens as particles are found in whatever configuration actualized. Randomness originates not in the fundamental sense of the Copenhagen interpretation but as an epistemic randomness, arising from not knowing the initial conditions specifying which trajectory was actualized. Nonlocal and other fundamentally quantum behavior arise from the state vector or "pilot wave", which may be nonlocal and change instantaneously over large distances, guiding the particles along trajectories which incorporate quantum features. The process of measurement contains an amplification mechanism which drastically alters the pilot wave, bifurcating it into the branches consistent with the outcome of the measurement and inactive branches which are never traversed again by the system, leading to effectively irreversible behavior from a dynamics which is purely Schrödinger.

2.3 Many Worlds Interpretation

The Many Worlds Interpretation dates back to the Ph.D thesis of H. Everett III, where it appeared under the name of the 'Relative State Interpretation'. The idea behind a relative state is that in a tensor product space, after the entangling interaction of a von Neumann measurement, as in equation (4), no individual state can be defined for the factor spaces of the tensor product space, only a state relative to a given arbitrarily chosen subsystem state can be defined. In particular, this has the consequence that a quantum jump is a relative proposition in a product space, depending on the decomposition of the total state and therefore not an essential feature. In fact, the state may be considered as a superposition over all definite states corresponding to the outcome of a measurement. Everett outlined a procedure, using only linear entangling interactions, in which he believed a consistent interpretation of quantum mechanics could be achieved. To describe measurement interactions he included only entangling interactions which give rise to more complicated vectors in ever increasing superpositions, the vectors which he identified as branches of the total or *universal* wavefunction, each giving rise to a consistent universe. The procedure starts with a vector in a generalized system space whose elements are linear combinations of N-fold tensor products of usual system state vectors, and in addition tensored with an observer wavefunction which encodes the outcome of measurements of observables in a set $\{A_i\}$, by changing its state according to the result of the measurement (an eigenvalue λ^i_j of A_i). Everett proposed a first rule which is a von Neumann type interaction with an observer record

Rule 1. *A system in an initial state* $|\psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle \otimes |\psi^O_{[\cdots]}\rangle$, *where* $|\psi_i\rangle$ *are initial tensor factors, and* $|\psi^O_{[\cdots]}\rangle$ *is an observer ready state which can encode eigenvalues of measurements, produces after a measurement type interaction, for an observable* A_1 *with eigenvectors* $\{|\phi^1_j\rangle\}$, *the following superposition*

$$|\psi\rangle \mapsto \sum_j \langle \phi^1_j|\psi_1\rangle\, |\phi^1_j\rangle \otimes \bigotimes_{i=2}^{N} |\psi_i\rangle \otimes |\psi^O_{[\cdots,\lambda^1_j]}\rangle.$$

We can see that after Rule 1 is applied, the initial state has been transformed into a superposition of branches, each corresponding to a single outcome of the measurement of observable A_1. The coefficients in the superposition $\left\langle \phi^1_j|\psi_1 \right\rangle$ correspond to the probability amplitudes of obtaining the result λ^1_j from the measurement process. The procedure continues with a second rule describing the next measurement.

Rule 2. *When measuring the next observable, we apply Rule 1 separately to the elements of the superposition resulting from its previous application. Applied to the observable* A_2, *this produces the state*

$$\sum_j \langle \phi^1_j|\psi_1\rangle |\phi^1_j\rangle \otimes \bigotimes_{i=2}^{N} |\psi_i\rangle \otimes |\psi^O_{[\cdots,\lambda^1_j]}\rangle$$
$$\mapsto \sum_{j_1,j_2} \langle \phi^1_{j_1}|\psi_1\rangle \langle \phi^2_{j_2}|\psi_2\rangle\, |\phi^1_{j_1}\rangle \otimes |\phi^2_{j_2}\rangle \otimes \bigotimes_{i=3}^{N} |\psi_i\rangle \otimes |\psi^O_{[\cdots,\lambda^2_{j_2},\lambda^1_{j_1}]}\rangle$$

The iteration of these rules results in a superposition of branches, each with a unique history corresponding to a particular sequence of measurements and their results. For the observer vector we may take as a model vector space $\mathscr{H}_O = \ell^2$, the space of square summable sequences of complex numbers. We may partition the time domain into countable or finite bins labeled by the index k. We can uniquely define orthogonal vectors for each measurement sequence by assigning a measurement number $M = \prod_k p_k^{j_k}$, where p_k is the k^{th} prime and j_i indexes the j^{th} eigenvalue of the observable measured in the i^{th} time bin, and $j_i \equiv 0$ for all time bins in which no measurement took place. Then $\psi^O_{[\cdots,\lambda^i_{j_i},\cdots,\lambda^1_{j_1}]}$ can be mapped to $|e_M\rangle$ being the vector in ℓ^2 with zeros in all places except for the M^{th} place. It is then clear that $\langle e_M|e_{M'}\rangle = \delta^{M'}_M$ so that the observer vectors form an orthonormal set.

Everett showed that there is a unique probability measure on the superpositions, satisfying natural regularity conditions, which describes Born rule. This demonstrates that measurement can be included in quantum mechanics by a purely linear procedure. Probability arises from a distribution on branches, although this is idealized to the case of N tensor factors. In reality, the process of decoherence does not exactly distinguish between branches making the procedure of assigning probabilities somewhat ambiguous. What is clear, however, is that because of the observer interaction, required to keep consistency along the branches of the universal wavefunction, the evolution is not described by a one-parameter unitary group and therefore is not a Schrödinger evolution. This is so because only vectors with consistent measurement numbers M and M' can be mapped to each other by the evolution operator. The evolution is more akin to a stochastic process, with the set of sequences of measurements as a complete stochastic ensemble. Thus, although the evolution is Markovian, the next value of the observer state depends on the current value in an essential manner. We will see that stochastic collapse models also have an observer state, albeit a hidden one which was previously not commented on and which may be compared to that of Everett's interpretation.

2.4 Copenhagen and Belavkin's Cut

The Copenhagen interpretation has been the dominating interpretation in physics for a hundred years. In this interpretation, interaction with a classical system leads to a collapse of a quantum system. This interpretation is slightly generalized to include histories in the interpretation known as consistent histories, where the projection operators signifying measurement are time ordered to include trajectories of the quantum system through projections acting upon the system. This is akin to the MWI with histories being analogous to branches. What causes the projection is still up for debate, so we return to the traditional Copenhagen interpretation and address the question of what causes collapse there. An essential part of the question is: when does a system become classical? A classical system is allowed to collapse the wavefunction but the question of when a system becomes classical was an open question until—in the opinion of the author—Belavkin laid it to rest with his non-demolition causality principle, which states [3],

> The quantum (hidden) properties of any dynamical system at each time t can be attributed only to anticipating questions, generating a present and future subalgebra $\mathbb{B}_t \subset \mathbb{M}_t$ consistent with classical (measurable) history algebras $\mathbb{A}_t = \mathbb{M}'_t$ [where $\mathbb{M}'_t$ denotes the commutant of $\mathbb{M}_t$,] of the total algebra $\mathbb{M}_t \supseteq \mathbb{A}_t$ under the current interest for this at t.

A classical system follows classical probabilistic logic. Classical events are sets in a σ-algebra, or equivalently indicator functions of those sets in a commutative von Neumann algebra. As the past may be described within this logic, it follows that the past events are classical, while future events must follow the expanded rules of quantum mechanics, allowing for events which are part of a noncommutative von Neumann algebra. Thus, the so called 'Heisenberg cut' between classical and quantum system must happen at the present moment, where the past is classical and the future is quantum. Typically, the Heisenberg cut may be described as that dividing two tensor factors $\mathscr{H}_1 \otimes \mathscr{H}_2$ with $\mathscr{H}_2$ representing a classical macroscopic system. We will see that in the Fock space formalism of quantum stochastic calculus we may factor the Hilbert space describing the noise as $\mathscr{H}_{t)} \otimes \mathscr{H}_{[t}$, with a moving factorization depending on the current time coordinate t. Then the 'Belavkin cut' would additionally require that events occurring in $\mathscr{H}_{t)}$ form an abelian center, $\mathbb{A}_t$ of the total Weyl algebra of operators on the Hilbert space, $\mathbb{M}_t$. These events are described by commuting operators as they have already happened and no quantum interferences may be allowed to occur, while future events are part of a fully quantum and noncommutative von Neumann algebra $\mathbb{B}_t$, acting on $\mathscr{H}_{[t}$.

2.5 GRW and CSL

Among the non-Schrödinger interpretations are GRW, for Ghirardi Rimini and Weber [14], and CSL [15], for Continuous Spontaneous Localization. These explicitly add the collapse to the dynamics of the trajectory, using localization operators taken to be Gaussians, which for a particle moving in a three-dimensional space are

$$L_{\mathbf{x}} = \left(\frac{a}{\pi}\right)^{3/4} e^{-a/2(\hat{q}-\mathbf{x})^2}.$$

where a is a localization radius, $\hat{q}$ is the position operator, and $\mathbf{x}$ is a position space parameter. This gives a new evolution given by a particular GKSL equation:

$$\frac{d}{dt}\rho(t) = -i[H,\rho(t)] + \lambda\left(T[\rho(t)] - \rho(t)\right) \tag{5}$$

with

$$T[\rho] = \int_{\mathbb{R}^3} d^3\mathbf{x} L_{\mathbf{x}} \rho L_{\mathbf{x}},$$

and where λ is a rate of localizations. The unravellings of this master equation model a quantum evolution with explicit randomly occurring collapses. These position space localizations compound via a so called 'trigger mechanism' to produce the classical states of macroscopic systems, while still maintaining the quantum nature of small systems. An important feature of GRW/CSL is that they are falsifiable as they offer the potential of different predictions than conventional quantum mechanics, although to date they have not been ruled out. In particular, energy is not conserved in these theories as they do not follow a unitary dynamics and as collapse happens in the position representation through Gaussian localization operators, momentum is injected into the system every time the wavefunction collapses. The collapse parameters for these theories are within a regime in which these effects are consistent with known experiments but potentially falsifiable by future ones. In particular, the continual process of collapse would inject energy on cosmological scales. It is interesting that these theories are a special case of the stochastic unravellings of the master equation discussed above and in the rest of this dissertation, and so our derivation of stochastic unravellings from a unitary dynamics (albeit one without the one-parameter group property), applies to the GRW theory, adding a new context for the theory.

There is an analogy between an open quantum system and a GRW/CSL type system. In particular, we may imagine that the field/bath in the open quantum system is also part of the GRW/CSL system and it is this field, with real physical status, which is responsible for collapse and thus may be called a *collapsaton* field. The framework of quantum stochastic calculus unifies descriptions of the collapsaton field, which was previously done in the classical context of Wiener and Poisson processes, with a properly quantum field theoretic context. However, the classically probabilistic case still gives a representation of the dynamics which makes evident certain features. For instance, the process which produces the norm, ϕ_t, of the unnormalized state vector process encodes the entire history of the system's events up to t and thus may be compared fruitfully with the observer in Everett's model, which we will elaborate on later. The process of normalization is an intricate one and an area where future work may flourish.

2.6 Conscious Collapse

It has been speculated that consciousness might be responsible for collapse of the state vector. The question of which organisms may collapse the wavefunction thus becomes equivalent to the question of how complex must an organism be to be conscious. This theory becomes problematic due to the absence of a theory of consciousness. The conundrum continues with the clearly subjective nature of consciousness. In physics, it is desired to have objective theories to predict result of experiments and so the subjective nature of consciousness is antithetical to the physical methodology. This was exemplified in the paradox of 'Wigner's Friend' [31] where Wigner's friend is in a laboratory conducting an experiment with a result at time t_1, and Wigner comes into the laboratory to ask the result of the experiment. Before Wigner consults his friend he may assign a superposition or purely quantum state to the system at time t_1, but after consultation at time t_2 he may conclude that the system has collapsed into one of the possible states. This means that at time t_1 the system did not have a wavefunction that the friends would agree upon. In the Many Worlds Interpretation, this would be resolved by both friends existing in multiple consistent realities with well defined branches of the wavefunction. In the Copenhagen interpretation, with Belavkin's modification this would be resolved by stating that the collapse happens at t_1 and Wigner had not assigned the correct wavefunction to the system, creating a consistent ontological reality for the two friends independent of their subjectivity. In stochastic collapse models, the collapse happens when the stochastic wavefunction collapses to the state independent of an observer and in de Broglie–Bohm theory the state is well defined regardless of an observer, as an integral along a momentum vector field.

3. Events and Observables in Quantum Probability

In quantum theory, a probability space is a triple, $(\mathscr{H}, \mathscr{P}(\mathscr{H}), \rho)$, consisting of a Hilbert space of state vectors $\mathscr{H}$, the set of all orthogonal projections on the Hilbert space, $\mathscr{P}(\mathscr{H})$, and a state $\rho \in \mathscr{S}(\mathscr{H})$, where $\mathscr{S}(\mathscr{H})$ is the set of all normalized trace-class operators on the Hilbert space. The space of events is the set of orthogonal projections, endowed with natural operations of multiplication and addition of linear operators. However, events may be *incompatible*: if, by analogy with the classical case we think of the joint event as the product of the two orthogonal projections, we have to remember that in the quantum case they may not commute. Thus, for two events $E, D \in \mathscr{P}(\mathscr{H})$, we have to allow for the possibility that

$$[E,D] = ED - DE \neq 0,$$

expressing the irresolvability of the joint event, and highlighting the essential difference between classical and quantum probability. For two events $E, D \in \mathscr{P}(\mathscr{H})$, we define the join,

$$E \wedge D = \inf_{\dim V} \{V : R(E) \cap R(D) \subseteq V\},$$

and the meet,

$$E \vee D = \inf_{\dim V} \{V : R(E) \cup R(D) \subseteq V\},$$

where $R(E)$ is the range of the event E, but these may generally differ from the algebraic operations natural for linear operators on $\mathscr{H}$. The probability of an event, E, is given via the state using the trace operation

$$P(E) = \langle E \rangle = \mathrm{Tr}\rho E.$$

We may identify the states $\rho = |u\rangle\langle u|$ with 1-dimensional range with pure states, or vectors $|u\rangle \in \mathscr{H}$, in which case the trace simplifies to

$$P(E) = \langle u|E|u\rangle .$$

We define observables as bounded self-adjoint operators on $\mathscr{H}$. For an observable X, let $\sigma(X)$ denote the spectrum of the operator X. X can be spectrally resolved, using a projection-valued measure ξ, associating an element of $\mathscr{P}(\mathscr{H})$ with every Borel subset of $\sigma(X)$, in such a way that $\xi(\sigma(X)) = I$ and for mutually disjoint $\{F_i : F_i \in \mathscr{F}\}$, with $\mathscr{F}$ the Borel σ-algebra, we have

$$\xi\left(\bigcup_i F_i\right) = \sum_i \xi(F_i),$$

where the (possibly infinite) sum, is strongly convergent, and $\xi(F \cap G) = \xi(F)\xi(G)$ for Borel sets F, G. The spectral resolution of X is

$$X = \int_\Omega \lambda \xi(d\lambda),$$

where the spectral projections are indicator functions of X: $\xi(F) = I_F(X)$—see [23] or [26] for details. In particular, for a pure state, $|u\rangle \in \mathscr{H}$ we define the measure.

$$\mu_u^X(E) = \int_E \lambda \langle u|\xi(d\lambda)|u\rangle$$

This is how one may obtain classical measures from quantum operators. In some sense, this mirrors the classical space, where an indicator is a simplified orthogonal projection operator on the space of observables.

