CENTRAL VALUES OF DEGREE SIX L-FUNCTIONS: THE CASE OF HILBERT MODULAR FORMS

by

Utkarsh Agrawal

A Dissertation Submitted to the Faculty of the
DEPARTMENT OF MATHEMATICS

In Partial Fulfilment of the Requirements
For the Degree of
DOCTOR OF PHILOSOPHY

In the Graduate College
THE UNIVERSITY OF ARIZONA

2022

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Dissertation Committee, we certify that we have read the dissertation prepared by: Utkarsh Agrawal

titled: Central values of degree six L-functions: The case of Hilbert Modular forms

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Hang Xue Date: May 5, 2022

Hang Xue

kirti joshi Date: May 5, 2022

kirti joshi

Brandon Levin Date: May 5, 2022

Brandon Levin

C. Douglas Haessig Date: May 5, 2022

C. Douglas Haessig

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Hang Xue Date: May 5, 2022

Hang Xue
Mathematics

To the beautiful and mystic Spirit of Mathematics, for choosing me as an instrument of its self-discovery.

ACKNOWLEDGMENTS

I am indebted to my advisor Hang Xue for accepting me as his student, his generosity of time and patience with me, his understanding and unassuming nature.

I would also like to thank my thesis committee members Doug Haessig, Kirti Joshi and Brandon Levin for sharing their mathematical knowledge and experiences, and for offering their invaluable guidance and support. I also take this opportunity to thank Maneesh Thakur, a friend and a mentor with whom I first started learning about automorphic representations, for his unflinching support all these years.

It is also my pleasure to thank my cohort from the number theory student seminar group, with special thanks to its organisers during my time here at the University of Arizona, Anthony Kling, Tristan Phillips and David Taylor for their diligent energy in keeping the momentum going. It is they who made my experience at grad school especially delightful by sharing their enthusiasm for number theory. Cheers also to Thomas Harris and Madhav Kaushish for some memorable times together.

Finally, I would like to thank my family and close friends for their unconditional support and for keeping my life light-spirited.

TABLE OF CONTENTS

ABSTRACT

We will prove an explicit formula (Eq. 1.5 or Thm. 4.4.4) for the central value of the L-function $L(s, \mathrm{Sym}^2 g \times f)$ when f and g are Hilbert newforms of level $\Gamma_0(1)$ by computing the local integrals appearing in the refined Gan-Gross-Prasad formula for $\mathrm{SL}_2 \times \widetilde{\mathrm{SL}_2}$ for some suitable choice of vectors. We will also work out the rationality of this value in the two extreme cases, the *purely balanced* and the *purely unbalanced* (Thm. 1.3). Our results in these cases are compatible with Deligne's conjecture on rationality of critical values of motivic L-functions. We will also give an explicit conjecture on the rationality of *all* critical values (cf. 1.2) of $L(s, \mathrm{Sym}^2 g \times f)$ in the general case without any restrictions on the weights or the levels (Conj. 1.2).

Part I

INTRODUCTION AND BACKGROUND MATERIAL

1

INTRODUCTION

1.1 SPECIAL VALUES OF L-FUNCTIONS AND RATIONALITY

In the mid-1700s, Euler proved that for each positive integer k,

$$\frac{\zeta(2k)}{\pi^{2k}} = (-1)^{k+1}\frac{B_{2k}}{2(2k)!},$$

where $\zeta(s)$ is the Riemann zeta function and $B_{2k} \in \mathbb{Q}$ is the $2k$th Bernoulli number. Using the functional equation of $\zeta(s)$, one can also find its values at the negative odd integers:

$$\zeta(1-2k) = \frac{-B_{2k}}{2k}.$$

Around the mid-1800s, Kummer showed that the rational numbers $B_{2k}/2k$ appearing in the above special values of $\zeta(s)$ encode interesting information about the arithmetic of the cyclotomic fields. Indeed, he showed that p does not divide the class number of the cyclotomic field $\mathbb{Q}(e^{2\pi i p})$ if and only if p does not divide the numerators of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$, in which case he could prove special cases of Fermat's Last Theorem.

The Riemann zeta function is the L-function associated to the trivial character. One may likewise associate L-functions to non-trivial algebraic characters and study their values at certain integer points. It is a fact that these values are again algebraic numbers up to well-defined transcendental factors. Moving higher to degree two

L-functions, we have the L-functions associated to elliptic curves. It is a famous conjecture that for an elliptic curve E defined over $\mathbb{Q}$, the value of its L-function at its center is, up to a *period* associated to E, an algebraic number which is a 'bouquet' of certain fundamental invariants associated to E.

All these examples and many more form the basis of the following philosophy pervading the theory of special values of L-functions: values of an L-function at certain *special* points can be expressed as a product of a 'nicely defined' transcendental factor and an algebraic factor which encodes interesting arithmetic information. This philosophy has taken a concrete shape in the form of some major conjectures such as those of Deligne [Del79] and Iwasawa [Iwa69a], [Iwa69b]. Deligne's conjecture captures the transcendental part of the values of L-functions attached to certain algebro-geometric objects, *motives*, at *critical* points in terms of a cohomologically defined complex number, *period*, and shows that the corresponding algebraic part behaves well under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; in short, it is a conjecture on rationality of critical values of motivic L-functions. On the other hand, Iwasawa's conjectures (and their extension to the set up of Deligne's conjecture by Greenberg) concern the arithmetic aspect of the theory and refine our understanding of the algebraic part of special values of L-functions. The work of this thesis can be viewed in the framework of Deligne's conjecture.

Let us now give a taste of results on rationality of L-values from the theory of modular forms. Let F be a totally real field of degree n. Let f resp. g be a Hilbert modular newform (function defined on a finite number of copies – not necessarily just one – of the upper half plane and satisfing properties similar to that of an elliptic modular newform) of weight $\kappa = (\kappa_1, \ldots, \kappa_n)$ resp $\kappa' = (\kappa'_1, \ldots, \kappa'_n)$. Assume that $\kappa_i - \kappa'_i$ is independent of i. Let S and S' be the two disjoint sets whose union is the set of archimedean places of F such that $\kappa_v > \kappa'_v$ for $v \in S$ and $\kappa'_v > \kappa_v$ for $v \in S'$. Let B be a quaternion algebra over F of signature (S, S'), that is, B is split at places in S and ramified at places in S'. Let B' be a quaternion algebra of signature (S', S). Assume that f resp. g correspond to nonzero modular forms f_B resp. $g_{B'}$ on B resp. B' via the Jacquet-Langlands correspondence (cf. §3.5.1). Then Shimura ([Shi83], Thm. 5.3) showed that

$$L_{\text{fin}}(m, f \times g) \sim_{\overline{\mathbb{Q}}} \pi^{\sum_{v \in S} \kappa_v + \sum_{v \in S'} \kappa'_v} \langle f_B, f_B \rangle \langle g_{B'}, g_{B'} \rangle,$$

where $L_{\text{fin}}(s, f \times g)$ is the (finite part) of the standard Rankin-Selberg convolution of f and g and m is any critical point $L_{\text{fin}}(s, f \times g)$.

Remark. *Shimura is a pioneer in the field of rationality of special values of L-functions associated to modular forms and has made many contributions to the field. For a good survey of many of Shimura's results, we refer the reader to his own paper [Shi88].*

Let us look at an example of a degree eight L-function that lies in the vicinity of the L-function we are interested in. Let f, g, h be elliptic modular newforms of weights k_1, k_2, k_3, respectively. Let $L_{\text{fin}}(s, f \times g \times h)$ denote the (finite part) of the standard triple product L-function associated to $f,\ g,\ h$; its center is $\frac{1}{2}(w+1)$, where $w = k_1 + k_2 + k_3 - 3$. Then Harris-Kudla ([HK91], Theorems 11.6 and 12.4) have showed that

$$\frac{L_{\text{fin}}(\frac{1}{2}(w+1), f \times g \times h)}{p(f, g, h)} \in \mathbb{Q}(f, g, h),$$

where $\mathbb{Q}(f, g, h)$ is the number field generated by the Hecke eigenvalues of $f,\ g$ and h, and $p(f, g, h)$ is the period factor

$$p(f, g, h) = \begin{cases} \langle f, f \rangle \langle g, g \rangle \langle h, h \rangle & \text{if } k_1 < k_2 + k_3 \\ \langle f, f \rangle^2 & \text{if } k_1 \geqslant k_2 + k_3. \end{cases}$$

The above result can be viewed as a special case of Deligne's conjecture since the center $s = \frac{1}{2}(w+1)$ is a critical point of $L_{\text{fin}}(s, f \times g \times g)$ in the sense of Deligne and the period $p(f, g, h)$ can be shown to be essentially equal to Deligne's period.

When $g = h$, the triple product L-function has the following factorization (assume trivial nebentypus for both f and g):

$$L_{\text{fin}}(s, f \times g \times g) = L_{\text{fin}}(s, \text{Sym}^2 g \times f) L_{\text{fin}}(s, f). \tag{1.1}$$

In this thesis we prove the rationality in some special cases of the central value of the degree six factor $L_{\text{fin}}(s, \text{Sym}^2 g \times f)$ appearing in the above factorization when f and g are newforms and the base field is any totally real number field (not just $\mathbb{Q}$). Note that this does not follow in any trivial manner from (1.1) and rationality of triple product L-function since it may very well happen that $L_{\text{fin}}(s, f \times g \times g)$ vanishes at its center even though $L_{\text{fin}}(s, \text{Sym}^2 g \times f)$ does not. Thus rationality of special values of $L_{\text{fin}}(s, \text{Sym}^2 g \times f)$ is a problem of independent interest.

All previous work on this problem has been in the case when $F = \mathbb{Q}$ and the special value of interest is the value of $L_{\text{fin}}(s, \text{Sym}^2 g \times f)$ at its center. A common idea underlying all the proofs is first deriving an explicit formula for the L-value and then working out the rationality of the various factors involved in the formula. Let f be of weight 2κ and level N, and g be of weight $\kappa' + 1$ and level N'. Ichino [Ich05] derived rationality of $L_{\text{fin}}(\text{center}, \text{Sym}^2 g \times f)$ via the explicit pullback formula for *Saito-Kurokawa lifts* assuming $\kappa = \kappa'$ and $N = N' = 1$. Xue [Xue19] dropped the assumption on weights and deduced rationality by explicitly calculating the periods in the *refined Gan-Gross-Prasad formula* for $\text{SL}_2 \times \widetilde{\text{SL}_2}$. Chen-Cheng [CC19] further relaxed the condition on levels, assuming them only to be square-free, and arrived at their result on rationality by deriving an explicit Ichino's central value formula for triple product L-functions.

We will work out the rationality of $L_{\text{fin}}(\text{center}, \text{Sym}^2 g \times f)$ when the base field is any totally real number field and f and g are (Hilbert) modular forms of trivial levels in some special cases of mutual relationship between the weights of f and g. We will do so by deriving an explicit formula (1.5) for the central value via the refined

Gan-Gross-Prasad formula for $\text{SL}_2 \times \widetilde{\text{SL}_2}$. Moreover, we will give an explicit conjecture (1.2) on the rationality of $L_{\text{fin}}(\text{center}, \text{Sym}^2 g \times f)$ at all its critical points without any assumption on weights, levels or nebentypus.

1.2 EXPECTED RATIONALITY OF L_{fin}(CRITICAL POINT, $\text{Sym}^2 g \times f$) VIA DELIGNE'S CONJECTURE

In [Del79], Deligne associates L-functions to certain objects in algebraic geometry known as *motives* and gives a conjecture on the rationality of values of such L-functions at *critical points*. In this section we will provide a conjecture on the rationality of critical values of $L_{\text{fin}}(s, \text{Sym}^2 g \times f)$ that is suggested by Deligne's conjecture.

Let M be a motive. Let $L(s, M)$ stand for the completed L-function associated to M and $L_{\text{fin}}(s, M)$ stand for its finite part; it satisfies a conjectural functional equation. Deligne defined an integer m to be a critical point of a motivic L-function if neither m nor the point symmetric to it with respect to the central point of the L-function are poles of the archimedean factor (essentially a product of Γ-functions) of the L-function (for more details, see Appendix B.1). Roughly speaking, Deligne's conjecture states that if an integer m is a critical point for a motive M, then the value $L(m, M)$ is a rational multiple of a *period* associated to M (an algebraic invariant of M coming from cohomology). More precisely, if m is a critical point of M then there exists a complex number $c^+(R_{F/\mathbb{Q}}(M(m)))$ such that for all $\sigma \in \text{Aut}(\mathbb{C})$,

$$\left(\frac{L_{\text{fin}}(m, M)}{c^+(R_{F/\mathbb{Q}}(M(m)))}\right)^\sigma = \frac{L_{\text{fin}}(m, M^\sigma)}{c^+(R_{F/\mathbb{Q}}(M^\sigma(m)))}.$$

Remark. *The complex number $c^+(M(m))$ is known as Deligne's period and is defined by a comparison of rational structures on the de Rham and Betti realizations of the motive M(cf. [Del79], §2).*

Let $F/\mathbb{Q}$ be a totally real field of degree n. Let f resp. g be a (holomorphic) Hilbert newform of weight $2\kappa = (2\kappa_1, \dots, 2\kappa_n)$ resp. $\kappa' + 1 = (\kappa'_1 + 1, \dots, \kappa'_n + 1)$; cf. §3.1.3 to recall the definition of a Hilbert modular form. Let Σ_{ub} and Σ_b be sets whose union is the set of all archimedean places, Σ_∞, of F, such that $\kappa_v > \kappa'_v$ for $v \in \Sigma_{ub}$ and $\kappa_v \leqslant \kappa'_v$ for $v \in \Sigma_b$. Motives for Hilbert modular forms are known to exist ([BR93]). Let $M = M(\mathrm{Sym}^2\, g \times f)$ denote the motive associated to $\mathrm{Sym}^2\, g \times f$ and let $L(s, \mathrm{Sym}^2\, g \times f)$ denote the associated motivic L-function.

Put

$$r_v = \begin{cases} \kappa_v - \kappa'_v - 1 & \text{if } v \in \Sigma_{ub} \\ \kappa'_v - \kappa_v & \text{if } v \in \Sigma_b. \end{cases}$$

and

$$t^0 = \min_{v \in \Sigma_{ub}} \{r_v\}, \quad a^0 = \min_{v \in \Sigma_b, 2\kappa_v > \kappa'_v} \{r_v\}, \quad b^0 = \min_{v \in \Sigma_b, \kappa' \geqslant 2\kappa} \{\kappa_v - 1\}.$$

Also, let

$$\kappa_0 = \max_{v \in \Sigma_\infty} \kappa_i \quad \text{and} \quad \kappa'_0 = \max_{v \in \Sigma_\infty} \kappa'_i.$$

Then

Proposition. *The set of all critical points of $L(s, \mathrm{Sym}^2\, g \times f)$ is*

$$\left\{ m \in \mathbb{Z} \mid \kappa_0 + \kappa'_0 - \min\{a^0, b^0, t^0\} \leqslant m \leqslant \min\{a^0, b^0, t^0\} + \kappa_0 + \kappa'_0 \right\}. \tag{1.2}$$

In particular, the center, $s = \kappa_0 + \kappa'_0$ is a critical point.

For a proof of the above proposition, see Prop. 5.3.2 and Lemma 5.3.1. We can also express Deligne's period $c^+(R_{F/\mathbb{Q}}(M(m)))$ associated to the motive M more explicitly in terms of the periods associated to the motives of g and f. We relegate the computations to Appendix B.2 and state here only the conjecture on the rationality of critical values of $L(s, \mathrm{Sym}^2\, g \times f)$ suggested by our computations. Let h be the half-integral weight Hilbert modular form associated to f via the Shimura correspondence. Let B be a quaternion algebra over F of signature (Σ_{ub}, Σ_b), that is, B is split at places in Σ_{ub} and ramified at places in Σ_b. Let B' be a quaternion algebra of signature (Σ_b, Σ_{ub}). Assume for simplicity that f resp. g correspond to nonzero modular forms f_B resp. $g_{B'}$ on B resp. B' via the Jacquet-Langlands correspondence (cf. 3.5.1).

Conjecture. *Let $\varepsilon_v = m \pmod 2$ for all $v \in \Sigma_\infty$ and put $\varepsilon = (\varepsilon_v)_{v \in \Sigma_\infty}$. If $m \in \mathbb{Z}$ is a critical point of $L(s, \text{Sym}^2 g \times f)$, then for all $\sigma \in \text{Aut}(\mathbb{C})$,*

$$\left(\frac{L_{\text{fin}}(m, \text{Sym}^2 g \times f)}{(2\pi i)^* i^{|\Sigma_{ub}|} u(\varepsilon, f) \nu^{\Sigma_{ub}}(f) \nu^{\Sigma_b}(g)^2} \right)^\sigma = \frac{L_{\text{fin}}(m, \text{Sym}^2 g^\sigma \times f^\sigma)}{(2\pi i)^* i^{|\Sigma_{ub}|} u(\varepsilon, f^\sigma) \nu^{\Sigma_{ub}}(f^\sigma) \nu^{\Sigma_b}(g^\sigma)^2},$$

where

$$* = |3m - 2\kappa_0 - 3\kappa_0' + 1|_{\Sigma_\infty} + |2\kappa|_{\Sigma_{ub}} + |2\kappa' + 2|_{\Sigma_b}, \quad (\textit{see §2.2 for notation})$$

$u(\cdot\,,\,\cdot)$ is Shimura's u-invariant (see Appendix C.1) and $\nu^{\cdot}(\,\cdot\,)$ is a period as defined in ([Har90], §1). In particular, the central critical value

$$L_{\text{fin}}(\kappa_0 + \kappa_0', \text{Sym}^2 g \times f) \sim_{\overline{\mathbb{Q}}} \pi^{*+(\kappa_0 - 1)n - \sum_{i=1}^n \kappa_i} \langle h, h \rangle \langle f_B, f_B \rangle \langle g_{B'}, g_{B'} \rangle^2, \tag{1.3}$$

where $*$ is as above with $m = \kappa_0 + \kappa_0'$.

Remark. *(1.3) follows from the properties of the u-invariant and the ν-invariant (see Appendix C).*

Remark. *Note that in the above conjecture f and g are any Hilbert newforms of levels of the 'Γ_0-type'. A slightly modified but similar assertion can also be made when either f_B or $g_{B'}$ does not exist.*

We will work out the rationality of the central critical value, $L_{\text{fin}}(\kappa_0 + \kappa_0', \text{Sym}^2 g \times f)$, in the two extreme cases, $\Sigma_b = \emptyset$, and $\Sigma_{ub} = \emptyset$, under the assumption that f and g

are of trivial level; our results are compatible with Deligne's conjecture. In order to do so, we will first establish an explicit formula for the central value.

1.3 THE MAIN FORMULA

Consider the following setup. Let f resp. g be a holomorphic Hilbert newform of weight $2\kappa = (2\kappa_v)_v$ resp. $\kappa' + 1 = (\kappa'_v + 1)_v \in \mathbb{Z}^{\Sigma_\infty}$ (identified with $\mathbb{Z}^n$). We will assume that f and g are of trivial levels and that $\kappa_v, \kappa'_v > 1$ for all v. Let Σ_{ub} and Σ_b be subsets of Σ_∞ whose union is Σ_∞, such that $\kappa_v > \kappa'_v$ for $v \in \Sigma_{ub}$ and $\kappa_v \leqslant \kappa'_v$ for $v \in \Sigma_b$. Put

$$r_v = \begin{cases} \kappa_v - \kappa'_v - 1 & \text{if } v \in \Sigma_{ub} \\ \kappa'_v - \kappa_v & \text{if } v \in \Sigma_b. \end{cases}$$

Let h be the Hecke eigenform of half-integral weight $\kappa + \frac{1}{2}$ associated to f by the Shimura correspondence. To h one can associate a holomorphic Jacobi form, F_h, which is a function on $\mathfrak{H}^n \times \mathbb{C}^n$. Furthermore, put

$$g_{ub}((\tau_v)_v) = g((\tau_v)_{v\in\Sigma_b}, (-\overline{\tau_v})_{v\in\Sigma_{ub}}), \quad h_{ub}((\tau_v)_v) = h((\tau_v)_{v\in\Sigma_b}, (-\overline{\tau_v})_{v\in\Sigma_{ub}});$$

that is, g_{ub} (resp. h_{ub}) is the modular form on $\mathfrak{h}^n$ corresponding to g (resp. h) which is holomorphic at places in Σ_b but anti-holomorphic at places in Σ_{ub}. Note that $F_{h_{ub}}$ is then a function on $\mathfrak{h}^n \times \mathbb{C}^n$ corresponding to F_h which is holomorphic at places in Σ_b but *skew-holomorphic* at places in Σ_{ub} (cf. [BS98], p.80).

Write the coordinates on $\mathfrak{h}^n \times \mathbb{C}^n$ as $((\tau_v)_v, (z_v)_v)$ and define the differential operator

$$\Delta_v = \frac{2i}{\pi}\Big(\frac{\partial}{\partial z_v} + 4\pi i \frac{z_v - \overline{z}_v}{\tau_v - \overline{\tau}_v}\Big) \tag{1.4}$$

on the space of smooth functions on $\mathfrak{H}^n \times \mathbb{C}^n$ for every archimedean place.

Put

$$\Delta^{(r)} = \prod_{v \in \Sigma_\infty} \Delta_v^{r_v}.$$

Let $L(s, \mathrm{Sym}^2 g \times f)$ denote the completed L-function whose central value we are interested in. It is an analytic function on the whole complex plane and satisfies a functional equation of the form

$$L(s, \mathrm{Sym}^2 g \times f) = \varepsilon(s, \mathrm{Sym}^2 g \times f) L(2\kappa_0 + 2\kappa_0' - s, \mathrm{Sym}^2 g \times f),$$

where $\kappa_0 = \max_v\{\kappa_v\}$ and $\kappa_0' = \max_v\{\kappa_v'\}$. Note that $s = \kappa_0 + \kappa_0'$ is the center of $L(s, \mathrm{Sym}^2 g \times f)$. The main formula (cf. Thm. 4.4.4) proved in this thesis is as follows.

Theorem. *We have,*

$$L(\kappa_0 + \kappa_0', \mathrm{Sym}^2 g \times f) = \frac{2^a}{\prod_{v \in \Sigma_\infty} \binom{2r_v}{r_v}} \xi_F(2)^2 D_F^{5/2} |\langle g_{ub}, \Delta^{(r)} F_{h_{ub}}|_{\mathfrak{h}^n} \rangle|^2 \frac{\langle f, f \rangle}{\langle h_{ub}, h_{ub} \rangle}, \quad (1.5)$$

where $a = |2\kappa + \kappa' + 6|_{\Sigma_\infty} + |2\kappa' - 2\kappa + 1|_{\Sigma_{ub}} + 2$. Here $\langle -, - \rangle$ is the normalized Petersson inner product, ξ_F is the completed Dedekind zeta function of F and D_F is the discriminant of F.

