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Do mathematicians interpret equations asymmetrically?

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ABSTRACT

In studies of children's reasoning about equations, a major finding is that many children understand equality to be asymmetric. In this study, we investigate how experts interpret equations in order to determine whether and why they interpret equations asymmetrically. We do so by using a breaching experiment in which we present nine mathematicians with equations that have been purposefully reordered to see if they critique or correct the ordering. We found clear evidence that they apply norms of ordering to critique the texts. We characterize their explanations for why they prefer or expect one order over another by using six rationales that express why experts read equations asymmetrically. We consider the implications for how we characterize sophisticated meanings of the equals sign. Our findings show that mathematicians attend to the coherence of a text, the communicational needs of the reader, and imagined context to determine appropriate equation order.

1. Introduction

Although = is a symmetric relation, it seems that many children do not view it as such (see, for example, Behr et al., 1980). We suspect that any mathematician would not hesitate to say that a = b[1] and b = a are truth-functionally equivalent. In this sense, mathematicians have truth-functionally symmetric meanings for the equals sign. Our study investigates whether mathematicians have *only* symmetric meanings for the equals sign by considering that mathematicians might also hold asymmetric meanings and usages of the equals sign. There are little or no studies of expert usage of the equals sign, which stands in stark contrast to the breadth of literature on children's reasoning (e.g., Behr et al., 1980; Denmark et al., 1976; Falkner et al., 1999; Kieran, 1980; Knuth et al., 2006). We observe regularities in the order of certain equations across various textbooks, but this does not mean that mathematicians would object to alternative orders. By presenting mathematicians with equations reordered from their typical presentations in textbooks, our study provides evidence that experts do have asymmetrical meanings for the equals sign. Furthermore, we identify the norms by which they object to reordered equations and the rationales by which they explain why the typical order is preferred. This adds to the literature on understanding of = by identifying relevant features of how experts interpret equations within mathematics texts.

Accordingly, our study addresses the existence of asymmetrical meanings by answering the question:

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[1] We are using "a" and "b" as schematic variables in the sense that "a" and "b" are stand-ins for any terms/noun phrases.

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(1) Do mathematicians assign asymmetrical meanings and usages of the equals sign?

Establishing the existence of this asymmetry paves the way to learn more about mathematicians' asymmetrical usage of the equals sign. Accordingly, we also address the following question:

(2) In what ways is the equals sign used asymmetrically? What rationales and norms govern the ordering of terms in equations?[2]

In answering (2), we consider that equations are a key part of mathematical text and they are used as a record of procedures, to accomplish communicative purposes, and to convey conceptual relationships (Shepherd & van de Sande, 2014). Why do equations better fulfill those purposes in one order and not another? We want to learn what norms (general rules) and rationales (functional purposes) guided expert preferences.

2. Literature and theoretical framework

In this section, we explore the literature that pertains to potential asymmetry regarding the equals sign in the mathematics community. Although not framed in terms of symmetry, Veel (1999) found instances in which mathematical text uses the equals sign asymmetrically in their discussion of *relational clauses.* A *relational clause* is any clause that express a relationship between two objects. An *attributive* relational clause involves an asymmetric relation, whereas an *identifying* relational clause is a statement about identity and typically involves some conjugation of "to be." Veel (1999, p.196) identifies the latter as "the linguistic equivalent of the equals sign." Although Veel contrasts attributive and identifying clauses in terms of symmetry, they also suggest an implicit asymmetry in identifying clauses. Veel (1999) notes that identifying clauses (in mathematical texts) often bridge something new to the students/readers with something they already know. Veel (1999) gives examples that involve the new information or term being introduced first, such as "The mean, or average, score is the sum of the scores divided by the number of scores" (p.195). This instance suggests a rationale for ordering to manage the unfolding of new information within the text, which entails a possible norm for preferring one order over another. Notice this preference is based on an implicit judgment of familiarity to the reader. As we will see in our results, the idea of least familiar term being on the left was mentioned by mathematicians.

The notion of *relevance* (to the listener) also governs human utterances (Grice, 1975). For instance, van der Henst et al. (2002) report on an empirical study that suggests that relevance motivates people to round when giving the time, unless a context suggests that a more precise time is relevant to the hearer. These relevance judgments require making inferences about others' knowledge. "Human communication involves the attribution of mental states by the interlocutors to one another" (van der Henst et al., 2002, p.465). This suggests that a mathematician might consider the communicative needs of the listener or reader, so the mathematician's beliefs about the reader might influence their decision to present an equation in a certain order.

Ernest (2008) also identifies relevance to a mathematical task as an important factor when communicating mathematics. We can easily imagine situations in which a particular ordering of an equation might be more relevant to the reader than the reversed ordering. For example, because we read from left to right, the distributive law written as $x(y+z) = xy+xz$ might be more relevant to someone who is doing a mathematics problem that requires distribution, since such a person would want to replace the term $x(y+z)$ with the term $xy + xz$. Similarly, the same law written as $xy+xz = x(y+z)$ might be more relevant for a student whose task is to factor an expression.

When we read a sentence, we are constrained by our language, space, and time. Ernest (2008) explains "in any form of representation, there is always an ordering present, and this structures the access and role of readers" (p.44). Halliday (1985) defines the notions of *theme* and *rheme* to highlight the fact that some words come before others, and hence a reader experiences some words before others: the "*theme* can be identified as the elements up to and including the first experiential element at the beginning of a clause" (Schleppegrell, 2004, p.68), whereas the *rheme* is the remainder of the clause. The theme serves as "point of departure" for the clause, whereas the rheme is "the part in which the theme is developed" (Halliday, 1994), as cited in (Schleppegrell, 2004). The structuring of theme and rheme can be viewed as a way of giving texts coherence to achieve communicative goals (Schleppegrell, 2004). Halliday and Matthiessen (2014) explain that there is a relationship between the theme-rheme structure and a given-new structure, wherein "given" information is information that is already part of the discourse between speaker and listener, and "new" information is less familiar in the discourse at hand. Halliday explains that what the speaker intends to be given (already part of the conversation) is frequently the theme, whereas what the speaker intends to be new aligns with the rheme. This relationship between given and new points to a relationship between theme and rheme in consecutive sentences. Often, the rheme of one clause becomes the theme of the following clause (Schleppegrell, 2004). Since the reader starts the second clause having already experienced the rheme of the previous clause, the theme of the second clause is now no longer "new" and is instead "given." This suggests we may have different communicative purposes in saying "a = b" or "b = a," especially when linking equalities to say "a = b = c", in which b shifts from rheme of "a = b" to theme of "b = c". Hence, we can see that the notions of theme and rheme provide some insight to account for the way that mathematicians order terms in equations. Importantly, the ideas of theme and rheme highlight the fact that in a sentence something must be written (and hence experienced) first, which imposes potential asymmetry of meaning and usage.

To summarize, prior literature provides several suggestions as to the communicative purposes at play in mathematical texts that might lead mathematicians to prefer one equation order over another. These relate to constructing a coherent text, organizing familiar and new information in particular ways, and conveying relevance to the reader. Often, the author makes choices about ordering by

[2] his particular study concerns the English-speaking mathematics community. It does not address issues of asymmetry regarding equations embedded in texts in other language, such as those that read right to left.

anticipating what will be relevant or familiar to their reader, which involves imagining a possible audience and communicative set of goals.

3. Methodology: a breaching experiment

As discussed earlier, there is a body of literature establishing that students tend to view equality as asymmetrical. Our study complements this work by considering expert equation reading: do experts have asymmetrical meanings and usages of the equals sign, and, if so, in what sense? What are the rationales and norms that govern this asymmetry? One particular study on children's meanings of the equals sign inspired our methodology; Behr et al. (1980) describe six children who read aloud $5 = 2 + 3$ as "two plus three equals five" (i.e., differently than how it was written symbolically). One student asked the interviewer "do you read backwards?" Translating between the symbolic and the verbal modalities suggests that these children viewed $5 = 2 + 3$ as a rule violation and assumed that it must have been an error. By breaking the "rule" that the "answer" should be to the right of the problem, the researchers gained data to suggest that this was indeed a rule for these children.