Classically, a probability space is a triple, $(\Omega, \mathscr{F}, \mathbb{P})$, consisting of a set (of probability parameters) Ω, a σ-algebra, $\mathscr{F}$, and a probability measure, $\mathbb{P}$. An elementary event may be equivalently thought of as either a set $E \in \mathscr{F}$ or its indicator function, $\mathbb{1}_E(\omega)$, taking value 1 or 0 according to whether ω is in E or not. The indicator is a useful object in studying the event. The probability of the event is equal to the integral of its indicator with respect to the probability measure.

$$\mathbb{P}(E) = \mathbb{E}\mathbb{1}_E = \int_\Omega \mathbb{1}_E(\omega) d\mathbb{P}(\omega).$$

Observables are measurable functions, $f : (\Omega, \mathscr{F}, \mathbb{P}) \to (S, \mathscr{G})$, from the probability space to some state space S, with σ-algebra, $\mathscr{G}$. The event that an observable takes a value in a set $A \in \mathscr{G}$, $E = f^{-1}(A)$ thus belongs to $\mathscr{F}$ and has a well-defined probability. In a classical probability space, the probability that two events occur is given by the probability of their intersection and the probability that at least one event occurs is given by the probability of their union. Thus we see that the quantum space is a more general probability space with classical measures derivable as spectral measures of certain self-adjoint operators.

4. Quantum Stochastic Calculus

The following brief introduction omits most of the proofs, except proofs of statements crucial for development Poisson stochastic calculus later on. We rely heavily on the canonical text [23] and refer the reader there for more details. Quantum stochastic calculus (QSC) uses the formalism of second quantization to define a more general theory of probability than the classical theory. Second quantization was developed to describe quantum systems with a variable number of particles rather than wavefunctions dependent on a fixed number of variables, and hence a fixed particle number. The space which allows for arbitrary particle number of a single species, represented by a Hilbert space $\mathfrak{h}$, is the Fock space over $\mathfrak{h}$,

$$\Gamma(\mathfrak{h}) = \bigoplus_n \mathfrak{h}^{\otimes n}.$$

The tensor product is symmetric or antisymmetric according to whether the particles are bosons or fermions, respectively. We will excusively consider the symmetric case. $\mathfrak{h}$ can be interpreted as a subspace of $\Gamma(\mathfrak{h})$—the *one-particle space*. A useful domain for QSC operators will be the exponential vectors, $|e(u)\rangle$, defined as

$$|e(u)\rangle = \bigoplus_n \frac{1}{\sqrt{n!}} u^{\otimes n}.$$

The linear span E of the exponential vectors is linearly dense in $\Gamma(\mathfrak{h})$. They have inner products

$$\langle e(u)|e(v)\rangle = \exp\langle u|v\rangle$$

We may act on the one-particle space with the Euclidean group of translations and unitary operators, $\{(u,U) \,:\, u \in \mathfrak{h}, U \in \mathscr{U}(\mathfrak{h})\}$, with the group action defined by

$$(u,U)|v\rangle = U|v\rangle + |u\rangle,$$

The above operators can be extended to *Weyl operators* on the Fock space ("second-quantized") by defining their action on the exponential vectors as follows:

$$W(u,U)|e(v)\rangle = \exp\left(-\frac{1}{2}\|u\|^2 - \langle u|Uv\rangle\right)|e(Uv+u)\rangle.$$

Two strongly continuous unitary groups on the Fock space are defined using the Weyl operators:

$$W(tu)|e(v)\rangle = \exp\left(-\frac{1}{2}t^2\|u\|^2 - t\langle u|v\rangle\right)|e(v+tu)\rangle$$

and

$$\Gamma(e^{-itH})\big|e(v)\rangle = \big|e\left(e^{-itH}v\right)\rangle.$$

These groups have Stone generators $p(u)$ and $\lambda(H)$, respectively, so that

$$W(tu) = e^{-itp(u)}$$

and

$$\Gamma\left(e^{-itH}\right) = e^{-it\lambda(H)}.$$

The generator $p(u)$ may be thought of as a momentum operator for generating translations in the direction of $\big|u\rangle$, while the second generator $\lambda(H)$ may be thought of as a differential second quantization of H. We can extend $\lambda(H)$ to non-selfadjoint operators, defining it on a bounded operator B as

$$\lambda(B) = \lambda\left(\frac{1}{2}\left(B+B^{\dagger}\right)\right) + i\lambda\left(\frac{1}{2i}\left(B-B^{\dagger}\right)\right).$$

Proposition 1. *The momentum operators satisfy the commutation relations*

$$[p(u), p(v)] = (\langle u|v\rangle - \langle v|u\rangle)$$

Proof. We begin by considering the momentum operators as Stone generators for the Weyl group $p(u) = -i\frac{d}{dt}W(tu)\Big|_{t=0}$. So

$$-\frac{\partial^2}{\partial s\partial t}\left\langle e(w')|e^{-isp(u)}e^{-itp(v)}|e(w)\right\rangle\Big|_{s=t=0} = -\frac{\partial^2}{\partial s\partial t}\left\langle e(w'-su)|e(w+tv)\right\rangle\Big|_{s=t=0} \tag{6}$$

$$= -\frac{\partial^2}{\partial s\partial t}\exp(\left\langle w'|w\right\rangle + t\left\langle w'|v\right\rangle - s\left\langle u|w\right\rangle + st\left\langle u|v\right\rangle)\Big|_{s=t=0} \tag{7}$$

$$= \langle u|v\rangle \exp(\left\langle w'|w\right\rangle) \tag{8}$$

By permuting u and v we get

$$[p(u), p(v)] = (\langle u|v\rangle - \langle v|u\rangle)$$

□

These are equivalent to the canonical commutation relations (CCR) of quantum field theory. Another important set of commutation relations is given in the following proposition.

Proposition 2. *The momentum operator and the differential second quantization operator satisfy*

$$i[p(u), \lambda(H)] = -p(Hu)$$

Proof. The calculation proceeds in three parts:

(I) The matrix elements of the momentum operator are

$$\langle e(w)|p(u)e(v)\rangle = i\frac{d}{ds}\langle e(w)|e(v+su)\rangle\Big|_{s=0}\exp\left(\frac{1}{2}s^2\|u\|^2 - s\langle u|v\rangle\right)\Big|_{s=0} \tag{9}$$

$$= i\frac{d}{ds}\exp\left(\langle w|u\rangle + s\langle w|u\rangle - \frac{1}{2}s^2\|u\|^2 - s\langle u|v\rangle\right)\Big|_{s=0}$$

$$= i(\langle w|u\rangle - \langle u|v\rangle)\exp\langle w|v\rangle$$

(II) Let $U_t = e^{-itH}$. Then

$$\frac{\partial^2}{\partial s \partial t} \left\langle e(w) | \Gamma(U_t) W(su) \Gamma(U_t^{-1}) e(v) \right\rangle \Big|_{s=t=0} = \left\langle e(U_t^{-1} w) | e(U_t^{-1} v + su) \right\rangle \times \tag{10}$$

$$\exp\left(-\frac{1}{2} s^2 \|u\|^2 - s \left\langle u | U_t^{-1} v \right\rangle \right) \tag{11}$$

$$= \frac{d}{dt} \left(\left\langle e^{itH} w | u \right\rangle - \left\langle u | e^{itH} v \right\rangle \right) \Big|_{t=0} \exp \langle w | v \rangle \tag{12}$$

$$= \left(\langle iHw | u \rangle - \langle u | iHv \rangle \right) \exp \langle w | v \rangle \tag{13}$$

$$= \left(\langle w | -iHu \rangle - \langle -iHu | v \rangle \right) \exp \langle w | v \rangle \tag{14}$$

(III) Lastly, we have

$$\frac{\partial^2}{\partial s \partial t} \left(\left\langle e(w) | \Gamma(U_t) W(su) \Gamma(U_t)^{-1} e(v) \right\rangle \right) \Big|_{s=t=0} = \frac{\partial^2}{\partial s \partial t} \left(\left\langle e^{itH} e(w) | e^{isp(u)} e^{-itH} e(v) \right\rangle \right) \Big|_{s=t=0} \tag{15}$$

$$= \frac{\partial}{\partial s} \left(\left\langle i\lambda(H) \Gamma(U_t)^{-1} e(w) | e^{-isp(u)} \Gamma(U_t)^{-1} e(v) \right\rangle \right) \tag{16}$$

$$+ \left\langle \Gamma(U_t^{-1}) e(w) | e^{-isp(u)} \left(i\lambda(H) \Gamma(U_t)^{-1} \right) e(v) \right\rangle \Big|_{s=t=0} \tag{17}$$

$$= \left(\langle i\lambda(H) e(w) | -ip(u) e(v) \rangle + \langle e(w) | -ip(u) i\lambda(H) e(v) \rangle \right) \tag{18}$$

$$= i \langle e(w) | [p(u), \lambda(H)] e(v) \rangle \tag{19}$$

It follows that

$$i[p(u), \lambda(H)] = -p(iHu)$$

□

We define the position operator $q(u) = -p(iu)$ and, in a way analogous to the canonical formalism of second quantization, the creation and annihilation operators

$$a(u) = \frac{1}{2}(q(u) + ip(u))$$

and

$$a^\dagger(u) = \frac{1}{2}(q(u) - ip(u)).$$

The exponential vectors are right eigenvectors for the annihilation operators and left eigenvectors for the creation operators

$$a(u) \big| e(v) \rangle = \langle u | v \rangle \, | e(v) \rangle \tag{20}$$

$$\langle e(v) \big| a^\dagger(u) = \langle v | u \rangle \, \langle e(v) | \tag{21}$$

while the differential second quantization operator has matrix elements

$$\langle e(u) | \lambda(H) e(v) \rangle = \langle u | Hv \rangle \exp \langle u | v \rangle .$$

The creation and annihilation operators satisfy the canonical commutation relations

$$\left[a(u), a^{\dagger}(v)\right] = \langle u|v\rangle\,.$$

In QSC, the aim is to describe a system, with Hilbert space $\mathscr{H}_0$ interacting with the environment which evolves in time, generating excitations whose energies are exchanged with the system. The evolution of the environment is modeled using the Fock space $\Gamma(\mathfrak{h})$, where elements of $\mathfrak{h}$ bear explicit time dependence. Here we will consider the bosonic stochastic calculus which uses the symmetric Fock space. Fermionic (antisymmetric) and free versions are also possible. The total Hilbert space is then $\widetilde{\mathscr{H}} = \mathscr{H}_0 \otimes \Gamma(\mathfrak{h})$. When we specify the one particle space as $\mathfrak{h} = L^2(\mathbb{R}_+;\mathbb{C}^m)$, with $\{\left|\alpha\right\rangle\}$ being an orthogonal basis of $\mathbb{C}^m$, the Fock space has the factorization:

$$\widetilde{\mathscr{H}} = \mathscr{H}_0 \otimes \Gamma(L^2([0,t_1)) \otimes \Gamma(L^2([t_1,t_2)) \otimes \cdots$$

This property is extremely important in QSC. It is the analogue of independence of increments in classical stochastic calculus, as we will see shortly. Denote the Hilbert space factors $\mathscr{H}_0 \otimes \Gamma(L^2([0,t))$, $\Gamma(L^2([t_i,t_{i+1}))$, $\Gamma(L^2([t,\infty)))$ by $\mathscr{H}_{t]}$, $\mathscr{H}_{[t_i,t_{i+1})}$ and $\mathscr{H}_{[t}$ respectively. Given a process $a(\mathbb{1}_{[t_i,t_{i+1})}\langle\alpha\right|)$ acting on the i-th factor only, we may extend its action to the entire Hilbert space by tensoring it with the identity. These operators are statistically independent for different i since the trace over $\widetilde{\mathscr{H}}$ factors as a product over different factors, mirroring the classical notion of independent increments. It is these increments we use to define stochastic processes. Proceeding analogously with the other fundamental operators, we may define new processes,

$$\begin{aligned} A_\alpha(t) &= a\left(\mathbb{1}_{[0,t)}\left|\alpha\right\rangle\right), \\ A^{\dagger}_\beta(t) &= a^{\dagger}\left(\mathbb{1}_{[0,t)}\left\langle\beta\right|\right), \\ \Lambda^{\beta}_\alpha(t) &= \lambda\left(\mathbb{1}_{[0,t)}\left|\alpha\right\rangle\left\langle\beta\right|\right), \end{aligned} \tag{22}$$

whose differentials satsify

$$\begin{aligned} \left\langle e(u)|dA^{\dagger}_\alpha(t)e(v)\right\rangle &= u^{\dagger}_\alpha(t)dt\,\langle e(u)|e(v)\rangle\,, \\ \langle e(u)|dA_\alpha(t)e(v)\rangle &= v_\alpha(t)dt\,\langle e(u)|e(v)\rangle\,, \\ \left\langle e(u)|d\Lambda^{\beta}_\alpha(t)e(v)\right\rangle &= u^{\dagger}_\beta(t)v_\alpha(t)dt\,\langle e(u)|e(v)\rangle\,. \end{aligned} \tag{23}$$