Remark. *Xue ([Xue19], Prop. 2.1) proved the above formula in the special case of $F = \mathbb{Q}$ and κ, κ' both odd; a factor of $\xi_F(2)^2$ appears in our formulation due to the fact that our inner product is normalized.*

We will derive (1.5) by explicitly computing the local period integrals appearing in the refined Gan-Gross-Prasad formula for $\mathrm{SL}_2 \times \widetilde{\mathrm{SL}_2}$. As an application of our formula, we will work out the rationality of $L(\kappa_0 + \kappa_0', \mathrm{Sym}^2 g \times f)$ when both f and g are Hilbert newforms of level $\Gamma_0(1)$ and either $\Sigma_b = \emptyset$ or $\Sigma_{ub} = \emptyset$. To do so, we will work out the rationality of the global period $\langle g, \Delta^{(r)} F_h|_{\mathfrak{h}^n} \rangle$ by using the theory of integral weight (Hilbert) modular forms (when $\Sigma_b = \emptyset$) and the theory of half-integral weight modular forms (when $\Sigma_{ub} = \emptyset$), and the rationality of the factor $\frac{\langle f,f\rangle}{\langle h,h\rangle}$ by a generalization of the Kohnen-Zagier formula for Hilbert modular forms and the fundamental result on the rationality of the central value of the standard L-function of a modular form (twisted by some character). Our results on rationality are as follows (cf. Theorems 5.1.2 and 5.2.2).

Theorem. *Let $\widetilde{F}$ denote the Galois closure of F. Put $\varepsilon = (\varepsilon_v) \in (\mathbb{Z}/2\mathbb{Z})^n$ with $\varepsilon_v = \kappa_0 + 1 \pmod 2$. Let $u(\cdot,\cdot)$ be Shimura's u-invariant.*

(i) Suppose $\Sigma_{ub} = \emptyset$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\widetilde{F})$,

$$\sigma\left(\frac{L(\frac{1}{2}, \mathrm{Sym}^2 g \times f)}{\pi^{(\kappa_0+2)n-\sum_{i=1}^n \kappa_i}\langle g,g\rangle^2 u(\varepsilon,f)}\right) = \frac{L(\frac{1}{2}, \mathrm{Sym}^2 g^\sigma \times f^\sigma)}{\pi^{(\kappa_0+2)n-\sum_{i=1}^n \kappa_i}\langle g^\sigma,g^\sigma\rangle^2 u(\varepsilon,f^\sigma)},$$

Moreover, we can replace $\mathrm{Aut}(\mathbb{C}/\widetilde{F})$ by $\mathrm{Aut}(\mathbb{C})$ if f is of parallel weight.

(ii) Suppose $\Sigma_b = \emptyset$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\widetilde{F})$,

$$\sigma\left(\frac{L(\frac{1}{2}, \mathrm{Sym}^2 g \times f)}{\pi^{(\kappa_0+1)n-\sum_{i=1}^n \kappa_i}\langle f,f\rangle u(\varepsilon,f)}\right) = \frac{L(\frac{1}{2}, \mathrm{Sym}^2 g^\sigma \times f^\sigma)}{\pi^{(\kappa_0+2)n-\sum_{i=1}^n \kappa_i}\langle f^\sigma,f^\sigma\rangle u(\varepsilon,f^\sigma)}.$$

Moreover, we can replace $\mathrm{Aut}(\mathbb{C}/\widetilde{F})$ by $\mathrm{Aut}(\mathbb{C})$ if f is of parallel weight.

2

NOTATIONS AND PRELIMINARIES

The reader should refer to this chapter for notations.

2.1 THE BASE FIELD F

F will denote a totally real number field of degree n over $\mathbb{Q}$. Let $\mathcal{O} = \mathcal{O}_F$ and $D = D_F$ denote, respectively, the ring of integers of F and the discriminant of F. Note that D_F is a positive integer since F is totally real. We will denote by Σ_∞ the set of archimedean places of F and by Σ_F the set of all places of F. Let $\mathbb{A} = \mathbb{A}_F$ denote the ring of adeles of F.

For each place v of F, let F_v denote the completion of F at v. Put $F_\infty = \prod_{v \in \Sigma_\infty} F_v$. For each non-archimedean place v of F, let $\mathfrak{o}_v$ be the ring of integers of F_v, ϖ_v a uniformizer of $\mathfrak{o}_v$ and $q_v = \#\mathfrak{o}_v/(\varpi_v)$ the order of the residue field of F_v.

2.2 SUM OVER A SUBSET S OF Σ_∞, $|\cdot|_S$

Let $S \subset \Sigma_\infty$. Define

$$\mathbb{Z}^S = \{\text{the sequences } (a_v)_v \mid v \in S \text{ and } a_v \in \mathbb{Z}\}.$$

For $\kappa = (\kappa_v)_v \in \mathbb{Z}^S$, put

$$|\kappa|_S = \sum_{v \in S} \kappa_v.$$

Moreover, if $a \in \mathbb{Z}$, then

$$|a|_S := \sum_{v \in S} a = a|S|,$$

where $|S|$ is the cardinality of the set S. If $a \in \mathbb{Z}$ and $\kappa = (\kappa_v)_v \in \mathbb{Z}^S$, then

$$|a + \kappa|_S := |a|_S + |\kappa|_S.$$

When $S = \Sigma_\infty$, we will often denote $|a+\kappa|_{\Sigma_\infty}$ simply $|a+\kappa|$.

2.3 THE COMPLETED DEDEKIND ZETA FUNCTION ξ_F

Let ξ_F be the completed Dedekind zeta function of F given by

$$\xi_F(s) = \Gamma_{\mathbb{R}}(s)^n \zeta_F(s),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$and ζ_F is the Dedekind zeta function of F. For Re(s)> 1, we have $\xi_F(s) = \prod_v \xi_{F_v}(s)$, where

$$\xi_{F_v}(s) = \Gamma_{\mathbb{R}}(s) \quad \text{if } v \text{ is archimedean,}$$

and

$$\zeta_{F_v}(s) = \frac{1}{1 - q_v^{-s}} \quad \text{if } v \text{ is non-archimedean.}$$

2.4 THE SETS Σ_{ub} AND Σ_b

Let $2\kappa = (2\kappa_v)_v$ and $\kappa' + 1 = (\kappa'_v + 1)_v \in \mathbb{Z}^{\Sigma_\infty}$ be weights of some Hilbert modular forms. For any archimedean place v, either (i) $\kappa_v > \kappa'_v$ or (ii) $\kappa_v \leqslant \kappa'_v$. The first case is called the *unbalanced case*, the second case is called the *balanced case*. Let Σ_{ub} and Σ_b be subsets of Σ_∞ whose union is Σ_∞, such that $\kappa_v > \kappa'_v$ for $v \in \Sigma_{ub}$ and $\kappa_v \leqslant \kappa'_v$ for $v \in \Sigma_b$. Put

$$r_v = \begin{cases} \kappa_v - \kappa'_v - 1 & \text{if } v \in \Sigma_{ub} \\ \kappa'_v - \kappa_v & \text{if } v \in \Sigma_b. \end{cases}$$

2.5 THE QUADRATIC CHARACTER χ_α

Let $\alpha \in F^\times$. We denote by χ_α denote the quadratic character associated to the extension K/F, where $K = F(\sqrt{\alpha})$, via class field theory. Explicitly,

$$\chi_\alpha : F^\times \backslash \mathbb{A}_F^\times / \mathrm{N}_{K/F}\, \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$
$$x \longmapsto \prod_{v \in \Sigma_F} \langle \alpha, x \rangle_v,$$

where $\langle -, - \rangle_v$ denotes the Hilbert symbol.

2.6 MEASURES

We fix the following measures. On $\mathbb{R}$ we take the usual Lebesgue measure. For any finite place v of F, let dx_v be the Haar measure on F_v so that $\mathrm{Vol}(\mathfrak{o}_v) = 1$. Then the measure on $\mathbb{A}$ will be $\prod_v dx_v$. On $\mathrm{SL}_2(\mathbb{R})$, let $dg = y^{-2}dxdyk$, where $g = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} k$ is the Iwasawa decomposition and dk is the measure on $\mathrm{SO}_2(\mathbb{R})$ so that $\mathrm{Vol}(\mathrm{SO}_2(\mathbb{R})) = 1$. For any finite place v of F, let dg_v be the Haar measure on $\mathrm{SL}_2(F_v)$ so that $\mathrm{Vol}(\mathrm{SL}_2(\mathfrak{o}_v)) = 1$. Then $D_F^{-3/2}\zeta_F(2)^{-1}\prod_v dg_v$ is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$; that is, the volume of $\mathrm{SL}_2(\mathbb{Q})\backslash \mathrm{SL}_2(\mathbb{A})$ with respect to this measure is 1. This also gives a measure on $\mathrm{J}(\mathbb{A})$ (cf. [BS98], Prop. 1.2.4). The inner products will always be defined using these measures, unless otherwise mentioned.

2.7 THE METAPLECTIC GROUP $\widetilde{\mathrm{SL}_2}$

Let k be the field of real numbers or a $\mathfrak{p}$-adic field. Consider the Heisenberg group $\mathrm{H}(k)$, which is k^3 as a set, and with multiplication defined by

$$(X, \kappa) \cdot (X', \kappa') = \left(X + X', \kappa + \kappa' + \det \begin{pmatrix} X \\ X' \end{pmatrix} \right),$$

The center of $\mathrm{H}(k)$ is $\{(0, 0, \kappa)\}$.

Fix a non-trivial additive character ψ of k. Then by a theorem of Stone and von Neumann, the Heisenberg group has a unique (up to isomorphism) irreducible unitary representation with central character ψ, the *Schrödinger representation*. Let us denote this representation by ω_ψ^S. Now, $\mathrm{SL}_2(k)$ acts on $\mathrm{H}(k)$ by

$$g.(X,\kappa) \mapsto (gXg^{-1},\kappa).$$

Since this action is trivial on the center of $\mathrm{H}(k)$, for a given $g \in \mathrm{SL}_2(k)$ it gives rise to another irreducible representation of $\mathrm{H}(k)$ with central character ψ:

$$(\mathrm{X},\kappa) \mapsto \omega_\psi^S(g.(X,\kappa)).$$

By the uniqueness property above, this representation must be isomorphic to ω_ψ^S, – in fact isometric since ω_ψ^S is unitary. Moreover, by Schur's lemma, the isomorphism is determined up to nonzero scalars of absolute value one. We fix one such isomorphism, $\omega_\psi^W(g)$, for every $g \in \mathrm{SL}_2(k)$. Now for $g, g' \in \mathrm{SL}_2(k)$ it follows from the definition of an intertwining operator that

$$\omega_\psi^W(g)\omega_\psi^W(g')\omega_\psi^S(h)\omega_\psi^W(g')^{-1}\omega_\psi^W(g)^{-1} = \omega_\psi^W(gg')\omega_\psi^S(h)\omega_\psi^W(gg')^{-1}$$

and again by Schur's lemma there must exist a scalar $\lambda(g,g')$ of absolute value one such that

$$\omega_\psi^W(gg') = \lambda(g,g')\omega_\psi^W(g)\omega_\psi^W(g').$$

From the associativity law in SL_2, it follows that

$$\lambda(gg',g'')\lambda(g,g') = \lambda(g,g'g'')\lambda(g',g''),$$

which just says that λ is a 2-cocyle for the trivial action of SL_2 on S^1. The freedom in multiplying the operators ω_ψ^W by scalars of absolute vale one amounts to changing λ by a coboundary. Hence the representation ω_ψ^S we started with determines in a unique way an element $\lambda \in H^2(\mathrm{SL}_2, S^1)$. It is a fact that $H^2(\mathrm{SL}_2(k), S^1)$ consists of only two elements if $k = \mathbb{R}$ or a $\mathfrak{p}$-adic field. It is further known that λ represents the nontrivial cocycle in these cases and can be so chosen that it satisfies the following properties.

- $\lambda(g, g') \in \{\pm 1\}$.
- $\lambda\Big|_{\mathrm{SL}_2(\mathfrak{o}_\mathfrak{p}) \times \mathrm{SL}_2(\mathfrak{o}_\mathfrak{p})} = 1$ if k is $\mathfrak{p}$-adic but not an extension of $\mathbb{Q}_2$.

We will take λ to be the above cocycle in all that follows when k is the real or a $\mathfrak{p}$-adic field. The metaplectic group, $\widetilde{\mathrm{SL}_2(k)}$, is the extension of $\mathrm{SL}_2(k)$ by the group $\{\pm 1\}$ determined by the cocycle λ. Explicitly, as a set

$$\widetilde{\mathrm{SL}_2(k)} = \mathrm{SL}_2(k) \times \{\pm 1\},$$

and the multiplication is given by

$$(g, \varepsilon) \cdot (g', \varepsilon') = (gg', \lambda(g, g')\varepsilon\varepsilon').$$

Remark. *It is clear from our choice of λ that $\mathrm{SL}_2(\mathfrak{o})$ is a subgroup of $\widetilde{\mathrm{SL}_2(k)}$ if k is a $\mathfrak{p}$-adic field but not an extension of $\mathbb{Q}_2$.*

The projective representation ω_ψ^W of $\mathrm{SL}_2(k)$ now becomes a representation (in the ordinary sense) of $\widetilde{\mathrm{SL}_2(k)}$:

$$(g, \varepsilon) \mapsto \omega_\psi^W(g)\varepsilon,$$

and is known as the Weil representation. We will continue to denote it by ω_ψ^W. By abuse of notation, we will usually write g for the element $(g, 1) \in \widetilde{\mathrm{SL}_2(k)}$. Set

$$m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}; \quad (a \in k^\times, b \in k).$$

Then the elements $(m(a), \varepsilon)$, $(n(b), \varepsilon)$ and (w, ε) for $a, b \in k^\times$, $\varepsilon \in \{\pm 1\}$ generate $\widetilde{\mathrm{SL}_2(k)}$.

For a global field F with ring of adeles $\mathbb{A}$, we can define the global metaplectic group $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ in a manner analogous to the local case. Explicitly, as a set

$$\widetilde{\mathrm{SL}_2(\mathbb{A})} = \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\},$$

and the multiplication is given by

$$(g, \varepsilon) \cdot (g', \varepsilon') = (gg', \lambda(g, g')\varepsilon\varepsilon'),$$

where λ is the product of local cocycles $\lambda_\mathfrak{p}$ for all non-archimedean places $\mathfrak{p}$ of F:

$$\lambda(g, g') = \prod_{\mathfrak{p}} \lambda_\mathfrak{p}(g, g').$$

Note that by our choice of the local cocycles the above product is well-defined.

2.8 THE JACOBI GROUP J

The action of SL_2 on the Heisenberg group H in the previous section defines the Jacobi group,

$$\mathrm{J} = \mathrm{SL}_2 \ltimes \mathrm{H}\,.$$

Explicitly it is given as follows. The the Jacobi group is the subgroup of GL_4 consisting of all matrices which can be written as

$$\begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \xi \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix}; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2 \,.$$

For brevity, a typical element of the Jacobi group will be written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\lambda, \mu, \xi)$, where (λ, μ, ξ) will be thought of as an element of H. Note that the Jacobi group is defined by polynomial conditions as a group of matrices and thus can be considered as an affine algebraic group. It is not reductive though, since as a subgroup of GL_4 it is not closed under transposition.

Let $j = g(X, \kappa)$ and $j' = g'(X', \kappa')$ be two elements in J with $g, g' \in \mathrm{SL}_2$ and $(X, \kappa), (X', \kappa') \in \mathrm{H}$. Then it can be shown that

$$jj' = gg'\left(Xg' + X', \kappa + \kappa' + \det \begin{pmatrix} Xg' \\ X' \end{pmatrix}\right).$$

A quick calculation shows that the center of the Jacobi group is $Z_J = \{(0, 0, \kappa)\}$. We denote by B_J the Borel subgroup of J consisting of elements of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(0, \mu, \xi)$.

For a local field k, the group $\widetilde{\mathrm{J}}(k) = \widetilde{\mathrm{SL}_2(k)} \ltimes \mathrm{H}(k)$ will denote the double cover of $\mathrm{J}(k)$. Similarly, for a global field F with ring of adeles $\mathbb{A}$, we have $\widetilde{\mathrm{J}}(\mathbb{A}) = \widetilde{\mathrm{SL}_2(\mathbb{A})} \ltimes \mathrm{H}(\mathbb{A})$. Moreover, putting together the Weil and the Schrödinger representations defines a representation of the Jacobi group which will be useful to us and will be described in more detail in the next chapter.

2.9 LIE ALGEBRA $\mathfrak{j}_{\mathbb{C}}$ OF THE JACOBI GROUP

Let $\mathfrak{j}$ be the Lie algebra of $J(\mathbb{R})$ and $\mathfrak{j}_{\mathbb{C}}$ its complexification. $\mathfrak{j}_{\mathbb{C}}$ is a six dimensional complex vector space spanned by the following elements

$$X_{\pm} = \frac{1}{2}\begin{pmatrix} 1 & 0 & \pm i & 0 \\ 0 & 0 & 0 & 0 \\ \pm i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Y_{\pm} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \pm i \\ 1 & 0 & \pm i & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Z_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Via the identification of $SL_2(\mathbb{R})$ as a subgroup of $J(\mathbb{R})$, we see that $\mathfrak{sl}_{2,\mathbb{C}}$ is a subalgebra of $\mathfrak{j}_{\mathbb{C}}$ and is spanned by $\{X_{\pm}, Z\}$. Moreover, Z spans the (complexified) Lie algebra of the maximal compact subgroup $SO_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. The Lie algebra of the Heisenberg group is spanned by $\{Y_{\pm}, Z_0\}$.

If π is a smooth representation of a Lie group and $d\pi$ the induced action of its Lie algebra, then for any Lie algebra element X, $d\pi(X)v$ will be simply written as Xv.

Remark. *At the level of representation theory,* $X_{\pm}$ *act as weight* ± 2 *raising/lowering operators, and* $Y_{\pm}$ *as weight* ± 1 *raising/lowering operators.*

2.10 MORE NOTATIONS

In this section we list down some more notations that we will be using throughout the text.

- The Schrödinger-Weil representation ω_ψ.....§3.6.1
- The Waldspurger packet $\mathrm{Wd}_\psi(\pi)$.....§3.5.4
- The adjoint L-function.....§4.1
- The L-function $L(s, \mathrm{Sym}^2 \tau \times \pi)$.....§4.2
- The u-invariant.....§C.1

3

REPRESENTATION THEORY

3.1 REPRESENTATION THEORY OF GL_2

We will briefly recall the representation theory of GL_2 over a local field and the theory of automorphic representation of GL_2. For more details, we refer the reader to [Bum97], [BCdS+03].

3.1.1 *Local Theory*

Let k be the real or a $\mathfrak{p}$-adic field. Let χ_1, χ_2 be two (multiplicative) characters of $k^\times$. Let $I(\chi_1, \chi_2)$ denote the representation of $\mathrm{GL}_2(k)$ induced from the character $\chi_1 \otimes \chi_2$ of the Borel subgroup $B(k)$ of $\mathrm{GL}_2(k)$ given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \chi_1(a)\chi_2(d).$$

$I(\chi_1, \chi_2)$ is 'generally' irreducible. When reducible, its Jordan-Hölder series has only two factors, one infinite dimensional and the other finite dimensional. To describe the representation theory of $\mathrm{GL}_2(k)$ in more detail, we now consider the cases $k = \mathbb{R}$ and k a $\mathfrak{p}$-adic field separately.

Suppose first that k is a $\mathfrak{p}$-adic field with ring of integers $\mathfrak{o}$. Then $I(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$; in this case it is called the principal series representation. When reducible, its (unique) infinite dimensional irreducible factor is a twist of the Steinberg (or special) representation, $St(|\cdot|^{1/2}, |\cdot|^{-1/2})$. The other factor is one-dimensional.

Below is the complete list of irreducible admissible representations of $\mathrm{GL}_2(k)$, up to equivalence.

- The principal series representations $I(\chi_1, \chi_2)$, where χ_1, χ_2 are characters of $k^\times$ such that $\chi_1\chi_2^{-1} \neq |\cdot|$.
- Twists of (special or) Steinberg representations, $St(|\cdot|^{1/2}, |\cdot|^{-1/2}) \otimes \chi$, where χ is a character of $k^\times$.
- The supercuspidal representations.
- The one-dimensional representations of the form $\chi \circ \det$.

An irreducible admissible representation of $\mathrm{GL}_2(k)$ is called *unramified* or *spherical* if it has a $\mathrm{GL}_2(\mathfrak{o})$-fixed vector. Such a representation is either a principal series representation of the form $I(|\cdot|^{s_1}, |\cdot|^{s_2})$ or a one-dimensional representation of the form $|\cdot|^{s} \circ \det$, where s, s_1, s_2 are complex numbers.

Suppose now that $k = \mathbb{R}$. Any character χ of $\mathbb{R}^\times$ looks like $\chi_{(s,\varepsilon)} : \mathbb{R}^\times \to \mathbb{C}^\times$ with

$$\chi_{(s,\varepsilon)}(t) = |t|^{s} \operatorname{sign}(t)^{\varepsilon},$$

where $s \in \mathbb{C}$ and $\varepsilon \in \{0, 1\}$. Now let χ_1, χ_2 be characters of $\mathbb{R}^\times$ such that $\chi_i = \chi_{(s_i,\varepsilon_i)}$. Assume that $\Re(s_1) \geqslant \Re(s_2)$. Let $I(\chi_1, \chi_2)$ be the representation of $\mathrm{GL}_2(\mathbb{R})$ induced from the character $\chi_1 \otimes \chi_2$ of the Borel subgroup as above. $I(\chi_1, \chi_2)$ is reducible if and only if $s_1 - s_2 = l - 1$ is a positive integer. In this case, its (unique) infinite dimensional irreducible factor is a twist of the discrete series representation $DS_l^{\pm}$

described below. The other factor is a finite dimensional representation, $J(\chi_1, \chi_2)$. When irreducible, $I(\chi_1, \chi_2)$ will be called a principal series representation.

For $l > 1$,

$$DS_l^{\pm} = \mathrm{Ind}_{\mathrm{GL}_2^+(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})} DS_l^+,$$

where DS_l^+ is the discrete series representation of $\mathrm{GL}_2^+(\mathbb{R})$ whose underlying $(\mathfrak{g}, \mathrm{SO}_2)$-module can be described by the following model. As a vector space,

$$DS_l^+ = \bigoplus_{i \in 2\mathbb{Z}_{\geqslant 0}} \mathbb{C} u_i, \tag{3.1}$$

and the action of $\mathfrak{g}$ is given by

$$Zu_i = (l+i)u_i, \qquad X_+ u_i = (l + \frac{i}{2}) u_{i+2}, \tag{3.2}$$

$$Z_0 u_i = l u_i, \qquad X_- u_i = \frac{-i}{2} u_{i-2}. \tag{3.3}$$

In particular, the set of K-types of $DS_l^{\pm}$ are $\{\pm k, \pm(k+2), \pm(k+4), \cdots\}$ and the central character is $a \mapsto \mathrm{sign}(a)^l$.