The technique of breaking rules in order to confirm their existence is known as a breaching experiment (Rafalovich, 2006). A breaching experiment is a technique in sociology research that involves breaking a purported social rule (without the subjects knowing that this is the intention of the researcher) and observing the subjects' reactions. The idea is that the subjects respond in such a way that reveals that they felt that there was a rule broken. Some mathematics educators use this research technique to confirm and explore the social norms governing classroom mathematics activity (see, for example, Chazan et al., 2012; Weiss et al., 2009).

In this study, we perform a breaching experiment as one technique within the context of individual task-based interviews with mathematicians. Halliday (1978) suggests that people might have an idea of how frequently linguistic items occur in particular contexts. He explains "A speaker of a language has a fairly clear idea of the probabilities attached to stored items; he 'knows' (in other words it is a property of the system) how likely a particular word or group or phrase is to occur" (p. xxii). Viewing $a = b$ and $b = a$ as different linguistic items, a speaker might have a sense of which occurs more frequently and hence have a sense of what ordering traditions exist. An expert participant, such as a mathematician, likely has some sense of which orderings are typical and thus can detect when ordering norms are breached (see also Reber, 1989).

Furthermore, whenever someone is reading a text, a social interaction is taking place – one between the author and the reader. It is inferred that the author has some communicative goal in mind for the text. These communicative goals provide the justification for why norms are in place; norms seek to ensure the text accomplishes its goals (c.f., Dawkins & Weber, 2017). We thus use the term *rationales* to describe the goals or purposes that mathematicians attribute to a text to justify why certain rules (norms) should be followed. We observed that these rationales often related to imagined contexts for the equations: records of problem solving such as a solution to an assigned problem, a pedagogical explanation for novices, or a mathematical justification. Other times the rationales were relative to producing effective texts in ways that did not seem to depend on a particular imagined context.

Norms are not inviolable rules (Herbst et al., 2011), but rather, tools for decision making. In our study, we are merely interested to discern the norms and rationales that mathematicians invoke when discussing breaches they perceived (in reordered equations), not to test whether these judgments are universally consistent across various applications of a rule or rationale. We expect that these norms are dependent upon the imagined contexts and rationales at play in a given text that contains equations. Inasmuch as possible, we note the contexts that mathematicians invoke to justify why a particular norm is relevant and appropriate regarding the order of an equation. For instance, when mathematicians invoke a pedagogical context for a text, that suggests the norm may be particularly operative in pedagogical contexts. If they invoke a norm interpreting the text as a transformation, that means the norm may primarily apply to that imagined activity (and equations that evoke it).

In summary, we use a breaching experiment to confirm that there are ordering norms (in the mathematical community) for equations, suggesting that the equals sign is not purely symmetric in its meaning and usage. By characterizing the rationales, we explain why one order of equation is preferred over another.

3.1. Subjects and methods

We recruited nine participants (Jacob, Larry, Kevin, Warren, Ben, Edgar, Patrick, Bing, Xena) all of whom have graduate mathematics degrees and currently teach or have taught mathematics at a university in the United States. Pseudonyms reflect the participants' perceived gender; all participants present as cisgender men, with the exception of Xena, who is a cisgender woman. Each participant took part in a 60 to 95 minute individual semi-structured task-based clinical interview performed by the first author of this manuscript. These interviews took place over Zoom using a shared whiteboard setting. Tasks were presented on the shared whiteboard, which the participants could then annotate. The virtual whiteboard and the audio were recorded. Following the interviews, we conducted an adaptation of member-checking interviews (Creswell, 2012) performed over email. We wrote a narrative summary of each individual participant's equation interpretations. We then contacted them to ask if they were interested in reading their summary. If they answered "yes," we then emailed them the summary and asked if it conformed with their interpretations of themselves. Four interviewees – Kevin, Patrick, Edgar, and Ben – participated in member-checking and confirmed that the narrative summaries were consistent with their interpretations of themselves.

3.2. Tasks and data collection

The tasks meant to breach norms were created based on hypothesized norms of ordering taken from the literature, our reflections

OSumRule	OEuler
State the sum rule for derivatives, and write it down.	Write down and state Euler's formula

OMice

Consider a population of field mice who inhabit a certain rural area. In the absence of predators, we assume that the mouse population increases at a rate proportional to the current population. Using t to denote time, $p(t)$ to denote the population, and r to represent the growth rate, write a differential equation expressing the relationship.

OIdentity

Suppose $\langle S,\star\rangle$ is a binary algebraic structure. What does it mean for an element $e \in S$ to be a left identity element?

Fig. 1. Selection of open-ended tasks.

DifferenceQuotient

The **difference quotient** of a function g is defined to be

$$\frac{g(x+h)-g(x)}{(x+h)-x}$$

where h is nonzero.
Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^2$.
The following shows the difference quotient:

$$\begin{aligned} 2x+h &= \frac{2xh+h^2}{h} \\ &= \frac{x^2+2xh+h^2-x^2}{h} \\ &= \frac{(x+h)^2-x^2}{h} \\ &= \frac{f(x+h)-f(x)}{h} \\ &= \frac{f(x+h)-f(x)}{(x+h)-x} \end{aligned}$$

Distributive

The distributive law tells us that for all numbers x,y, and z,

$$xy+xz=x(y+z)$$

ProductRule

The *product rule* for derivatives says that if f and g are differentiable functions, then

$$fg'+f'g=(fg)'$$

Idempotent

Suppose $\langle S,\star\rangle$ is a binary algebraic structure. An element $p \in S$ is said to be idempotent under $\star$ if and only if

$$p = p \star p$$

Fig. 2. Selection of breaching tasks.

on equation use, and research experience. The interviewees were not told that this study is about equations or symmetry; they were instead told that the study is about "how experts read, write, and interpret mathematics." We created multiple tasks related to each hypothesis and the efficacy of each individual task in breaching a norm was not itself part of the study design. As such, no single task was essential for answering the research questions and not every task was done with every participant (see Table 1 in the Section 4). Our goal was to establish the norms' existence and rationales, not to quantify them in terms of strength, frequency, or scope. This flexibility was important for two reasons. First, some tasks, such as *SetTheory*, were unfamiliar enough that participants did not sense a breach (had no apparent expectations). Second, at some point in each interview the study participants became aware of the breaching intent of the interview, at which point we generally shifted the interview structure as described hereafter.

We partitioned the interviews into three phases. Phase 1 involved open-ended questions that ask participants to write an equation. We did this to confirm whether the mathematicians *produced* equations that conformed to our hypothesized norms of ordering. For example, we hypothesized that all the subjects would write the sum rule for derivatives with the derivative of the sum on the left side of the equals sign. The interviewee was asked to read the text out loud, carry out the task, and then explain their answer. Follow-up questions included prompts for elaboration or explaining in other words. Every open-ended task was completed with every participant; these tasks all have a label starting with "O." Examples of such tasks appear in Fig. 1, and all the tasks are in Appendix A.

The second phase of the interview is the portion in which the breaching experiment took place. A sample of the tasks involved is shown in Fig. 2. All breaching tasks are included in Appendix B. In these tasks, the interviewee read mathematics text that included an

MVTCompare

There are various ways that textbooks state the mean value theorem.

Theorem 1. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 2. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Theorem 3. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f'(c)(b - a) = f(b) - f(a)$$

Theorem 4. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a)$$

Fig. 3. A comparison task used in the third portion of the interview.

equation. That equation was reversed reordered from the way we observed it in textbooks. Our hypotheses about ordering expectations were the following: *ordering should abide by tradition, equations should go left to right from complex to simple, calculations go left-to-right-top-to-bottom, when proving $x = y$ one should start with x and end with y, some equations are rules for calculation.* For example, the task *Idempotent* displays $p = p \star p$, whereas $p \star p = p$ appears to be the equation texts typically use. This violates the potential norm that a calculation or operation should be on the left side, or that a simplified expression be on the right. In *Homomorphism*, the equation is written with $\phi(x \star y)$ on the right, whereas it usually appears on the left. While tradition was the only norm we were attempting to violate with this particular task, the results reveal that there were other norms violated.

Other tasks were designed to reveal the norm that calculations go left-to-right-top-to-bottom: that is, that on the left or beginning of a running equation is something that is given or presented to the problem-solver and on the right is something derived or calculated. In this case, the rationale for the norm is that the text expresses the calculational process of the problem solver in the order they solved it. For example, *DifferenceQuotient* includes a string of equations that starts with $2x + h$ and ends with $\frac{f(x+h)-f(x)}{(x+h)-x}$. Similarly, *Induction* includes a string of equations that starts with $(n^3 - n) + 3n(n+1)$ and ends with $(n+1)^3 - (n+1)$. This string of equalities was obtained by reversing the string of equalities that appears in a textbook (Stankova et al., 2008).