Note that if we define the vacuum as $\left|\Omega\right\rangle = \left|e(0)\right\rangle$ by virtue of these definitions

$$dA_\alpha(t)\left|\Omega\right\rangle = 0, \tag{24}$$

$$d\Lambda^{\beta}_\alpha(t)\left|\Omega\right\rangle = 0. \tag{25}$$

These processes in addition to the scalar time process t, form the four fundamental processes of QSC. An important property of exponential vectors is that they behave well under the factorization of Fock space so that

$$\left|e(u\oplus v)\right\rangle = \left|e(u)\right\rangle \otimes \left|e(v)\right\rangle.$$

Analogously to classical stochastic calculus, we may define quantum stochastic integrals by using simple functions. To begin, we define an algebra of adapted processes $\mathscr{A}_t = \left\{X \otimes I_{[t} | X \in \mathscr{O}(\mathscr{H}_{t)})\right\}$

denoting the space of operators on $\mathscr{H}_{t)}$ tensored with the identity on $\mathscr{H}_{[t}$. The analogue of simple functions in classical stochastic calculus are simple operator processes

$$X(s) = \sum_i X_i \mathbb{1}_{[t_i,t_{i+1})}(s)$$

with $X_i \in \mathscr{A}_t$ and we define quantum stochastic integrals of simple processes as, for example

$$\int_0^t X(t) dA_\alpha(t) = \sum_i X_i a\left(\mathbb{1}_{[t_i,t_{i+1})}\langle\alpha|\right).$$

We choose their domains to be the domain generated by the span of the operators in $\mathscr{H}_0$ tensored with exponential vectors. We expand the notion of simple processes via their L^2 closure in the space of operators on $\widetilde{\mathscr{H}}$. This means we look for sequences of simple processes defined over partitions of $\mathbb{R}_+$, $\{0,t_1,t_2,\ldots,t\}$ with cardinality n, and define a general stochastic integral $\int_0^t X(s)dA_\alpha(s)$ to exist provided that there exists a sequence $\int_0^t X^n(s)dA_\alpha(s) = \sum_{i=0}^{n-1} X_i a(\mathbb{1}_{[t_i,t_{t+1})})$ of simple processes such that

$$\left\| \int_0^t X(s) dA_\alpha(s) - \int_0^t X^n(s) dA_\alpha(s) \right\| \to 0.$$

These stochastic integrals are well defined and they have product rules associated with them, which are derived rigorously and comprehensively in [23]. Just as in classical stochastic calculus, we may heuristically define differential processes instead of integrals and all the information in the products of integrals may be captured by the quantum Itô rules:

$$\begin{aligned} dA_\alpha(t) dA_\beta^\dagger(t) &= \delta_\alpha^\beta dt, \\ d\Lambda_\alpha^\beta(t) d\Lambda_\gamma^\sigma(t) &= \delta_\gamma^\beta d\Lambda_\alpha^\sigma(t), \\ d\Lambda_\alpha^\beta(t) dA_\gamma^\dagger(t) &= \delta_\gamma^\beta dA_\alpha^\dagger(t), \\ dA_\alpha(t) d\Lambda_\beta^\gamma(t) &= \delta_\alpha^\gamma dA_\beta(t). \end{aligned} \tag{26}$$

Hudson and Parthasarathy [18] introduced a class of processes which are unitary-valued. They showed that in the case of constant coefficient operators L_α and $S_{\beta\alpha}$, this class of unitary valued stochastic processes, U_t satsify a QSDE, now known as the Hudson-Parthasarathy (HP) equation:

$$dU_t = \left(\sum_\alpha L_\alpha dA_\alpha^\dagger(t) - \sum_\beta L_\beta^\dagger S_{\beta\alpha} dA_\alpha(t) + \sum_\beta (S_{\beta\alpha} - \delta_{\alpha\beta}) d\Lambda_\alpha^\beta(t) - \left(iH + \frac{1}{2} \sum_\alpha L_\alpha^\dagger L_\alpha \right) dt \right) U_t. \tag{27}$$

where U_t are unitary operators precisely when $\sum_\alpha L_\alpha^\dagger L_\alpha$ is a bounded operator, $H = H^\dagger$, and $\sum_\alpha S_{\alpha\beta}^* S_{\alpha\gamma} = \sum_\alpha S_{\beta\alpha} S_{\gamma\alpha}^* = \delta_{\beta\gamma}$. The following lemma will be useful in what follows.

Lemma 1. *The partial trace on a tensor product Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$ satisfies the following identity*

$$\mathrm{Tr}_1[(A \otimes I)(B \otimes C)] = \mathrm{Tr}_1[(B \otimes C)(A \otimes I)]$$

for A, B bounded linear operators on $\mathscr{H}_1$ and C a bounded linear operator on $\mathscr{H}_2$.

Proof. The partial trace, $\mathrm{Tr}_1[\cdot]$ is the linear operator which maps trace class operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$ to trace class operators $\mathscr{H}_2$ satisfying

$$\mathrm{Tr}_1[B \otimes C] \mapsto \mathrm{Tr}_1[B]C.$$

We see then that

$$\mathrm{Tr}_1[(A \otimes I)(B \otimes C)] = \mathrm{Tr}_1[(AB) \otimes C] = \mathrm{Tr}_1[AB]C = \mathrm{Tr}_1[BA]C = \mathrm{Tr}_1[(B \otimes C)(A \otimes I)]$$

□

We can extend the result via linearity to include general trace class operators on $\mathscr{H}_1 \otimes \mathscr{H}_2$. The next theorem says that HP equation serves as a unitary dilation of the Lindblad semigroup [18, 27].

Theorem 1. *Let $T_t = e^{t\mathscr{L}}$ be the Lindblad semigroup generated by $\mathscr{L}$ as in equation 1. Let U_t be a solution of the HP equation, for Hamiltonian, H, Lindblad operators L_α, and scattering operators $S_{\alpha\beta}$. Then U_t is a unitary dilation of T_t to the tensor product space of system and noise:*

$$\rho(t) = T_t\rho(0) = \mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[U_t \big|\psi_0 \otimes \Omega\rangle\langle\psi_0 \otimes \Omega\big| U_t^\dagger\right],$$

Proof. Let $\rho_0 = \big|\psi_0 \otimes \Omega\rangle\langle\psi_0 \otimes \Omega\big|$. We calculate the differential using the quantum Itô rule

$$d\rho(t) = \mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[dU_t\rho_0U_t^\dagger + U_t\rho_0U_t^\dagger + dU_t\rho_0 dU_t^\dagger\right].$$

Noting that $dA_\alpha(t)\big|\Omega\rangle = \langle\Omega\big|dA_\alpha^\dagger(t) = \langle\Omega\big|d\Lambda_\alpha^\beta(t) = d\Lambda_\alpha^\beta(t)\big|\Omega\rangle = 0$ and using lemma 1 we reduce the equation to get

$$\begin{aligned} d\rho(t) = \mathrm{Tr}_{\Gamma(\mathfrak{h})}\Bigg[& -\left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\right) dtU_t\rho_0U_t^\dagger + U_t\rho_0U_t^\dagger\left(iH - \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\right) dt \\ & + \sum_{\alpha\beta} L_\alpha U_t\rho_0U_t^\dagger L_\beta^\dagger dA_\beta(t) dA_\alpha^\dagger(t)\Bigg] \\ & = \mathscr{L}[\rho(t)]dt \end{aligned}$$

where in the second to last line we have used lemma 1 to move the operator $I \otimes dA_\alpha^\dagger(t)$ to the right and use the quantum Itô rules to get a factor of dt. □

It was shown by Hudson and Parthasarathy [18, 23] that if we let η be a projection-valued measure on $\mathfrak{h}$ then $\lambda(\eta)$ gives rise to a quantum Poisson distribution. This means that given a coherent state $\big|\widetilde{e}(u)\rangle = \exp\left(-\frac{1}{2}\|u\|^2\right)\big|e(u)\rangle$ the measure given by $\langle\widetilde{e}(u)|\lambda(\eta)|\widetilde{e}(u)\rangle$ is Poissonian with parameter $\langle u|\eta|u\rangle$. In particular, if $\eta(t) = \mathbb{1}_{[0,t)}$ we have the standard Poisson process. It was shown in [23] that if we want to use $\big|\widetilde{e}(0)\rangle = \big|\Omega\rangle$ we may use the Weyl operator $W(u)$ to generate the same Poisson distribution, $\langle\Omega|W^\dagger(u)\lambda(\eta)W(u)|\Omega\rangle$. The following two propositions make this precise.

Proposition 3. *In the state $\left|\widetilde{e}(u)\right\rangle = \exp\left(-\frac{1}{2}\|u\|^2\right)\left|e(u)\right\rangle$, the distribution of $\lambda(\eta)$ for an arbitrary projection η is Poissonian with parameter $\langle u|\eta|u\rangle$.*

Proof. We calculate the characteristic function

$$\left\langle \widetilde{e}(u) \middle| e^{is\eta} \middle| \widetilde{e}(u) \right\rangle = \left\langle e(u)|e(e^{is\eta}u)\right\rangle = \exp\left(-\|u\|^2 + \left\langle u|e^{is\eta}u\right\rangle\right)$$
$$= \exp\left(-\|u\|^2 + \langle u|\eta|u\rangle\left(e^{is}-1\right) + \langle u|u\rangle\right)$$
$$= \exp\left(\langle u|\eta|u\rangle\left(e^{is}-1\right)\right)$$

where in the second line we used the fact that all powers of a projection are the same. The expression in the last line is the characteristic function of a Poisson distribution with parameter $\langle u|\eta|u\rangle$. □

Proposition 4. *Let $\lambda(H,u) := W(u)^\dagger\lambda(H)W(u)$. Then $\lambda(H,u)$ has the same distribution in the vacuum state $\left|\Omega\right\rangle$ as $\lambda(H)$ does in the coherent state $\left|\widetilde{e}(u)\right\rangle$.*

Proof. We begin by proving an identity:

$$e^{-tB}[A,e^{tB}] = \int_0^t dse^{-sB}[A,B]e^{sB}.$$

Taking the derivative of the left hand side we see that

$$\begin{aligned}\frac{d}{dt}e^{-tB}[A,e^{tB}] &= -Be^{-tB}Ae^{tB} + e^{-tB}ABe^{tB} \\ &= -e^{-tB}BAe^{tB} + e^{-tB}ABe^{tB} \\ &= e^{-tB}[A,B]e^{tB}.\end{aligned} \tag{28}$$

Upon integration the identity is proved. We may calculate

$$\begin{aligned}e^{itp(u)}\lambda(H)e^{-itp(u)} &= e^{itp(u)}[\lambda(H),e^{-itp(u)}] + \lambda(H) \\ &= \int_0^t dse^{isp(u)}[\lambda(H),-ip(u)]e^{-isp(u)} + \lambda(H) \\ &= -\int_0^t dse^{isp(u)}p(iHu)e^{-isp(u)} + \lambda(H) \\ &= \int_0^t ds\left(e^{isp(u)}[p(iHu),e^{-isp(u)}]\right) - tp(iHu) + \lambda(H) \\ &= -i\int_0^t ds\int_0^s dwe^{iwp(u)}\left(\langle iHu|u\rangle - \langle u|iHu\rangle\right)e^{-iwp(u)} - tp(iHu) + \lambda(H) \\ &= t^2\langle u|Hu\rangle - tp(iHu) + \lambda(H).\end{aligned} \tag{29}$$

Evaluating at $t=1$ we see that $\lambda(H,u) = \langle u|Hu\rangle + p(iHu) + \lambda(H) = \langle u|Hu\rangle + q(Hu) + \lambda(H)$ □

Using Proposition 3 with $\eta(t) = \left|\alpha\right\rangle\left\langle\alpha\right|\mathbb{1}_{[0,t)}$ and $u = \left|\alpha\right\rangle$ we see that the quantum process

$$X_\alpha(t) = \Lambda_\alpha^\alpha(t) + A_\alpha(t) + A_\alpha^\dagger(t) + dt \tag{30}$$

can thus be thought of as a quantum version of the standard Poisson process. We now want to represent the HP equations as SDEs driven by classical Poisson processes, starting from the theory of classical Poisson measures.

5. Classical Poisson Measures

The concept of a Poisson measure (see e.g. [20] from which much of the following is taken) is developed in the same way as that of a Wiener measure. A Wiener measure $W(\omega,A)$ is defined on $\Omega\times\mathcal{T}$ where $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space and $(T,\mathcal{T})$ is a parameter space with a σ-algebra $\mathcal{T}$. For example, T may be the time semiaxis $\mathbb{R}_+$ (with the Borel σ-algebra $\mathcal{T}$). The Wiener measure assigns to each element of $\mathcal{T}$ a normal random variable. Assume that the space $(T,\mathcal{T})$ is endowed with a positive measure μ. We say that W is a Wiener measure with the control measure μ if

$$\mathbb{E}W(\omega,A)W(\omega,B)=\mu(A\cap B).$$

Analagously to the Wiener measure, we define a counting measure $N(\omega,A)$ which maps sets in the parameter space to integer-valued random variables $N(A)$. Let ν be a positive measure on $(T,\mathcal{T})$. We say that N is a Poisson measure with the control measure ν if $\{N(A_i)\}$ are independent random variables for mutually disjoint A_i and if, for any A, $N(\cdot,A)$ is a Poisson measure with

$$\mathbb{E}N(\omega,A)=\nu(A).$$

so that

$$\mathbb{P}(N(A)=n)=\frac{\nu(A)^n}{n!}e^{-\nu(A)}$$

If ν is the Lebesgue measure on $\mathbb{R}_+$ then $N(\omega,A)$ becomes the standard Poisson process. For a T-valued random variable x on Ω, a single jump measure, $\delta_x(\omega,A)=\mathbb{1}_A(x(\omega))$ describes a random jump at parameter x, randomly distributed over T. We can also define a mean zero measure $\widetilde{N}=N-\nu$ known as the compensated Poisson measure. In standard Malliavin calculus based on a Wiener measure W, one studies square integrable functionals of W by decomposing them into multiple stochastic integrals with respect to W, see [22]. We will now develop the Poisson analogue of the Wiener Malliavin calculus. Both versions can be used to reinterpret HP equations as classical stochastic differential equations. Here we will develop the Poisson interpretation of the quantum stochastic calculus.