The following is the complete list of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{R})$ up to equivalence.

- The principal series representation $I(\chi_1, \chi_2)$, where $\chi_i = \chi_{(s_i, \varepsilon_i)}$ such that $\Re(s_1) \geqslant \Re(s_2)$ and $s_1 - s_2$ is not an integer.
- Twists of discrete series representations, $DS_l^{\pm} \otimes |\cdot|_{\mathbb{R}}^t$, where $l > 1$ is an integer and $t \in \mathbb{C}$.
- The finite dimensional representations $J(\chi_1, \chi_2)$ with χ_i as above and $s_1 - s_2$ a positive integer.

3.1.2 *Global theory*

We will restrict ourselves to automorphic representations. Let φ be an automorphic form on $\mathrm{GL}_2(\mathbb{A})$. Suppose that the span of φ under the action of $\mathrm{GL}_2(\mathbb{A})$ by right translation is an irreducible representation π of $\mathrm{GL}_2(\mathbb{A})$. Then

Theorem (Tensor product theorem)**.** *The irreducible representation π of $\mathrm{GL}_2(\mathbb{A})$ decomposes as*

$$\pi = \otimes_{v \in \Sigma_F} \pi_v$$

such that

- *π_v is an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(F_v)$ and,*
- *At all but finitely many non-archimedean places π_v is an unramified principal series representation.*

We will term π as the irreducible *automorphic* representation of $\mathrm{GL}_2(\mathbb{A})$ generated by φ.

Theorem (Strong multiplicity-one for GL_2)**.** *Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be two irreducible automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ such that $\pi_v \cong \pi'_v$ at all but finitely many places v. Then $\pi \cong \pi'$.*

3.1.3 *Modular forms as automorphic representations*

In this subsection we briefly explain how to attach an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ to a Hilbert newform. To keep the exposition simple, we will assume that F has narrow class number one. For the general theory we refer the reader to ([RT11],§4) and ([Bum97]).

We first recall the definition of a Hilbert modular form. Let $\mathfrak{h}$ denote the upper half plane. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be an element of $\mathrm{GL}_2(\mathbb{R})^n$ and write $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ for each $j = 1, \ldots, n$. Then γ acts on $\mathfrak{h}^n$ by

$$\gamma.z = \Big(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\Big),$$

with $z = (z_1, \ldots, z_n) \in \mathfrak{h}^n$. For a holomorphic function f on $\mathfrak{h}^n$, an element $\gamma \in \mathrm{GL}_2(\mathbb{R})^n$, and $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{Z}^n$, define

$$f \,||_\kappa\, \gamma(z) = \det(\gamma)^{\kappa/2}(cz+d)^{-\kappa} f(\gamma.z) \tag{3.4}$$

where $(cz+d)^{-\kappa} = \prod_{j=1}^n (c_j z_j + d_j)^{-\kappa_j}$ and $\det(\gamma)^{\kappa/2} = \prod_{j=1}^n \det(\gamma_j)^{\kappa_j/2}$. Let $\mathfrak{n}$ be an integral ideal in $\mathcal{O}$. Let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of $\mathrm{GL}_2^+(F)$, the group of 2×2 matrices with entries in F and totally positive determinants, defined as

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathcal{O}) : c \in \mathfrak{n} \right\}.$$

A Hilbert modular form f of weight $\kappa = (\kappa_1, \ldots, \kappa_n)$ and level $\Gamma_0(\mathfrak{n})$ is a function that is holomorphic on $\mathfrak{h}^n$ and at the cusps, and satisfies

$$f \,||_\kappa\, \gamma(z) = f(z) \tag{3.5}$$

for all $\gamma \in \Gamma_0(\mathfrak{n})$ (viewed inside $\mathrm{GL}_2(\mathbb{R})^n$). It has a Fourier expansion of the form

$$f(z) = \sum_{\xi} a(\xi) e^{2\pi i \xi z},$$

where $e^{2\pi i\xi z} = \exp(2\pi i \Sigma_{j=1}^{n} \xi_j z_j)$, and $\xi \in \{0\} \cup \{\text{totally positive elements in } \mathcal{O}\}$. We say that f is normalized if $a(\xi) = 1$. We call f a Hilbert cusp form if, for all $\gamma \in \mathrm{GL}_2^+(F)$, the constant term of $f\,||_\kappa\,\gamma$ in its Fourier expansion is zero.

As in the case of (elliptic) modular forms, we have for each prime ideal $\mathfrak{p}$ in $\mathcal{O}$ a Hecke operator $T_\mathfrak{p}$ that acts on the space of Hilbert modular forms and preserves the subspace of cusp forms. A Hilbert cusp form will be called an eigenform if it is an eigenvector of all the Hecke operators. A normalized eigenform of level $\Gamma_0(\mathfrak{n})$ will be called a newform if it "doesn't come from a lower level".

Remark. *The transformation identity (3.5) forces that*

$$\kappa_1 \equiv \cdots \equiv \kappa_n \equiv 0 \ (\mathit{mod}\ 2).$$

Now, we pass to automorphic forms. For each finite place v of F, define a subgroup $K_v(\mathfrak{n})$ of $\mathrm{GL}_2(F_v)$ as

$$K_v(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{o}_v) : c \in \mathfrak{n}_v \right\},$$

and put

$$K_0(\mathfrak{n}) = \prod_{v<\infty} K_v(\mathfrak{n}).$$

Then $\Gamma_0(\mathfrak{n}) = K_0(\mathfrak{n})\,\mathrm{GL}_2^+(F_\infty) \cap \mathrm{GL}_2(F)$, where $\mathrm{GL}_2^+(F_\infty) = \mathrm{GL}_2^+(\mathbb{R})^n$. Also, let $K_\infty = \mathrm{O}_2(\mathbb{R})^n$ denote the maximal compact subgroup of $\mathrm{GL}_2(F_\infty)$.

Let f be a Hilbert cusp form of weight $\kappa = (\kappa_v)_v$ and level $\Gamma_0(\mathfrak{n})$ with Fourier expansion $f(z) = \sum_\xi a(\xi) e^{2\pi i \xi z}$ at infinity. To f one can attach a complex-valued function $\boldsymbol{f}$ on $\mathrm{GL}_2(\mathbb{A})$ defined as

$$\boldsymbol{f}(\gamma g_\infty k_0) = f||_\kappa g_\infty(i); \quad \gamma \in \mathrm{GL}_2(F), g_\infty \in \mathrm{GL}_2^+(F_\infty), k_0 \in K_0(\mathfrak{n}),$$

where the operator $||_\kappa$ is as defined in 3.4. $\boldsymbol{f}$ satisfies the following properties:

(i) $\boldsymbol{f}(\gamma g_\infty k_0) = \boldsymbol{f}(g_\infty); \quad \gamma \in GL_2(F), g_\infty \in GL_2^+(F_\infty), k_0 \in K_0(\mathfrak{n})$.

(ii) $\boldsymbol{f}(gk_\theta) = e^{i\kappa\theta}\boldsymbol{f}(g), \quad g \in GL_2(\mathbb{A}), k_\theta \in K_\infty$.

(iii) $X_- \boldsymbol{f} = 0$.

(iv) $\boldsymbol{f}$ is of 'moderate growth'.

(v) $\int_{F\backslash\mathbb{A}} \boldsymbol{f}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0$ for all $g \in GL_2(\mathbb{A})$.

The function $\boldsymbol{f}$ is the *adelization* of the Hilbert modular form f. Indeed, $(i) \equiv$ the transformation law of f with level $\Gamma_0(\mathfrak{n})$, $(ii) \equiv$ weight of f is κ, $(iii) \equiv f$ is holomorphic on $\mathfrak{h}^n$, $(iv) \equiv f$ holomorphic at the cusps and $(v) \equiv$ the constant term of Fourier expansion of $f||_\kappa\gamma$ is zero for all $\gamma \in GL_2^+(F)$.

$\boldsymbol{f}$ is the automorphic cusp form corresponding to the Hilbert modular form f. By abuse of terminology we will call $\boldsymbol{f}$ a newform if f is a newform. The span of a newform $\boldsymbol{f}$ under the action of $GL_2(\mathbb{A})$ by right translation is an irreducible representation, π, and will be termed the irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ generated by $\boldsymbol{f}$. By the tensor product theorem π decomposes as a product of local representations, $\pi = \otimes_v \pi_v$, such that

(i) $\pi_\infty = \otimes_v DS^\pm_{\kappa_v}$, where $DS^\pm_{\kappa_v}$ is the discrete series representation of $GL_2(\mathbb{R})$ of minimal weight $\pm\kappa_v$.

(ii) At precisely the finite places $v = \mathfrak{p}_v \nmid \mathfrak{n}$, π_v is an unramified principal series representation $I(\chi_{1,v}, \chi_{2,v})$ of $GL_2(F_v)$ with Satake parameters $\alpha_{1,v} = \chi_{1,v}(\varpi_v)$, $\alpha_{2,v} = \chi_{2,v}(\varpi_v)$ of π_v satisfying the relation

$$(1 - \alpha_{1,v}X)(1 - \alpha_{2,v}X) = 1 - q^{1/2}c(\mathfrak{p}_v, \boldsymbol{f})X + X^2,$$

where $c(\mathfrak{p}_v, \boldsymbol{f}) = a(\xi)\xi^{-\kappa/2}$ for some totally positive element ξ in F such that $\mathfrak{p}_v = \xi\mathcal{O}$. (Note that $c(\mathfrak{p}_v, \boldsymbol{f})$ is well-defined because the right hand side of the expression is invariant under the totally positive elements in $\mathcal{O}^\times$.)

Moreover, the newform $\boldsymbol{f} \in \pi = \otimes_v \pi_v$ admits a factorization $\boldsymbol{f} = \otimes_v f_v$.

Remark. *By the strong multiplicity-one theorem for* GL_2, $f \leftrightarrow \pi$ *is a bijection between Hilbert modular newforms of weight* κ *and level* $\mathfrak{n}$, *and irreducible cuspidal automorphic* $\mathrm{GL}_2(\mathbb{A})$*-representations satisfying (i) and (ii).*

For any $\varphi, \varphi' \in \pi$, the Petersson inner product of φ and φ' is defined by

$$\langle \varphi, \varphi' \rangle = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}dg,$$

where dg is the Tamagawa measure on $\mathrm{SL}_2(\mathbb{A})$. The automorphic representation π is unitary with respect to this inner product.

3.2 REPRESENTATION THEORY OF $\widetilde{\mathrm{SL}_2}$

In this section, we will briefly discuss the representation theory of $\widetilde{\mathrm{SL}_2}$ over $\mathbb{R}$ or a $\mathfrak{p}$-adic field. For details, we refer the reader to [BS98]. For an overview of the theory of automorphic representations on $\widetilde{\mathrm{SL}_2}$ and Hilbert modular forms of half-integral weight, we refer the readers to ([HI13],§8).

Let k be a $\mathfrak{p}$-adic field. The representation theory of the metaplectic group depends on a nontrivial additive character of k, so we fix one, ψ. We have already seen the Weil representation, ω_ψ^W, of $\widetilde{\mathrm{SL}_2(k)}$. The Weil representation is a direct sum of two irreducible representations, the even (or positive) Weil representation, $\omega_\psi^{W,+}$, and the odd (or negative) Weil representation, $\omega_\psi^{W,-}$. Let χ be a character of $k^\times$ and consider the genuine character of the Borel subgroup of the metaplectic group defined by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \frac{\gamma_\psi(1)}{\gamma_\psi(a)}\chi(a).$$

Let $\widetilde{I}_\psi(\chi)$ be the representation of $\widetilde{\mathrm{SL}_2(k)}$ induced from the above character. $\widetilde{I}_\psi(\chi)$ is reducible if and only if $\chi^2 = |\cdot|^{\pm 1}$. In this case, one can find a $\xi \in k^\times$ such that $\chi = |\cdot|^{\pm 1/2}\langle \cdot, \xi \rangle$, where $\langle \cdot, \cdot \rangle$ is the Hilbert symbol. Then $\widetilde{I}_\psi(\chi)$ has exactly two irreducible factors, one of which is isomorphic to the even Weil representation $\omega_{\psi_\xi}^{W,+}$ where $\psi_\xi(x) := \psi(x\xi)$. The other factor is called special representation and denoted $\widetilde{\sigma}_{\xi,\psi}$.

The following is the complete list of genuine irreducible admissible representations of $\widetilde{\mathrm{SL}_2(k)}$ with respect to a fixed nontrivial additive character ψ of k.

- The principal series representation $\widetilde{I}_\psi(\chi)$, where χ is a character of $k^\times$ such that $\chi^2 \neq |\cdot|^{\pm 1}$.
- The even Weil representation $\omega_{\psi_\xi}^{W,+}$ with $\xi \in F^\times/F^{\times 2}$.
- The odd Weil representation $\omega_{\psi_\xi}^{W,-}$ with $\xi \in F^\times/F^{\times 2}$.
- The special representations $\widetilde{\sigma}_{\xi,\psi}$ with with $\xi \in F^\times/F^{\times 2}$.
- The supercuspidal representations which are not equal to the odd Weil representations.

If $\mathfrak{p} \nmid 2$, we will call a principal series representation unramified if it has a $\mathrm{SL}_2(\mathfrak{o})$-fixed vector.

When $k = \mathbb{R}$, a representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ induces a representation of $\mathfrak{sl}_{2,\mathbb{C}}$, the complexified Lie algebra of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. We will only be interested in those irreducible unitary representations of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ whose underlying $\mathfrak{sl}_{2,\mathbb{C}}$-representation has half-integral weights, that is, the element Z spanning the Lie algebra of $\mathrm{SO}_2(\mathbb{R})$ acts by half-integers. For a complete list of irreducible representations of $\mathfrak{sl}_{2,\mathbb{C}}$ with half-integral weights, see ([BS98], p. 34). We only mention here that they are of two kinds,

- The principal series representations, and
- the discrete series representations.

3.3 THE SHIMURA-WALDSPURGER CORRESPONDENCE

The correspondence between modular forms of integral weight and those of half-integral weight was first considered by Shimura ([Shi73]). Waldspurger treated the Shimura correspondence in terms of automorphic representations via the theta correspondence (cf. §3.5.2,§3.5.3).

Fix an integral ideal, $\mathfrak{n}$, of $\mathcal{O}_K$. Assume $\mathfrak{n}$ is coprime to the ideal $2\mathcal{O}_K$. Also fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Let $\pi = \otimes\pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ satisfying the following two conditions:

(1) π_v is an unramified principal series representation $I(|\cdot|^{s_v}, |\cdot|^{-s_v})$, $s_v \in i\mathbb{R}$, at precisely the non-archimedean places v for which $\mathfrak{p}_v \nmid \mathfrak{n}$.

(2) π_v is the discrete series representation of minimal weight $\pm 2\kappa_v$ if v is an archimedean place.

For π as above, let $\eta \in F^\times$ be such that

$$(\operatorname{sign} \mathrm{N}_{F/\mathbb{Q}}\,\eta) \cdot \prod_{v\in S} \langle -1, \eta \rangle_v = (-1)^{|\kappa|},$$

where S is the set of places v of F at which π_v is a supercuspidal representation. We shall show that $\varepsilon(1/2, \pi \otimes \chi_\eta) = 1$. In fact, $\varepsilon(1/2, \pi_{\infty_i} \otimes \chi_\eta) = (-1)^{\kappa_i}$ and $\varepsilon(1/2, \pi_v \otimes \chi_\eta) = \langle -1, \eta \rangle_v$ for every non-archimedean place v for which π_v is not a supercuspidal representation. If π_v is supercuspidal, then $\varepsilon(1/2, \pi_v \otimes \chi_\eta) = 1$. The assertion now follows from the Hilbert product formula. Furthermore, by ([Wal91], Thm. 4) there exists a totally positive element $\xi \in F^\times$ such that $L(1/2, \pi \otimes \chi_{\xi\eta}) \neq 0$. Then the theta correspondence (with respect to ψ_ξ; see §3.5.3 for more details) of $\pi \otimes \chi_{\xi\eta}$ is a genuine irreducible cuspidal automorphic representation, σ, of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ satisfying the following three conditions:

(i) σ_v is isomorphic to the principal series representation $\widetilde{I}_{\psi_\eta}(|\cdot|^{s_v})$ at precisely the non-archimedean places v for which $\mathfrak{p}_v \nmid \mathfrak{n}$.

(ii) σ_v is the discrete series representation of minimal weight $\kappa_v + \frac{1}{2}$ if v is an archimedean place.

(iii) The element $(\mathrm{diag}(-1,-1), 1)$ in the center of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ acts by $\chi_\eta(-1)$.

Remark. *By the strong multiplicity-one for $\widetilde{\mathrm{SL}_2}$ (cf. [Wal91], Theorem 3), σ is uniquely determined by the above three conditions conditions.*

Theorem. *Suppose that $\kappa_v > 1$ for some archimedean place v. Then there is a one-to-one correspondence between the following two sets:*

(a) The set of irreducible cuspidal automorphic representations π of $\mathrm{PGL}_2(\mathbb{A})$ with properties (1) and (2).

(b) The set of genuine irreducible cuspidal automorphic representations of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ with properties (i), (ii) and (iii).

3.4 NEARLY HOLOMORPHIC AUTOMORPHIC FORMS

On the way to working out the rationality of $L(s, \mathrm{Sym}^2 g \times f)$ in the purely balanced resp. purely unbalanced case, we would require the notion of *nearly holomorphic modular forms* of integral resp. half-integral weight. They were first defined by Shimura in [Shi76] and play an important role in the theory of modular forms. We will first recall their definition in the classical language of modular forms and then reinterpret them in automorphic language. For more details, we refer the reader to [Shi87].

Let $F/\mathbb{Q}$ be a totally real field of degree n. By a nearly holomorphic Hilbert modular form of (integral or half-integral) weight κ and level $\mathfrak{n}$, we mean a real analytic function f on $\mathfrak{h}^n$ such that

$$f||_\kappa \gamma = f \quad \text{for every } \gamma \in \Gamma_0(\mathfrak{n})$$

and that

$$f(z) = \sum_{0 \leqslant a \leqslant A} (\pi y)^{-a} f_a(z),$$

where $A \in \mathbb{Z}^n$ and $f_a(z) = \sum_\xi c_a(\xi) e^{2\pi i \xi z}$ are holomorphic functions. Here it is understood that $e^{2\pi\xi z} = e^{\sum_{i=1}^n 2\pi i \xi_i z_i}$ and that $(\pi y)^{-a} = \prod_{i=1}^n (\pi y_i)^{-a_i}$. Let us denote the space of all nearly holomorphic modular forms of weight κ by $\mathcal{N}_\kappa$.

Define the Maass-Shimura differential operators $\delta_v(c)$ and $\delta_k^{(a)}$ on the space of smooth functions on $\mathfrak{h}^n$ for $v \in \Sigma_\infty$, $c \in \mathbb{R}$, and $0 \leqslant a \in \mathbb{Z}^n$ by

$$\delta_v(c) = (2\pi i)^{-1} \Big(\frac{\partial}{\partial z_v} + \frac{c}{(z_v - \overline{z_v})} \Big),$$

$$\delta_\kappa^{(a)} = \prod_{v \in \Sigma_\infty} \prod_{j=1}^{a_v} \delta_v(\kappa_v + 2j - 2).$$

It is not hard to see that $\delta_\kappa^{(a)} \mathcal{N}_\kappa \subset \mathcal{N}_{\kappa+2a}$. In particular, $\delta_\kappa := \delta_\kappa^{(1)}$ is a weight raising operator, raising the weight by 2.

By ([Shi87], Lemma 8.2), every nearly holomorphic Hilbert modular form of weight κ can be uniquely written in the form

$$f = \sum_{0 \leqslant p \leqslant \kappa/2} \delta^p_{\kappa-2p} g_p + \begin{cases} c\delta_2^{(\kappa/2)-1} E_2 & \text{if } F = \mathbb{Q} \text{ and } \kappa \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases},$$

where g_p is a Hilbert modular form of weight $\kappa - 2p$, $c \in \mathbb{C}$, and

$$E_2(z) = (4\pi y)^{-1} - 12^{-1} + 2\sum_{n=1}^{\infty} \Big(\sum_{0<d|n} d \Big) e^{2\pi i n z}.$$

It is clear that the converse is also true: if a function on $\mathfrak{h}^n$ can be written in the above form then it must be a nearly holomorphic Hilbert modular form; hence we can and will take this to be the definition.

Now we will reinterpret a nearly holomorphic modular form in the language of automorphic forms. Since a function on $\mathfrak{h}^n$ can be realized as a function on $G = \mathrm{GL}_2^+(\mathbb{R})^n$, we may think of $\delta_\kappa^{(a)}$ as a differential operator on the space of smooth functions on G. It is not hard to check that $\delta_\kappa^{(a)}$ is in fact a left-invariant differential operator and hence, under the identification of left-invariant differential operators on G with he universal enveloping algebra $\mathcal{U}(\mathrm{Lie}(G))$, may be viewed as an element $\boldsymbol{\delta}_\kappa^{(a)} \in \mathcal{U}(\mathrm{Lie}(G))$. Recall also the element X_+ of $\mathrm{Lie}(\mathrm{J}(\mathbb{R}))$ (see §2.9) which as an element of $\mathrm{Lie}(\mathrm{GL}_2^+(\mathbb{R}))$ looks like

$$\mathrm{X}_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

It is a fact that X_+ as a differential operator on $\mathrm{GL}_2^+(\mathbb{R})$ is a weight raising operator, raising the weight by 2. It follows from the structure of irreducible $(\mathrm{Lie}(\mathrm{GL}_2^+(\mathbb{R})), SO_2(\mathbb{R}))$-modules (cf. p. 24) that if g_p is a Hilbert modular form of weight $\kappa - 2p$ and $\boldsymbol{g}_p$ denotes its adelization, then $\boldsymbol{\delta}_{\kappa-2p}^{(p)}\boldsymbol{g}_p$ must be a scalar multiple of $\prod_{v\in\Sigma_\infty} \mathrm{X}_+^{p_v}\,\boldsymbol{g}_p$. Hence, we may call an automorphic form φ a nearly holomorphic automorphic form of weight κ if (assume $F \neq \mathbb{Q}$)

$$\varphi = \sum_{0\leqslant p\leqslant \kappa/2}\ \prod_{v\in\Sigma_\infty} \mathrm{X}_+^{p_v}\,\boldsymbol{g}_p,$$

where $\boldsymbol{g}_p$ is a Hilbert modular automorphic form of weight $\kappa - 2p$.