The *Proofs* task was more complex since it showed five expressions being equal, which allowed us to reorder the expressions in a more complex way. This relates to the theme/rheme structure and textual coherence discussed above. A goal of this task was to verify the following related norms: mathematicians do *not* read a = b = c = d = e as “a–e are all equal to each other”, and mathematicians are concerned with not only truth but also deducibility of each inference expressed in an equation.

In these Phase 2 tasks, the interviewee was prompted to read the text aloud and asked follow-up questions for elaboration. They were additionally asked for opinions such as “what do you think of this equation?” and “would you write it differently?” These latter questions were included for participants who do not mention breaches on their own. It seems reasonable to believe that some mathematicians might notice a breach but just not mention it – they might instead only focus on mathematical correctness. Just because you would have done it differently or find it unconventional does not mean you would necessarily remark on that observation. Indeed, we found evidence of these two standards of judgment about various breaches.

In some cases where the participant mentioned a breach (in particular, in the tasks *DifferenceQuotient* and *Induction*), the interviewer made attempts to repair the breach and then inquired further. This investigated whether the breach could be amended without reversing the order of the equations, or whether the breach was fundamentally tied to the equation order. This provided the opportunity to learn more about what, exactly, was being breached, which therefore revealed more information about reasons for asymmetry. For example, in *Induction*, several participants remarked that the term 3n(n + 1) appears to “come out of nowhere.” To repair this breach, the interviewer changed the task to move the sentence “Since either n or n + 1 is even, 3n(n + 1) is divisible by 6” to appear before the string of equations; this way, the 3n(n + 1) in the string of equations no longer “came out of nowhere.”

Phase 3 of the interview consisted of explicit discussions about ordering of equations, which usually began when the interviewees became aware of our manipulating the equations. Phase 3 involved “Comparison Tasks” in which the interviewee is asked to explicitly compare equations (Fig. 3 and Appendix C). In these tasks, participants were shown equations from previous tasks juxtaposed with reordered equations. The interviewer asked questions such as:

- "I have found that textbooks usually present the equation this way. Do you think there is a reason for that?",
- "Is there a difference in meaning between these equations?",
- "Is there an advantage to writing the equation one way over the other?", and
- "Can you give an example where writing it this way would be preferable and explain why?"

3.3. Data analysis methods

After each interview, the interviewer wrote up brief notes on her interpretations of the participants' reasons for preferring some orderings over others. Specifically, the interviewer took notes of any situation in which the participants answered the open-ended tasks in atypical ways, any situation in which they mentioned a breach, and a list of considerations regarding asymmetry (e.g., "this participant seemed to focus on the idea that terms shouldn't come out of nowhere"). In the second phase of each interview, the interviewer took note of situations in which the participant mentioned the right-hand side of the equation first in their explanation (e.g., explaining *Idempotent* by saying "p multiplied by itself results in p"). This is a form of repairing the breach in the translation into spoken language. Relatedly, the interviewer took note of language that appeared to entail an asymmetric interpretation of equality, such as "e times x *becomes* x" (emphasis added).

Once all the interviews were completed, the interviewer took notes on each individual interview. This involved re-watching the interview recording and transcribing the portions that pertained to ordering or asymmetry. We then began retrospectively coding the explanations for asymmetry using constant comparison inquiry (Creswell, 2012). We began with an initial list of codes to account for asymmetry, which reflected our hypothesized norms for asymmetry together with initial data-based interpretations of participant meanings. We then applied these codes to the interview data and took note of any reason for asymmetry not captured by these codes. All the reasons that we had expected people to have for ordering appeared, but reviewing the interview notes led to additional codes.

While the majority of the analysis involved looking across multiple participants for the norms and reasons for asymmetry, we also found it beneficial to analyze each person's interpretations individually. This involved writing a narrative interpretation of each individual's reasons for asymmetry, their rationales, and the tasks that evoked such asymmetry. These are the narratives that were shared with participants in the member-checking emails. This proved important because some individual participants used different words to mean the same thing, and the same word was used with different meanings across participants. Hence, an individual participant had to be interpreted across multiple tasks in order to discern their word meanings. For example, one participant used the words "mysterious" and "complicated" synonymously to describe a mathematical idea that would be relatively new to his envisioned reader. The meaning of the word "new" was dependent upon the participant. Some used the word "new" to mean "new to the conversation" while others used the word "new" to mean "mathematically new or unfamiliar to the reader." We note that these two notions of "new" mirror the two different notions of "new" discussed in the literature – while Halliday and Matthiessen (2014) use "new" to mean "new to the conversation" and suggests that "new" things are in the rheme (the right-hand part) of a sentence, Veel (1999) uses "new" to describe information that is educationally new to the reader and is on the left (in the theme). By doing the narrative analysis, we were able to discern how participants were using these words. The product of the individual analysis was a narrative summary together with an account of the various norms and rationales the participant gave for explaining asymmetry.

4. Results

We begin by answering research question (1): Do mathematicians assign symmetrical meanings and usages of the equals sign? In doing so, we provide evidence that there are indeed norms and rationales surrounding the usage of the equals sign that prioritize one order over another, meaning the equals sign operates asymmetrically. This is accomplished primarily with the first two interview phases: the open-ended tasks and the breaching tasks. We then later discuss specifically what these norms and rationales are (research question 2), which draws primarily on the latter two interview phases.

4.1. Evidence of asymmetry (research question 1)

The results of the open-ended tasks suggest that there are norms for writing equations in certain orders inasmuch as participants gave relatively consistent answers on all the tasks.

- On *OMice*, all nine participants wrote the differential equation the same way, with the derivative on the left-hand side ($\mathrm{dp}/\mathrm{dt} = \mathrm{rp}$).
- On *OEuler*, all eight of the participants who wrote the intended correct answer put the exponential term on the left ($e^{i\pi} = -1$ or $e^{i\,\theta} = cos\ (\theta) + i\,sin\ (\theta)$.
- All eight of the participants who wrote the correct equation for *OMVT* put $f'(c)$ on the left ($f'(c) = \frac{f(b)-f(a)}{b-a}$).
- On *OSumRule*, all nine participants wrote their equation with the derivative of the sum on the left ($(f + g)' = f' + g'$).
- On *OIdentity*, every participant wrote the identity element on the left ($e \star s = s$).

To summarize, participants ordered their equations consistently on the open-ended tasks; the only exceptions corresponded to inaccurate equations. This makes sense that norms of expectation may not apply as readily to information not easily recalled.

The results of the breaching tasks (in which we presented equations purposefully reordered) also confirmed that there are norms regarding asymmetry. This happened in three ways: they called out the breach of expected order, they implicitly repaired the breach by

Table 1
Results of breaching experiment by task and participant.

	DifferenceQuotient		Homomorphism	Exponents	Induction
Jacob	Y		N	Y	Y
Larry	Y		N	Y	Y
Warren	Y		Y		Y
Edgar	NY		N		NY
Patrick	Y		N		Y
Xena	Y		N		Y
Kevin	Y		Y	NN	
Ben	Y		N		Y
Bing	Y		N		Y
Total	Y(8), NY(1)		Y(2), N(7)	Y(2), NN(1)	Y(7), NY(1)
	SetTheory	ProductRule	Idempotent	Proof	Distributive
Jacob	N	Y	N	Y	N
Larry		Y	N	Y	Y
Warren		Y	*Y	Y	Y
Edgar	N	N	NN	Y	N
Patrick		Y	N	Y	NN
Xena		NY	N	Y	N
Kevin	N	Y	N		
Ben		Y	N	Y	Y
Bing	N	NY	N	Y	N
Total	N(4)	Y(6), N(1), NY(2)	*Y(1),N(7), NN(1)	Y(8)	Y(3), N(4), NN(1)

Note. Y means they encountered it as a breach task and mentioned an order breach. N means they encountered it as a breach task and did not mention an order breach. NN means they encountered it as a breach task and a comparison task and did not find there to be an order breach with either. NY means they encountered it as a breach task but didn't find there to be an order breach but encountered it again as a comparison task and did find there to be an order breach. *Y means they did not encounter it as a breach task, did encounter it as comparison task, and found there to be an order breach as a comparison task (when asked explicitly about ordering).

discussing the equation in the expected order, or they later reported that they perceived a breach but did not say anything. They explained that this is because they viewed the breach as either unproblematic or relatively unproblematic compared to a different breach.