The n-fold stochastic integrals of symmetric functions in $L^2(T^n)$ are orthogonal for different n. They thus form subspaces, known as chaoses and denoted $\mathcal{C}^n$, of the space of square integrable functionals of the Poisson measure

$$\mathcal{P}:=L^2(\mathbb{P}_{\widetilde{N}})$$

We will see that if we let $\mathfrak{h}=L^2(T)$, then the $\mathcal{C}^n\cong\mathrm{Sym}(\mathfrak{h}^{\otimes n})$, with Sym denoting the symmetric subspace. As the chaoses span the space of square integrable random variables we see that $\mathcal{P}\cong\Gamma(\mathfrak{h})$.

We define the difference operator acting on $\mathcal{P}$

$$D_xf(\widetilde{N})=f(\widetilde{N}+\delta_x)-f(\widetilde{N})$$

and we inductively define

$$D^n_{x_1,\ldots,x_n}f(\widetilde{N})=D_{x_n}D^{n-1}_{x_1,\ldots,x_{n-1}}f(\widetilde{N}).$$

This is the jump process equivalent of the Malliavin derivative for Wiener processes since, as we will see shortly, it acts on the domain of exponential vectors in exactly the same manner as the derivative operator [22]. Let $T_n:\mathcal{P}\to L^2(T^n,\nu^{\otimes n})$

$$[T_nf](x_1,\ldots,x_n)=\mathbb{E}D^n_{x_1,\ldots,x_n}f(\widetilde{N})$$

T_n are symmetric measurable functions, with $T_0 f = \mathbb{E} f(\widetilde{N})$. The Hilbert space $\mathscr{P}$ has inner product

$$(f,g)_{\mathscr{P}} = \mathbb{E} f(\widetilde{N}) g(\widetilde{N}).$$

Let $f, g \in \mathscr{P}$, then we have the following isometry

$$\mathbb{E} f(\widetilde{N}) g(\widetilde{N}) = \mathbb{E} f(\widetilde{N}) \mathbb{E} g(\widetilde{N}) + \sum_{n=1}^{\infty} \langle T_n f, T_n g \rangle_{L^2(T^n)}.$$

This isometry shows that

$$L^2(\mathbb{P}_{\widetilde{N}}) \cong \bigoplus_{n=0}^{\infty} L^2(T^n) \tag{31}$$

as Hilbert spaces. This is known as the chaotic expansion property, by analogy with the Wiener chaos expansion [22]. Define the n-fold iterated stochastic integral as

$$I_n(\hat{f}) = \int_0^a \cdots \int_0^{x_2} f(x_1, \ldots, x_n) d\widetilde{N}(x_1) \cdots d\widetilde{N}(x_n),$$

where for simplicity we've chosen $T = [0,a]$. Then we have the following orthogonality relation

$$\mathbb{E} I_n(f) I_m(g) = \delta_n^m n! \left\langle \widetilde{f}, \widetilde{g} \right\rangle_{L^2(T^n)}, \tag{32}$$

where $\widetilde{f}$ denotes the symmetrization, i.e. average over the group of permutations $\mathfrak{S}_n$ of arguments of $\hat{f}$, given by

$$\widetilde{f} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \hat{f}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This leads to the following chaos expansion of f,

$$f(\widetilde{N}) = \sum_{n=0}^{\infty} I_n(T_n f)$$

with $I_0(T_0 f) = \mathbb{E} f(\widetilde{N})$, showing that

$$\mathscr{P} = \bigoplus_{n=0}^{\infty} \mathscr{C}_n.$$

The Malliavin derivative acts as a lowering operator on the chaoses; that is, it acts on random variables $F = \mathbb{E}F + \sum_{n=1}^{\infty} I_n(f_n)$ such that $\sum_{n=0}^{\infty} n n! \|f_n\|^2_{L^2(T^n)} < \infty$ by

$$D_x F = \sum_{n=1}^{\infty} I_{n-1}(\widetilde{f}_n(x, \ldots, x_n)).$$

(See Theorem 3.3 of [20]). Analogously to the classical exponential function, we can define exponential random variables for $f \in L^2(T)$ as

$$\mathscr{E}(f) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}),$$

and see that we also have that $\mathscr{E}(f)$ is an eigenvector of the derivative operator, so that

$$D_x\mathscr{E}(f) = f(x)\mathscr{E}(f).$$

Under the isomorphism 31, the vectors $\mathfrak{E} = \left\{\mathscr{E}(f) \mid f \in L^2(T)\right\}$ map to the exponential vectors in the Fock space over $L^2(T)$. They are total which can be proved in exact analogy to the totality of exponential vectors as in [23]. We may construct processes from the random variables in $\mathscr{P}$ by restricting f via products with the indicators $\mathbb{1}_{[0,t)}$; in this case the differential of $I_n(f_t^{\otimes n})$ is

$$dI_n(f_t^{\otimes n}) = nf_tI_{n-1}(f_t^{\otimes n-1})d\widetilde{N}(t)$$

so that for $\mathscr{E}(f_t)$, we have

$$d\mathscr{E}(f_t) = f_t\mathscr{E}(f_t)d\widetilde{N}(t). \tag{33}$$

The process $\mathscr{E}(f_t)$ is called the Dolèans-Dade exponential process. It follows from the above equation that it is a local martingale. Note that $\mathscr{E}(f)$ is a random variable, and $\mathscr{E}(f_t)$ is a stochastic process equivalently defined via the conditional expectation

$$\mathscr{E}(f_t) = \mathbb{E}[\mathscr{E}(f)|\mathscr{F}_t].$$

If X_t and Y_t are two arbitrary processes, then the corresponding Dolèans-Dade exponential processes are known to satisfy a product rule,

$$\mathscr{E}(X_t)\mathscr{E}(Y_t) = \mathscr{E}(X_t + Y_t + [[X,Y]]_t),$$

where $[[X,Y]]_t$ is the quadratic covariation process. For two Poissonian martingales $X_t = \displaystyle\int_0^t f_s d\widetilde{N}(s)$ and $Y_t = \displaystyle\int_0^t g_s d\widetilde{N}(s)$, we have

$$[[X,Y]]_t = \int_0^t f_s g_s dN(s) = \int_0^t f_s g_s (d\widetilde{N}(s) + d\nu(s)).$$

This means that the product rule for a compensated Poisson-driven Dolèans-Dade process is

$$\mathscr{E}(X_t)\mathscr{E}(Y_t) = \exp\left(\int_0^t f_s g_s d\nu(s)\right)\mathscr{E}\left(\int_0^t f_s + g_s + f_s g_s d\widetilde{N}(s)\right). \tag{34}$$

6. An Isomorphism of Hilbert Spaces

As mentioned before, for Wiener or compensated Poisson measures, $M = W, \widetilde{N}$, we have a chaotic decomposition property that allows us to decompose $L^2(M)$ into chaoses spanned by the n-fold stochastic integrals of $f \in L^2(T^n;\mathbb{C}^m)$, where $(T,\mathscr{T},\nu)$ is a parameter space, taken to be compact in what folows. Considering m independent Poisson processes and for each one of them identifying the nth chaos with the n-particle subspace of the Fock space, we obtain the following isomorphism, explored by Segal [29],

$$\Theta : \Gamma(\mathfrak{h}) \to \mathscr{P},$$

$$\begin{array}{ccc} \mathscr{H}_0 \otimes \mathscr{P} & \xrightarrow{\widetilde{U}_t} & \mathscr{H}_0 \otimes \mathscr{P} \\ \big\downarrow \Theta^{-1} & & \Theta \big\uparrow \\ \mathscr{H}_0 \otimes \Gamma(\mathfrak{h}) & \xrightarrow{U_t} & \mathscr{H}_0 \otimes \Gamma(\mathfrak{h}) \end{array}$$

Figure 1: A diagram showing how unitary evolution is mapped to a stochastic evolution

where $\mathfrak{h} = L^2(T, \nu; \mathbb{C}^m)$ denotes the one-particle space, and M denotes a vector of m independent stochastic processes (Wiener or Poisson). This means that for multi-index α

$$\Theta\big|f(x_1,\ldots,x_n)\rangle = \sum_{|\alpha|=n} \frac{1}{n!}\int_T \cdots \int_T f_\alpha(x_1,\ldots,x_n) dM_{\alpha_1}(x_1)\cdots dM_{\alpha_n}(x_n),$$

and therefore

$$\Theta\big|e(f)\rangle = \mathscr{E}(f).$$

This isomorphism is diagrammed in Figure 1. Both exponential vectors and exponential martingales are total in their respective spaces so they are convenient for calculations involving Θ. Recall that we have defined a quantum version of the Poisson process, which we will now show is isomorphic to the classical Poisson process. From now on $\mathscr{P}$ will denote the space $L^2\left(\mathbb{P}_{\otimes_\alpha \widetilde{N}_\alpha}\right)$ for multiple compensated Poisson processes $\widetilde{N}_\alpha$ $\alpha = 1,\ldots,n$.

Theorem 2. *The operator of multiplication by the Poisson process $N_\alpha(t)$ in $\mathscr{P}$ is isomorphic to the quantum Poisson process $X_\alpha(t)$, that is*

$$N_\alpha(t) = \Theta X_\alpha(t) \Theta^{-1}.$$

Proof. As exponential vectors and exponential martingales are total we may proceed using matrix elements. For the Fock space a simple calculation shows that

$$\langle e(f)|dX_\alpha|e(g)\rangle = \langle e(f)|e(g)\rangle\,(h(t)dt + dt).$$

with $h_\alpha(t) = f^*_\alpha(t) + g_\alpha(t) + f^*_\alpha(t) g_\alpha(t)$, In the case of $\mathscr{P}$ we have

$$\begin{aligned} \mathbb{E}N_\alpha(t)\mathscr{E}(f)^*\mathscr{E}(g) &= \exp(f,g)_{\mathfrak{h}} \mathbb{E}N_\alpha \mathscr{E}(h) \\ &= \exp(f,g)_{\mathfrak{h}} \mathbb{E}(I_1(\mathbb{1}_{[0,t)}) + t)\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n\left(h(t)^{\otimes n}\right)\right). \end{aligned} \tag{35}$$

Here we used the fact that $\widetilde{N}(t) = \int_T \mathbb{1}_{[0,t)}(s) d\widetilde{N}(s) = I_1(\mathbb{1}_{[0,t)}) = N(t) - t$, since in this case ν is the Lebesgue measure. Using the orthogonality relation equation (32), together with the fact that $\mathbb{E}I_n(f) = 0$ for any $f \in L^2(\nu^n)$, we reduce equation (35) to obtain

$$\mathbb{E}N_\alpha(t)\mathscr{E}(f)^*\mathscr{E}(g) = \exp(f,g)_{\mathfrak{h}} \int_0^t h_\alpha(s) + 1 ds.$$

and we see that the matrix elements of both operators are the same. □

7. Derivation of the Poisson Belavkin Equation

We will now derive the Poisson unravelling of the Lindblad master equation in the form expressed in [1]. We will follow the derivation of [19] based on [24]. This unravelling and its Wiener equivalent are relevant to stochastic collapse models, but were originally thought of in the context of measurement with respect to open quantum systems. We start with the HP equation (27) and express it in terms of the quantum Poisson process by adding and subtracting $L_\alpha dA_\alpha(t) + L_\alpha d\Lambda_\alpha^\alpha + L_\alpha dt$, and map to $\mathcal{H}_0 \otimes \mathcal{P}$ using the isomorphism Θ,

$$\begin{aligned} d\big|\psi_t\big\rangle &= \Theta\left(\sum_\alpha L_\alpha dX_\alpha - \left(\sum_\beta L_\beta^\dagger S_{\beta\alpha} + L_\alpha\right) dA_\alpha(t) + \sum_\beta (S_{\beta\alpha} - \delta_{\alpha\beta} - \delta_{\alpha\beta} L_\alpha) d\Lambda_\alpha^\beta(t) - L_\alpha dt \right. \\ &\quad \left. - \left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\right) dt\right) U_t\big|\psi_0 \otimes \Omega\big\rangle \\ &= \Theta\left(\sum_\alpha L_\alpha dX_\alpha - L_\alpha dt - \left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\right) dt\right) \Theta^{-1}\Theta U_t\big|\psi_0 \otimes \Omega\big\rangle. \end{aligned}$$

with $\big|\psi_0\big\rangle$ denoting the initial system vector and $\big|\Omega\big\rangle$ denoting the vacuum state in Fock space. This results in a linear SDE for $\big|\psi_t\big\rangle$ (equation (36) below), after using Theorem 2 and dilating the isomorphism Θ to $I_{\mathcal{H}_0} \otimes \Theta$.

Theorem 3. *The following linear SDE is an unravelling of the GKSL equation*

$$d\big|\psi_t\big\rangle = \left(\sum_\alpha L_\alpha d\widetilde{N}_\alpha - \left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\right) dt\right)\big|\psi_t\big\rangle. \tag{36}$$

Proof.

$$\begin{aligned} \mathbb{E} d\big|\psi_t\big\rangle\big\langle\psi_t\big| &= \sum_\alpha L_\alpha\big|\psi_t\big\rangle\big\langle\psi_t\big| d\widetilde{N}_\alpha - iH\big|\psi_t\big\rangle\big\langle\psi_t\big| dt - \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha\big|\psi_t\big\rangle\big\langle\psi_t\big| dt \\ &\quad + \big|\psi_t\big\rangle\big\langle\psi_t\big|\sum_\alpha L_\alpha^\dagger d\widetilde{N}_\alpha + \big|\psi_t\big\rangle\big\langle\psi_t\big| iH dt - \frac{1}{2}\big|\psi_t\big\rangle\big\langle\psi_t\big|\sum_\alpha L_\alpha^\dagger L_\alpha + \\ &\quad + \sum_\alpha L_\alpha\big|\psi_t\big\rangle\big\langle\psi_t\big| L_\alpha^\dagger d\widetilde{N}_\alpha + \sum_\alpha L_\alpha\big|\psi_t\big\rangle\big\langle\psi_t\big| L_\alpha^\dagger dt \\ &= \mathcal{L}[\mathbb{E}\big|\psi_t\big\rangle\big\langle\psi_t\big|] dt \end{aligned}$$

□

While this equation is an unravelling of the GKSL equation, it does not preserve the norm and so its solution does not represent the physical wavefunction. To obtain a properly normalized solution, we express the evolution equation for the norm squared of this process via the Itô rule

$$d\left\langle\psi_t|\psi_t\right\rangle = (d\langle\psi_t\big|)\big|\psi_t\big\rangle + \langle\psi_t\big| d\big|\psi_t\big\rangle + d\langle\psi_t\big| d\big|\psi_t\big\rangle.$$

The next lemma shows that the process satisfying this equation can be written as an exponential martingale.