Lemma 3.4.1. *Let φ be an automorphic form on $\mathrm{GL}_2(\mathbb{A})$. Then φ is a linear combination of nearly holomorphic automorphic forms if and only if $\prod_{v\in\Sigma_\infty} \mathrm{X}_-^{s_v}\,\varphi = 0$ for some $s = (s_v)_v \in \mathbb{Z}^{\Sigma_\infty}$, where X_- is the element of $\mathrm{Lie}(\mathrm{J})$ defined in §2.9 but viewed as an element of $\mathrm{Lie}(\mathrm{SL}_2)$.*

Proof. Without loss of generality assume that φ is nearly holomorphic of the form $\varphi = \prod_{v\in\Sigma_\infty} \mathrm{X}_+^{p_v}\,\boldsymbol{g}_p$, where $\boldsymbol{g}_p$ is a Hilbert modular automorphic form of weight $\kappa - 2p$. It follows from the theory of $(\mathrm{Lie}(\mathrm{GL}_2(\mathbb{R})), O_2(\mathbb{R}))$-modules that X_- acts as a weight lowering operator, lowering the weight by 2, and that $\mathrm{X}_-\,\mathrm{X}_+$ acts as a scalar (cf. [Bum97], Prop. 2.5.2). Therefore, $\prod_{v\in\Sigma_\infty} \mathrm{X}_-^{p_v+1}\,\varphi = 0$.

For the converse, write φ as a linear combination of linearly independent pure tensors ϕ_i. If $\prod_{v\in\Sigma_\infty} \mathrm{X}_-^{s_v}\,\varphi = 0$, then linear independence implies that $\prod_{v\in\Sigma_\infty} \mathrm{X}_-^{s_v}\,\varphi_i = 0$ for all i. Hence, without loss of generality, we may assume φ is a pure tensor. If $\prod_{v\in\Sigma_\infty} \mathrm{X}_-^{s_v}\,\varphi = 0$ and $s = (s_v)_v$ is smallest such, then $\boldsymbol{f} = \prod_{v\in\Sigma_\infty} \mathrm{X}_-^{s_v-1}\,\varphi$ must be a holomorphic Hilbert modular automorphic form. It then follows from the theory of $(\mathrm{Lie}(\mathrm{GL}_2(\mathbb{R})), O_2(\mathbb{R}))$-modules that

$$\prod_{v\in\Sigma_\infty} \mathrm{X}_+^{s_v-1}\,\boldsymbol{f} = c\phi \quad \text{for some } c \in \mathbb{C}.$$

□

3.5 THE WALDSPURGER PACKETS

Waldspurger has used global theta correspondence to divide the set $\mathcal{A}_{00}(\widetilde{\mathrm{SL}_2})$ of genuine cuspidal automorphic forms on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ orthogonal to theta series associated to one-dimensional quadratic forms into packets parametrized by irreducible cuspidal automorphic $\mathrm{PGL}_2(\mathbb{A})$-representations. We will briefly describe how this is done. First, we recall the Jacquet-Langlands correspondence and the theta correspondence.

3.5.1 *Jacquet-Langlands correspondence*

In this section we will state the Jacquet-Langlands correspondence. For more details, we refer the readers to [GJ79], [JL70].

3.5.1.1 *Local statement*

Let k be the real or a $\mathfrak{p}$-adic field and D denote the unique quaternion division algebra over k. The local Jacquet-Langlands correspondence is a bijection between irreducible smooth representations of $D^\times$ with central character ω and irreducible discrete series representations of $\mathrm{GL}_2(k)$ with central character ω. Let us make this more explicit.

Suppose $k = \mathbb{R}$. The irreducible smooth representations of $D^\times$ are of the form $|\cdot|^t \otimes \mathrm{Sym}^n(\mathbb{C}^2)$ and the irreducible discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ are of the form $|\cdot|^t \otimes DS_n^\pm$, where $t \in \mathbb{C}$. Then the Jacquet-Langlands correspondence for $\mathrm{GL}_2(\mathbb{R})$ is the bijection

$$|\cdot|^t \otimes \mathrm{Sym}^n(\mathbb{C}^2) \leftrightarrow |\cdot|^t \otimes DS_{n+2}^\pm,$$

where $t \in \mathbb{C}$.

Suppose now that k is a $\mathfrak{p}$-adic field. The irreducible smooth representations of $D^\times$ are all finite dimensional. The irreducible discrete series representations of $\mathrm{GL}_2(k)$ are the twisted Steinberg representations and the supercuspidal representations. Under the Jacquet-Langlands correspondence,

$$\chi \circ \mathrm{Nm}_{D/k} \leftrightarrow \mathrm{St}(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2}),$$

where χ is a character of $k^\times$, and

$$\left\{ \begin{array}{c} \text{irreducible finite dimensional representations} \\ \text{of } D^\times \text{ of dimension} > 1 \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{supercuspidal representations} \\ \text{of } \mathrm{GL}_2(k) \end{array} \right\}.$$

3.5.1.2 *Global statement*

Let F be a number field and D a quaternion algebra over F ramified exactly at places S. Let ω be a smooth character of $F^\times \backslash \mathbb{A}_F^\times$.

Theorem. *Let $\pi' = \otimes \pi'_v$ be an irreducible automorphic representation of $\mathbb{A}_D^\times$ of dimension>1. Consider $\pi = \otimes \pi_v$ where $\pi_v \cong \pi'_v$ if $v \notin S$ and $\pi_v \leftrightarrow \pi'_v$ under the local Jacquet-Lnaglands correspondence if $v \in S$. Then π is an automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ and the map*

$$\pi' \mapsto \pi :$$

$$\{\textit{irreducible automorphic representations of } \mathbb{A}_D^\times \textit{ of dimension >1 and central character } \omega\}$$

$$\to \{\textit{irreducible cuspidal automorphic representations of } \mathrm{GL}_2(\mathbb{A}_F) \textit{ with central character } \omega\}$$

is injective.

3.5.2 *Howe duality or the Local theta correspondence*

Let k be a real or a $\mathfrak{p}$-adic field. Let (V, q) be a 3-dimensional quadratic space over k. Consider the pair of groups $(\mathrm{SO}(V, q), \widetilde{SL_2(k)})$. The Howe correspondence or the local theta correspondence for the above pair is a correspondence between irreducible representations of $\mathrm{SO}(V, q)$ and $\widetilde{SL_2(k)}$. Explicitly, it is given as follows.

There exists a quaternion algebra D over k and an element $a \in k^\times$ such that $(V, q) \cong (V_D, aq_D)$, where

$$V_D = \{x \in D | Tr_D(x) = 0\} \quad \text{and} \quad q_D(x) = -Nm_D(x),$$

where Tr and Nm are the standard trace and norm maps on D. The group of units, $D^\times$ of D, acts on V_D by $b.x = bxb^{-1}$ and this leads to an isomorphism

$$D^\times / k^\times = PD^\times \cong \mathrm{SO}(V, q).$$

We will identify $\mathrm{SO}(V, q)$ with $PD^\times$ via the above isomorphism. Now, fix an additive character ψ of k. Then the Howe correspondence for $(PD^\times, \widetilde{SL_2(k)})$ is as follows.

(i) If $D = M_2(k)$, then there is a bijection between the set of all irreducible infinite dimensional representations of $\mathrm{PGL}_2(k)$ and the set of all genuine representations of $\widetilde{SL_2(k)}$ (i.e not factoring through $\mathrm{SL}_2(k)$) which have a ψ-Whittaker model.

(ii) If D is the unique division algebra, then there is a bijection between the set of all irreducible representations of $D^\times / k^\times$ and the set of all genuine irreducible representations of $\widetilde{SL_2(k)}$ not accounted for by the *odd Weil representation* (cf. [BS98],§2.5) and which are square integrable but do not have a ψ-Whittaker model.

Remark. *The bijection is via the Weil representation of the metaplectic group which depends on the additive character ψ.*

3.5.3 *The Global theta correspondence*

Let F be a totally real number field. Let (V, q) be a 3-dimensional quadratic space over F and consider the pair of groups $(\mathrm{SO}(V, q), \widetilde{\mathrm{SL}_2})$. The global theta correspondence is a correspondence between irreducible cuspidal automorphic representations of $\mathrm{SO}(V, q)$ and $\widetilde{\mathrm{SL}_2}$. We describe it below.

A key ingredient in the construction of the global theta correspondence is a function on $\mathrm{SO}(V, q)(\mathbb{A}) \times \widetilde{\mathrm{SL}_2(\mathbb{A})}$, the so-called theta-kernel, which allows one to pass from automorphic forms on one group of the pair to the other. To define it, consider the embedding $\mathrm{SO}_3 \times \widetilde{\mathrm{SL}_2} \subset \widetilde{\mathrm{Sp}_6}$, where Sp_6 is the symplectic group associated to a 6-dimensional symplectic space and $\widetilde{\mathrm{Sp}_6}$ is its double cover. For a fixed nontrivial additive character ψ of $F \backslash \mathbb{A}$, $\widetilde{\mathrm{Sp}_6}$ is endowed with a Weil representation ω_ψ which has a realization on the Schwarz space $\mathcal{S}(\mathbb{A}^3)$. For $\phi \in \mathcal{S}(\mathbb{A}^3)$, we can define a function on $\widetilde{\mathrm{Sp}_6(\mathbb{A})}$ by

$$\theta_\phi(g) = \sum_{x \in F^3} (\omega_\psi(g)\phi)(x).$$

The function θ_ϕ is known to be an automorphic form on $\mathrm{Sp}_6(F) \backslash \widetilde{\mathrm{Sp}_6(\mathbb{A})}$; in particular it is a slowly-increasing function. The restriction of θ_ϕ to $\mathrm{SO}_3(\mathbb{A}) \times \widetilde{\mathrm{Sp}_6(\mathbb{A})}$ is called the theta-kernel.

As in the local case, we can identify $\mathrm{SO}(V, q)$ with $PD^\times$ for some quaternion algebra D over F, and thus consider the pair of groups $(PD^\times, \widetilde{\mathrm{SL}_2})$ instead. Put

$$[PD^\times] = PD^\times(F) \backslash PD^\times(\mathbb{A}) \quad \text{and} \quad [\mathrm{SL}_2] = \mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A}).$$

Let $\mathcal{A}_{cusp}(PD^\times)$ be the set of all cuspidal automorphic forms on $PD^\times(\mathbb{A}) = D^\times(\mathbb{A})/\mathbb{A}^\times$. Let $\mathcal{A}_{00}(\widetilde{\mathrm{SL}_2})$ be the set of all genuine cuspidal automorphic forms on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}_2(\mathbb{A})}$ (i.e not factoring through $\mathrm{SL}_2(\mathbb{A})$) orthogonal to theta series associated to one-dimensional quadratic forms. For an irreducible $PD^\times(\mathbb{A})$-representation $\pi \subset \mathcal{A}_{cusp}(PD^\times)$ and an irreducible $\widetilde{\mathrm{SL}_2(\mathbb{A})}$-representation $\sigma \subset \mathcal{A}_{00}(\widetilde{\mathrm{SL}_2})$, define the global theta lifts of π and σ with respect to a fixed additive character ψ of $F\backslash\mathbb{A}$ by

$$\Theta(\pi;\psi) = \{\theta_\psi(f,\phi)|f \in \pi, \phi \in \mathcal{S}(\mathbb{A})\} \quad \text{and} \quad \Theta(\sigma;\psi) = \{\theta_\psi(h,\phi)|h \in \sigma, \phi \in \mathcal{S}(\mathbb{A})\},$$

where

$$\theta_\psi(f,\phi)(g') = \int_{[PD^\times]} \theta_\phi(g,g')f(g)dg \quad \text{and} \quad \theta_\psi(h,\phi)(g) = \int_{[\mathrm{SL}_2]} \theta_\phi(g,g')h(g')dg'.$$

Then it is known that ([Wal80],[Wal91]) (i) if $\Theta(\pi;\psi)$ is nonzero, then it is irreducible cuspidal and belongs to $\mathcal{A}_{00}(\widetilde{\mathrm{SL}_2})$; (ii) if $\Theta(\sigma;\psi)$ is nonzero, then it is irreducible cuspidal; (iii) if the global theta lift is nonzero, then it is the product of local theta lifts; (iv) $\Theta(\pi;\psi) = \sigma \iff \Theta(\sigma;\psi) = \pi$; (v) if $\Theta(\pi;\psi)$, $\Theta(\pi;\psi_\alpha)$ are nonzero, then $\Theta(\pi;\psi_\alpha) = \Theta(\pi;\psi) \otimes \chi_\alpha$, where χ_α is the quadratic attached to $F(\sqrt{\alpha})/F$; (vi) $\Theta(\pi;\psi) \neq 0 \iff L(1/2,\pi) \neq 0$.

Since modular forms of integral weight can be viewed as automorphic forms on GL_2 and modular forms of half-integral weight can be viewed as automorphic forms on $\widetilde{\mathrm{SL}_2}$, the theta correspondence for the pair $(\mathrm{PGL}_2, \widetilde{\mathrm{SL}_2})$ should yield a correspondence between these two classes of modular forms. In the next section, we lay down this correspondence explicitly.

3.5.4 *The Waldspurger packet* $\mathrm{Wd}_{\psi}(\pi)$

Fix a nontrivial additive character $\psi = \otimes \psi_v$ of $F \backslash \mathbb{A}$. For each place v of F consider the pair $(PD^{\times}, \widetilde{\mathrm{SL}_2})$, where D is a quaternion algebra (either a matrix algebra or the unique division algebra) over F_v. Let $\Theta(\,\cdot\,; \psi_v)$ stand for the local theta correspondence with respect to ψ_v for the above pair.

Now, consider an irreducible $\pi = \otimes \pi_v \subset \mathcal{A}_{cusp}(\mathrm{PGL}_2)$. Put $\sigma_v^+ = \Theta(\pi_v; \psi_v)$. If π_v is square integrable, that is, either a discrete series representation if v is archimedean or, a Steinberg or supercuspidal representation if v is non-archimedean, let $\mathrm{JL}(\pi_v)$ be the Jacquet-Langlands lift (cf. [JL70],§16) of π_v; it is a representation of $PD^{\times}$, where D is the unique quaternion division algebra over F_v. Put $\sigma_v^- = \Theta(\mathrm{JL}(\pi_v); \psi_v)$. The local Waldspurger packet of π_v with respect to ψ_v is given by

$$\mathrm{Wd}_{\psi_v}(\pi_v) := \begin{cases} \{\sigma_v^+\} & \text{if } \pi_v \text{ is not square integrable} \\ \{\sigma_v^+, \sigma_v^-\} & \text{if } \pi_v \text{ is square integrable.} \end{cases}$$

The local Waldspurger packets can be described explicitly (cf. [Wal91], Propositions 4,5,7, and 8 and Lemma 20).

Theorem. *Let* $\pi = \otimes \pi_v$ *be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$ *and* ψ *be a fixed nontrivial additive character of* $F \backslash \mathbb{A}$*. Write* $\psi = \otimes \psi_v$*, where* $\psi_v = \exp(2\pi i \alpha_v)$ *for some real number* α_v *when* $F_v = \mathbb{R}$*. Then:*

π_v	σ_v^+	σ_v^-
$DS_{2\kappa}^{\pm}$	$\begin{cases} DS^+_{\kappa+\frac{1}{2}} & \textit{if } \alpha_v > 0 \\ DS^-_{\kappa+\frac{1}{2}} & \textit{if } \alpha_v < 0 \end{cases}$	$\begin{cases} DS^-_{\kappa+\frac{1}{2}} & \textit{if } \alpha_v > 0 \\ DS^+_{\kappa+\frac{1}{2}} & \textit{if } \alpha_v < 0 \end{cases}$
$I(\mu, \mu^{-1}),\ \mu^2 \neq \lvert\cdot\rvert$	$\widetilde{I}_{\psi_v}(\mu)$	-
$St(\lvert\cdot\rvert^{1/2}, \lvert\cdot\rvert^{-1/2})$	$\omega_{\psi_v}^{W,-}$	$\widetilde{\sigma}_{1,\psi_v}$
$St(\mu, \mu^{-1}),\ \mu = \lvert\cdot\rvert^{1/2}\langle\cdot,\xi\rangle,\ \xi \notin F^{\times 2}$	$\widetilde{\sigma}_{\xi,\psi_v}$	$\omega_{\psi_{\xi,v}}^{W,-}$
supercuspidal	*supercuspidal*	*supercuspidal*($\ncong$ *odd Weil rep.)*

To recall the notations in the table above, see §3.1.1, §3.2.

For an irreducible $\pi \subset \mathcal{A}_{cusp}(\mathrm{PGL}_2)$, the global Waldspurger packet of π with respect to a nontrivial additive character ψ of $F\backslash\mathbb{A}$ is defined as

$$\mathrm{Wd}_\psi(\pi) := \{\Theta(\pi \otimes \chi_\xi; \psi_\xi) \mid \xi \in F^\times/F^{\times 2}\}.$$

For any $\alpha \in F^\times$, one has

$$\mathrm{Wd}_{\psi_\alpha}(\pi) = \mathrm{Wd}_\psi(\pi) \otimes \chi_\alpha.$$

Moreover,

$$\mathcal{A}_{00}(\widetilde{\mathrm{SL}_2}) = \bigsqcup_{\text{irred. } \pi \subset \mathcal{A}_{cusp}(\mathrm{PGL}_2)} \mathrm{Wd}_\psi(\pi),$$

The global Waldspurger packets can be conveniently described in terms of the local ones (cf. [Wal91], §6).

Theorem 3.5.1. *([Wal91], Lemma 40) For each $\varepsilon = (\varepsilon_v)$ such that ε_v belongs to $\{\pm 1\}$ and takes value 1 when π_v is not square-integrable, put $\sigma^\varepsilon = \otimes \sigma_v^{\varepsilon_v}$. Then the Waldspurger packet of π with respect to ψ consists of automorphic representations of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ and is given by*

$$\mathrm{Wd}_\psi(\pi) = \left\{ \sigma^\varepsilon \,\middle|\, \prod_v \varepsilon_v = \varepsilon\left(\frac{1}{2}, \pi\right) \right\}.$$

3.6 REPRESENTATION THEORY OF THE JACOBI GROUP

3.6.1 *The Schrödinger-Weil representation ω_ψ*

Let k the real or a $\mathfrak{p}$-adic field. Fix a nontrivial additive character ψ of k. In section §2.7 we looked at the Schrödinger representation ω_ψ^S of the Heisenberg group $\mathrm{H}(k)$ and the Weil representation ω_ψ^W of the metaplectic group $\widetilde{\mathrm{SL}_2(k)}$. Putting together these two representations we get a (genuine) representation of the double cover, $\widetilde{\mathrm{J}}(k) = \widetilde{\mathrm{SL}_2(k)} \ltimes H(k)$, of the Jacobi group. This is the so-called Schrödinger-Weil representation of $\mathrm{J}(k)$ and we will denote it by ω_ψ. It has a realization on the Schwarz space $\mathcal{S}(k)$ and is given by the following formulae.

$$\begin{aligned} \omega_\psi(m(a))\phi(t) &= \frac{\gamma_\psi(1)}{\gamma_\psi(a)}|a|^{1/2}\phi(at), \\ \omega_\psi(n(b))\phi(t) &= \psi(bt^2)\phi(t), \\ \omega_\psi(w)\phi(t) &= \overline{\gamma_\psi(1)}|2|^{1/2}\widehat{\phi}(-2t), \\ \omega_\psi((\lambda,\mu,\xi))\phi(t) &= \psi(\xi + (2t+\lambda)\mu)\phi(x+\lambda) \\ \omega_\psi((1,\varepsilon))\phi(t) &= \varepsilon\phi(t). \end{aligned} \tag{3.6}$$

The last formula ensures that the Schrödinger-Weil representation is indeed a genuine representation of $\widetilde{\mathrm{J}}(k)$. Here $\widehat{\phi}$ denotes the Fourier transform of ϕ;

$$\widehat{\phi}(x) = \int_k \phi(y)\psi(xy)dy,$$

with dy the self-dual additive Haar measure on k. Furthermore, $\gamma_\psi : k^\times \to \mathbb{C}^\times$ denotes the Weil constant: for any $a \in k^\times$, $\gamma_\psi(a)$ is an eight root of unity which depends only on the class of a in $k^\times/k^{\times 2}$ and satisfies

$$\int_k \phi(x)\psi(ax^2)dx = \gamma_\psi(a)|2a|^{-1/2} \int_k \widehat{\phi}(x)\psi\Big(-\frac{x^2}{4a}\Big)dx.$$

The Schrödinger-Weil representation is unitary with respect to the inner product

$$\langle \phi_1, \phi_2 \rangle = \int_k \phi_1(y)\overline{\phi_2(y)}dy \qquad (\phi_1, \phi_2 \in \mathcal{S}(k)).$$

Just as in the local case, we have for a global field F with ring of adeles $\mathbb{A}$, the global Schrödinger-Weil representation ω_ψ of $\mathrm{J}(\mathbb{A})$ for any nontrivial additive character ψ of $F\backslash\mathbb{A}$. It is a genuine representation of $\widetilde{\mathrm{J}}(\mathbb{A})$. It has a realization on the Schwarz space $\mathcal{S}(\mathbb{A})$ and is defined by formulae similar but slightly simpler (due to the fact that there is a product formula for the Weil constant) to those in the local cases.

$$\begin{aligned} \omega_\psi(m(a))\phi(t) &= |a|^{1/2}\phi(at), \\ \omega_\psi(n(b))\phi(t) &= \psi(bt^2)\phi(t), \\ \omega_\psi(w)\phi(t) &= |2|^{1/2}\widehat{\phi}(-2t), \\ \omega_\psi((\lambda,\mu,\xi))\phi(t) &= \psi(\xi + (2t+\lambda)\mu)\phi(x+\lambda) \\ \omega_\psi((1,\varepsilon))\phi(t) &= \varepsilon\phi(t). \end{aligned} \tag{3.7}$$

Here w denotes as usual the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(F) \subset \mathrm{SL}_2(\mathbb{A})$, and $|\cdot|$ is the product over all places of absolute values of F_v, normalized such that the product formula[1] holds. ω_ψ is unitary with respect to the inner product

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{A}} \phi_1(y)\overline{\phi_2(y)}dy \qquad (\phi_1, \phi_2 \in \mathcal{S}(\mathbb{A})).$$

Moreover, if $\psi = \otimes_v \psi_v$, then $\omega_\psi = \otimes_v \omega_{\psi_v}$.

3.6.2 *Correspondence between representations of* J *and* $\widetilde{\mathrm{SL}_2}$

The Schrödinger-Weil representation gives a 1-1 correspondence between genuine irreducible representations of the metaplectic group and irreducible representations of the Jacobi group with non-trivial central character. We state the results here. For full theory, see [BS98].

3.6.2.1 *Local Theory*

Let k be the real or a $\mathfrak{p}$-adic field.

Theorem. *Suppose $k = \mathbb{R}$. Let ψ be a non-trivial additive character of k and ω_ψ the Schrödinger-Weil representation of $\mathrm{J}(k)$. Then the map*

$$\tilde{\pi} \mapsto \tilde{\pi} \otimes \omega_\psi$$

gives a 1-1 correspondence between genuine irreducible unitary representations of $\widetilde{\mathrm{SL}_2(k)}$ and irreducible unitary representations of $\mathrm{J}(k)$ with central character ψ.