Table 1 presents each participant's responses on the breaching tasks. Empty cells denote when a participant did not see a task, and double entries refer to when the participant did not mention a breach in phase 2 (breaching experiment) and had the opportunity to identify the ordering as a breach in phase 3 (reflective discussion and comparison). Examining the totals at the bottom of each column, we can see the trend that all but one or two participants gave the same initial judgment on most of the tasks. However, on tasks like *Proof*, *DifferenceQuotient*, and *Induction* they uniformly reported a breach while on tasks like *Homomorphism*, *SetTheory*, and *Idempotent* they uniformly did not report a breach. The only tasks with more mixed response were *ProductRule* and *Distributive*. We explain the disagreement on *Distributive* in light of the fact that both orderings of that equation are common, though one is usually called distribution and the other factoring. In line with this explanation, those who reported a breach pointed out that the name "distributive" was inappropriate. Responses to *ProductRule* were mixed because two of the three who did not report an order breach instead focused on their dislike of the use of function notation without variables ($(fg)'$ instead of $(f(x)g(x))'$).

In breach tasks, it was common for participants to explain the meaning of the equation from right to left. For example, in *Idempotent*, seven of the eight participants explained the meaning of the equation by mentioning the right-hand side of the equation first. Though they did not report an order breach on that task, they repaired the breach as they verbalized the equation. For instance, Larry said, "if you take this element and you apply it to itself, you obtain the original element." Mentioning the right side first occurred also in *ProductRule*, *Homomorphism*, and *Exponents*.

4.2. Reasons for asymmetry (research question 2)

The strong uniformity of participant response suggests that norms indeed render these equations as asymmetric. As we explained above, norms are in place to help texts achieve certain goals, which we call rationales. In this section, we present our characterizations of these rationales that emerged from our coding of participant explanations. These codes are not intended to accomplish a strict partition of the reasons given; that is, there are some overlapping codes. Some of the codes were closely related, but were tied to different rationales, such as pedagogical purposes or mathematical purposes for the text. We identify six primary rationales that the ordering norms expressed:

1. Texts Should be Coherent.
2. The Right Side Explains.
3. Transformations and Substitution: a Produces or Becomes b.
4. Proofs Should Reflect the Prover's Process for Creating the Proof.
5. Ordering Should be Pedagogically Optimal.

CProofs

Theorem 1. *Suppose $\langle S, \star\rangle$ and $\langle S', \star'\rangle$ are binary algebraic structures, and ϕ is an isomorphism from $\langle S, \star\rangle$ onto $\langle S', \star'\rangle$. Further suppose that e is a left identity element in $\langle S, \star\rangle$. Then $\phi(e)$ is a left identity element in $\langle S', \star'\rangle$.*

Proof. **First Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) = \phi(e \star s) = \phi(s) = s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star\rangle$ □

Proof. **Second Proof** Suppose $s' \in S'$. Since ϕ is onto, there exists $s \in S$ such that

$$s' = \phi(s) \tag{1}$$

By (1) we also know that

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) \tag{2}$$

Since e is a left identity element,

$$\phi(e \star s) = \phi(s) \tag{3}$$

and since ϕ is a homomorphism,

$$\phi(e) \star' \phi(s) = \phi(e \star s). \tag{4}$$

Therefore, by equations (1)-(4),

$$s' = \phi(e) \star' s'. \tag{5}$$

□

Proof. **Third Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(s) = \phi(e) \star' \phi(s) = s' = \phi(e) \star' s' = \phi(e \star s)$$

□

Proof. **Fourth Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$s' = \phi(s) = \phi(e \star s) = \phi(e) \star' \phi(s) = \phi(e) \star' s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star\rangle$ □

Fig. 4. The CProof task used in phase 3 of the interview.

6. Equations in Proofs Should Express Justifiable Inferences.

4.2.1. Texts should be coherent

Our first overarching rationale is textual coherence. This captures the idea that mathematical text, like any text, ought to be structured in a way that is coherent, cohesive, and organized. The choice of term ordering in equations is thus used to create a more coherent text. This is, perhaps, the most common rationale that related the norms that our participants cited. It occurred on multiple tasks with every participant. Within this textual coherence umbrella were various other common ideas or rules for ordering. These norms, although listed separately, are interrelated.

4.2.1.1. Norm 1: ordering should be consistent and match expectation. The notions of consistency of order appeared in a few situations. Both consistency across contexts and consistency within a context were suggested as reasons for particular orderings. For example, some participants expressed that the way that an equation is written in a theorem should reflect the way that it appears most often in proofs or computations (within text). Some participants cited "tradition" as a reason for certain orderings (across texts). Naturally, the latter is sensitive to experience. Ben explained that he had no strong preference for ordering in *Idempotent* because he hasn't "used it enough times to have a sense of the frequency with which each [ordering] occurs."

As an example of consistency within a text, in *CProofs* some participants explained that because the definition of left identity has the operations on the left, the proof should mirror this same structure. Warren explained that the equation in *Homomorphism* should be reversed to match the function direction:

> For the simple fact that the homomorphism maps S to S'. it's directional, we start with S and we want to get S'. And, also, the two algebraic structures are presented in that order, S then S'. And then when we write the equation, we're actually kind of, reading left to right anyway, we're actually going backwards from the way everything is presented.

4.2.1.2. Norm 2: theme-rheme structure should be respected. SFL posits that theme/rheme structure is a key way texts build coherence across sentences. Participants used this as a reason for ordering in the task *CProofs* (Fig. 4); specifically, some participants explained that they preferred the fourth proof because it starts with s', which is what the previous sentence ended with (rheme becomes theme). The notions of theme and rheme also help account for why participants tended to dislike the second proof and would rewrite it. The first four equations in this proof are of the form (stacked) "a = b, e = d, c = b, d = c," and participants tended to take objection with this presentation and wanted to rewrite as a string of equalities with a first, b second, c third, d fourth, and e last, which repairs the pattern that the rheme of one equation becomes the theme of the next. In *CMVT*, one reason participants cited for preferring $f'(c)$ on the left is that the sentence above ended with c. Xena explained "If I say there exists a point c such that, I'd wanna then say something about c." In other words, while "a = b, e = d, c = b, d = c" is truth-functionally equivalent to "a =b, b = c, c = d, d = e," the latter ordering is preferable since it creates a more coherent text. This is evidence for and explanation of an asymmetric meaning for the equals sign.

4.2.1.3. Norm 3: things shouldn't come out of nowhere. Perhaps the strongest norm related to coherence is the rule that "things should not come out of nowhere." The tasks *Induction* and *DifferenceQuotient* evoked these responses very strongly. *Induction* consists of a proof of the inductive step in showing that $k^3 - k$ is divisible by six for all k and begins with a string of equations starting with $(n^3 - n) + 3n(n+1)$ and ending with $(n+1)^3 - (n+1)$. All participants except Edgar disliked this presentation on the grounds that $3n(n+1)$ was introduced out of nowhere. For example, Warren remarked that he disliked that it was "summoned out of thin air." Larry explained "we have this mysterious $n^3 - n$. I don't know where this is coming from. Then the $3n$, that haunts me as well." Ben said, "I'm annoyed." The norm of introducing new expressions is part of how mathematical texts maintain coherence and clarity, and our participants responded to breaches of this norm quite strongly. This notion of "things not coming out of nowhere" relates closely to the notion of theme and rheme in Halliday's SFL; recall that the theme is assumed to be familiar in the conversation, whereas the rheme is less familiar.