Lemma 2. *The norm-squared process satisfies the SDE*

$$d\langle\psi_t|\psi_t\rangle = \sum_\alpha R_\alpha(t)d\widetilde{N}_\alpha(t)\langle\psi_t|\psi_t\rangle,$$

where $R_\alpha = \frac{\left\langle \psi_t|L_\alpha + L_\alpha^\dagger + L_\alpha^\dagger L_\alpha|\psi_t\right\rangle}{\langle\psi_t|\psi_t\rangle}$*. Thus the norm-squared process satisfies the implicit exponential equation*

$$\langle\psi_t|\psi_t\rangle = \mathscr{E}\left(\sum_\alpha \int_0^t R_\alpha(s)d\widetilde{N}_\alpha(s)\right).$$

Proof. This follows from the Itô rules for compensated Poisson processes, which are

$$\begin{aligned} d\widetilde{N}_\alpha(t)d\widetilde{N}_\beta(t) &= \delta_\alpha^\beta\left(d\widetilde{N}_\alpha(t)+dt\right)\\ d\widetilde{N}_\alpha(t)dt &= 0. \end{aligned} \tag{37}$$

Comparing with equation 33, we see that the solution to this SDE is the Dolèans-Dade process. □

To get the Belavkin equation, we will construct a process Φ_t such that $\Phi_t^*\Phi_t = \langle\psi_t|\psi_t\rangle^{-1}$, and multiply the unnormed solution by Φ_t to get a normalized solution, $\left|\Psi_t\right\rangle = \Phi_t\left|\psi_t\right\rangle$. We invert the Dolèans-Dade process in the following lemma.

Lemma 3. *Let* $X_t = \int_0^t f(s)d\widetilde{N}(s)$*. Assume that with probability 1, the jumps of* X_t *are strictly greater than* -1*. Then, the inverse of* $\mathscr{E}(X_t)$ *is*

$$\mathscr{E}(X_t)^{-1} = \exp\left(\int_0^t \frac{f^2}{1+f}ds\right)\mathscr{E}\left(\int_0^t \frac{-f}{1+f}d\widetilde{N}(s)\right). \tag{38}$$

Proof. Using the product rule for Dolèans-Dade processes, equation (34), we have that if $\mathscr{E}(Y_t)$ with $Y_t = \int_0^t g(s)d\widetilde{N}(s)$ is to invert $\mathscr{E}(X_t)$ we must have $f+g+fg=0$ so that $g = \frac{-f}{1+f}$. The exponential term then comes from re-expressing the quadratic variation in terms of the compensated Poisson process and it follows that $\mathscr{E}(X_t)\mathscr{E}(X_t)^{-1} = \mathscr{E}(0) = 1$. □

If we let $S_\alpha = \frac{R_\alpha}{1+R_\alpha}$, then we may express the inverse norm-squared process as

$$\langle\psi_t|\psi_t\rangle^{-1} = \exp\left(\sum_\alpha \int_T R_\alpha S_\alpha ds\right)\mathscr{E}\left(-\sum_\alpha \int_T S_\alpha d\widetilde{N}_\alpha\right).$$

We note that $R_\alpha = \frac{\left\langle \psi_t|(L_\alpha^\dagger+I)(L_\alpha+I)|\psi_t\right\rangle}{\langle\psi_t|\psi_t\rangle} - 1 \geq -1$ with equality in the case of $\left\langle \psi_t|(L_\alpha^\dagger+I)(L_\alpha+I)|\psi_t\right\rangle = 0$. This would imply that $L_\alpha\left|\psi_t\right\rangle = -\left|\psi_t\right\rangle$ or that $\left|\psi_t\right\rangle$ is in the eigenspace of -1. To avoid this possibility we exclude the possibility of an eigenvalue of -1 from L_α. The inverse norm-squared process satisfies the following SDE

$$d\langle\psi_t|\psi_t\rangle^{-1} = \left[\sum_\alpha R_\alpha S_\alpha dt - S_\alpha d\widetilde{N}_\alpha\right]\langle\psi_t|\psi_t\rangle .$$

If we then multiply this process by the unnormalized density matrix $\rho_\psi(t) = |\psi_t\rangle\langle\psi_t|$ we obtain a normalized density matrix $\rho_\Psi(t) = |\Psi_t\rangle\langle\Psi_t|$. We can view the norm-squared process as a Radon-Nikodym derivative for a change of measure

$$\mathbb{E}\rho_\psi(t) = \mathbb{E}\langle\psi_t|\psi_t\rangle\,|\Psi_t\rangle\langle\Psi_t| = \mathbb{E}'|\Psi_t\rangle\langle\Psi_t|$$

where we change from the probability space $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P}_{\widetilde{N}})$ to $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P}'_{\widetilde{N}})$, and where $\langle\psi_t|\psi_t\rangle = \mathbb{E}\left[\langle\psi_T|\psi_T\rangle|\mathscr{F}_t\right] = \mathbb{E}\left[\frac{d\mathbb{P}'_{\widetilde{N}}}{d\mathbb{P}_{\widetilde{N}}}\middle|\mathscr{F}_t\right]$. The Girsanov-Meyer theorem for jump processes allows us to construct new martingales in the primed space,

$$\widetilde{N}'_\alpha(t) = \widetilde{N}_\alpha - \int_0^t \langle\psi_{s-}|\psi_{s-}\rangle\, d\left\langle\widetilde{N}_\alpha(s),\langle\psi_s|\psi_s\rangle\right\rangle = \widetilde{N}_\alpha(t) - \int_0^t R_\alpha(s)ds,$$

(See e.g. [25]) with $\left\langle\widetilde{N}_\alpha(s),\langle\psi_s|\psi_s\rangle\right\rangle$ being the angle bracket process, which is the compensator process, i.e. the process such that $[[\widetilde{N},\langle\psi_s|\psi_s\rangle]] - \left\langle\widetilde{N},\langle\psi_s|\psi_s\rangle\right\rangle$ is a martingale. This allows us to calculate the new expected values, $\mathbb{E}'d\widetilde{N}_\alpha = \mathbb{E}'R_\alpha dt$ and $\mathbb{E}'dN_\alpha = \mathbb{E}(1+R_\alpha)dt$. It follows from a calculation using Itô rules that the normalized density matrix satisfies the SDE

$$\begin{aligned} d\rho_\Psi(t) = {} & \mathscr{L}[\rho_\Psi(t)]dt + \left(\sum_\alpha R_\alpha S_\alpha dt - S_\alpha d\widetilde{N}_\alpha\right)\rho_\Psi(t) \\ & + \left(\sum_\alpha L_\alpha\rho_\Psi(t) + \rho_\Psi(t)L_\alpha^\dagger + L_\alpha\rho_\Psi(t)L_\alpha^\dagger\right)(d\widetilde{N}_\alpha - S_\alpha dN_\alpha), \end{aligned}$$

This density matrix is normalized but is not the most suitable method for integrating the GKSL equation yet. We still have to solve for the $n^2/2$ density matrix elements, whereas if we had a vector unravelling we would only have to solve for the n components of the vector. Also it is the normalized vector unravelling which we will compare to GRW and CSL. In order to get a vector unravelling, we must still take the square root of the inverse norm-squared process to normalize the vector $|\psi_t\rangle$. We do so in the following lemma.

Lemma 4. *Let* $c_\alpha = \frac{1}{\sqrt{1+R_\alpha}}$, *and define*

$$\Phi_t = \exp\left(\sum_\alpha\int_0^t\left[\frac{1}{2}(R_\alpha-2)+c_\alpha\right]ds\right)\mathscr{E}\left(\sum_\alpha\int_0^t(c_\alpha-1)d\widetilde{N}_\alpha\right). \tag{39}$$

Then we have $\Phi_t^2 = \langle\psi_t|\psi_t\rangle^{-1}$.

Proof. We can use the exponential product rule in reverse. Let $\Phi_t = \mathscr{N}_t \mathscr{E}(X_t)$, where $X_t = \sum_\alpha \int_0^t f_\alpha d\widetilde{N}_\alpha$ is an unknown martingale and $\mathscr{N}_t$ is a finite-variation coefficient process. The requirement is then that

$$|\mathscr{N}_t|^2 \mathscr{E}(X_t^*)\mathscr{E}(X_t) = \mathscr{M}_t \mathscr{E}(Y_t),$$

where

$$\mathscr{M}_t = \exp\left(\sum_\alpha \int_0^t R_\alpha S_\alpha ds\right) \text{ and } Y_\alpha = -\int_0^t S_\alpha d\widetilde{N}_\alpha.$$

Then we must have that $f_\alpha + f_\alpha^* + f_\alpha^2 = -S_\alpha$. Solving the quadratic results in $f_\alpha = -1 \pm \sqrt{1-S_\alpha} = -1 \pm c_\alpha$, with $c_\alpha = \frac{1}{\sqrt{1+R_\alpha}}$. If we take $f_\alpha = -1 + c_\alpha$, then to compensate $\mathscr{M}$ along with the terms that come from re-expressing the quadratic variation process $[[X_t, X_t]] = \sum_\alpha \int_0^t 1 - 2c_\alpha + c_\alpha^2 dN_\alpha$ in terms of $\widetilde{N}$, we must have

$$\mathscr{N}_t = \exp\left(\sum_\alpha \int_0^t \frac{1}{2}(R_\alpha - 2) + c_\alpha ds\right),$$

which is the required result. □

The expression we derived for Φ_t implied the SDE

$$d\Phi_t = \left[\sum_\alpha (\frac{1}{2}(R_\alpha - 2) + c_\alpha)dt + (c_\alpha - 1)d\widetilde{N}_\alpha\right]\Phi_t.$$

Finally, the SDE for the normalized unraveling can be obtained.

Theorem 4. *The process $|\Psi_t\rangle = \Phi_t|\psi_t\rangle$, satisfying the SDE,*

$$\begin{aligned} d|\Psi_t\rangle = &-\left(iH + \frac{1}{2}\sum_\alpha L_\alpha^\dagger L_\alpha + 2L_\alpha + 1 - (R_\alpha + 1)\right)dt|\Psi_t\rangle \\ &+ \sum_\alpha (c_\alpha(L_\alpha + I) - I)\,dN_\alpha(t)|\Psi_t\rangle, \end{aligned}$$

is a normalized unravelling of the GKSL equation, in the sense that

$$\mathbb{E}d|\Psi_t\rangle\langle\Psi_t| = \mathscr{L}[\mathbb{E}|\Psi_t\rangle\langle\Psi_t|]dt$$

Proof. That $|\Psi_t\rangle$ is an unraveling follows from theorem 36, the Itô product rule, and the Meyer-Girsanov Theorem with Φ_t as the square root of the Radon-Nikodym derivative. We thus have

$$\mathbb{E}\Phi_t^*\Phi_t|\psi_t\rangle\langle\psi_t| = \mathbb{E}\langle\psi_t|\psi_t\rangle^{-1}|\psi_t\rangle\langle\psi_t| = \mathbb{E}'|\Psi_t\rangle\langle\Psi_t|.$$

□

This equation is different in form from the Belavkin equation but it is equivalent to it, as the following corollary shows.

Corollary 1. *The normalized unravelling SDE can be written as the canonical Belavkin equation*

$$d\big|\Psi_t\rangle = -\left(iH' + \frac{1}{2}\sum_\alpha \left(M_\alpha^\dagger M_\alpha - \|M_\alpha \Psi_t\|^2\right)\right) dt\big|\Psi_t\rangle + \sum_\alpha \left(\frac{M_\alpha}{\|M_\alpha \Psi_t\|} - I\right) dN_\alpha\big|\Psi_t\rangle. \quad (40)$$

Proof. Define $M_\alpha = L_\alpha + I$ and use invariance of $\mathscr{L}$ and the HP equation under the substitions $L_\alpha \mapsto L_\alpha + f_\alpha I$ and $H \mapsto H + \frac{1}{2i}\sum_\alpha f_\alpha^* L_\alpha - f_\alpha L_\alpha^\dagger$, for f_α locally square-integrable in time. The claimed equation follows with $H' = H + \frac{1}{2i}\sum_\alpha M_\alpha - M_\alpha^\dagger$. This change of Hamiltonian is sometimes called a "Lamb shift". □

8. GRW from HP Evolution

We can arrive at the GRW equation if we know that the Lindblad operators are complete, in the sense that $\int_{\mathbb{R}^3} d\mathbf{x} L_\mathbf{x}^\dagger L_\mathbf{x} = I$, and that they represent Gaussian localizations in position space

$$L_\mathbf{x} = \left(\frac{a}{\pi}\right)^{3/4} e^{-a/2(\hat{q}-\mathbf{x})^2}.$$

It is not clear in what sense the completeness integral is defined, so we may simplify the notion by defining a countable net in $\mathbb{R}^3$, $\left\{\mathbf{x}_\alpha^{(k)}\right\}$. In this way, we can view the integral in terms of a Riemann integral of countable terms

$$M_\alpha^{(k)} \equiv \left(\frac{a}{\pi}\right)^{3/4} e^{-a/2(\hat{q}-\mathbf{x}_\alpha^{(k)})^2} \sqrt{\delta^{(k)}}.$$

with $\delta^{(k)}$ approximating a differential, so that

$$\int_{\mathbb{R}^3} d\mathbf{x} L_\mathbf{x}^\dagger L_\mathbf{x} = \lim_{k\to\infty} \sum_\alpha M_\alpha^{(k)} M_\alpha^{(k)}. \quad (41)$$

We get the following equation

$$d\big|\Psi_t\rangle = -iH\big|\Psi_t\rangle dt + \sum_\alpha \left(\frac{M_\alpha^{(k)}}{\left\|M_\alpha^{(k)}\big|\Psi_t\rangle\right\|} - I\right) dN_\alpha\big|\Psi_t\rangle. \quad (42)$$

The relationship to this equation and GRW is shown in the following theorem

Theorem 5. *Equation 42 is an unravelling of the approximate GRW master equation.*

$$\frac{d}{dt}\rho(t) = -iHdt + \lambda\left(T[\rho] - \rho\right)$$

with

$$T[\rho] = \sum_\alpha M_\alpha^{(k)\dagger} \rho M_\alpha^{(k)}$$

and λ, a collapse rate equal to 1.