1 If $a \in$F, then $\prod_{v \in \Sigma_F} a_v = 1$, where Σ_F is the set of all places of F.

Remark. *A representation of* $\mathrm{J}(\mathbb{R})$ *induces a representation of* $\mathfrak{j}_{\mathbb{C}}$*, the complexified Lie algebra of* $\mathrm{J}(\mathbb{R})$*. We will be only interested in those irreducible unitary representations of* $\mathrm{J}(\mathbb{R})$ *whose underlying* $\mathfrak{j}_{\mathbb{C}}$*-representation has integral weights, that is, the element* Z *spanning the Lie algebra of* $\mathrm{SO}_2(\mathbb{R})$ *acts by integers. For a complete list of irreducible representations of* $\mathfrak{j}_{\mathbb{C}}$ *with integral weights, see ([BS98], Thm. 3.1.9).*

Theorem. *Suppose* k *is a* $\mathfrak{p}$*-adic field. Let* ψ *be a non-trivial additive character of* k *and* ω_ψ *the Schrödinger-Weil representation of* $\mathrm{J}(k)$*. Then the map*

$$\widetilde{\pi} \mapsto \widetilde{\pi} \otimes \omega_\psi$$

gives a 1-1 correspondence between genuine irreducible smooth representations of $\widetilde{\mathrm{SL}_2(k)}$ *with respect to* $\overline{\psi}$ *(ref. §3.2) and irreducible smooth representations of* $\mathrm{J}(k)$ *with central character* ψ*.*

3.6.2.2 *Global theory*

Let F be a global field with ring of adeles $\mathbb{A}$. Let ψ be a non-trivial additive character of $F\backslash\mathbb{A}$ and ω_ψ the Schrödinger-Weil representation of $\mathrm{J}(\mathbb{A})$. It is possible to lift the Schrödinger-Weil representation from $L^2(\mathbb{A})$ to functions living on the Jacobi group. Indeed, assign to $\phi \in L^2(\mathbb{A})$ the theta function

$$\theta_\phi(gh) = \sum_{a\in F}(\omega_\psi(g)\phi)(a); \quad g \in \mathrm{SL}_2(\mathbb{A}),\ h \in \mathrm{H}(\mathbb{A}),$$

and let $\mathrm{J}(\mathbb{A})$ act on θ_ϕ via right translation. Then the above map is an intertwining map for representations of $\mathrm{J}(\mathbb{A})$ and is moreover an isometry if the norm on the image is given by integration on the Heisenberg group only:

$$\int_{H(F)\backslash H(\mathbb{A})/Z(\mathbb{A})} |\theta_\phi(h)|^2 dh.$$

Theorem 3.6.1 ([BS98], Thm. 7.3.3)**.** *There is a natural isometry of Hilbert spaces*

$$L^2(\widetilde{SL_2(F)}\backslash\widetilde{SL_2(\mathbb{A})}\otimes L^2(\mathbb{A}) \longrightarrow L^2(\mathrm{J}(F)\backslash\mathrm{J}(\mathbb{A})),$$
$$f\otimes\phi \longmapsto (gh\mapsto f(g,1)\theta_\phi(gh)).$$

This isometry is an intertwining map for representations of the Jacobi group on both sides, where the action of $\mathrm{J}(\mathbb{A})$ *on* $L^2(\mathbb{A})$ *is the Schrödinger-Weil representation, and right translation on other parts. Restriction to cuspidal functions yields another isometry,*

$$L^2_0(\widetilde{SL_2(F)}\backslash\widetilde{SL_2(\mathbb{A})})\otimes L^2(\mathbb{A}) \longrightarrow L^2_0(\mathrm{J}(F)\backslash\mathrm{J}(\mathbb{A})).$$

Corollary 3.6.2 ([BS98], Cor. 7.3.5)**.** *Let* F *be a global field with ring of adeles* $\mathbb{A}$*. Let* ψ *be a non-trivial additive character of* $F\backslash\mathbb{A}$ *and* ω_ψ *the Schrödinger-Weil representation of* $\mathrm{J}(\mathbb{A})$*. Then the map*

$$\widetilde{\pi}\mapsto\pi := \widetilde{\pi}\otimes\omega_\psi$$

gives a 1-1 correspondence between the genuine automorphic representations of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ *(with respect to* $\overline{\psi}$*) and automorphic representations of* $J(\mathbb{A})$ *with central character* ψ*. The cuspidal representations correspond to cuspidal ones. Further, this correspondence is compatible with local data: if* $\omega = \otimes\omega_{\psi_v}$ *and*

$$\widetilde{\pi} = \otimes_v\widetilde{\pi}_v, \qquad \pi = \otimes_v\pi_v$$

are the decompositions of $\widetilde{\pi}$ *and* π*, then*

$$\pi_v = \widetilde{\pi}_v\otimes\omega_{\psi_v}.$$

Part II

THE MAIN RESULTS

4

THE MAIN FORMULA

4.1 THE ADJOINT L-FUNCTION $L(s,\tau,\mathrm{Ad})$

Let τ be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ of weight $\kappa'+1$. For simplicity, we will assume that τ is unramified at all the finite places. For each prime $\mathfrak{p}$, let $\{\alpha_{\mathfrak{p}}, \alpha_{\mathfrak{p}}^{-1}\}$ denote the Satake parameters of τ at $\mathfrak{p}$. Put

$$L_{\mathrm{fin}}(s,\tau,\mathrm{Ad}) = \prod_{\mathfrak{p}} \{(1-\alpha_{\mathfrak{p}}^2 N\mathfrak{p}^{-s})(1-N\mathfrak{p}^{-s})(1-\alpha_{\mathfrak{p}}^{-2}N\mathfrak{p}^{-s})\}^{-1}$$

and

$$L_\infty(s,\tau,\mathrm{Ad}) = \Gamma_{\mathbb{R}}(s+1)^n \prod_{v\in\Sigma_\infty} \Gamma_{\mathbb{C}}(s+\kappa'_v),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$. Then (cf. [Hid16], §5.1)

$$L(s,\tau,\mathrm{Ad}) = L_\infty(s,\tau,\mathrm{Ad})L_{\mathrm{fin}}(s,\tau,\mathrm{Ad})$$

is the adjoint L-function of τ. It is a fact that (cf. [IP21], Prop. 6.6, Lemma 6.1)

$$L(1,\tau,\mathrm{Ad}) = 2^{|\kappa'+3|_{\Sigma_\infty}} D_F^{1/2}\zeta_F(2)\langle \boldsymbol{g},\boldsymbol{g}\rangle, \tag{4.1}$$

where $\boldsymbol{g}$ is the newform generating τ.

4.2 THE L-FUNCTION $L(s, \mathrm{Sym}^2 \tau \times \pi)$

Let π resp. τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ of weight 2κ resp. $\kappa' + 1$ whose central character ω_π resp. ω_τ we assume to be unitary. Let Σ_{ub} and Σ_b be subsets of Σ_∞ whose union is Σ_∞, such that $\kappa_v > \kappa'_v$ for $v \in \Sigma_{ub}$ and $\kappa_v \leqslant \kappa'_v$ for $v \in \Sigma_b$. The L-function $L(s, \mathrm{Sym}^2 \tau \times \pi)$ can be written down explicitly as follows.

For each prime $\mathfrak{p}$ outside the set S of all finite places where either π or τ is not an unramified principal series representation, let $\{\alpha_{1,\mathfrak{p}}, \alpha_{2,\mathfrak{p}}\}$ resp. $\{\beta_{1,\mathfrak{p}}, \beta_{2,\mathfrak{p}}\}$ denote the Satake parameters of τ resp. π at $\mathfrak{p}$. Put

$$A_\mathfrak{p} = \begin{pmatrix} \alpha_{1,\mathfrak{p}}^2 & & \\ & \alpha_{1,\mathfrak{p}}\alpha_{2,\mathfrak{p}} & \\ & & \alpha_{2,\mathfrak{p}} \end{pmatrix}, \quad B_\mathfrak{p} = \begin{pmatrix} \beta_{1,\mathfrak{p}} & \\ & \beta_{2,\mathfrak{p}} \end{pmatrix},$$

and

$$L_\mathfrak{p}(s, \mathrm{Sym}^2 \tau \times \pi) = \det(I_6 - (A_\mathfrak{p} \otimes B_\mathfrak{p}) N\mathfrak{p}^{-s})^{-1}.$$

It is also possible to define $L_\mathfrak{p}(s, \mathrm{Sym}^2 \tau \times \pi)$ for $v \in S$. Let

$$L_{\mathrm{fin}}(s, \mathrm{Sym}^2 \tau \times \pi) = \prod_\mathfrak{p} L_\mathfrak{p}(s, \mathrm{Sym}^2 \tau \times \pi),$$

where the product runs over all the finite places of F.

We can also define explicitly the L-factor at infinity. Put $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. By applying Lemma 16.3 of [Jac09], we see that

$$L_v(s, \tau \times \tau \times \pi) = \Gamma_\mathbb{C}\Big(s + \kappa'_v + \kappa_v - \frac{1}{2}\Big) \cdot \Gamma_\mathbb{C}\Big(s + \kappa'_v - \kappa_v + \frac{1}{2}\Big) \cdot \Big(\Gamma_\mathbb{C}\Big(s + \kappa_v - \frac{1}{2}\Big)\Big)^2$$

if $v \in \Sigma_b$; and

$$L_v(s, \tau \times \tau \times \pi) = \Gamma_{\mathbb{C}}\Big(s + \kappa_v' + \kappa_v - \frac{1}{2}\Big) \cdot \Gamma_{\mathbb{C}}\Big(s + \kappa_v - \kappa_v' - \frac{1}{2}\Big) \cdot \Big(\Gamma_{\mathbb{C}}\Big(s + \kappa_v - \frac{1}{2}\Big)\Big)^2$$

if $v \in \Sigma_{ub}$. Moreover, for any $v \in \Sigma_F$,

$$L_v(s, \pi \otimes \omega_\tau) = \Gamma_{\mathbb{C}}\Big(s + \kappa_v - \frac{1}{2}\Big).$$

Since

$$L_\infty(s, \mathrm{Sym}^2\,\tau \times \pi) = \frac{L_\infty(s, \tau \times \tau \times \pi)}{L_\infty(s, \pi \otimes \omega_\tau)},$$

we get that

$$\begin{aligned} L_\infty(s, \mathrm{Sym}^2\,\tau \times \pi) = &\prod_{v \in \Sigma_\infty} \Gamma_{\mathbb{C}}\Big(s + \kappa_v - \frac{1}{2}\Big)\Gamma_{\mathbb{C}}\Big(s + \kappa_v + \kappa_v' - \frac{1}{2}\Big) \\ &\cdot \prod_{v \in \Sigma_{ub}} \Gamma_{\mathbb{C}}\Big(s + \kappa_v - \kappa_v' - \frac{1}{2}\Big) \cdot \prod_{v \in \Sigma_b} \Gamma_{\mathbb{C}}\Big(s + \kappa_v' - \kappa_v + \frac{1}{2}\Big). \end{aligned} \tag{4.2}$$

Then the completed L-function $L(s, \mathrm{Sym}^2\,\tau \times \pi)$ is given by

$$L(s, \mathrm{Sym}^2\,\tau \times \pi) = L_\infty(s, \mathrm{Sym}^2\,\tau \times \pi) L_{\mathrm{fin}}(s, \mathrm{Sym}^2\,\tau \times \pi).$$

It converges for $\mathrm{Re}(s) >> 0$ and has an analytic continuation to the whole complex plane. It satisfies a functional equation,

$$L(s, \mathrm{Sym}^2\,\tau \times \pi) = \varepsilon(s, \mathrm{Sym}^2\,\tau \times \pi) L(1 - s, \mathrm{Sym}^2\,\tau \times \pi),$$

where $\varepsilon(s, \mathrm{Sym}^2\,\tau \times \pi)$ is a function of exponential type, $ab^{s-\frac{1}{2}}$, for some constants $a \in \mathbb{C}^\times$ of absolute value one and $b \in \mathbb{R}$. Moreover, $\varepsilon(1/2, \mathrm{Sym}^2\,\tau \times \pi) \in \{\pm 1\}$.

4.3 THE REFINED GAN-GROSS-PRASAD FORMULA FOR $\widetilde{\mathrm{SL}_2} \times \mathrm{SL}_2$

The Gan-Gross-Prasad conjecture is a fundamental conjecture in the study of the problem of restriction of irreducible representations from a group G to a smaller group $G' \subset G$. We will state it in the case when G is the Jacobi group, $\mathrm{J} = \mathrm{SL}_2 \ltimes \mathrm{H}$, and $G' = \mathrm{SL}_2$. In this case the conjecture is referred to in the literature as the GGP conjecture for $\mathrm{SL}_2 \times \widetilde{\mathrm{SL}_2}$ and is known to be true ([Qiu14], Thm. 4.5).

Recall that F is a totally real number field with ring of adeles $\mathbb{A}$. Fix a nontrivial additive character $\psi = \otimes_v \psi_v$ of $F \backslash \mathbb{A}$ and let $\omega_\psi = \otimes_v \omega_{\psi_v}$ be the Schrödinger-Weil representation with respect to ψ of the double cover $\widetilde{\mathrm{J}}(\mathbb{A})$ of the Jacobi group realized on the Schwarz space $\mathcal{S}(\mathbb{A})$. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$; let $\sigma = \otimes_v \sigma_v \in \mathrm{Wd}_{\overline{\psi}}(\pi)$ be a genuine irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}_2(\mathbb{A})}$ in the Waldspurger packet of $\pi = \otimes \pi_v$ relative to $\overline{\psi}$. Then $\sigma \otimes \omega_\psi$ is an irreducible cuspidal automorphic representation of $\mathrm{J}(\mathbb{A})$. Let $\tau = \otimes_v \tau_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ whose central character ω_τ is unitary. We can view τ as a representation of $\mathrm{SL}_2(\mathbb{A})$ via the pullback $\mathrm{SL}_2(\mathbb{A}) \to \mathrm{GL}_2(\mathbb{A})$. Under this pullback τ decomposes as a direct sum of finitely many irreducible automorphic representations of $\mathrm{SL}_2(\mathbb{A})$ each appearing with multiplicity one (cf. [LL79]).

Let $\boldsymbol{g} = \otimes_v g_v \in \tau$ and $\boldsymbol{h} = \otimes_v h_v \in \sigma$ be factorizable automorphic forms. Let $\boldsymbol{\phi} = \otimes_v \phi_v \in \mathcal{S}(\mathbb{A})$ be a Schwarz function on $\mathbb{A}$. Let $\theta_{\boldsymbol{\phi}}(g) = \sum_{a \in F} (\omega_\psi(g)\boldsymbol{\phi})(a)$ be the theta function on $\widetilde{\mathrm{J}}(\mathbb{A})$ associated to $\boldsymbol{\phi}$ and ω_ψ and denote by $\boldsymbol{h} \otimes \boldsymbol{\phi}$ the automorphic form on $\mathrm{J}(\mathbb{A})$ defined by $\boldsymbol{h} \otimes \boldsymbol{\phi}(gh) = \boldsymbol{h}(g)\theta_{\boldsymbol{\phi}}(gh)$ for $g \in SL_2(\mathbb{A})$, $h \in \mathrm{H}(\mathbb{A})$.

Now consider the space, $\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{A})}(\tau \otimes \overline{\sigma \otimes \omega_\psi}, \mathbb{C})$, of $\mathrm{SL}_2(\mathbb{A})$-invariant linear forms on $\tau \otimes \overline{\sigma \otimes \omega_\psi}$. Furthermore, consider the canonical element – the global period functional – of this space defined by

$$\mathrm{P}(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{\phi}) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \boldsymbol{g}(\alpha)\overline{\boldsymbol{h} \otimes \boldsymbol{\phi}(\alpha)} d\alpha.$$

This is well-defined since the automorphic forms $\boldsymbol{g}$ and $\boldsymbol{h} \otimes \boldsymbol{\phi}$ are cuspidal. Then the GGP conjecture for $\widetilde{\mathrm{SL}_2} \times \mathrm{SL}_2$ asserts that:

Proposition. *For a fixed ψ, there is a unique $\sigma \in \mathrm{Wd}_{\overline{\psi}}(\pi)$ such that $\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{A})}(\tau \otimes \overline{\sigma \otimes \omega_\psi}, \mathbb{C}) \neq 0$. In this case,*

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(\mathbb{A})}(\tau \otimes \overline{\sigma \otimes \omega_\psi}, \mathbb{C}) = 1$$

and

$$\mathrm{P} \neq 0 \quad \textit{if and only if} \quad L(1/2, \mathrm{Sym}^2 \tau \times \pi) \neq 0.$$

The refined GGP conjecture makes (ii) more precise. For each place v, pick some inner product on σ_v, τ_v and ω_{ψ_v} so that for each of the representations $\sigma,\ \tau,\ \omega_\psi$, the product of the local inner products equals the global inner product. Let $\boldsymbol{g} = \otimes_v g_v$, $\boldsymbol{h} = \otimes_v h_v$ and $\boldsymbol{\phi} = \otimes_v \phi_v$ be factorizable vectors as before. Define for each place v of F the local period, P_v, by integration of matrix coefficients:

$$\mathrm{P}_v(g_v, h_v, \phi_v) = \int_{\mathrm{SL}_2(F_v)} \langle \tau_v(X) g_v, g_v \rangle \overline{\langle \sigma_v(X) h_v, h_v \rangle \langle \omega_{\psi_v}(X)\phi_v, \phi_v \rangle} dX.$$

By Lemma 4.3 in [Qiu14], P_v is absolutely convergent. We normalize it by setting

$$\mathrm{P}_v^\sharp(g_v, h_v, \phi_v) = \left(\frac{\xi_{F_v}(2) L(1/2, \mathrm{Sym}^2 \tau_v \times \pi_v)}{L(1, \pi_v, \mathrm{Ad}) L(1, \tau_v, \mathrm{Ad})} \right)^{-1} \frac{\mathrm{P}_v(g_v, h_v, \phi_v)}{\langle g_v, g_v \rangle \langle h_v, h_v \rangle \langle \phi_v, \phi_v \rangle}.$$

By Lemma 4.4 in [Qiu14], $\mathrm{P}_v^\sharp(g_v, h_v, \phi_v)$ equals 1 for all but finitely many places v, and hence the product

$$\prod_{v\in\Sigma_F} \mathrm{P}_v^\sharp(g_v, h_v, \phi_v)$$

is well-defined. The refined Gan-Gross-Prasad formula for $\mathrm{SL}_2 \times \widetilde{\mathrm{SL}_2}$ asserts that

Theorem. *For any factorizable vectors $\boldsymbol{g}$, $\boldsymbol{h}$ and $\boldsymbol{\phi}$,*

$$\frac{|\,\mathrm{P}(\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{\phi})|^2}{\langle \boldsymbol{g}, \boldsymbol{g}\rangle \langle \boldsymbol{h}, \boldsymbol{h}\rangle \langle \boldsymbol{\phi}, \boldsymbol{\phi}\rangle} = \frac{D_F^{-3/2}}{4} \times \frac{L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)}{L(1, \pi, \mathrm{Ad}) L(1, \tau, \mathrm{Ad})} \times \prod_{v\in\Sigma_F} \mathrm{P}_v^\sharp(g_v, h_v, \phi_v). \quad (4.3)$$

This is Theorem 4.5 in [Qiu14]; in [Qiu14], the first factor on the right is different due to different choice of measures.

Our main theorem (Theorem 4.4.4) is a formula for the central L-value, $L(1/2, \mathrm{Sym}^2 \tau \times \pi)$, that we will arrive at by explicitly computing the local integrals appearing in (4.3) for an appropriate choice of ψ, σ and the vectors $\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{\phi}$. For a fixed ψ there is at most one such choice by the Gan-Gross-Prasad conjecture. The representation-theoretic consequence of the dichotomy of the balanced and the unbalanced places will play a key role in determining the correct data, as we will see.

4.4 THE MAIN THEOREM

4.4.1 *The basic setup*

Let $\pi = \otimes \pi_v$ resp. $\tau = \otimes \tau_v$ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ of weight 2κ resp. $\kappa' + 1$. We will assume that both π and τ are of trivial levels, that is, at all non-archimedean places of F both π and τ are unramified principal series representations. Moreover, assume that $\kappa_v, \kappa_v' > 1$ at all places $v \in Sigma_\infty$.

Recall that Σ_{ub} and Σ_b are subsets of Σ_∞ whose union is Σ_∞, such that $\kappa_v > \kappa'_v$ for $v \in \Sigma_{ub}$ and $\kappa_v \leqslant \kappa'_v$ for $v \in \Sigma_b$.

Our goal is to find a formula for the central (critical) value of $L(s, \mathrm{Sym}^2 \tau \times \pi)$ via the refined Gan-Gross-Prasad formula (4.3). For this purpose, fix ψ to be the nontrivial additive character of $F\backslash\mathbb{A}$ whose infinity component is given by $x \mapsto \exp(2\pi i x)$ for any real place. For any non-archimedean place of F, let δ_v denote the conductor of ψ_v, that is, $\delta_v^{-1}\mathfrak{o}_v$ is the largest subgroup of F_v on which ψ_v is trivial. We also need to choose appropriately an automorphic representation $\sigma \in \mathrm{Wd}_{\overline{\psi}}(\pi)$ and vectors $\mathbf{g} \in \tau$, $\mathbf{h} \in \sigma$, $\boldsymbol{\phi} \in \omega_\psi$ so that the local integrals appearing in the GGP formula are nonzero. We will do so in the next section.

4.4.2 *Correct data for refined GGP*

We first make an obseravtion.

Lemma 4.4.1. *$L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi) = 0$, unless $|\kappa| \equiv |\Sigma_b| \pmod 2$.*

Proof. By ([HK91], §8),

$$\varepsilon_v\left(\frac{1}{2}, \tau \times \tau \times \pi\right) = \begin{cases} 1 & \text{if } v \in \Sigma_{ub} \\ -1 & \text{if } v \in \Sigma_b. \end{cases}$$

Also, $\varepsilon(\frac{1}{2}, \pi) = (-1)^{|\kappa|}$. Since

$$\varepsilon\left(\frac{1}{2}, \tau \times \tau \times \pi\right) = \varepsilon\left(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi\right)\varepsilon\left(\frac{1}{2}, \pi\right),$$

we see that if $|\kappa|$ is odd (resp. even) and $|\Sigma_b|$ is even (resp. odd), then $\varepsilon(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi) = -1$ and thus $L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi) = 0$. $\square$

Hence, let us assume that

$$|\kappa| \equiv |\Sigma_b| \pmod 2 \tag{4.4}$$

Now we will make our choices.

Suppose that $v \in \Sigma_b$.

- π_v is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of minimal weight $\pm 2\kappa_v$. Let σ_v be the discrete series representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of lowest weight $\kappa_v + \frac{1}{2}$ and h_v denote a lowest weight vector of σ_v.
- τ_v is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of lowest weight $\pm(\kappa'_v + 1)$. Let $g_v \in \tau_v$ be a minimal weight vector.

Suppose that $v \in \Sigma_{ub}$.