4.2.1.4. Norm 4: the reader should know why a term is being introduced. The norm that "things shouldn't come out of nowhere" overlaps with other aspects of textual coherence. Generally speaking, participants wanted it to be evident to the reader why a term is being introduced. In proofs, this means that it needed to be clear how introducing a term contributed to the proof at hand. Participants invoked this issue commonly on *Induction* and *DifferenceQuotient*. For example, in *DifferenceQuotient*, participants explained that they should know why the term $2x + h$ is introduced to begin with and its relation to difference quotients. Edgar explains his reaction in the following exchange:

> Edgar: I did not like that. Like, I kinda got to when it said, "the following shows the difference quotient" and I'm like, where are you going with this? What are you gonna tell me, right? And then you start with 2x plus… and what I had to do is I read 2x plus h and I was like, I don't even care about the steps, I'm like what's the last step? I jumped directly from there to the last step to figure out what in the hell you were talking about. I didn't make a big deal out of it, but maybe I should have. I didn't like it.
>
> Interviewer: OK, tell me more about how you were feeling. Like, what you didn't like about it. What was the issue?
>
> Edgar: Well, it's like, what is your point? Like, what are you trying to tell me?
>
> Interviewer: Was it clear once you finished?
>
> Edgar: Yes, but then I was annoyed at having had to like, go around your presentation. Like, literally I went around it. I jumped from 2x plus h to the bottom to see what the hell you were talking about.

Notice that Edgar gave more of a reason than simply $2x + h$ coming out of nowhere. Beyond the term not previously appearing, he expects that the relevance of the term be explained.

4.2.1.5. Norm 5: the topic should go on the left. Another frequently mentioned norm surrounding textual coherence is that *the topic should go on the left*. This expresses a sense that while equations show two equivalent expressions, they are not typically of equal status in the text itself. One reason participants gave for preferring $f'(c)$ to be on the left-hand side in *CMVT* (see Fig. 3) is that the Mean Value Theorem is a theorem about derivatives. Jacob explains: "the $f'(c)$ is almost like, the topic of this theorem". Some participants made comparisons to the grammatical notion of a subject of a sentence. For example, in the *CProofs* task, Edgar explained why he preferred to write $\phi(e) \star$ *'s'* on the very left of the string of equalities:

> But uh, the thing that you are saying something about, that you feel like you're talking about should be on the left-hand side. And are we making a statement about $\phi(e) \star$'s' or are we making a statement about s'? And I feel like because we're trying to establish that $\phi(e)$ is the identity element, that's the subject of the sentence, so it should be on the left.

Though we are focused on equations, we anticipate this same norm applies to mathematical statements such as inequalities.

4.2.2. The right side explains

This rationale relates to the goal that a mathematical text should not only be true, but in some sense informative or insightful to the reader. Hence, these norms are contingent upon the envisioned context of a mathematical text teaching the reader something about mathematics. Since the left side is the topic of interest, it makes sense that the right-hand side would be explanatory; it should give information about the topic. The equation a=b can be interpreted as "a has the property of being b." This interpretation lends asymmetry in meaning to relational clauses; in this sense, such a clause (e.g., "2 is the only even number") can be understood as attributive rather than relational even though they are logically symmetric in a way strictly attributive claims ("2 is an even number") are not. This rationale was often expressed in the asymmetric language participants used in explaining the meaning of the equations, such as "ends up being," "happens to be," and "turns out to be." For example, in *Ceuler*, Warren explained that "$e^{i\theta}$ turns out to be $isin(\theta) + cos(\theta)$." Under this umbrella of the right side being explanatory, there are several closely interrelated norms for ordering of the form "Equations should read from…":

(a) *unknown to known:* an unknown thing is on the left while a known is on the right.
(b) *sophisticated to less sophisticated*: the more mathematically sophisticated thing is on the left.
(c) *question to answer*: a question is on a left, whose answer is on the right.
(d) *less expanded to more expanded*: the left is shorter or less expanded than the right.
(e) *defined to definition*: the concept being defined goes on the left, and its definition goes on the right.

Recall the idea that the left side is the topic of discussion or interest, while the right side explains what is on the left. Notice that norm (c) in particular may invoke a pedagogical context for the text while norm (e) is somewhat connected to the context of mathematics. Sometimes the order preference was relative to the broader discussion or problem-solving process. In *OMice*, Kevin explained that if he knew the value of p' then he would write the equation as $rp = p'$, whereas if he knew the value of rp, he would write the equation as $p' = rp$. To him, this was linked to the topic of discussion; the topic of discussion is something mathematically unknown to us that we want to find out about (which is on the left), and we have information that tells us something about it (which is on the right). Notice that, here we have another connection to the theme and rheme constructs from SFL. For Kevin, the notion of p' being first is linked to the notion that p' was already being discussed in the imagined discussion. It therefore makes sense that p' plays the role of the theme, whereas the object "newer" to the discussion (rp) is positioned in the rheme.

The notion of the right side being more known than the left side parallels the idea that the right side is more easily understood or less sophisticated than the left-hand side. In the task *CEuler*, Larry explained that $isin(\theta) + cos(\theta)$ is easier to understand: "You want to find real and imaginary parts, so decomposing in that way makes life simpler. You see what's going on." In *CMVT*, Bing explained his rationale for having $f'(c)$ on the left side: "formulas often have the form something we don't understand equals something we understand."

Equations should read from question to answer also involves the unfolding of knowledge. We ask questions about things that we do not fully understand, and our answers should be easier to understand or more known than the question. It bears mentioning that an analogous norm appears in the literature on young students' reasoning about equations; Denmark et al. (1976) characterizes young students as understanding the equals sign as "a one-directional operator separating a problem from its answer" (p.31). For mathematicians, the notion of problem and answer is less of a strict rule and relates more to context.

Jacob used the word "simple" to mean "mathematically less sophisticated" or "easier to understand." For example, in *CMVT*, he explained: "the left-hand side is the more mysterious quantity, and here we're giving, it's like the question, what is this mysterious quantity? And the answer is the right-hand side." Jacob also argued that the number of stipulations needed for the object in question to exist is another aspect of what makes one side of an equation more sophisticated. In *CMVT* and *OMice*, he explained that the stipulations needed for a derivative to exist indicate that the side with the derivative is more "sophisticated." The first consideration invokes the context of reasoning while the second invokes considerations of mathematical definition.

Some participants explained that the "newer" object should be on the left. This might seem to conflict with things not coming out of nowhere, but here participants were concerned with "educationally new" rather than "new to the conversation." In a pedagogical explanation, something globally unfamiliar to the learner (the topic of instruction) will often present as a unifying topic in the text. In a lesson first introducing derivatives, readers learning about derivatives for the first time will recognize the fact that derivatives are being discussed. Hence, it makes sense that the topic familiar to the conversation plays the role of theme while simultaneously being educationally new. The rheme, then, gives us information about this educationally new idea. These two senses of "new" or "familiar" distinguish these norms that may become operative either for creating textual coherence (familiar thing on the left) or for constructing an explanation (novel thing on the left). In *ProductRule*, Kevin used the ideas of "unknown," "new," and "problem" being on the left:

> Kevin: The question is um, if we knew how fast each of the quantities grows or function um, how could we uh, get some information about the um, uh, growth rate of or the rate of change of the product
>
> Interviewer: So, you're saying this would be in a problem where this (points to (fg)') is –
>
> Kevin: Something that I'm interested in, yes.
>
> Interviewer: Then what role would the other side (points to $fg' + f'g$) play?
>
> Kevin: Um, that it's basically using things that we already know about (…) the typical question is we know the rates of change of f and g and now we're interested in the rate of change of the product

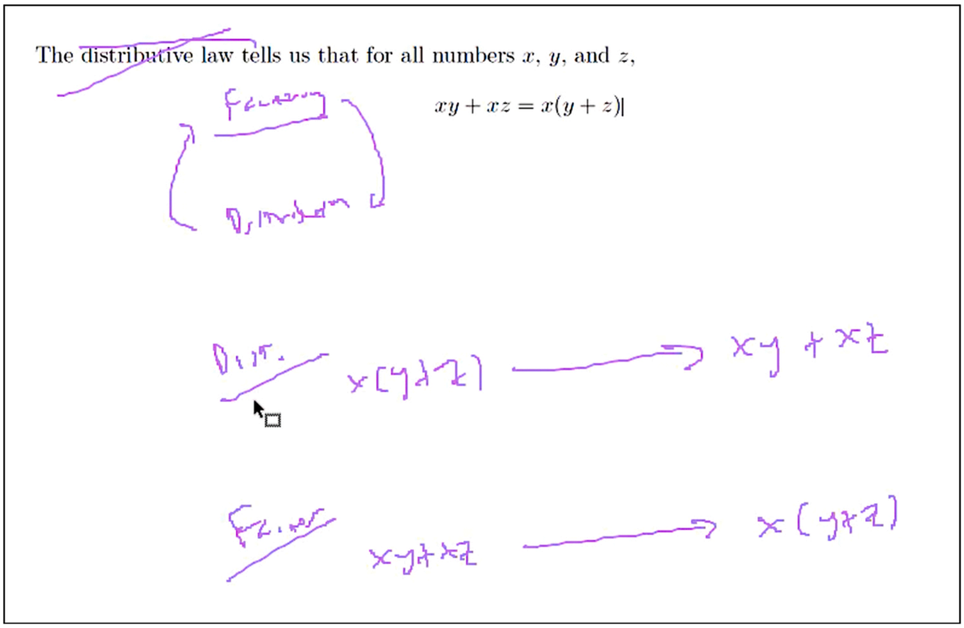

Fig. 5. Ben's illustration of his meaning for the distributive/factoring law(s).