Proof. Using the standard Itô rule, we arrive at the following calculation for the stochastic density operator $\rho_\Psi(t) = |\Psi_t\rangle\langle\Psi_t|$,

$$d\rho_\Psi^{(k)}(t) = -i[H,\rho_\Psi(t)] + \sum_\alpha \left(\frac{M_\alpha^{(k)}}{\left\| M_\alpha^{(k)} |\Psi_t\rangle \right\|} - I \right) dN_\alpha^{(k)} \rho_\Psi(t) \tag{43}$$

$$+ \rho_\Psi(t) \sum_\alpha \left(\frac{M_\alpha^{(k)}}{\left\| M_\alpha^{(k)} |\Psi_t\rangle \right\|} - I \right) dN_\alpha^{(k)} + \tag{44}$$

$$+ \sum_{\alpha\gamma} \left(\frac{M_\alpha^{(k)}}{\left\| M_\alpha^{(k)} |\Psi_t\rangle \right\|} - I \right) \rho_\Psi(t) \left(\frac{M_\gamma^{(k)}}{\left\| M_\gamma^{(k)} |\Psi_t\rangle \right\|} - I \right) dN_\alpha^{(k)} dN_\gamma^{(k)} \tag{45}$$

Due to the independence of the inhomogeneous Poisson processes, the last sum collapses to a single sum. When we take into account that $\mathbb{E} dN_\alpha^{(k)} = \mathbb{E} \left\| M_\alpha^{(k)} |\Psi_t\rangle \right\|^2 dt$, we see that the $\left\| M_\alpha^{(k)} |\Psi_t\rangle \right\|^2$ terms in the denominator cancel out in expected value and we get the GRW master equation for $\rho(t) = \mathbb{E}\rho_\Psi(t)$. Due to the completeness of the localization operators the GRW master equation is achieved. □

This net forms a set upon which localizations can be achieved and in the limit as $k \to \infty$ where the distance between localization points converges to zero, we will achieve the effect of the GRW equation where the Lindblad operators are indexed by an arbitrary $\mathbf{x} \in \mathbb{R}^3$. It is important that the sequence of HP equations corresponding to these Lindblad operators also converges and we have the following theorem.

Theorem 6. *Consider all processes to be defined on a compact time interval* $[0,T]$. *Let* $\{L_\alpha^{(k)}\}$ *be the Lindblad localization operators associated to the sequence* $\{\mathbf{x}_\alpha^{(k)}\}$ *which fills* $\mathbb{R}^3$. *Then if* $\sum_\alpha {L_\alpha^{(k)}}^2 \to I$, *and we choose noises* $A_\alpha^{(k)}(t) = a\left(\langle c_\alpha^{(k)}| \mathbb{1}_{[0,t)}\right)$ *and* ${A^{(k)}}^\dagger_\alpha(t) = a^\dagger\left(|c_\alpha^{(k)}\rangle \mathbb{1}_{[0,t)}\right)$, *with* $\{|c_\alpha^{(k)}\rangle\} \in \mathscr{H}_1$ *real and orthonormal, such that*

$$\sum_\alpha L_\alpha^{(k)} \sum_\beta L_\beta^{(l)} \left\langle c_\alpha^{(k)} \middle| c_\beta^{(l)} \right\rangle \to I \tag{46}$$

in the strong sense as $k,l \to \infty$, *then the sequence of solutions to the HP equations* $U_t^{(k)}$ *corresponding to the Lindblad operators* $\{L_\alpha^{(k)}\}$ *(with no scattering terms) strongly converges.*

Proof. Let $|\xi\rangle = |\phi \otimes e(u)\rangle \in \mathscr{H}_0 \otimes \Gamma(\mathfrak{h})$, and $\phi \in \mathscr{H}_0$, $u \in \mathfrak{h} := L^2(T) \otimes \mathscr{H}_1$. We've chosen $|\xi\rangle$ to be a member of a dense set in $\mathscr{H}_0 \otimes \Gamma(\mathfrak{h})$. Then the condition for strong convergence is that $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ such that if $k,l > N$ we have

$$\left\| (U_t^{(k)} - U_t^{(l)}) |\xi\rangle \right\|^2 = 2\langle\xi|\xi\rangle - \left\langle \xi \middle| {U_t^{(k)}}^\dagger U_t^{(l)} \middle| \xi \right\rangle - \left\langle \xi \middle| {U_t^{(l)}}^\dagger U_t^{(k)} \middle| \xi \right\rangle < \varepsilon \langle\xi|\xi\rangle$$

where we have used that $U_t^{(k)}$ and $U_t^{(l)}$ are solutions to HP equations and therefore unitary. For strong convergence then we require that $\left\langle \xi \middle| U_t^{(k)\dagger} U_t^{(l)} \middle| \xi \right\rangle$ and its conjugate must converge to $\langle \xi | \xi \rangle$. If we examine the differential $d\left\langle \xi \middle| U_t^{(k)\dagger} U_t^{(l)} \middle| \xi \right\rangle$, we can use the Itô calculus to show that

$$\begin{aligned} d\left\langle \xi \middle| U_t^{(k)\dagger} U_t^{(l)} \middle| \xi \right\rangle &= \left\langle \xi \middle| \left[U_t^{(k)\dagger} \sum_\alpha L_\alpha^{(k)} (dA_\alpha^{(k)} - dA_\alpha^{(k)\dagger}) U_t^{(l)} \right] \middle| \xi \right\rangle \\ &+ \left\langle \xi \middle| U_t^{(k)\dagger} \left[\sum_\beta L_\beta^{(l)} (dA_\beta^{(l)\dagger} - dA_\beta^{(l)}) \right] U_t^{(l)} \middle| \xi \right\rangle \\ &+ \left\langle \xi \middle| U_t^{(k)\dagger} \left[\sum_\alpha L_\alpha^{(k)} \sum_\beta L_\beta^{(l)} \left\langle c_\alpha^{(k)} | c_\beta^{(l)} \right\rangle - \frac{1}{2} \sum_\alpha L_\alpha^{(k)2} - \frac{1}{2} \sum_\beta L_\beta^{(l)2} \right] U_t^{(l)} \middle| \xi \right\rangle dt \end{aligned} \tag{47}$$

where we have explicitly used the fact that the Lindblad operators are self-adjoint and $S_{\beta\alpha} = I$ in the HP equation, indicating the absense of scattering. It is important to note that due to the future pointing nature of of the quantum differentials, they commute with adapted processes, thus the operators $dA_\alpha^\dagger(t)$ and $dA_\alpha(t)$ may operate on the vectors $|\xi\rangle$, passing through the processes $U_t^{(k)}$ and $U_t^{(l)}$. The first term in equation 47 can be evaluated as

$$\sum_\alpha \left\langle \xi \middle| U_t^{(k)\dagger} L_\alpha^{(k)} U_t^{(l)} \middle| \xi \right\rangle \left(\left\langle u \middle| c_\alpha^{(k)} \right\rangle - \left\langle c_\alpha^{(k)} \middle| u \right\rangle \right) dt,$$

and we see that as $\mathscr{H}_1$ was taken to be real, this term is zero, and similarly for the second term in equation (47). For the last term, we can choose an N for each $\varepsilon' = \dfrac{\varepsilon}{T}$ such that for all $k, l > N$ we have, via the Cauchy-Schwarz inequality, that

$$\left| \left\langle \zeta \middle| I - \sum_\alpha L_\alpha^{(k)} \sum_\beta L_\beta^{(l)} \left\langle c_\alpha^{(k)} | c_\beta^{(l)} \right\rangle \middle| \eta \right\rangle \right| \le \frac{\varepsilon'}{6} \langle \zeta | \zeta \rangle^{1/2} \langle \eta | \eta \rangle^{1/2}$$

and

$$\left| \left\langle \zeta \middle| I - \sum_\alpha L_\alpha^{(k)2} \middle| \eta \right\rangle \right| \le \frac{\varepsilon'}{3} \langle \zeta | \zeta \rangle^{1/2} \langle \eta | \eta \rangle^{1/2}, \text{ and } \left| \left\langle \zeta \middle| I - \sum_\alpha L_\alpha^{(l)2} \middle| \eta \right\rangle \right| \le \frac{\varepsilon'}{3} \langle \zeta | \zeta \rangle^{1/2} \langle \eta | \eta \rangle^{1/2}$$

for $|\zeta\rangle, |\eta\rangle \in \mathscr{H}_0 \otimes \Gamma(\mathfrak{h})$, because these sums are positive contractions. Taking $\langle \zeta | = \langle \xi U_t^{(k)} |$ and $|\eta\rangle = |U_t^{(l)} \xi\rangle$, and writing $X_t = \left\langle \xi \middle| U_t^{(k)\dagger} U_t^{(l)} \middle| \xi \right\rangle$ we have upon integration that $|X_t - X_0| \le \frac{\varepsilon'}{2} t \langle \xi | \xi \rangle$, from which it follows that $-X_t \le \frac{\varepsilon'}{2} t \langle \xi | \xi \rangle - \langle \xi | \xi \rangle$, where we've used that $U_0^{(k)} = U_0^{(l)} = I$. Doing the same for $X_t = \left\langle \xi \middle| U_t^{(l)\dagger} U_t^{(k)} \middle| \xi \right\rangle$, we get that $\left\| (U_t^{(k)} - U_t^{(l)}) |\xi\rangle \right\|^2 = 2 \langle \xi | \xi \rangle - 2 \langle \xi | \xi \rangle + t\varepsilon' \langle \xi | \xi \rangle < \varepsilon \langle \xi | \xi \rangle$. □

9. The Weyl Operator and Change of State

In the derivation of the Belavkin equation it was necessary to apply a Girsanov transformation to retain a normalized state. The norm-squared process was used as a Radon-Nikodym derivative for the change of measure. We now calculate its square root.

Proposition 5. *The square root of the process* $\langle \psi_t | \psi_t \rangle$ *is given by*

$$\phi_t = \exp\left(-\frac{1}{2}\sum_\alpha \int_0^t (\|M_\alpha \Psi_s\| - 1)^2 \, ds\right) \mathscr{E}\left(\sum_\alpha \left(\int_0^t \|M_\alpha \Psi_s\| - 1\right) d\widetilde{N}_\alpha\right). \tag{48}$$

Proof. The calculation to obtain this expression is analogous to the one in the proof of Lemma 4. Alternatively, one can apply the inversion formula of Lemma 3 to the process Φ_t. □

The second factor is an analogue of an exponential vector in the Fock space. The normalization $\mathbb{E}\phi_t^* \phi_t = \mathbb{E}\langle \psi_t | \psi_t \rangle = 1$ thus makes ϕ_t an analogue of a coherent state. We will call such random variables coherent. In Fock space, the operator which creates coherent states is the Weyl operator $W(u, I)$. It creates the coherent state $|\widetilde{e}(u)\rangle$ by acting on the vacuum $|\widetilde{e}(0)\rangle = |\Omega\rangle$. This change of state has the effect of changing the distribution of a quantum stochastic process as was seen in the case of the quantum Poisson process in Proposition 4. Let us adopt an abbreviated notation for the stochastic integral $\int_T f(s) dM(s)$, denoting it by $f \cdot M$. We see that the stochastic analogue of the operator which changes states is the stochastic Weyl process $\widetilde{W}(f, U)$, which, for a square-integrable process f and a unitary-valued process U, is defined uniquely by its action on random variables $\mathscr{E}(g \cdot M)$:

$$\widetilde{W}(f, U)\mathscr{E}(h \cdot M) = \exp\left(-\frac{1}{2}[[f^\dagger \cdot M, f \cdot M]] - [[f \cdot M, Uh \cdot M]]\right) \mathscr{E}(Uh \cdot M + f \cdot M).$$

Note that the right-hand side of the above equation is a coherent random variable. In the case of the vacuum, i.e. $h \equiv 0$ and $U \equiv I$, the resulting vector is the square root of a Doléans-Dade exponential random variable, $\mathscr{E}(g \cdot M)$ with $g = 2\mathfrak{Re} f \cdot M$ in the Wiener case and $(f + f^\dagger + f^\dagger f) \cdot M$ in the Poisson case. We thus see that general Girsanov transformations of measure may be expressed in terms of stochastic Weyl operators. For example, we may effect a change in distribution of the process M by acting with a Weyl operator as illustrated by the relation

$$\mathbb{E}\mathscr{E}(0)\widetilde{W}^\dagger(f) M \widetilde{W}(f)\mathscr{E}(0) = \mathbb{E}\mathscr{E}(g \cdot M) M = \mathbb{E}' M,$$

so that the effect of the Weyl operator in the unprimed space is the same as the change of distribution resulting from passage to the primed space (as in the case of standard Weyl operators, we are abbreviating $\widetilde{W}(f, I)$ to $\widetilde{W}(f)$).

As we have said, the norm-squared random variables $\langle \psi_t | \psi_t \rangle$ form an exponential process which constitutes the Radon-Nikodym derivative necessary to normalize the HP equation. A simplified version of this occurs when $f \in \mathfrak{h}$, and therefore is nonstochastic. We have two measures, the unprimed measure and the primed one with Radon-Nikodym derivative $\mathscr{E}(f)$, on the underlying probability space, together with the corresponding expected value operations. Under the Segal isomorphism, these operations become trace operations, namely, $A \mapsto \mathrm{Tr}\left(|\Omega\rangle\langle\Omega| A\right)$ and

$A \mapsto \mathrm{Tr}\left(\left|\tilde{e}(u_t)\right\rangle\langle\tilde{e}(u_t)\big|A\right)$, where u_t chosen so as to make the $\left|\tilde{e}(u_t)\right\rangle$, the image of the square root of $\mathscr{E}(f)$ under Θ^{-1}.