- π_v is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of minimal weight $\pm 2\kappa_v$. Let σ_v be the discrete series representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of highest weight $-(\kappa_v + \frac{1}{2})$ and h_v denote a highest weight vector of σ_v.
- τ_v is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of mimimal weight $\pm(\kappa'_v + 1)$. Let $g_v \in \tau_v$ be a highest weight vector.

Suppose that $v < \infty$.

- π_v is an unramified principal series representation of $\mathrm{PGL}_2(F_v)$. If $v \nmid 2$, then σ_v is an unramified principal series representation of $\widetilde{\mathrm{SL}_2(F_v)}$. Let $h_v \in \sigma_v$ be a $\mathrm{SL}_2(\mathfrak{o}_v)$-fixed vector. If $v|2$, then σ_v is a principal series representation and contains a distinguished vector h_v which will be defined in §6.2. Let $h_v^{(2)} = \sigma(m(2))h_v$ if $v|2$.
- τ_v is an unramified principal series representation of $\mathrm{PGL}_2(F_v)$. Let $g_v \in \tau_v$ be the $\mathrm{PGL}_2(\mathfrak{o}_v)$-fixed vector such that $g_v(1) = 1$. Let $g_v^{(\delta_v)} = \tau \begin{pmatrix} \delta_v & \\ & 1 \end{pmatrix} g_v$.

Put

$$\sigma = \otimes \sigma_v,$$

and

$$\boldsymbol{g}^{(\delta)} = (\otimes_{v|\infty} g_v) \otimes (\otimes_{v<\infty} g_v^{(\delta_v)}), \qquad \boldsymbol{h}^{(2)} = (\otimes_{v|2} h_v^{(2)}) \otimes (\otimes_{v\nmid 2} h_v).$$

Note that by our assumption (4.4), σ indeed lies in the Waldspurger packet $\mathrm{Wd}_{\overline{\psi}}(\pi)$ (cf. §3.5.4).

We will now fix a vector in the Schrödinger-Weil representation $\omega_\psi = \otimes \omega_{\psi_v}$. Let $\boldsymbol{\phi} = \otimes \phi_v \in \mathcal{S}(\mathbb{A})$ be the Schwarz function with

$$\phi_v = \begin{cases} \exp(-2\pi x^2) & \text{if } v \in \Sigma_\infty \\ \mathbb{1}_{\frac{1}{2}\mathfrak{o}_v} & \text{if } v|2 \\ \mathbb{1}_{\mathfrak{o}_v} & \text{otherwise} \end{cases}$$

Recall that

$$r_v = \begin{cases} \kappa_v - \kappa'_v - 1 & \text{if } v \in \Sigma_{ub} \\ \kappa'_v - \kappa_v & \text{if } v \in \Sigma_b. \end{cases}$$

Set

$$Y_+^{(r)} \boldsymbol{\phi} = (\otimes_{v\in\Sigma_\infty} Y_+^{r_v} \phi_v) \otimes (\otimes_{v<\infty} \phi_v),$$

where $Y_+ = \left(\frac{2}{\pi}\right) \cdot \mathrm{Y}_+$ and Y_+ is the element of $\mathrm{Lie}(\mathrm{J})$ as defined in §2.9 that acts as a weight raising differential operator, raising the weight by 1.

Remark. *By ([BS98], Remark 3.5.1), $\boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}$ is the adelization of $\Delta^{(r)} F_h$ mentioned in §1.3.*

Lemma 4.4.2. *We have,*

$$\langle Y_+^{(r)} \boldsymbol{\phi}, Y_+^{(r)} \boldsymbol{\phi} \rangle = \prod_{v\in\Sigma_\infty} \frac{(2r_v)!}{r_v! \pi^{r_v}}. \tag{4.5}$$

Proof. For a non-archimedean place v of odd residue characteristic it is not hard to see that $\langle \phi_v, \phi_v \rangle = 1$. Moreover, if $v|2$, then $\langle \phi_v, \phi_v \rangle = 2^d$, since $[\frac{1}{2}\mathfrak{o}_v : \mathfrak{o}_v] = 2^d$ and $\mathrm{Vol}(\mathfrak{o}_v) = 1$. Let $v \in \Sigma_\infty$. Then by Lemma 3.2.1 in [BS98], we have

$$\begin{aligned} Y_+^{r_v} \exp(-2\pi x^2) &= \left(\frac{2}{\pi}\right)^{r_v} \left(\frac{1}{2}\frac{d}{dx} - 2\pi x\right)^{r_v} \exp(-2\pi x^2) \\ &= \left(\frac{2}{\pi}\right)^{r_v} \cdot (-1)^{r_v} \left(\frac{\pi}{2}\right)^{r/2} H_{r_v}(\sqrt{2\pi}x) \exp(-2\pi x^2), \end{aligned}$$

where H_r is the rth Hermite polynomial. It is a fact that (cf. [GR00], §7.375, #1)

$$\int_{-\infty}^{\infty} H_r(y) H_r(y) e^{-2y^2} dy = \frac{(2r)!}{2^r r!} \sqrt{\frac{\pi}{2}}.$$

Therefore

$$\langle Y_+^{r_v} \phi_v, Y_+^{r_v} \phi_v \rangle = \frac{(2r_v)!}{2\pi^{r_v} r_v!}.$$

Since $\langle Y_+^{(r)} \boldsymbol{\phi}, Y_+^{(r)} \boldsymbol{\phi} \rangle = \prod_{v|\infty} \langle Y_+^{r_v} \phi_v, Y_+^{r_v} \phi_v \rangle \prod_{v<\infty} \langle \phi_v, \phi_v \rangle$, the lemma follows . □

4.4.3 *The Theorem*

To state concisely the next proposition we introduce some new notation. Let

$$(g'_v, h'_v, \phi'_v) = \begin{cases} (g_v, h_v, Y_+^{r_v} \phi_v) & \text{if } v \in \Sigma_\infty \\ (g_v^{(\delta_v)}, h_v^{(2)}, \phi_v) & \text{if } v|2 \\ (g_v^{(\delta_v)}, h_v, \phi_v) & \text{otherwise} \end{cases}$$

For each place v, pick some inner product on σ_v, τ_v and ω_v and define

$$\mathrm{P}_v(g'_v, h'_v, \phi'_v) = \int_{\mathrm{SL}_2(F_v)} \langle \tau_v(\alpha_v) g'_v, g'_v \rangle \overline{\langle \sigma_v(\alpha_v) h'_v, h'_v \rangle \langle \omega_{\psi_v}(\alpha_v) \phi'_v, \phi'_v \rangle} d\alpha_v.$$

By Lemma 4.3 in [Qiu14], P_v is a finite number. We normalize it by setting

$$\mathrm{P}_v^\sharp(g_v', h_v', \phi_v') = \left(\frac{\xi_{F_v}(2) L(1/2, \mathrm{Sym}^2 \tau_v \times \pi_v)}{L(1, \pi_v, \mathrm{Ad}) L(1, \tau_v, \mathrm{Ad})}\right)^{-1} \frac{\mathrm{P}_v(g_v', h_v', \phi_v')}{\langle g_v', g_v'\rangle \langle h_v', h_v'\rangle \langle \phi_v', \phi_v'\rangle}.$$

Note that in the notation introduced above,

$$\boldsymbol{g}^{(\delta)} = \otimes_v g_v', \quad \boldsymbol{h}^{(2)} = \otimes_v h_v', \quad Y_+^{(r)} \boldsymbol{\phi} = \otimes_v \phi_v'.$$

In §6.1 and §6.2 we'll show that,

Proposition 4.4.3.

$$\mathrm{P}_v^\sharp(g_v', h_v', \phi_v') = \begin{cases} \frac{2^{2r_v+1}}{r_v!} \pi^{r_v} & \text{if } v \in \Sigma_{ub} \\ \frac{1}{r_v!} \pi^{r_v} & \text{if } v \in \Sigma_b \\ 2^{-[F_v:\mathbb{Q}_2]} & \text{if } v | 2 \\ 1 & \text{otherwise} \end{cases}$$

We can now state and prove our main theorem.

Theorem 4.4.4. *With notation as in the previous two subsections, we have*

$$L\left(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi\right) = \frac{2^a D_F^2}{\prod_{v \in \Sigma_\infty} \binom{2r_v}{r_v}} \xi_F(2) |\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}\rangle|^2 \frac{L(1, \pi, \mathrm{Ad})}{\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)}\rangle}, \tag{4.6}$$

where $a = |\kappa' + 4|_{\Sigma_\infty} + |2\kappa' - 2\kappa + 1|_{\Sigma_{ub}} + 2$.

Proof. The theorem follows by combining Proposition 4.4.3, Lemma 4.4.2 and formula (4.1) with Theorem 4.3 wherein we take the factorizable vectors to be $\boldsymbol{g}^{(\delta)}$, $\boldsymbol{h}^{(2)}$ and $Y_+^{(r)} \boldsymbol{\phi}$. □

Corollary 4.4.5. $L\left(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi\right) \geqslant 0.$

Proof. Follows immediately from our theorem noting that $L(1, \pi, \mathrm{Ad}) \geqslant 0$. □

Remark. *The above corollary is already known ([Lap03]).*

5

AN APPLICATION: RATIONALITY OF $L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)$

To work out the rationality of $L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)$ we need to work out the rationality of the factors $|\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})} \rangle|^2$ and $\frac{L(1,\pi,\mathrm{Ad})}{\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)} \rangle}$ appearing in our formula (4.6). We will first consider the second factor.

Let $\sum_{\xi \in \mathcal{O}} c_h(\xi) e^{2\pi i \xi z}$ be the Fourier expansion of $\boldsymbol{h}^{(2)}$ (or more precisely, the half-integral weight modular form h whose adelization is $\boldsymbol{h}^{(2)}$). Fix a totally positive element $\xi \in F^\times$ such that the Fourier coefficient $c_h(\xi)$ of $\boldsymbol{h}^{(2)}$ is nonzero and then normalize $\boldsymbol{h}^{(2)}$ so that $c_h(\xi) = 1$. Furthermore, let $\eta \in \mathcal{O}_F^\times$ be a unit such that $\mathrm{N}_{F/\mathbb{Q}}(\eta) = (-1)^{|\kappa|}$. Since the automorphic representation generated by $\boldsymbol{h}^{(2)}$ is an element of $\mathrm{Wd}_{\overline{\psi}}(\pi)$, therefore the automorphic representation generated by $\boldsymbol{h}^{(2)} \otimes \chi_{-\eta}$ is an element of $\mathrm{Wd}_{\psi_\eta}(\pi)$. Clearly, $\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)} \rangle = \langle \boldsymbol{h}^{(2)} \otimes \chi_{-\eta}, \boldsymbol{h}^{(2)} \otimes \chi_{-\eta} \rangle$. We have the following generalization of the Kohnen-Zagier formula.

Proposition.

$$\frac{L(1, \pi, \mathrm{Ad})}{\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)} \rangle} = D_F^{1/2} 2^{-1+3|\kappa|} \zeta_F(2) L(\frac{1}{2}, \pi \otimes \chi_{\eta\xi}) \Psi_h(\eta\xi) \prod_{v \in \Sigma_\infty} \xi_v^{\kappa_v - \frac{1}{2}},$$

where $\chi_{\eta\xi}$ *is the character attached to* $F(\sqrt{\eta\xi})/F$ *and* $L(s, \pi \otimes \chi_{\eta\xi})$ *is the completed L-function of* $\pi \otimes \chi_{\eta\xi}$. $\Psi_h(\eta\xi)$ *is a complex number such that* $\Psi_h(\eta\xi)^\sigma = \Psi_{h^\sigma}(\eta\xi)$ *for any* $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Proof. This is Eqn. (12.1) in [HI13] with $\Psi_h(\eta\xi) = \prod_{v<\infty} \Psi_v(\eta\xi, \alpha_v)$. Note that $\Psi_h(\eta\xi)^\sigma = \Psi_{h^\sigma}(\eta\xi)$ follows from the definition ([HI13], Defn. 4.1) of Ψ_v. □

Next, put

$$\kappa_0 = \max_{v \in \Sigma_\infty} \{\kappa_v\} \quad \text{and} \quad \varepsilon = (\varepsilon_v) \in (\mathbb{Z}/2\mathbb{Z})^n \text{ with } \varepsilon_v = \kappa_0 + 1 \pmod 2. \tag{5.1}$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $r \in (\mathbb{Z}/2\mathbb{Z})^n$, let $u(r, \pi^\sigma)$ denote Shimura's u-invariant (see §C.1; here it is understood that $u(r, \pi^\sigma) = u(r, f^\sigma)$ where f is the modular form corresponding to π). Then it is well-known (cf. [Shi78], Thm. 4.3; see also [Shi87], Lemma 9.3) that

Proposition. *For any* $\sigma \in \mathrm{Aut}(\mathbb{C})$, *we have*

$$\left(\frac{L(\frac{1}{2}, \pi \otimes \chi_{\eta\xi})}{\pi^{|\kappa_0 - \kappa|} |\mathrm{N}_{F/\mathbb{Q}}\, \eta\xi|^{1/2} u(\varepsilon + \Sigma_{ub}, \pi)} \right)^\sigma = \frac{L(\frac{1}{2}, \pi^\sigma \otimes \chi_{\eta\xi})}{\pi^{|\kappa_0 - \kappa|} |\mathrm{N}_{F/\mathbb{Q}}\, \eta\xi|^{1/2} u(\varepsilon + \Sigma_{ub}, \pi^\sigma)}, \tag{5.2}$$

where ε *is as in* (5.1) *and for a subset* S *of* Σ_∞, $\varepsilon + S = ((\varepsilon + S)_v)$ *with*

$$(\varepsilon + S)_v = \begin{cases} \varepsilon_v + 1 & \text{if } v \in S \\ \varepsilon_v & \text{otherwise.} \end{cases}$$

$\mathrm{N}_{F/\mathbb{Q}}$ *denotes the norm map.*

We now have the following result on the rationality of $\frac{L(1,\pi,\mathrm{Ad})}{\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)} \rangle}$.

Proposition 5.0.1. *Let $\widetilde{F}$ be the Galois closure of F. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\widetilde{F})$,*

$$\left(\frac{L(1,\pi,\mathrm{Ad})}{\pi^{|\kappa_0-\kappa+1|}\langle \boldsymbol{h}^{(2)},\boldsymbol{h}^{(2)}\rangle u(\varepsilon+\Sigma_{ub},\pi)}\right)^{\sigma}=\frac{L(1,\pi^{\sigma},\mathrm{Ad})}{\pi^{|\kappa_0-\kappa+1|}\langle \boldsymbol{h}^{(2),\sigma},\boldsymbol{h}^{(2),\sigma}\rangle u(\varepsilon+\Sigma_{ub},\pi^{\sigma})}, \tag{5.3}$$

where ε is as in (5.1). Moreover, if π is of parallel weight, that is $\kappa_1=\kappa_2=\ldots\kappa_n$, then we can replace $\mathrm{Aut}(\mathbb{C}/\widetilde{F})$ *by* $\mathrm{Aut}(\mathbb{C})$.

Proof. Follows from the previous two propositions together with the well known fact that

$$D_F^{1/2}\xi_F(2)\in\pi^n\mathbb{Q}. \tag{5.4}$$

The second assertion of the proposition follows from the fact that in the Kohnen-Zagier formula $\prod_{v\in\Sigma_\infty}\xi_v^{\kappa_v}\in\mathbb{Q}$ if π is of parallel weight. □

Next, we will work out the rationality of the global period in the two extreme cases, the *purely balanced* and the *purely unbalanced*, and as a consequence derive the rationality of the central value $L(\frac{1}{2},\mathrm{Sym}^2\tau\times\pi)$ in these cases.

5.1 THE PURELY BALANCED CASE

We will now workout the rationality of the global period in the special case of $\Sigma_\infty=\Sigma_b$ and as a consequence the rationality of the central L-value in this case.

Note that, when $\Sigma_\infty=\Sigma_b$, the automorphic form $\boldsymbol{h}^{(2)}\otimes Y_+^{(r)}\boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}$ is a nearly holomorphic Hilbert modular form of integral weight by Lemma 3.4.1, since

$$X_-^{(r)}(\boldsymbol{h}^{(2)}\otimes Y_+^{(r)}\boldsymbol{\phi})=0.$$

Denote its holomorphic projection by $\boldsymbol{g}_0$. Then by ([Shi87], Prop. 9.4(i)),

$$\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})} \rangle = \langle \boldsymbol{g}^{(\delta)}, \boldsymbol{g}_0 \rangle.$$

The Fourier coefficients of $\boldsymbol{g}^{(\delta)}$ are all totally real since the central character of τ is trivial. Recall that we have normalized $\boldsymbol{h}^{(2)}$ so that $c_h(\xi) = 1$ for some totally positive element $\xi \in F^\times$. Therefore, the Fourier coefficients of $\boldsymbol{h}^{(2)}$ are totally real. Hence, by the rationality of the holomorphic projection (ref. [Shi87], Prop. 9.4(ii)), the Fourier coefficients of $\boldsymbol{g}_0$ are also totally real. Therefore,

$$|\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})} \rangle|^2 = \langle \boldsymbol{g}^{(\delta)}, \boldsymbol{g}_0 \rangle^2. \tag{5.5}$$

Proposition 5.1.1.

$$\sigma\left(\frac{|\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})} \rangle|^2}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle^2} \right) = \frac{|\langle \boldsymbol{g}^{(\delta),\sigma}, \boldsymbol{h}^{(2),\sigma} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})} \rangle|^2}{\langle \boldsymbol{g}^\sigma, \boldsymbol{g}^\sigma \rangle^2}.$$

Proof. Let $\boldsymbol{g}$ denote the newform $\otimes_v g_v$ with g_v's as defined in §4.4.2. Then we can write $\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{g}_0 \rangle$ as $\langle \boldsymbol{g}, \boldsymbol{g}_0' \rangle$ for some $\boldsymbol{g}_0'$ since we are working with unitary representations. Now, by ([Shi78], Prop. 4.15), for any $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\sigma\left(\frac{\langle \boldsymbol{g}, \boldsymbol{g}_0' \rangle}{\langle \boldsymbol{g}, \boldsymbol{g} \rangle} \right) = \frac{\langle \boldsymbol{g}^\sigma, \boldsymbol{g}_0'^\sigma \rangle}{\langle \boldsymbol{g}^\sigma, \boldsymbol{g}^\sigma \rangle}. \tag{5.6}$$

The proposition now follows from (5.5). □

Theorem 5.1.2. *Let π and τ be irreducible cuspidal automorphic representations of* $\mathrm{GL}_2(\mathbb{A})$ *of weights 2κ and $\kappa'+1$ respectively. Assume that both π and τ are of trivial levels. Let $\widetilde{F}$ be the Galois closure of F. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\widetilde{F})$,*

$$\sigma\left(\frac{L(\frac{1}{2},\mathrm{Sym}^2\tau\times\pi)}{\pi^{|\kappa_0-\kappa+2|}\langle \boldsymbol{g},\boldsymbol{g}\rangle^2 u(\varepsilon,\pi)}\right)=\frac{L(\frac{1}{2},\mathrm{Sym}^2\tau^\sigma\times\pi^\sigma)}{\pi^{|\kappa_0-\kappa+2|}\langle \boldsymbol{g}^\sigma,\boldsymbol{g}^\sigma\rangle^2 u(\varepsilon,\pi^\sigma)},$$

where $\boldsymbol{g}$ is the newform generating τ and $\varepsilon=(\varepsilon_v)\in(\mathbb{Z}/2\mathbb{Z})^n$ with $\varepsilon_v=\kappa_0+1$ (mod 2). Moreover, we can replace $\mathrm{Aut}(\mathbb{C}/\widetilde{F})$ by $\mathrm{Aut}(\mathbb{C})$ if π is of parallel weight.

In particular,

$$L(\frac{1}{2},\mathrm{Sym}^2\tau\times\pi)\sim_{\mathbb{Q}(\pi,\tau)} i^{|\kappa_0-1|}\pi\langle \boldsymbol{g},\boldsymbol{g}\rangle^2\langle \boldsymbol{h}^{(2)},\boldsymbol{h}^{(2)}\rangle,$$

where $Q(\pi,\tau)$ is the number field generated by the Fourier coefficients of the Hilbert modular forms corresponding to π and τ, and $\sim_{\mathbb{Q}(\pi,\tau)}$ means up to an element of $\mathbb{Q}(\pi,\tau)$.

Proof. The first assertion follows by combining Propositions 5.1.1, 5.0.1 and fact 5.4 with Thm. 4.4.4. The last assertion follows from the first and the fact that (cf. (5.8))

$$i^{|\kappa_0-1|}\pi^{|\kappa_0-\kappa-1|}u(\varepsilon,\pi)\sim_{\mathbb{Q}(\pi)}\langle \boldsymbol{h}^{(2)},\boldsymbol{h}^{(2)}\rangle.$$

□

5.2 THE PURELY UNBALANCED CASE

In this section we will work out the rationality of the factor $|\langle \boldsymbol{g}^{(\delta)},\boldsymbol{h}^{(2)}\otimes Y_+^{(r)}\boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}\rangle|^2$ appearing in our main formula in the special case of $\Sigma_\infty=\Sigma_{ub}$ and as a consequence work out the rationality of the central L-value in this case. In contrast with the purely balanced case, this time we will make use of the theory of nearly holomorphic Hilbert modular forms of *half-integral* weight.