> Interviewer: So, you're saying the known part is on the right, and the thing we are trying to find out about is on the left? So why do you think that is?
>
> Kevin: That's just how we read things from left to right. It's similar to when I write a computer program and I make an assignment, or I make a definition that usually the new object is written on the left-hand side

Notice that Kevin related the notion of "new" to that of "definition," specifically in a programming context. Similarly, in *CEuler*, he explained that because Euler's formula is actually a definition of irrational exponents, the exponential term must be on the left.

Closely related to the idea of the definition being on the left is the norm that the right-hand side is more expanded or verbose than the left. An explanation tends to be more verbose than the thing being explained. In the context of *ProductRule*, Patrick said, "the less expanded form first and then more expanded form later seems to be the unwritten rule of mathematical writing."

4.2.3. Transformations and substitution: a produces or becomes b

The next rationale concerns transformation, which invokes asymmetry because $a = b$ is interpreted as a transforms into b or b may be substituted for a (in the record of some problem-solving process). Ernest (2008) explains that transformations induce directionality (and hence asymmetry): "The transformation of signs in semiotic systems is directional" (p.43). These norms reflect the general idea that, when an equation represents a transformation, the directionality of that transformation (from a starting point to a finishing point) should be reflected in a left-to-right ordering. Note that norms are contingent upon viewing equations as transformations. Transformations tend to take place in problem-solving contexts (Ernest, 2008). Accordingly, we observed that many participants invoking this rationale envisioned a problem-solving context, even when not presented in the task.

The first way that a transforms into b is in the sense that *operations produce a result*, and therefore an ordering norm asserts that *operations go on the left while the result they produce goes on the right*. Warren, in the context of *Idempotent*, explains his reasoning for this norm: "I start with the operation, then a second later, if you will, I have a result. With our thinking, that's what happens. So, when you're reading left to right, that should mimic as it happens in your brain, too." For at least some participants, the notion of operation producing a result was an essential aspect of the meaning of a left identity element. Their explanation in *CProofs*, in favor of First Proof ($\phi(e) \star$'s' $= \ldots = s$') over Fourth Proof (s' $= \ldots = \phi(e) \star$'s'), was that the first proof shows that ($\phi(e) \star$'s') ends up resulting in or producing s'. Xena explains: "If I could do it starting with the star operation with the identity element and show it doesn't do anything, then that would be my preference. The other order, although equivalent, seems less natural."

A related norm for an equation as a transformation is that *what is given should be on the left, whereas the goal or the result should be on the right*. Notice that, here, we again have some overlap between the theme-rheme constructs. Recall that the theme is often what is *given*, whereas what we end up with in a transformation is in some sense *new*. This interpretation differs from those previously described because the reader imagines that the left-hand side actually becomes (transforms to) the right-hand side via some (mental or possible) actions. In *ProductRule*, Edgar explains that "what you *have* is on the left, and what you *can have if you want* is on the right," and Ben similarly explains "the thing I got already is on the left, and the rule is telling me what I should do with the thing." In other words, theorems of the form "a = b" suggest that, in problem-solving contexts, one might want to transform a to b. Ben illustrated this idea in the context of *Distributive* (the distributive law) by explaining "we're presenting rules for operating on symbols" and using an arrow in place of the equals sign to indicate transformation (see Fig. 5). His point was that $xy + xz = x(y + z)$ suggests a different transformation than $x(y+z) = xy + xz$. Since these suggest different transformations, these different transformations suggest opposite orders. Hence, the application of this norm is contingent upon envisioning an equation as a transformation and the order it prioritizes depends upon which transformation is imagined.

This notion generalizes to the idea of substitution; that in an equation, *the thing on the left can be substituted with the thing on the right*. For example, both Ben and Luke explained that in *CProofs* they wanted the equations to be written to express a substitution. Ben said:

> My favorite thing about the equals sign is that it literally means that the two things on either side of it are the same thing, and therefore when you see one of them come up later you can replace it with what's on the other side of the equals sign…like if I see $\phi(s)$ and I know $\phi(s)$ is s', then I can write s' whenever I see $\phi(s)$.

These transformational interpretations were clearly context specific. Transforming in one direction or another requires assuming there is some goal in mind. A common theme expressed by participants is that *simplification is preferable to the opposite*, and simplification occurs as a transformation from left to right. Participants use "simplification" in two different senses. One is the sense in which the simplified thing (the thing on the right) is easier to work with or more understandable. This idea overlaps with the idea of substitution; if b is "simpler" than a in the sense of "easier to work with," then one might want to replace a with b. Ernest (2008) refers to an "implicit heuristic of simplification" which "seeks to reduce the complexity of terms in an equation en route to a solution" (p. 40).

Kevin expressed the other meaning for simplification as "try(ing) to write this in as few symbols as possible." For example, in *DifferenceQuotient*, $\frac{f(x+h)-f(x)}{(x+h)-x}$ simplifies to $2x + h$. Participants explained that simplification is preferable to "messification" (Xena's term) because it is easier to perform. Two reasons were given for simplification being easier than messification: 1) school mathematics trains people to perform simplification procedures and 2) the simplification is more deterministic than messification. While discussing *DifferenceQuotient*, Warren uses both of these reasons to explain why simplification is easier:

> Students are mathematically trained to simplify expressions. They're not trained to complicate expressions in order to fit a predefined structure. And, the steps of factoring, reducing, simplifying, cancelling, whatever in more simple terms, these are all directions students are comfortable with. And, the other direction, you know when you say, why would I want to change $\frac{2}{3}$ to $\frac{4}{6}$ or change it to $\frac{2x}{3x}$? There's multiples of those steps, where you're complicating in the dark not knowing what steps you need to do so that the chain of applications leads to the difference quotient. That is not, there's no series of, there's no sequence of instructions available to a student to know how to do that backwards I think, especially when there's multiple steps.

Here again we see a possible conflict between norms depending upon which is evoked in reading an equation. The norm *less expanded to more expanded* applied to certain formulas while norms of *simplification* applied to the record of a problems solving process. The first suggested we would expect less symbols on the left while the latter prioritized less symbols on the right. This again shows how the norms evoked depended upon the rationales at play, which are tied to imagined contexts and activities.

4.2.4. Proofs and solutions express the author's process for creating them

The tasks *DifferenceQuotient* and *Induction* both evoked the rationale that a proof or a problem's solution should, when possible, communicate how the prover came up with the proof or solution. Unsurprisingly, this invokes asymmetry, due to the nature of a prover coming up with their proof or solution over time. In this sense, there are ordering norms stipulating that *a proof or solution should be a record of a problem-solving process*. This rationale invokes obligations the author has to their reader in conveying their reasoning process. For example, if the prover performed a process to transform a to b, then the equation a=b should be in the proof rather than b=a. The participants read both *DifferenceQuotient* and *Induction* as records of mental processes and transformations. For example, on *Induction*, participants explained that at first they found the equation $(n^3 - n) + (3n^2 + 3n) = (n^3 + 3n^2 + 3n + 1) - (n+1)$ to be "clever" because it suggested that the prover performed a transformation of adding 1, subtracting 1, and regrouping. Xena explained: "I would have done this from the other way. I would have started here (underlines $(n+1)^3 - (n+1)$) and expanded it and gotten something else and then tried to figure out how to prove it." Her remarks indicate that she conflated the presentation containing the proof with how the prover came up with the proof.

At other times, participants disbelieved the author reasoned in the order presented in the equations and thus objected to the presented ordering. In *DifferenceQuotient*, Ben remarked "no human would go this way." Objecting to this ordering suggest that not only is a proof supposed to have true and deducible statements, but also it should also communicate something about the prover's intellectual process. Literature on $\varepsilon - \delta$ proofs discuss how this norm is notoriously violated in such proofs (Schaub, 2021; Selden et al., 2018). Nevertheless it seems that mathematicians in other contexts maintain a norm that textual order and reasoning process be consistent whenever possible.