In the situation which concerns us here, the argument of the Doléans-Dade process is not from $\mathfrak{h}$, it is itself stochastic. This is a significant generalization because the Weyl process is not defined for stochastic arguments in the quantum case. The quantum equivalent of the random variable $\langle\psi_t|\psi_t\rangle$ is

$$\rho_{\langle\psi_t|\psi_t\rangle}(t) = \mathrm{Tr}_{\mathscr{H}_0} U_t\big|\psi_0\otimes\Omega\rangle\langle\Omega\otimes\psi_0\big|U_t^\dagger,$$

which is a reduced density operator on Fock space. As states are analogous to classical probability measures, we see that this may be viewed as a change from the vacuum state $\big|\Omega\rangle\langle\Omega\big|$ to a new state, which is analogous to modifying the original measure, multiplying it by a density equal to the classical norm-squared process. To see this, note that the norm-squared process is the quantum expectation of the identity $\langle\psi_t|\psi_t\rangle = \langle\psi_t|I|\psi_t\rangle$, and this is mirrored in $\rho_{\langle\psi_t|\psi_t\rangle}$ with the quantum expectation changing to the partial trace of the identity in the pure state $\big|\psi_t\rangle\langle\psi_t\big|$. The change from the unprimed to the primed space is mirrored by the change in trace from $\mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[\big|\Omega\rangle\langle\Omega\big|\cdot\right]$ to $\mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[\rho_{\langle\psi_t|\psi_t\rangle}\cdot\right]$, on the von Neumann algebra of operators on $\Gamma(\mathfrak{h})$.

The classical derivation of the Belavkin equation suggests that the change of state from a standard Poisson process to the one which preserves the norm satisfies

$$\left\langle\Omega|W(u)^\dagger\lambda\left(\big|\alpha\rangle\langle\alpha\big|\mathbb{1}_{[0,t)}\right)W(u)|\Omega\right\rangle = \int_0^t \mathrm{Tr}\left[U_s\big|\psi_0\otimes\Omega\rangle\langle\Omega\otimes\psi_0\big|U_s^\dagger M_\alpha^\dagger M_\alpha\right]ds,$$

for some u, hopefully in the one-particle space so that we may define the Weyl transformation $W(u)$. This suggests that we choose

$$u_t^\alpha = \mathrm{Tr}\left[U_t\big|\psi_0\otimes\Omega\rangle\langle\Omega\otimes\psi_0\big|U_t^\dagger M_\alpha^\dagger M_\alpha\right]^{1/2}.$$

We can change the distribution of the Poisson process using an arbitrary Weyl operator with vector z_t as long as we make the corresponding change of the state vector evolution. This can be seen by taking the expected value of $X_\alpha = \Lambda_\alpha^\alpha + A_\alpha^\dagger + A_\alpha + t$,

$$\begin{aligned}\mathrm{Tr}\left[\big|\psi_t\rangle\langle\psi_t\big|X_\alpha\right] &= \mathrm{Tr}\left[W(z_t)^\dagger W(z_t)\big|\psi_t\rangle\langle\psi_t\big|W(z_t)^\dagger W(z_t)X_\alpha\right]\\ &= \mathrm{Tr}\left[W(z_t)\big|\psi_t\rangle\langle\psi_t\big|W(z_t)^\dagger\left(W(z_t)X_\alpha W(z_t)^\dagger\right)\right]\end{aligned}$$

The desired change in distribution corresponds to $z_t = u_t - w_t$ with $\big|w_t\rangle = \sum_\alpha\big|\alpha\rangle\mathbb{1}_{[0,t)}$ and $V_t = W(z_t)U_t$, as the next proposition makes precise.

Proposition 6. *Conjugation of the Weyl operator $W(z_t)$, with $z_t = u_t - w_t$ and $w_t^\alpha = \mathbb{1}_{[0,t)}\big|\alpha\rangle$, results in a change of X_α from the standard Poisson distribution to one with parameter $\int_0^t |u_s^\alpha|^2 ds$.*

Proof. Using the fact that the Weyl operator satisfies the QSDE [23],

$$dW(z_t) = \left[dA_{z_t}^\dagger - dA_{z_t} - \frac{1}{2}d\langle z_t|z_t\rangle\right]W(z_t),$$

and $dX_\alpha = d\Lambda_\alpha^\alpha + dA_\alpha^\dagger(t) + dA_\alpha(t) + dt$, we calculate

$$\begin{aligned} d(W(z_t)X_\alpha W^\dagger(z_t)) &= dW(z_t)X_\alpha W^\dagger(z_t) + W(z_t)dX_\alpha W^\dagger(z_t) + W(z_t)X_\alpha dW^\dagger(z_t) + W(z_t)dX_\alpha dW^\dagger(z_t) \\ &+ dW(z_t)dX_\alpha W^\dagger(z_t) + dW(z_t)X_\alpha dW(z_t) + dW(z_t)dX_\alpha dW^\dagger(z_t) \\ &= d\Lambda_\alpha^\alpha + dA^\dagger_{u_t^\alpha} + dA_{u_t^\alpha} + |u_\alpha|^2 dt \end{aligned}$$

□

The corresponding evolution equation for the state vector, which will be called the quantum Belavkin equation, is $|\hat{\Psi}_t\rangle = W(z_t)U_t|\psi_0 \otimes \Omega\rangle = V_t|\psi_0 \otimes \Omega\rangle$, where V_t satisfies the QSDE

$$\begin{aligned} dV_t = \Bigg[\sum_\alpha \left(\frac{M_\alpha}{u_t^\alpha} - I\right)\left(dA^\dagger_{u_t^\alpha} + |u_t^\alpha|^2 dt\right) - \left(\frac{M_\alpha^\dagger}{u_t^\alpha} - I\right) dA_{u_t^\alpha} \\ + \frac{1}{2}\sum_\alpha \left(M_\alpha^\dagger M_\alpha - |u_t^\alpha|^2\right) dt - iH' dt\Bigg] V_t, \end{aligned} \quad (49)$$

with $|u_t^\alpha|^2 = \mathrm{Tr}\left[V_t|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0|V_t^\dagger M_\alpha^\dagger M_\alpha\right]$ due to the cyclic property of the trace and the commutation of the Weyl operator with the system operators M_α, and where the Hamiltonian H' has incorporated the Lamb shift terms as was performed in equation (40). Equation (49) is a closed but nonlinear quantum stochastic differential equation. Its solution preserves the norm of the system's state as shown in the next theorem.

Theorem 7. *The quantum Belavkin equation, equation (49), preserves the system norm state,*

$$\rho_{\langle\psi_t|\psi_t\rangle} = \mathrm{Tr}_{\mathscr{H}_0}\left[V_t|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0|V_t^\dagger\right].$$

Proof. This follows from the quantum Itô rule,

$$dA_{u_t^\alpha} dA^\dagger_{u_t^\alpha} = |u_t^\alpha|^2 dt,$$

and lemma 1 on the system partial trace. We apply these rules to the differential

$$\begin{aligned} d\rho_{\langle\psi_t|\psi_t\rangle} = \mathrm{Tr}_{\mathscr{H}_0}\Big[dV_t|\psi_0 \otimes \Omega\rangle\langle\psi_0 \otimes \Omega|V_t^\dagger + V_t|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0|dV^\dagger \\ + dV_t|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0|dV_t^\dagger\Big] \end{aligned}$$

to get that $d\rho_{\langle\psi_t|\psi_t\rangle} = 0$ with initial condition $|\Omega\rangle\langle\Omega|$. □

We now discuss a different quantum equation, corresponding to the Wiener interpretation of the quantum evolution. Analogous to the z_t^α above, in the discussion leading to the quantum Belavkin equation, we define the vectors

$$v_t^\alpha = \mathrm{Tr}\left[L_\alpha V_t|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0|V_t^\dagger\right].$$

We define the quantum Gisin-Percival evolution as

$$V_t = W(v_t)U_t$$

and arrive at the following equation,

$$dV_t = -iHdt + \left[\sum_\alpha (L_\alpha - v_t^\alpha)\, dA_\alpha^\dagger(t) + \left(L_\alpha^\dagger - v_t^{\alpha *}\right) dA_\alpha(t) \right.$$
$$\left. -\frac{1}{2}\sum_\alpha \left(L_\alpha^\dagger L_\alpha - 2L_\alpha v_t^{\alpha *} + |v_t^\alpha|^2\right) dt\right] V_t \tag{50}$$

Theorem 8. *The quantum Gisin-Percival equation preserves the system norm state.*

$$\rho_{\langle\psi_t|\psi_t\rangle} = \mathrm{Tr}_{\mathscr{H}_0}\left[V_t\big|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0\big|V_t^\dagger\right].$$

Proof. The proof follows along the same lines as that of Theorem 7. □

Just as the HP equation is a dilation of the Lindblad semigroup so are equations 49 and 50.

Theorem 9. *The quantum Belavkin equation, equation (49), and the quantum Gisin-Percival equation, equation (50), are nonlinear unitary dilations of the Lindblad semigroup, in the sense that*

$$\mathrm{Tr}_{\Gamma(\mathfrak{h})} d\rho(t) = \mathscr{L}\left[\mathrm{Tr}_{\Gamma(\mathfrak{h})}[\rho(t)]\right] dt,$$

with $\rho(t) = V_t\big|\psi_0 \otimes \Omega\rangle\langle\psi_0 \otimes \Omega\big|V_t^\dagger$.

Proof. Using lemma 1, we have for $V_t = W_t U_t$, with $W_t = W(z_t)$ or $W_t = W(v_t)$,

$$\begin{aligned}\mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[V_t\big|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0\big|V_t^\dagger\right] &= \mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[W_t U_t\big|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0\big|U_t^\dagger W_t^\dagger\right] \\ &= \mathrm{Tr}_{\Gamma(\mathfrak{h})}\left[U_t\big|\psi_0 \otimes \Omega\rangle\langle\Omega \otimes \psi_0\big|U_t^\dagger\right]\end{aligned}$$

By theorem 1, the last line represents the HP dilation of the Lindblad semigroup. □

Note that the quantum Belavkin equations and the classical Belavkin equations differ in a fundamental way. In the classical Belavkin equation, we change the state in a manner dependent on a random variable f, while in the quantum Belavkin equation the Fock space component has been traced out to make for a change of state dependent on a vector belonging to the one particle space, z_t or v_t. The Poisson processess used in the two cases are not directly isomorphic as seen from the fact that the rate in the classical case is itself a random variable. In fact, calling such a process a Poisson process is a bit of an abuse of language.

Finally, we note that in both the quantum and classical Belavkin equations, the Weyl operator is encoding the history of the system into the current state of the system. In the classical case, this results in the norm process, ϕ_t, dependent on the history of the system up to time t, while in the quantum case, the Weyl operator with the appropriate one-particle vector encodes the history of the system and can be compared to Everett's observer in MWI.

10. Expected Values Derived from the Belavkin Equation

The stochastic Schrödinger equation, equation (40), allows us to calculate the stochastic expectation of observables. (The expression “stochastic expectation” refers to the fact that we are taking quantum expected values of system observables, calculated in a state, which depends on a random parameter.) This follows from a calculation using the Itô rule and results in an SDE for the stochastic expectation $\langle X\rangle_t$ of an observable X:

$$d\langle X\rangle_t = \left\langle \mathscr{L}^\dagger[X]\right\rangle_t dt - \sum_\alpha \left(\frac{\left\langle M_\alpha^\dagger X M_\alpha \right\rangle_t}{\|M_\alpha \Psi\|^2} - \langle X\rangle_t \right) d\widetilde{N}_\alpha, \tag{51}$$

where $\mathscr{L}^\dagger$ is the adjoint Lindbladian

$$\mathscr{L}^\dagger[X] = i[H,X] + \frac{1}{2}\sum_\alpha 2M_\alpha^\dagger X M_\alpha - M_\alpha^\dagger M_\alpha X - X M_\alpha^\dagger M_\alpha,$$

which for any two bounded operators on the state space satsifies

$$\mathrm{Tr}\left[A\mathscr{L}[B]\right] = \mathrm{Tr}\left[\mathscr{L}^\dagger[A]B\right].$$

We see that the stochastic expectation is a sum of two terms, the adjoint Lindbladian which is typical of the expectation of an observable in an open system, and a martingale term whose expected value is zero. Thus the classical expectation of the $\langle X\rangle_t$ satisfies

$$d\mathbb{E}\langle X\rangle_t = \mathbb{E}\left\langle \mathscr{L}^\dagger[X]\right\rangle_t dt.$$

In particular for the Hamiltonian, we have that

$$d\mathbb{E}\langle H\rangle_t = \mathbb{E}\left(\sum_\alpha \left\langle M_\alpha^\dagger H M_\alpha\right\rangle_t - \langle H\rangle_t\right) dt = \left(\mathbb{E}\langle H'\rangle_t - \mathbb{E}\langle H\rangle_t\right) dt,$$

with decohered Hamiltonian $H' = \sum_\alpha M_\alpha^\dagger H M_\alpha$. This is a linear inhomogeneous differential equation, which can be solved using the integrating factor e^t to get

$$\mathbb{E}\langle H\rangle_t = \int_0^t e^{s-t}\mathbb{E}\langle H'\rangle_s ds + \mathbb{E}\langle H\rangle_0.$$

We see that $\langle H\rangle_t$ is not conserved as generally $\langle H'\rangle$ will be different from $\langle H\rangle$ so that $\langle H\rangle$ can grow, as it does in GRW and CSL, or decay depending on the Lindblad operators.

11. Construction of Solutions and a Monte Carlo Algorithm

The GRW unravelling

$$d|\Psi_t\rangle = -iH|\Psi_t\rangle dt + \sum_\alpha \left(\frac{M_\alpha}{\left\|M_\alpha|\Psi_t\rangle\right\|} - I\right) dN_\alpha |\Psi_t\rangle \tag{52}$$

has a natural procedure for constructing solutions. There is a time component and a Poisson-driven component. As a reminder, the total rate of jumps is assumed to be $\lambda = 1$. In the following we keep the parameter λ in the notation, to make its role explicit. The solution can be constructed in the following way:

(i) Generate a random variable τ which is distrubuted exponentially with the rate parameter λ.

(ii) Integrate the Schrödinger equation $d\big|\Psi_t\rangle = -iH\big|\Psi_t\rangle dt$ up to time τ.