Notice that, when $\Sigma_\infty = \Sigma_{ub}$, $\boldsymbol{g}^{(\delta)}$ and $\boldsymbol{h}^{(2)}$ are anti-holomorphic. Let $\widehat{\boldsymbol{g}^{(\delta)}}$ and $\widehat{\boldsymbol{h}^{(2)}}$ be the holomorphic forms corresponding to $\boldsymbol{g}^{(\delta)}$ and $\boldsymbol{h}^{(2)}$. Then it is not hard to see that,

$$\langle \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \boldsymbol{g}^{(\delta)} \rangle = \langle \widehat{\boldsymbol{g}^{(\delta)}} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \widehat{\boldsymbol{h}^{(2)}} \rangle,$$

where $\widehat{\boldsymbol{g}^{(\delta)}} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}$ is now nearly holomorphic of half-integral weight, again by Lemma 3.4.1. Let $\boldsymbol{h}_0$ be its holomorphic projection. Then, by ([Shi87], Prop. 9.4),

$$\langle \widehat{\boldsymbol{g}^{(\delta)}} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \widehat{\boldsymbol{h}^{(2)}} \rangle = \langle \boldsymbol{h}_0, \widehat{\boldsymbol{h}^{(2)}} \rangle.$$

By the reasoning similar to that in the purely balanced case, the Fourier coefficients of $\widehat{\boldsymbol{g}^{(\delta)}}, \widehat{\boldsymbol{h}^{(2)}}, \boldsymbol{h}_0$ are all totally real. Therefore,

$$|\langle \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \boldsymbol{g}^{(\delta)} \rangle|^2 = |\langle \widehat{\boldsymbol{g}^{(\delta)}} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \widehat{\boldsymbol{h}^{(2)}} \rangle|^2 = \langle \boldsymbol{h}_0, \widehat{\boldsymbol{h}^{(2)}} \rangle^2. \tag{5.7}$$

Proposition 5.2.1. *For all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{|\langle \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \boldsymbol{g}^{(\delta)} \rangle|^2}{\pi^{2|\kappa_0 - \kappa - 1|} u(\varepsilon, \pi)^2} \right) = \frac{|\langle \boldsymbol{h}^{(2),\sigma} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}, \boldsymbol{g}^{(\delta),\sigma} \rangle|^2}{\pi^{2|\kappa_0 - \kappa - 1|} u(\varepsilon, \pi^\sigma)^2}.$$

Proof. By ([Shi87], Thm. 10.5), for all $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$\sigma\left(\frac{\langle \boldsymbol{h}_0, \widehat{\boldsymbol{h}^{(2)}} \rangle}{i^{|\kappa_0 - 1|} \pi^{|\kappa_0 - \kappa - 1|} u(\varepsilon, \pi)} \right) = \frac{\langle \boldsymbol{h}_0^\sigma, (\widehat{\boldsymbol{h}^{(2)}})^\sigma \rangle}{i^{|\kappa_0 - 1|} \pi^{|\kappa_0 - \kappa - 1|} u(\varepsilon, \pi^\sigma)} \tag{5.8}$$

The proposition now follows from (5.7). □

Theorem 5.2.2. *Let π and τ be irreducible cuspidal automorphic representations of* $\mathrm{GL}_2(\mathbb{A})$ *of weights 2κ and $\kappa' + 1$ respectively. Assume that both π and τ are of trivial levels. Let $\widetilde{F}$ be the Galois closure of F. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\widetilde{F})$,*

$$\sigma\left(\frac{L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)}{\pi^{|\kappa_0 - \kappa + 1|}\langle \boldsymbol{f}, \boldsymbol{f}\rangle u(\varepsilon, \pi)}\right) = \frac{L(\frac{1}{2}, \mathrm{Sym}^2 \tau^\sigma \times \pi^\sigma)}{\pi^{|\kappa_0 - \kappa + 1|}\langle \boldsymbol{f}^\sigma, \boldsymbol{f}^\sigma\rangle u(\varepsilon, \pi^\sigma)},$$

where $\boldsymbol{f}$ is the newform generating π and $\varepsilon = (\varepsilon_v) \in (\mathbb{Z}/2\mathbb{Z})^n$ with $\varepsilon_v = \kappa_0 + 1 \pmod 2$. Moreover, we can replace $\mathrm{Aut}(\mathbb{C}/\widetilde{F})$ by $\mathrm{Aut}(\mathbb{C})$ if π is of parallel weight.

In particular,

$$L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi) \sim_{\mathbb{Q}(\pi)} i^{|\kappa_0 - 1|}\langle \boldsymbol{f}, \boldsymbol{f}\rangle\langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)}\rangle,$$

where $Q(\pi)$ is the number field generated by the Fourier coefficients of the Hilbert modular form corresponding to π and $\sim_{\mathbb{Q}(\pi)}$ means up to an element of $Q(\pi)$.

Proof. It is well-known that (cf. [Shi87], Thm 10.2(II))

$$\sigma\left(\frac{u(\varepsilon, \pi)u(1+\varepsilon, \pi)}{(2\pi i)^{|2\kappa - 2\kappa_0 + 1|}\langle \boldsymbol{f}, \boldsymbol{f}\rangle}\right) = \frac{u(\varepsilon, \pi^\sigma)u(1+\varepsilon, \pi^\sigma)}{(2\pi i)^{|2\kappa - 2\kappa_0 + 1|}\langle \boldsymbol{f}^\sigma, \boldsymbol{f}^\sigma\rangle}. \tag{5.9}$$

The first assertion now follows by combining Propositions 5.2.1, 5.0.1, Eqn. (5.9) and fact 5.4 with Thm. 4.4.4. The last assertion follows from the first by noting that (5.8) implies

$$i^{|\kappa_0 - 1|}\pi^{|\kappa_0 - \kappa - 1|}u(\varepsilon, \pi) \sim_{\mathbb{Q}(\pi)} \langle \boldsymbol{h}^{(2)}, \boldsymbol{h}^{(2)}\rangle.$$

□

Remark. *Comparing Propositions 5.1.1 and 5.2.1 on the rationality of the (square of the) global period $|\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}|_{\mathrm{SL}_2(\mathbb{A})}\rangle|^2$ in the purely balanced and the purely unbalanced cases, we see that the periods in the two cases are different. This results in different periods of $L(\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)$ in these two cases.*

5.3 A CONJECTURE FOR THE GENERAL CASE

Let f resp. g be Hilbert newforms of weight $2\kappa = (2\kappa_v)_v$ resp. $\kappa' + 1 = (\kappa'_v + 1)_v \in \mathbb{Z}^{\Sigma_\infty}$ and $L(s, \mathrm{Sym}^2 g \times f)$ be the completed (motivic) L-function of $\mathrm{Sym}^2 g \times f$. Let $\pi = \otimes_v \pi_v$ resp. $\tau = \otimes_v \tau_v$ be the automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ generated by f resp. g and $L(s, \mathrm{Sym}^2 \tau \times \pi)$ be the completed (automorphic) L-function of $\mathrm{Sym}^2 \tau \times \pi$.

Lemma 5.3.1. *We have,*

$$L(s, \mathrm{Sym}^2 g \times f) = L(s - \kappa_0 - \kappa'_0 + \frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi).$$

Proof. Let $\{\alpha_{1,\mathfrak{p},f}, \alpha_{2,\mathfrak{p},f}\}$ and $\{\alpha_{1,\mathfrak{p},\pi}, \alpha_{2,\mathfrak{p},\pi}\}$ be the Satake parameters of f and π, respectively. It is well known (cf. [RT11], Thm. 4.16) that $L_{\mathrm{fin}}(s, f) = L_{\mathrm{fin}}(s - \frac{2\kappa_0 - 1}{2}, \pi)$. Therefore, by comparing the Euler product expansions, we can see that $\alpha_{i,\mathfrak{p},f} = (N\mathfrak{p})^{(2\kappa_0 - 1)/2} \alpha_{i,\mathfrak{p},\pi}$. Analogously, $\alpha_{i,\mathfrak{p},g} = N\mathfrak{p}^{\kappa'_0/2} \alpha_{i,\mathfrak{p},\tau}$. The lemma now follows from the definition of $L_{\mathrm{fin}}(s, \mathrm{Sym}^2 \tau \times \pi)$ (cf. §4.2) and the analogous definition for $L_{\mathrm{fin}}(s, \mathrm{Sym}^2 g \times f)$. □

By Lemma 5.3.1 and definition B.1.1 of a critical point for $L(s, \mathrm{Sym}^2 g \times f)$, a critical point for $L(s, \mathrm{Sym}^2 \tau \times \pi)$ must be a half-integer $m + \frac{1}{2}$ such that neither $L_\infty(s, \mathrm{Sym}^2 \tau \times \pi)$ nor $L_\infty(1 - s, \mathrm{Sym}^2 \tau \times \pi)$ has a pole at $s = m + \frac{1}{2}$. Put

$$t^0 = \min_{v \in \Sigma_{ub}} \{r_v\}, \quad a^0 = \min_{v \in \Sigma_b, 2\kappa_v > \kappa'_v} \{r_v\}, \quad b^0 = \min_{v \in \Sigma_b, \kappa' \geqslant 2\kappa} \{\kappa_v - 1\}.$$

Proposition 5.3.2. *The set of critical points for $L(s, \mathrm{Sym}^2 \tau \times \pi)$ is*

$$\left\{ m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2} \mid -\min\{a^0, b^0, t^0\} \leqslant m \leqslant \min\{a^0, b^0, t^0\} \right\}$$

In particular, $s = \frac{1}{2}$ is a critical point.

Proof. Recall that (Eqn. (4.2)),

$$L_\infty(s, \mathrm{Sym}^2 \tau \times \pi) = \prod_{v \in \Sigma_\infty} \Gamma_\mathbb{C}\Big(s + \kappa_v - \frac{1}{2}\Big)\Gamma_\mathbb{C}\Big(s + \kappa_v + \kappa_v^{'} - \frac{1}{2}\Big) \cdot \prod_{v \in \Sigma_{ub}} \Gamma_\mathbb{C}\Big(s + \kappa_v - \kappa_v^{'} - \frac{1}{2}\Big) \cdot \prod_{v \in \Sigma_b} \Gamma_\mathbb{C}\Big(s + \kappa_v^{'} - \kappa_v + \frac{1}{2}\Big)$$

and $\Gamma_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. Also recall that,

$$r_v = \begin{cases} \kappa_v - \kappa_v' - 1 & \text{if } v \in \Sigma_{ub} \\ \kappa_v' - \kappa_v & \text{if } v \in \Sigma_b. \end{cases}$$

Now, let $m + \frac{1}{2}$ be a critical point of $L(s, \mathrm{Sym}^2 \tau \times \pi)$. For $L_\infty(s, \mathrm{Sym}^2 \tau \times \pi)$ not to have a pole at $s = m + \frac{1}{2}$, m must satisfy the following three conditions:

- $m > -r_v - 1$ for all $v \in \Sigma_{ub}$.
- $m > -r_v - 1$ for all $v \in \Sigma_b$ for which $2\kappa_v > \kappa_v'$.
- $m > -\kappa_v$ for all $v \in \Sigma_b$ for which $\kappa' \geqslant 2\kappa$.

Similarly, for $L_\infty(1 - s, \mathrm{Sym}^2 \tau \times \pi)$ not to have a pole at $s = m + \frac{1}{2}$, m must satisfy the following three conditions:

- $m < r_v + 1$ for all $v \in \Sigma_{ub}$.
- $m < r_v + 1$ for all $v \in \Sigma_b$ for which $2\kappa_v > \kappa_v'$.
- $m < \kappa_v$ for all $v \in \Sigma_b$ for which $\kappa' \geqslant 2\kappa$.

The lemma now follows by taking the intersection of the above two sets of conditions.

□

In §1.2 we stated an explicit conjecture for the rationality of the critical values of $L_{\text{fin}}(s, \mathrm{Sym}^2 g \times f)$. We will now restate Conj. 1.2 in terms of automorphic representations. In fact, we will do so for the completed L-function, $L(s, \mathrm{Sym}^2 \tau \times \pi)$. By (4.2),

$$
\begin{aligned}
L_\infty\left(m+\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi\right) &= \prod_{v \in \Sigma_\infty} \Gamma_{\mathbb{C}}(m+\kappa_v)\Gamma_{\mathbb{C}}(m+\kappa_v+\kappa'_v) \cdot \prod_{v \in \Sigma_{ub}} \Gamma_{\mathbb{C}}(m+\kappa_v-\kappa'_v) \\
&\quad \cdot \prod_{v \in \Sigma_b} \Gamma_{\mathbb{C}}(m+\kappa'_v-\kappa_v+1) \\
&\quad \sim_{\mathbb{Q}^\times} (2\pi)^{-|2m+2\kappa+\kappa'|_{\Sigma_\infty}+|\kappa'-\kappa-m|_{\Sigma_{ub}}+|\kappa-\kappa'-m-1|_{\Sigma_b}}
\end{aligned}
\tag{5.10}
$$

Therefore, by Conj. 1.2 we have,

Conjecture. *If $m+\frac{1}{2} \in \mathbb{Z}+\frac{1}{2}$ is a critical point of the automorphic L-function $L(s, \mathrm{Sym}^2 \tau \times \pi)$, then, for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$
\left(\frac{L(m+\frac{1}{2}, \mathrm{Sym}^2 \tau \times \pi)}{\pi^* i^{|m+\kappa_0|+|\Sigma_b|} u(\varepsilon, \pi) \nu^{\Sigma_{ub}}(\pi) \nu^{\Sigma_b}(\tau)^2}\right)^\sigma = \frac{L(m+\frac{1}{2}, \mathrm{Sym}^2 \tau^\sigma \times \pi^\sigma)}{\pi^* i^{|m+\kappa_0|+|\Sigma_b|} u(\varepsilon, \pi^\sigma) \nu^{\Sigma_{ub}}(\pi^\sigma) \nu^{\Sigma_b}(\tau^\sigma)^2}, \tag{5.11}
$$

where

$$
* = |\kappa_0 - \kappa + 1| + |\Sigma_b|,
$$

$\varepsilon = (\varepsilon_v)$ with $\varepsilon_v = m + \kappa_0 + 1 \pmod 2$ for all v, and $u(\cdot\,,\,\cdot)$ and $\nu^{\cdot}(\,\cdot\,)$ are respectively, Shimura's u-invariant and Harris' period defined in ([Har90], §1).

When $m = 0$, Prop. 5.0.1 together with properties of Shimura's invariants (cf. Appendix C) suggest that the above conjecture is equivalent to

Conjecture. *For every $S \subset \Sigma_\infty, \varepsilon \in (\mathbb{Z}/2\mathbb{Z})^S$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ there exists a complex number $P(\pi^\sigma, S, r)$ such that, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\left|\frac{\langle \boldsymbol{g}^{(\delta)}, \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}\rangle}{P(\pi, \Sigma_{ub}, \varepsilon)\nu^{\Sigma_b}(\tau)}\right|^\sigma = \left|\frac{\langle (\boldsymbol{g}^{(\delta)})^\sigma, (\boldsymbol{h}^{(2)})^\sigma \otimes Y_+^{(r)} \boldsymbol{\phi}\rangle}{P(\pi^\sigma, \Sigma_{ub}, \varepsilon)\nu^{\Sigma_b}(\tau^\sigma)}\right|.$$

The complex number $P(\pi^\sigma, S, r)$ should be thought of as a cohomological interpretation of Shimura's P-invariant defined in [Yos95].

Remark (Brief outline of a strategy for proving the above conjecture)**.** *There are two key ingredients:*

(i) Defining the complex number $P(\pi, S, r)$.

(ii) Proving a result similar to Prop. 2.6 of [Har90].

For (i), let $\sigma \otimes \omega_\psi$ be the automoprhic representation of $\mathrm{J}(\mathbb{A})$ corresponding to π (with respect to a fixed character ψ). Now consider the subspace of $\sigma \otimes \omega_\psi$ of automorphic forms skew-holomoprhic at places in S and holomorphic at others. This space admits a Whittaker model which carries a natural rational structure. On the other hand, this space has a cohomological realization via cohomology of automorphic vector bundles (on mixed Shimura varieties) and this endows it with another rational structure. $P(\pi, S, r)$ should then be defined by playing-off these rational structures against each other (r should correspond to ψ).

Prop. 2.6 of [Har90] gives a formula for the integral of the product of three automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ (modified by certain weight-raising differential operators to ensure that the integral is not trivially zero) each of which is holomorphic at some places and anti-holomorphic at others. The places at which they are anti-holomorphic are moreover mutually disjoint and their union is the set of all archimedean places. This situation is quite similar to our situation where we are considering the integral $\langle \boldsymbol{h}^{(2)} \otimes Y_+^{(r)} \boldsymbol{\phi}, \boldsymbol{g}^{(\delta)}\rangle$ of the product of an automorphic form on $\mathrm{J}(\mathbb{A})$ and an automorphic form on $\mathrm{GL}_2(\mathbb{A})$ with similar holomorphy conditions on archimedean places as in the proposition of [Har90]. To prove such a result in our situation *we would first need to prove a result similar to Proposition-Construction 2.3 of [Har90] by considering automorphic vector bundles on Shimura variety for* GL_2 *and (mixed) Shimura variety for* J.

In the remainder of this thesis we will prove Proposition 4.4.3.

Part III

EVALUATION OF THE LOCAL INTEGRALS

6

PROOF OF PROP. 4.4.3

6.1 COMPUTATIONS AT THE ARCHIMEDEAN PLACES

6.1.1 *The Unbalanced Case*

Let $v \in \Sigma_{ub}$. For simplicity we will suppress all subscripts v throughout this section. Recall (ref. §4.4.2) the following notation at v.

- τ is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of mimimal weight $\pm(\kappa' + 1)$ and $g \in \tau$ is a highest weight vector. Furthermore, let $\widehat{\tau}$ denote the contragradient of τ and $\widehat{g} \in \widehat{\tau}$ be the (holomorphic) vector corresponding to g.
- σ is a discrete series representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of highest weight $-(\kappa + \frac{1}{2})$ and $h \in \sigma$ is a highest weight vector. Furthermore, let $\widehat{\sigma}$ denote the contragradient of σ and $\widehat{h} \in \widehat{\sigma}$ be the (holomorphic) vector corresponding to h.
- ψ is the additive character of $\mathbb{R}$ given by $x \mapsto e^{2\pi i x}$. Furthermore, ω_ψ is the Schrödinger-Weil representation of $\mathrm{SL}_2(\mathbb{R})$ and $\phi = e^{-2\pi i x^2} \in \omega_\psi$ is a lowest weight vector of weight $1/2$.
- We have $\kappa > \kappa'$ and $r = \kappa - \kappa' - 1$.

The goal of this subsection is to compute

$$P^{\sharp}(g,h,\phi) = \left(\frac{\xi_F(2)L(1/2, \mathrm{Sym}^2 \tau \times \pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})} \right)^{-1} \frac{P(g,h,\phi)}{\langle g,g\rangle \langle h,h\rangle \langle \phi,\phi\rangle},$$

where

$$P(g,h,Y_+^r\phi) = \int_{\mathrm{SL}_2(\mathbb{R})} \langle \tau(X)g,g\rangle \overline{\langle \sigma(X)h,h\rangle \langle \omega_\psi(X)Y_+^r\phi, Y_+^r\phi\rangle} dX.$$

Proposition 6.1.1. *We have,*

$$P^{\sharp}(g,h,Y_+^r\phi) = \frac{2^{2r_v+1}}{r_v!}\pi^{r_v}.$$

Firstly, note that we can rewrite

$$P(g,h,Y_+^r\phi) = \int_{\mathrm{SL}_2(\mathbb{R})} \overline{\langle \widehat{\tau}(X)\widehat{g},\widehat{g}\rangle} \langle \widehat{\sigma}(X)\widehat{h},\widehat{h}\rangle \overline{\langle \omega_\psi(X)Y_+^r\phi, Y_+^r\phi\rangle} dX,$$

Now, the idea of the proof of the above proposition is to realize the holomorphic discrete series $\widehat{\sigma}$ of weight $\kappa + \frac{1}{2}$ as a subrepresentation of $\widehat{\tau} \otimes \omega_\psi|_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}$.

We make use of explicit models for the representations $\widehat{\tau}$ and ω_ψ. For the full description of these models we refer the readers to (3.1) and ([BS98], Prop. 3.2.3), respectively. Let the vector space underlying the model for ω_ψ be $\bigoplus_{j \in \mathbb{N}_0} \mathbb{C} v_j$. Then the action of Z and X_- is given by

$$Z\,v_j = (j + \frac{1}{2})v_j, \quad X_-\,v_j = \pi j(j-1)v_{j-2}. \tag{6.1}$$

$\widehat{\tau}$ is a discrete series representation of lowest weight $\kappa' + 1$. The vector space underlying the model for $\widehat{\tau}$ is $DS_{\kappa'+1} = \bigoplus_{i \in 2\mathbb{N}_0} \mathbb{C} u_i$ and the action of Z and X_- is given by (cf. 3.2)

$$Z\,u_i = (i + \kappa' + 1)u_i, \quad X_-\,u_i = -\frac{i}{2}u_{i-2}. \tag{6.2}$$

Tensoring the representations $\widehat{\tau}$ and $\omega_\psi|_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}$, we get a (genuine) representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ whose underlying vector space is

$$D_{\kappa'+\frac{3}{2}} = \bigoplus_{l\in 2\mathbb{N}_0,\ k\in\mathbb{N}_0} \mathbb{C}u_l \otimes v_k,$$

and the action of Z and X_- can be calculated using (6.1) and (6.2);

$$\mathrm{Z}(u_i \otimes v_j) = (i+j+\kappa'+\frac{3}{2})u_i \otimes v_j, \qquad \mathrm{X}_-(u_i \otimes v_j) = -\frac{i}{2}u_{i-2} \otimes v_j + \pi j(j-1)u_i \otimes v_{j-2}.$$

There is an inner product on $D_{\kappa'+\frac{3}{2}}$ such that $(u_i \otimes v_j)$'s form an orthogonal basis. Denote this inner product by $\langle -, - \rangle$ and write $\langle v, v \rangle = ||v||^2$. Then by ([BS98], p. 46-47) it follows

$$||u_i \otimes v_{j+1}||^2 = 2\pi(j+1)||u_i \otimes v_j||^2, \quad (i+2\kappa')||u_i \otimes v_j||^2 = i||u_{i-2} \otimes v_j||^2.$$

We may normalize the inner product so that $||u_0 \otimes v_t|| = 1$. Then for any $2 \leqslant i \leqslant r$, i even, we have

$$||u_i \otimes v_{r-i}||^2 = (2\pi)^{-i} \prod_{\substack{0\leqslant l\leqslant i-2 \\ l\in 2\mathbb{N}_0}} \frac{(i-l)}{(i+2\kappa'-l)(r-i+l+1)(r-i+l+2)}. \tag{6.3}$$

The space

$$D_{\kappa'+\frac{3}{2}}(r) = \bigoplus_{i+j=r, i\in 2\mathbb{N}_0, j\in\mathbb{N}_0} \mathbb{C}u_i \otimes v_j$$

is the largest subspace on which Z acts by $r + \kappa' + \frac{3}{2} = \kappa + \frac{1}{2}$.

Lemma 6.1.2. *There is a unique (upto a scalar) vector* v_t^{hol} *in* $D_{\kappa'+\frac{3}{2}}(r)$ *with the property that* $X_- \, v_t^{\text{hol}} = 0$*. It is given by*

$$v_r^{\text{hol}} = \sum_{\substack{0 \leqslant i \leqslant r \\ i \in 2\mathbb{N}_0}} c_i u_i \otimes v_{r-i}; \quad c_0 = 1, \quad c_i = (2\pi)^{i/2} \prod_{\substack{0 \leqslant l \leqslant i-2 \\ l \in 2\mathbb{N}_0}} \frac{(r-l)(r-l-1)}{(i-l)}, \quad (i \geqslant 2).$$

In particular,

$$||v_r^{\text{hol}}||^2 = 2^{-r} \frac{\binom{2\kappa-2}{r}}{\binom{\kappa-1}{r}} \tag{6.4}$$

Proof. Suppose that

$$v_r^{\text{hol}} = \sum_{\substack{0 \leqslant i \leqslant r \\ i \in 2\mathbb{N}_0}} c_i u_i \otimes v_{r-i}$$

and $X_- \, v_r^{\text{hol}} = 0$. Then by the formula for action of X_-, we conclude that for any $0 \leqslant i \leqslant r-2$ and i even, we have

$$-c_{i+2}\frac{(i+2)}{2} = \pi(r-i)(r-i-1)c_i.$$

Let $c_0 = 1$. Then we may recursively solve for c_i's. Furthermore,

$$\begin{aligned} ||v_r^{\text{hol}}||^2 &= \sum_{\substack{0 \leqslant i \leqslant r \\ i \in 2\mathbb{N}_0}} |c_i|^2 ||u_i \otimes v_{r-i}||^2 \\ &= 1 + \sum_{\substack{2 \leqslant i \leqslant r \\ i \in 2\mathbb{N}_0}} \prod_{\substack{0 \leqslant l \leqslant i-2 \\ l \in 2\mathbb{N}_0}} \frac{(r-l)^2(r-l-1)^2}{(i-l)(i+2\kappa-2r-l-2)(r-i+l+1)(r-i+l+2)} \end{aligned}$$

The lemma now follows from Lemma A.0.3 given in the appendix. □

Thus, we realize $\widehat{\sigma}$ as a subrepresentation of $\widehat{\tau} \otimes \omega_\psi|_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}$ generated by v_r^{hol}. We may assume that the inner product on $\widehat{\sigma}$ is given by the restriction of that of $\widehat{\tau} \otimes \omega_\psi|_{\widetilde{\mathrm{SL}_2(\mathbb{R})}}$. Since $\mathrm{P}_v^\sharp(g, h, \phi)$ does not change if we replace h, ϕ or g by a scalar multiple of them, we may assume that $h = v_r^{\text{hol}}$ and $g \otimes Y_+^r \phi = u_0 \otimes v_r$.