4.2.5. Ordering should be pedagogically helpful

Another rationale governing ordering is that *equations should be ordered in a way that is pedagogically helpful.* This rationale suggests asymmetry, since some orderings are understood to be more pedagogically helpful than others. Participants used this rationale to critique equation orders that they anticipated would be confusing or less helpful to students. This shows the influence of context in textual choices (the mathematicians assumed a pedagogical context not given in the tasks) and the principle that speakers formulate texts by attributing mental states to the listener or reader. Lai and Weber (2014) also observe that mathematicians consider the role of their audience when presenting a proof. Several participants justified earlier norms such as things not coming out of nowhere by invoking the confusion it might cause students.

This rationale frequently overlapped with the previous rationale; a proof or solution to a problem communicates to the student something that they should learn about. In other words, some of the same expectations for ordering (norms) were justified in terms of

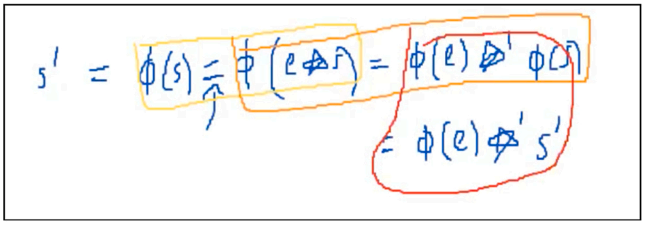

Fig. 6. Ben's illustration of how he reads a string of equations in a proof.

these slightly different rationales (they uphold multiple goals for communication). At times, what students needs to learn is the process of reasoning or the sequence of transformations considered in the previous two rationales. Bing explained, "Part of teaching math isn't teaching theorems that are true, it's teaching students how they could have done it themselves." This rationale appeared prominently on both *Induction* and *DifferenceQuotient*. On *Induction*, some participants objected on the grounds that students would not be able to produce the proof in the order that it is presented. On *DifferenceQuotient*, participants similarly explained that the equation should be presented with the difference quotient form first because it was communicating to the student a way of dealing with difference quotients that could be generalized to other functions.

Ben mentioned that there are cases, such as epsilon-delta proofs in analysis, where we do present a proof in the opposite order in which we created it. However, he explained that we have "culturally decided not to share this scratchwork" (tradition allows breaching the norm). This is consistent with Herbst et al.'s (2011) characterization of norms in which they are not always inviolable rules, but default expectations whose breaches often invite attention and justification. The fact that Ben was aware of sanctioned breaches of these ordering norms is evidence that he was keenly aware of the norm's existence more broadly.

4.2.6. Equations in proofs should express justifiable inferences

For proof texts, our participants expressed the rationale that equation order should express justified inferences and make such inferences easy to follow. This rationale suggests asymmetry; "a = b = c" is understood to express an inference *from* "a = b" *to* "b = c," which is directed. This rationale strongly reflects the context of mathematical practice. In proofs, the goal is not just to say true things, but to warrant the claims made. In proof tasks, participants explained that they verified each equation pairwise and thus the order of equations mattered for justification. The *Proofs* task (see Fig. 4) evoked these responses quite often since it violated norms associated with this rationale. Fig. 6 displays how Ben illustrated his reading of the equation string from left to right, attempting to verify each equation pairwise.

For example, participants took issue with the Third Proof on the grounds that it was not of the form "a = b =c = d = e." While each individual equation was true, they were not inferable from the previous equations. The *Proofs* task was somewhat unique in that it is the only task in which participants described the presentation as mathematically incorrect or wrong, rather than merely having a preference or an expectation for a different ordering. In addition, some participants objected to proof presentations that were justifiable, but did not make the inference easy to follow (such as the second proof). In this manner, this rationale was also framed within an author's obligations to the reader. Ben suggested that this breached a norm by not building the work of justification into the text itself: "you left all the work on the table!"

5. Discussion and future directions

Overall, the results of this study suggest that mathematicians assign asymmetrical meanings and usages to the equals sign. This is evident by the consistent responses to the open-ended tasks. The norms surrounding ordering of terms in equations are strong enough that the majority of the participants (eight out of nine) pointed out order breaches (either explicitly or by verbally reordering). The one participant who did not point out a breach during the breaching experiment phase, when asked explicitly his opinions on ordering, did explain that he found there to be breaches that he had just not mentioned. In other words, all nine participants found there to be ordering breaches. Thus, we see that equations do not operate symmetrically due to the rationales and norms we described in Section 4.2.

Consistent with prior literature about mathematical language and texts, mathematicians perceive equations asymmetrically because they interpret the text as pursuing implicit goals (rationales), they attend to the context of communication (e.g., pedagogical), and they reason about the communicational needs of the reader to judge textual choices. They often interpret the text attending to the unfolding of information, inferring the text reflects a record of a reasoning process, desiring coherence across the text, and pursuing mathematical goals such as substituting, simplifying, or justifying. All the ordering norms seem reflect the general notion that from Halliday's SFL that the reader experiences the left side of a sentence first. Mostly, these norms are about style and not mathematical accuracy, but our participants nevertheless expressed emotional responses to some breaches of ordering norms (such as frustration or annoyance).

We proposed a two-part structure for our coding of the mathematicians' preferences for one equation ordering over another: norms and rationales. The norms express the particular rules that a text may violate such as "the reader should know why a term is being introduced." These norms are not absolute laws and we saw within our data places where it might appear they were applied inconsistently. This does not undermine our findings because we merely claim that the norms were operative in at least some settings, since

Table 2
Summary of rationales, norms, and imagined contexts.

Rationale	Norms
Texts Should be Coherent	– Ordering Should be Consistent and Match Expectation – Theme-Rheme Structure Should Be Respected – Things Shouldn't Come Out of Nowhere – The Reader Should Know Why a Term is Being Introduced – The Topic Should Go on the Left
The Right Side Explains	– Equations should read from known to unknown – Equations should read from sophisticated to less sophisticated – Equations should read from less expanded to more expanded – Equations should read from defined to definition
Transformations and Substitution: *a* Produces or Becomes *b*	– Equations should be ordered in the direction of transformation or of intended substitution
Proofs and Solutions Express the Author's Process for Creating Them	– Equations in proofs or problem solutions should follow the order in which they were produced
Ordering Should be Pedagogically Helpful	– Equations in proofs or problem solutions should demonstrate how a student might reason through the task or imitate the procedure on a similar task
Equations in Proofs Should Express Justifiable Inferences	– Equations in proofs should be ordered so each equality can be clearly justified, and that justification should be accessible to the reader

our study's participants invoked them. Furthermore, these norms were contingent upon the rationales. The norms express the (implicit or explicit) rule that the reader perceives the text to have violated. This is consistent with prior literature that notes how norms are tools for decision making and evaluating people's activity (as well as one's own), but they are not binding (e.g., Dawkins & Weber, 2017; Herbst et al., 2011). These rules exist to help ensure that the text meets its goals, which we have called rationales. These express the virtues that an effective text should exhibit, either to be effective as a text, to communicate appropriately, to abide by standards of mathematical practice, or to be useful for instruction (see Table 2). In a sense, these rationales express purposes and meanings for equations in mathematical texts. These constructs could be leveraged for future work exploring expert understanding. Alternatively, they could be used as a starting point for evoking and analyzing novice interpretations of equations. Future work could also do more to confirm the breadth, strength, and interrelationships between these norms and rationales more fully. Our study provides evidence for their existence among experts.