(iii) Choose a jump of type α, with probability $p_\alpha = \big\|M_\alpha\big|\psi_t\rangle\big\|^2$.

(iv) Apply jump and normalization.

$$\big|\psi_{t-}\rangle \mapsto \big|\psi_t\rangle = \frac{M_\alpha\big|\psi_{t-}\rangle}{\big\|M_\alpha\big|\psi_{t-}\rangle\big\|}$$

(v) Go back to step (i) and repeat the procedure.

The function defined piecewise in the above way is a solution of the Belavkin equation, with uniqueness following from the uniqueness of solutions to ordinary differential equations, provided the jump maps preserve uniqueness, as they do in GRW (for normalized states), so that the initial condition for the next time interval of length τ follows uniquely from the previous jump. The use of an exponential random variable is justified because Poisson jumps are exponentially distributed for a homogeneous jump parameter, and the total rate of jumps is λ for a complete set of Lindblad operators,

$$\sum M_\alpha^\dagger M_\alpha = I,$$

allowing us to integrate up to the next jump in time and simply choose which jump happens at the prescribed time. This differs from a direct integration of the Belavkin equation where we have to check at each time step whether a jump happens or whether to integrate. The use of exponential variable jump times greatly reduces computation time by prescribing the next jump times after each jump so that only direct numerical integration in time needs to be carried out between jumps

This explicit method of constructing solutions not only proves their existence but also leads to a Monte Carlo algorithm which generates unravellings. These serve to integrate the Lindblad equation via the formula

$$\mathbb{E}d\big|\psi_t\rangle\langle\psi_t\big| = \mathscr{L}[\mathbb{E}\big|\psi_t\rangle\langle\psi_t\big|]dt.$$

Thus generating an ensemble of trajectories according to the prescription above and taking their expected value over realizations at each time gives an approximate solution to the Lindblad equation. This has advantages over the direct integration of the Lindblad equation for large n-dimensional systems as it involves the generation of ensembles of n-dimensional trajectories, versus the integration of an $n^2/2$-dimensional differential equation. The use of a running expectation can significantly decrease memory requirements. The Poisson algorithm described above has significant stability advantages over an equivalent Gisin-Percival unravelling as integration of the Schrödinger equation with white noise added can lead to rapidly growing errors and require exceedingly small time steps, versus the Poisson algorithm which inherits the stability of the integrator. The Gisin-Percival algorithm is simply a direct integration of the Gisin-Percival equation where at each time step the drift terms are integrated following a regular ODE integrator such as Runge-Kutta, and the stochastic terms are added at each time step, with dW_α modelled as a complex noise with independent components, each a normal random variable with variance $(\Delta t)^{1/2}$. This can be expected to generally take longer as at each time step we are doing more computations than the corresponding Belavkin equation.

12. Entropy

The use of stochastic unravellings allows for calculations beyond the Lindblad equation, in which no randomness is present, and whose solution is a density operator which depends on time only. Any nonlinear function of the stochastic solution leads to corrections arising from the Itô formula, which we recall below.

Theorem 10 (Itô's Formula)**.** *Let $f:\mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and let $X_t = (X_t^1,\dots,X_t^n)$ be an n-tuple of semimartingales. Then $f(X_t)$ is a semimartingale with*

$$f(X_t)-f(X_0)=\sum_{i=1}^{n}\int_{0+}\frac{\partial f}{\partial x_i}(X_{s-})dX_s^i+\frac{1}{2}\sum_{1\le i,j\le n}^{n}\int_{0+}\frac{\partial^2 f}{\partial x_i\partial x_j}(X_{s-})d[[X^i,X^j]]_s^c$$
$$+\sum_{0\le s\le t}\left(f(X_s)-f(X_{s-})-\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X_{s-})\Delta X_s^i\right)$$

where $[[X,X]]_t^c$ is the continuous part of the quadratic variation process, $[[X,X]]_t$ and $\Delta X_s = X_s - X_{s-}$ is the jump part of the semimartingale.

For a proof see [25]. This can be viewed as a modification of the fundamental theorem of calculus. The so called 'Itô correction' comes from the quadratic variation, where we write the heuristic $dXdY$ for $d[[X,Y]]$, from which we derive the Itô rule for both Wiener and Poisson stochastic calculus.

The evolution of probabilities under the Gisin-Percival equation is given by the following SDE

$$dp_i(t)=\left\langle \mathscr{L}^\dagger\left[M_i^\dagger M_i\right]\right\rangle_t dt+\frac{1}{\sqrt{2}}\left[\sum_j\left\langle M_i^\dagger M_i\left(L_j-\langle L_j\rangle\right)\right\rangle_t dW_j(t)+h.c.\right], \tag{53}$$

where $h.c.$ stands for Hermitian conjugate, and with the Poisson-driven Belavkin equations the corresponding equation is

$$dp_i(t)=\left\langle \mathscr{L}^\dagger\left[M_i^\dagger M_i\right]\right\rangle_t dt+\sum_j\left(\frac{\left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_{t-}}{\left\langle L_j^\dagger L_j\right\rangle_{t-}}-p_i(t-)\right)d\widetilde{N}_j(t). \tag{54}$$

The Lindblad equation describes the evolution of the density operator in the Schrödinger picture (generalized to open quantum systems), while its adjoint describes the Heisenberg evolution of observables (in this case $M_i^\dagger M_i$) as in Section 10. It is interesting that both equations (53) and (54) consist of the usual (adjoint) Lindbladian evolution plus a martingale which has expected value zero, so on average the probabilities are what we obtain from the Lindbladian evolution. However, these martingales play a nontrivial role in the evolution of the entropy functional because of Theorem 10, where they will give rise to corrections to the entropy of the density operator described by Lindbladian evolution.

To see this we first calculate the logarithms using Theorem 10, to get an SDE

$$d\log p_i(t) = \frac{1}{p_i(t)}\left[\left\langle \mathscr{L}^\dagger\left[M_i^\dagger M_i\right]\right\rangle_t dt + \frac{1}{\sqrt{2}}\left(\sum_j \left\langle M_i^\dagger M_i(L_j - \langle L_j\rangle)\right\rangle_t dW_j(t) + h.c\right)\right]$$
$$-\frac{1}{p_i^2(t)}\sum_j \left|\left\langle M_i^\dagger M_i(L_j - \langle L_j\rangle)\right\rangle_t\right|^2 dt$$

in the Wiener case and

$$d\log p_i(t) = \frac{1}{p_i(t)}\left[\left\langle \mathscr{L}^\dagger\left[M_i^\dagger M_i\right]\right\rangle_t dt - \sum_j\left(\left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_t - p_i(t)\left\langle L_j^\dagger L_j\right\rangle_t\right)dt\right]$$
$$+\sum_j\left[\log\left(\frac{\left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_{t-}}{\left\langle L_j^\dagger L_j\right\rangle_{t-}}\right) - \log p_i(t-)\right]dN_j(t)$$

in the Poisson case. We may then calculate the SDE satisfied by the entropy functional using the Itô product rule, which follows as a consequence of Theorem 10, and applying it to p_i times $\log p_i$. If we denote the by dS_{vN} the standard Lindbladian evolution of the von Neumann entropy, with

$$\frac{dS_{vN}(t)}{dt} = -\sum_i \left\langle \mathscr{L}^\dagger[M_i^\dagger M_i]\right\rangle_t \log p_i(t)$$

then the resulting SDEs are

$$dS(t) = dS_{vN}(t) - \frac{1}{\sqrt{2}}\sum_{ij}\left[\left\langle M_i^\dagger M_i(L_j - \langle L_j\rangle)\right\rangle_t \log p_i(t) dW_j(t) + h.c.\right]$$
$$-\sum_i \frac{1}{p_i(t)}\sum_j\left|\left\langle M_i^\dagger M_i(L_j - \langle L_j\rangle)\right\rangle_t\right|^2 dt, \tag{55}$$

for the Wiener case and

$$dS(t) = dS_{vN}(t) - \sum_i \log p_i(t)\left(\sum_j p_i(t)\left\langle L_j^\dagger L_j\right\rangle_t - \left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_t\right)dt$$
$$-\sum_{ij}\left(\frac{\left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_{t-}}{\left\langle L_j^\dagger L_j\right\rangle_{t-}}\log\left(\frac{\left\langle L_j^\dagger M_i^\dagger M_i L_j\right\rangle_{t-}}{\left\langle L_j^\dagger L_j\right\rangle_{t-}}\right) - p_i(t-)\log p_i(t-)\right)dN_j(t). \tag{56}$$

for the Poisson case. Here simplifications have been made by using the completeness relation, $\sum_i M_i^\dagger M_i = I$ and the fact that $\mathscr{L}^\dagger[I] = 0$, which follows from the trace-preserving property of the Lindbladian evolution.

We see that the there are non-martingale corrections to the entropy, and so making use of the unravellings themselves—viewing them as real trajectories instead of just using the expectation to unravel the Lindblad equation—has real consequences for the entropy as it does for many different nonlinear functionals of the processes.

12.1 Special Case: Localization Entropy

The first simplification that can be made is to assume that the Lindblad operators, L_j, are themselves carrying out a measurement so that

$$\sum_j L_j^\dagger L_j = I.$$

This is the case for the unravelling describing GRW. But a further simplification, that L_j and M_j are both indicators of the same partition of space $\{\Delta_i\}$, and so, in particular, are orthogonal decompositions of the identity, is quite illustrative. In this case we have

$$L_j M_i = \delta_{ij} M_i.$$

Applying this to equations (55) and (56) gives the following SDEs

$$dS(t) = dS_{vN}(t) - \left(1 - \sum_j p_j^2(t)\right) dt \tag{57}$$

for the Wiener case and

$$\begin{aligned} dS(t) &= dS_{vN}(t) + \sum_i p_i(t-) \log p_i(t-) \sum_j dN_j(t) \\ &= dS_{vN}(t) + \sum_i p_i(t-) \log p_i(t-) dN(t) \\ &= dS_{vN}(t) - S(t-) dN(t) \end{aligned} \tag{58}$$

for the Poisson case, where we have denoted by N the Poisson process with $dN = \sum_j dN_j$ which is the process which jumps when any of the N_j jump, with rate equal to the total rate of jumps (in our case this is 1).

In the Wiener case, the localization entropy correction is a term equivalent to the linear entropy. Entropy is decreasing as localization is occuring which is decreasing the number of possible states accessible to the system. In the Poisson case, a complete localization is occuring which extracts from the system an amount of information equivalent to the entropy in the distribution of probabilities over states. This happens any time a single localization occurs, with a rate equal to the total jump rate. A similar process can be expected to happen for the case of Gaussian localizations, as they are decreasing the allowable position states. Whether or not this is observable in a system would be an interesting subject for future study as if the effect is observed it may prove that the corresponding quantum trajectories exist.

13. Discussion

A nonlinear stochastic differential equation, the Belavkin equation, has been derived from purely quantum machinery describing an unravelling of the Lindblad equation, including the GRW master equation. This unravelling can be used as a Monte Carlo integrator of the Lindblad equation or interpreted in the sense of foundational quantum mechanics as a new fundamental dynamics for all quantum systems. In the former case, we have a mathematical tool which can be used purely for computation whereas in the latter case we have an equation, containing N (or in the CSL case W), an object which may be called the collapse or collapsaton field, with a new ontological interpretation. This field is a classical stochastic process, which endows the state space of the extended system with a dual nature—partially quantum and partially classically stochastic. It has been shown in Section 9 that the collapse field can be put into a purely quantum formalism using the HP equation and a Weyl operator, which enacts the quantum analogue of a change of distribution. Although the purely quantum formalism is not directly analogous to the Belavkin equations used in stochastic collapse, due to the absence of the fully quantum analog of the stochastic Weyl operator, which in the probabilistic interpretation may take arguments outside the one-particle space, the quantum Belavkin equations do exhibit some of the properties we would require of a stochastic collapse equation such as preservation of norm (in this case a generalized norm defined as the reduced Fock state) and unravelling of the correct master equation. In these cases, the Weyl operator is what differentiates between CSL and GRW, via the different one-particle states u_t^α and v_t^α in Section 9. Note that to derive the CSL and GRW equation from HP evolution, the natural initial Fock vector was chosen as the vacuum, $|\Omega\rangle$. This eliminates all quantum noise processes except for the creation operator, which is shared between the quantum white noise $A_\alpha(t)+A_\alpha^\dagger(t)$ and the quantum compensated Poisson process $\Lambda_\alpha^\alpha(t)+A_\alpha^\dagger(t)+A_\alpha(t)$, so that the HP evolution for the CSL and GRW derivations remains the same.

Since in stochastic collapse models, the trajectories are taken to be real stochastic system trajectories, it becomes necessary to test whether these trajectories exist. In this sense, the nonlinear functionals of the state vector become essential in differentiating betweeen physically real stochastic trajectories and mathematical objects used to unravel the master equation. In Section 12, one such functional was considered: the entropy. It was shown that, although the probabilities associated with measurements on the system do not differ from those obtained from the master equation, in the case of real trajectories the entropy is different, due to the process of localization (or action of the Lindblad operators), from the standard von Neumann entropy, a result which could potentially have thermodynamic consequences for the system due to the decrease in entropy resulting from localizations which decreases the spread of possible position states.

One important distinction to be made between the classical stochastic and quantum stochastic pictures, is the appearance of the probability space, which, via the probability parameter ω, defines the trajectories, not apparent in Fock space. The Segal isomorphism leads to a representation of Fock space vectors as functionals of a stochastic process. This means that the probabilistic interpretation is actually essential for the unravellings to be considered as real trajectories. In the quantum picture, we have a field which is considered as any other physical bosonic field, while in the classical case, we have a formalism that allows us to speak of individual trajectories for each probability parameter. The quantum picture alone cannot account for dynamical collapse, as in GRW, because of the status of vectors in Fock space as functionals with no concept of collapse until a probabilistic picture is introduced.

Finally, new closed nonlinear quantum stochastic Belavkin equations were derived based on the preservation of the system norm, defined as a partial trace over the system space of the total density matrix. Because of the reasons given above this cannot be seen as equivalent to the GRW unravelling due to differences in the change of measure. It does, however, provide a fully quantum unravelling of the GRW and CSL equations, bringing them satisfactorily into the fold of quantum evolutions.

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