The results of this study are especially interesting as compared with studies on children's reasoning about the equals sign. Those studies largely establish that students *lack* symmetrical meanings of the equals sign in a way that is problematic; Byrd, McNeil, Chesney, and Matthews (2015) suggest that lacking symmetric meanings of equality predicts of difficulties in early algebra. Those studies provided powerful insights about children's purely asymmetrical meanings for equations that constrained their understanding, while our study seeks to broaden our awareness of the breadth of expert interpretations. Unlike with students, the experts in our study do not lack symmetric meanings (none of them questioned whether an equation changed truth-value after reordering). What our study contributes is evidence that experts also have asymmetric understandings. Unlike many students, experts' asymmetrical meanings are context-dependent and help them accomplish key mathematical goals rather than constraining them. Writing $x = e \star x$ is not considered "wrong" by experts, but connotatively different and in some contexts less preferable than writing $e \star x = x$. Interestingly, there is some overlap between experts' and children's asymmetrical meanings and usages of the equals sign. For instance, the transformation norm suggests operations should be on the left. Many of the "rule violation" tasks presented to children in Oksuz (2007), which researchers take to be appropriate and to which students object, do not have the operations on the left. McNeil and Alibali (2005) report that even college students have this norm of operations being on the left (they call this an "operational pattern"). Similarly, the norm that equations should appear with the left as a problem and the right as an answer appears also amongst children (Behr et al., 1980; Denmark et al., 1976). We find it interesting that some of the same communicative frames that children adopt for equations reflect the rationales we observed among experts. One area of future investigation is to explore older students' ordering norms and the ways that they do or do not overlap with those documented in this study.

CRediT authorship contribution statement

Alison Mirin selected the interview candidates, served as primary designer of the study, conducted the interviews, and contributed to the analysis and writing of results. Paul Dawkins helped design the study and contributed to the analysis and writing of results.

Declarations of interest

None.

Appendix A. Open-ended tasks on experts' use of the equals sign

OSumRule

State the sum rule for derivatives, and write it down.

OEuler

Write down and state Euler's formula.

OMice[1]

Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. Using t to denote time, $p(t)$ to denote the population, and r to represent the growth rate, write a differential equation expressing this relationship.

OIdentity[2]

Suppose $\langle S, \star\rangle$ is a binary algebraic structure. What does it mean for an element $e \in S$ to be a left identity element?

OMVT

Finish the following statement of the Mean Value Theorem by writing an equation:

Theorem 1. *Suppose f is a continuous function on $[a, b]$ and is differentiable on (a, b). Then there exists a point c in (a, b) such that...*

[1]This task was borrowed from Boyce and DiPrima (2009).

[2]The wording for this task (e.g. "binary algebraic structure") and other related abstract algebra tasks is from Fraleigh (2003).

Appendix B. Breaching tasks used on experts' use of the equals sign

DifferenceQuotient

The **difference quotient** of a function g is defined to be

$$\frac{g(x+h)-g(x)}{(x+h)-x}$$

where h is nonzero.

Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=x^2$. The following shows the difference quotient:

$$2x+h=\frac{2xh+h^2}{h}$$

$$=\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\frac{(x+h)^2-x^2}{h}$$

$$=\frac{f(x+h)-f(x)}{h}$$

$$=\frac{f(x+h)-f(x)}{(x+h)-x}$$

Homomorphism

The following is a portion of an introductory Abstract Algebra text.

Let $\langle S,\star\rangle$ and $\langle S',\star'\rangle$ be binary algebraic structures. A **homomorphism from** $\langle S,\star\rangle$ **to** $\langle S',\star'\rangle$ is a function $\phi:S\to S'$ such that for all $x,y\in S$,

$$\phi(x)\star'\phi(y)=\phi(x\star y)$$

(continued on next page)

(*continued*)

Exponents

The following is a portion of a Precalculus text.

Recall the Properties of Exponents:

$$b^{x+y} = b^x \cdot b^y$$

$$\frac{b^x}{b^y} = b^{x-y}$$

$$b^{xy} = (b^x)^y$$

Induction

The following is a portion of a proof by induction that for all natural numbers k, $k^3 - k$ is divisible by 6. At this point in the proof, it has been assumed that $n^3 - n$ is divisible by 6, and it is being shown that $(n+1)^3 - (n+1)$ is therefore also divisible by 6.

$$(n^3 - n) + 3n(n+1) = (n^3 - n) + (3n^2 + 3n)$$

$$= (n^3 + 3n^2 + 3n + 1) - (n+1)$$

$$= (n+1)^3 - (n+1)$$

Since either n or $n+1$ is even, $3n(n+1)$ is divisible by 6. By assumption, $n^3 - n$ is divisible by 6. Hence, $(n^3 - n) + 3n(n+1)$ is divisible by 6, and therefore $(n+1)^3 - (n+1)$ is divisible by 6.

SetTheory[1]

The following is a proof in a set theory textbook that if a is a transitive set, then $\bigcup(a^+) = a$. Note that a transitive set is defined to be a set a such that all members of a are subsets of a, and a^+ is defined to be $a \cup \{a\}$.

Proof.

$$a = (\bigcup a) \cup a$$

$$= (\bigcup a) \cup (\bigcup \{a\})$$

$$= \bigcup (a \cup \{a\})$$

$$= \bigcup (a^+)$$

(*continued on next page*)

(*continued*)

Productrule

The *product rule* for derivatives says that if f and g are differentiable functions, then

$$fg' + f'g = (fg)'$$

Proof

Theorem 1. *Suppose $\langle S, \star \rangle$ and $\langle S', \star' \rangle$ are binary algebraic structures, and ϕ is an isomorphism from $\langle S, \star \rangle$ onto $\langle S', \star' \rangle$. Further suppose that e is a left identity element in $\langle S, \star \rangle$. Then $\phi(e)$ is a left identity element in $\langle S', \star' \rangle$.*

Proof. Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(s) = \phi(e) \star' \phi(s) = s' = \phi(e) \star' s' = \phi(e \star s)$$

□

Distributive

The following is a portion of a precalculus text.

The distributive law tells us that for all numbers x, y, and z,

$$xy + xz = x(y + z)$$

[1]This proof is a reversed version of the proof given in Enderton (1977).

Appendix C. Comparison tasks on experts' use of the equals sign

CIdentity

An element $e \in S$ is a *left identity element* for $\langle S, \star \rangle$ if and only if for all $x \in S$

$$x = e \star x$$

CProofs

Theorem 1. *Suppose $\langle S, \star \rangle$ and $\langle S', \star' \rangle$ are binary algebraic structures, and ϕ is an isomorphism from $\langle S, \star \rangle$ onto $\langle S', \star' \rangle$. Further suppose that e is a left identity element in $\langle S, \star \rangle$. Then $\phi(e)$ is a left identity element in $\langle S', \star' \rangle$.*

Proof. **First Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) = \phi(e \star s) = \phi(s) = s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star \rangle$ □

Proof. **Second Proof** Suppose $s' \in S'$. Since ϕ is onto, there exists $s \in S$ such that

$$s' = \phi(s) \quad (1)$$

By (1) we also know that

$$\phi(e) \star' s' = \phi(e) \star' \phi(s) \quad (2)$$

Since e is a left identity element,

$$\phi(e \star s) = \phi(s) \quad (3)$$

and since ϕ is a homomorphism,

$$\phi(e) \star' \phi(s) = \phi(e \star s). \quad (4)$$

Therefore, by equations (1)-(4),

$$s' = \phi(e) \star' s'. \quad (5)$$

□

Proof. **Third Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$\phi(s) = \phi(e) \star' \phi(s) = s' = \phi(e) \star' s' = \phi(e \star s)$$

□

Proof. **Fourth Proof** Let s' be an element of S'. Since ϕ is onto, there exists some $s \in S$ such that $\phi(s) = s'$. Hence

$$s' = \phi(s) = \phi(e \star s) = \phi(e) \star' \phi(s) = \phi(e) \star' s'$$

by the properties of homomorphism and the fact that e is a left identity element in $\langle S, \star \rangle$ □

(*continued on next page*)

(*continued*)

CEuler

Euler's formula states that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

CSumRule

The *sum rule* for derivatives says that if f and g are differentiable functions, then

$$f' + g' = (f+g)'$$

CMVT

There are various ways that textbooks state the mean value theorem.

Theorem 1. *Suppose f is a continuous function on $[a,b]$ and is differentiable on (a,b). Then there exists a point c in (a,b) such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem 2. *Suppose f is a continuous function on $[a,b]$ and is differentiable on (a,b). Then there exists a point c in (a,b) such that*

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Theorem 3. *Suppose f is a continuous function on $[a,b]$ and is differentiable on (a,b). Then there exists a point c in (a,b) such that*

$$f'(c)(b-a) = f(b)-f(a)$$

Theorem 4. *Suppose f is a continuous function on $[a,b]$ and is differentiable on (a,b). Then there exists a point c in (a,b) such that*

$$f(b)-f(a) = f'(c)(b-a)$$

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