Now we can prove the proposition. The orthogonal projection of $u_0 \otimes v_r$ to the line generated by v_r^{hol} is $||v_r^{\mathrm{hol}}||^{-2} v_r^{\mathrm{hol}}$. It follows that

$$P(g,h,Y_+^r\phi) = \frac{1}{||v_r^{\mathrm{hol}}||^4} \int_{\mathrm{SL}_2(\mathbb{R})} |\langle \widehat{\sigma}(X) v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}} \rangle|^2 dX.$$

As $\widehat{\sigma}$ is the discrete series representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with lowest weight $\kappa + \frac{1}{2}$, it is known that (ref. [Xue18], Lemma 5.2),

$$|\langle \widehat{\sigma}(\mathrm{diag}[e^t, e^{-t}]) v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}} \rangle| = ||v_r^{\mathrm{hol}}||^2 \times (\cosh t)^{-(\kappa+\frac{1}{2})}, \quad r \geqslant 0.$$

Let $X = k_1 \,\mathrm{diag}[e^t, e^{-t}] k_2$ be the Cartan decomposition. Then $dX = 2\pi \sinh 2t dt dk_1 dk_2$, where dk_1, dk_2 are the measure on $\mathrm{SO}_2(\mathbb{R})$ so that its volume is one and dt is the usual Lebesgue measure on $\mathbb{R}$. Therefore

$$\mathrm{P}_v^\sharp(g,h,Y_+^r\phi) = \left(\frac{\xi(2) L(1/2, \mathrm{Sym}^2 \tau \times \pi)}{L(1,\pi,\mathrm{Ad}) L(1,\tau,\mathrm{Ad})} \right)^{-1} \frac{\int_0^\infty (\cosh t)^{-(2\kappa+1)} 2\pi \sinh 2t dt}{||v_r^{\mathrm{hol}}||^2}. \tag{6.5}$$

By definition,

$$\begin{aligned} \frac{\xi(2) L(1/2, \mathrm{Sym}^2 \tau \times \pi)}{L(1,\pi,\mathrm{Ad}) L(1,\tau,\mathrm{Ad})} &= \frac{\pi^{-1}\Gamma(1) \cdot 2^2 (2\pi)^{-2\kappa} \Gamma(\kappa+\kappa')\Gamma(\kappa-\kappa') 2(2\pi)^{-\kappa}\Gamma(\kappa)}{2(2\pi)^{-\kappa'-1}\Gamma(\kappa'+1)\pi^{-1}\Gamma(1) \cdot 2^2(2\pi)^{-2\kappa}\Gamma(2\kappa)\pi^{-1}\Gamma(1)} \\ &= (2\pi)^{1-r} \frac{t!}{2\kappa-1} \frac{\binom{\kappa-1}{r}}{\binom{2\kappa-2}{r}}. \end{aligned}$$

Moreover (cf. [GR00], §2.433, #11),

$$\int_0^\infty (\cosh t)^{-(2\kappa+1)} 2\pi \sinh 2t dt = 4\pi (2\kappa-1)^{-1}.$$

Proposition 6.1.1 is now proved by plugging (6.4) and the formulae above into (6.5).

6.1.2 *The Balanced Case*

Let $v \in \Sigma_b$. For simplicity we will suppress all subscripts v throughout this section. Recall (ref. §4.4.2) the following notation at v.

- τ is a discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ of mimimal weight $\pm(\kappa'+1)$ and $g \in \tau$ is a lowest weight vector.
- σ is a discrete series representation of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of lowest weight $\kappa+\frac{1}{2}$ and $h \in \sigma$ is a lowest weight vector.
- ψ is the additive character of $\mathbb{R}$ given by $x \mapsto e^{2\pi i x}$. Furthermore, ω_ψ is the Schrödinger-Weil representation of $\mathrm{SL}_2(\mathbb{R})$ and $\phi = e^{-2\pi i x^2} \in \omega_\psi$ is a lowest weight vector of weight $1/2$.
- We have $\kappa' \geqslant \kappa$ and $r = \kappa' - \kappa$.

The goal of this subsection is to compute

$$\mathrm{P}^\sharp(g,h,\phi) = \left(\frac{\xi_F(2)L(1/2,\mathrm{Sym}^2\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\right)^{-1}\frac{\mathrm{P}(g,h,\phi)}{\langle g,g\rangle\langle h,h\rangle\langle\phi,\phi\rangle},$$

where

$$\mathrm{P}(g,h,Y_+^r\phi) = \int_{\mathrm{SL}_2(\mathbb{R})}\langle\tau(X)g,g\rangle\overline{\langle\sigma(X)h,h\rangle\langle\omega_\psi(X)Y_+^r\phi,Y_+^r\phi\rangle}dX.$$

Proposition 6.1.3. *We have,*

$$\mathrm{P}_v^\sharp(g_v,h_v,Y_+^{r_v}\phi_v) = \frac{1}{r_v!}\pi^{r_v}.$$

This proposition has been proved in ([Xue19],§3.4) (except that there the rational part of $\mathrm{P}_v^\sharp$ is not explicitly computed). We include the proof here for completeness. The proof is similar to the one in the unbalanced case except that this time we will realize the *integral* weight representation $\tau_v|_{SL_2(\mathbb{R})}$ as a subrepresentation of $\sigma_v \otimes \omega_{\psi_v}|_{SL_2(\mathbb{R})}$.

We make use of an explicit model for the representation $\omega_\psi \otimes \sigma$ of $J(\mathbb{R})$. For the full description of this model, we refer the readers to ([BS98], Prop. 3.1.7). We denote this model by $D_{\kappa+1}$. As a vector space,

$$D_{\kappa+1} = \bigoplus_{k,l \geqslant 0,\ l \text{ even}} \mathbb{C} v_{k,l},$$

and the action of Z and X_- is given by

$$Z v_{k,l} = (\kappa + 1 + k + l) v_{k,l}, \quad X_- v_{k,l} = \pi k(k-1) v_{k-2,l} - \frac{l}{2}(\kappa - \frac{1}{2} + \frac{l}{2}) v_{k,l-2}.$$

There is an inner product on $D_{\kappa+1}$ for which the vectors $\{v_{k,l}\}$ form an orthogonal basis. Denote this inner product by $\langle -, - \rangle$. Then by ([BS98],p. 46-47) we have

$$||v_{k,l+2}||^2 = \frac{l+2}{2}(\kappa + \frac{l+1}{2})||v_{k,l}||^2, \quad ||v_{k+1,l}||^2 = 2\pi(k+1)||v_{k,l}||^2.$$

We may normalize the inner product so that $||v_{r,0}|| = 1$. Then for any $2 \leqslant l \leqslant r$, l even, we have

$$||v_{r-l,l}||^2 = (4\pi)^{-l} \prod_{\substack{0 \leqslant j \leqslant l-2 \\ j \in 2\mathbb{N}_0}} \frac{(j+2)(2\kappa + j + 1)}{(r-j)(r-j-1)}. \tag{6.6}$$

The

$$D_{\kappa+1}(r) = \bigoplus_{k+l=r, l even} \mathbb{C} v_{k,l}$$

is the largest subspace on which Z acts by the scalar $\kappa' + 1$.

Lemma 6.1.4. *There is a unique (up to a scalar) vector v_r^{hol} in $D_{\kappa+1}(r)$ with the property that $\mathrm{X}_- v = 0$. It is given by*

$$\sum_{\substack{0\leqslant l\leqslant r\\ l\in 2\mathbb{N}_0}} c_l v_{r-l,l}, \quad c_0 = 1, \quad c_l = (2\pi)^{l/2} \prod_{\substack{0\leqslant j\leqslant l-2\\ j\in 2\mathbb{N}_0}} \frac{(r-j)(r-j-1)}{(j+2)(2\kappa+j+1)}, \quad (l \geqslant 2).$$

Moreover,

$$||v_r^{\mathrm{hol}}||^2 = 2^r \frac{\binom{r+\kappa-1}{r}}{\binom{r+2\kappa-1}{r}} \tag{6.7}$$

Proof. Suppose that

$$v_r^{\mathrm{hol}} = \sum_{\substack{0\leqslant l\leqslant r\\ l\in 2\mathbb{N}_0}} c_l v_{r-l,l}$$

and $\mathrm{X}_- v = 0$. Then by the formula for the action of X_-, we conclude that for any $0 \leqslant l \leqslant r-2$ and l even, we have

$$c_{l+2}\frac{l+2}{2}(\kappa - \frac{1}{2} + \frac{l+2}{2}) = \pi(r-l)(r-l-1)c_l.$$

Let $c_0 = 1$. Then we may recursively solve for c_l's. Furthermore,

$$\begin{aligned} ||v_r^{\mathrm{hol}}||^2 &= \sum_{\substack{0\leqslant l\leqslant r\\ l\in 2\mathbb{N}_0}} |c_l|^2 ||v_{r-l,l}||^2 \\ &= 1 + \sum_{\substack{2\leqslant l\leqslant r\\ l\in 2\mathbb{N}_0}} \prod_{\substack{0\leqslant j\leqslant l-2\\ j\in 2\mathbb{N}_0}} \frac{(r-j)(r-j-1)}{(j+2)(2\kappa+j+1)}. \end{aligned}$$

The lemma now follows from Lemma A.0.2 given in the appendix. □

Thus, we realize τ as a subrepresentation of $\sigma \otimes \omega_\psi|_{\mathrm{SL}_2(\mathbb{R})}$ generated by v_r^{hol}. We may assume that the inner product on τ is given by the restriction of that of $\sigma \otimes \omega_\psi$. Since $\mathrm{P}_v^\sharp(g, h, Y_+^r\phi)$ does not change if we replace h, ϕ or g by a scalar multiple of them, we may assume that $g = v_r^{\mathrm{hol}}$ and $Y_+^r\phi \otimes h = v_{r,0}$.

Now we can prove the proposition. The orthogonal projection of $v_{r,0}$ to the line generated by v_r^{hol} is $||v_r^{\mathrm{hol}}||^{-2}v_r^{\mathrm{hol}}$. By Schur orthogonality it follows that,

$$\mathrm{P}_v(g,h,Y_+^r\phi) = \frac{1}{||v_r^{\mathrm{hol}}||^4}\int_{\mathrm{SL}_2(\mathbb{R})} |\langle \tau(X)v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}}\rangle|^2 dX.$$

As τ is the discrete series representation of $\mathrm{SL}_2(\mathbb{R})$ with lowest weight $\kappa'+1$, it is well known that (cf. [Xue18], Lemma 5.2)

$$|\langle \tau(\mathrm{diag}[e^t,e^{-t}])v_r^{\mathrm{hol}}, v_r^{\mathrm{hol}}\rangle| = ||v_r^{\mathrm{hol}}||^2 \times (\cosh t)^{-(\kappa'+1)}, \quad t \geqslant 0.$$

Let $X = k_1\,\mathrm{diag}[e^t,e^{-t}]k_2$ be the Cartan decomposition. Then $dX = 2\pi \sinh 2t dt dk_1 dk_2$ where dk_1, dk_2 are the measure on $\mathrm{SO}_2(\mathbb{R})$ so that its volume is one and dt is the usual Lebesgue measure on $\mathbb{R}$. Therefore

$$\mathrm{P}_v^\sharp(g,h,Y_+^r\phi) = \left(\frac{\xi(2)L(1/2,\mathrm{Sym}^2\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}\right)^{-1}\frac{1}{||v_r^{\mathrm{hol}}||^2}\int_0^\infty (\cosh t)^{-(2\kappa'+2)}2\pi\sinh 2t dt. \tag{6.8}$$

By definition,

$$\begin{aligned}\frac{\xi(2)L(1/2,\mathrm{Sym}^2\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})} &= \frac{\pi^{-1}\Gamma(1)\cdot 2^2(2\pi)^{-2\kappa'-1}\Gamma(\kappa'+\kappa)\Gamma(\kappa'-\kappa+1)2(2\pi)^{-\kappa}\Gamma(\kappa)}{2(2\pi)^{-\kappa'-1}\Gamma(\kappa'+1)\pi^{-1}\Gamma(1)\cdot 2^2(2\pi)^{-2\kappa}\Gamma(2\kappa)\pi^{-1}\Gamma(1)} \\ &= (2\pi)^{1-r}\frac{r!}{r+\kappa}\frac{\binom{r+2\kappa-1}{r}}{\binom{r+\kappa-1}{r}}.\end{aligned}$$

Moreover (cf. [GR00], §2.433, #11),

$$\int_0^\infty (\cosh t)^{-(2\kappa'+2)}2\pi\sinh 2t dt = 4\pi(2\kappa')^{-1}.$$

Proposition 6.1.3 is now proved by plugging (6.7) and the formulae above into (6.8).

6.2 COMPUTATION AT NONARCHIMEDEAN PLACES

6.2.1 *Odd residue characteristic places*

Recall the following notation. For simplicity, we suppress all subscripts v.

- F is a nonarchimedaen local field with ring of integers $\mathfrak{o}$, uniformizer ϖ and size of residue field $q = \#\mathfrak{o}/(\varpi)$.
- ψ is a nontrivial additive character of F with conductor δ, i.e $\delta^{-1}\mathfrak{o}$ is the largest subgroup of F on which ψ is trivial. For each $a \in F^\times$, we denote by $\gamma_\psi(a)$ the Weil constant.
- τ is an unramified principal series representation of $\mathrm{PGL}_2(F)$ and contains a $\mathrm{PGL}_2(\mathfrak{o})$-fixed element g such that $g(1) = 1$. Let

$$g^{(\delta)} = \tau \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} g.$$

- $B(F)$ is the Borel subgrop of $\mathrm{SL}_2(F)$ consisting of upper triangular matrices of the form $n(b)m(a)$, where

$$m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}; \quad (a \in F^\times,\ b \in F)$$

Let $\widetilde{B(F)}$ be the inverse image of $B(F)$ in $\widetilde{\mathrm{SL}_2(F)}$. Then $\sigma = \mathrm{Ind}_{\widetilde{B(F)}}^{\widetilde{\mathrm{SL}_2(F)}} \chi_{\psi,s}$ is the unramified principal series representation of $\widetilde{\mathrm{SL}_2(F)}$ induced from the character

$$\chi_{\psi,s}((n(b)m(a), \varepsilon) = \varepsilon \frac{\gamma_\psi(1)}{\gamma_\psi(a)} |a|^s.$$

of $\widetilde{B(F)}$ for some purely imaginary complex number s. Thus the underlying vector space $\widetilde{I}_\psi(s)$ of σ is the space of complex-valued functions f on $\widetilde{\mathrm{SL}_2(F)}$ such that

$$f((n(b)m(a),\varepsilon)g) = \varepsilon\frac{\gamma_\psi(1)}{\gamma_\psi(a)}|a|^{s+1}f(g) \qquad (g \in \widetilde{\mathrm{SL}_2(F)}),$$

and $\widetilde{\mathrm{SL}_2(F)}$ acts by right translation on $\widetilde{I}_\psi(s)$. It is a unitary representation with inner product

$$(f, f') = \int_{x\in F} f(w(1)n(x))\overline{f'(w(1)n(x))}dx,$$

where $w(1) = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. Moreover, σ contains an $\mathrm{SL}_2(\mathfrak{o})$-fixed vector h.

- ω_ψ is the Schrödinger-Weil representation of $\widetilde{\mathrm{J}(F)}$.

In this subsection we prove the following:

Proposition 6.2.1.

$$\langle g^{(\delta)}, g^{(\delta)}\rangle^{-1}\langle h,h\rangle^{-1}\langle \phi,\phi\rangle^{-1}\int_{\mathrm{SL}_2(F)}\langle \tau(X)g^{(\delta)}, g^{(\delta)}\rangle\overline{\langle\sigma(X)h,h\rangle\langle\omega_\psi(X)\phi,\phi\rangle}dX$$

$$= \xi_F(2)\frac{L(1/2, \mathrm{Sym}^2\,\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}.$$

We first explain how to reduce the proposition to the case $\delta = 1$. To stress the dependence of σ on ψ, we temporarily denote it by σ_ψ. For any $a \in F^\times$, we write

ψ_a for the character $x \mapsto \psi(ax)$. Note that the conductor of $\psi_{\delta^{-1}}$ is one. Define an automorphism r_δ of $\widetilde{\mathrm{SL}_2(F)}$ as

$$r_\delta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon\right) = \begin{cases} \left(\begin{pmatrix} a & \delta^{-1}b \\ \delta c & d \end{pmatrix}, \epsilon\right) & c \neq 0 \\ \left(\begin{pmatrix} a & \delta^{-1}b \\ & d \end{pmatrix}, (\delta, d)\epsilon\right) & c = 0 \end{cases}$$

Define a map

$$\imath_\delta : I_\psi(s) \to I_{\psi_{\delta^{-1}}}(s), \quad h' \mapsto h'^\delta, \quad h'^\delta(g) = h'(r_\delta(g)).$$

Then,

- $\imath_\delta \circ \sigma_{\psi(r_\delta(g))} = \sigma_{\psi_{\delta^{-1}}}(g) \circ \imath_\delta$
- h^δ is fixed by $\mathrm{SL}_2(\mathfrak{o})$.
- $\omega_\psi \circ r_\delta = \omega_{\psi_{\delta^{-1}}}$
- $(h', h') = |\delta|^{2s+3}(\imath_\delta h', \imath_\delta h')$.

These properties are not hard to verify. In the left-hand side of the proposition, we now make a change of variable $g \mapsto r_\delta(g)$. Then we are reduced to the case of $\delta = 1$. Thus we may assume that $\delta = 1$. Proposition 6.2.1 then follows from ([Qiu14], Lemma 4.4).

6.2.2 *Dyadic places*

Recall the following notation. For simplicity, we suppress all subscripts v.

- F is a degree d extension of $\mathbb{Q}_2$ with ring of integers $\mathfrak{o}$, uniformizer ϖ and $q = \#\mathfrak{o}/(\varpi)$. Let e be the integer such that $|2|^{-1} = q^e$.

- ψ is a nontrivial additive character of F with conductor δ, i.e, $\delta^{-1}\mathfrak{o}$ is the largest subgroup of F on which ψ is trivial; for each $a \in F^\times$, $\gamma_\psi(a)$ denotes the Weil constant. ω_ψ is the (local) Schrödinger-Weil representation of $\widetilde{\mathrm{J}(F)}$ with inner product
$$\langle \phi_1, \phi_2 \rangle = \int_F \phi_1(y)\overline{\phi_2(y)}dy \qquad (\phi_1, \phi_2 \in \mathcal{S}(F)).$$
Our chosen vector in ω_ψ is $\phi = \mathbb{1}_{\frac{1}{2}\mathfrak{o}_v}$.
- $\tau = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(|\cdot|^{s'}, |\cdot|^{-s'})$ is the unramified principal series representation of $\mathrm{GL}_2(F)$ induced from the character of $B(F)$,
$$\begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto |ad^{-1}|^{s'}$$
for some purely imaginary complex number s'. It contains a $\mathrm{GL}_2(\mathfrak{o})$-fixed element g such that $g(1) = 1$. Moreover,
$$g^{(\delta)} = \tau \begin{pmatrix} \delta & \\ & 1 \end{pmatrix} g.$$
- $\sigma = \mathrm{Ind}_{\widetilde{B(F)}}^{\widetilde{\mathrm{SL}_2(F)}} \chi_{\psi,s}$ is a principal series representation of $\widetilde{\mathrm{SL}_2(F)}$ acting on the space $\widetilde{I}_\psi(s)$ with definition same as the one given in §6.2. Note that this time $\mathrm{SL}_2(\mathfrak{o})$ is not a subgroup of $\widetilde{\mathrm{SL}_2(F)}$, so σ cannot have an $\mathrm{SL}_2(\mathfrak{o})$-fixed vector, but it does contain a distinguished vector defined below.

We also introduce some new notation.

- For any fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F, put
$$\Gamma[\mathfrak{a}, \mathfrak{b}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, d \in \mathfrak{o}, b \in \mathfrak{a}, c \in \mathfrak{b} \right\}.$$

For any subgroup Γ of $\mathrm{SL}_2(F)$, we denote by $\widetilde{\Gamma}$ its inverse image in $\widetilde{\mathrm{SL}_2(F)}$. Fix $\Gamma = \Gamma[\delta^{-1}\mathfrak{o}, 4\delta\mathfrak{o}]$. There exists a genuine character $\varepsilon : \widetilde{\Gamma} \to \mathbb{C}^\times$ such that $\omega_\psi(\gamma)\phi_0 = \varepsilon(\gamma)^{-1}\phi_0$, where $\gamma \in \widetilde{\Gamma}$ and ϕ_0 is the characteristic function of $\mathfrak{o}$ ([HI13], Lemma 1.1). Consider the Hecke algebra $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}(\widetilde{\Gamma}\backslash\widetilde{\mathrm{SL}_2(F)}/\widetilde{\Gamma}; \varepsilon)$ which is the space of compactly supported genuine functions ϕ on $\widetilde{\mathrm{SL}_2(F)}$ such that $\phi(\gamma_1\widetilde{X}\gamma_2) = \varepsilon(\gamma_1)\varepsilon(\gamma_2)\phi(\widetilde{X})$, for $\gamma_1, \gamma_2 \in \widetilde{\Gamma}$ and $\widetilde{X} \in \widetilde{\mathrm{SL}_2(F)}$ together with a convolution product. The Hecke algebra $\widetilde{\mathcal{H}}$ acts on the space $\widetilde{I}_\psi(s)$ by

$$\sigma(\phi)f(\widetilde{X}) = \int_{\widetilde{\mathrm{SL}_2(F)}} f(\widetilde{X}\widetilde{X}')\phi(\widetilde{X}')d\widetilde{X}'.$$

- Define

$$E^K(\widetilde{X}) = \begin{cases} q^e\langle\phi_0, \omega_\psi(\gamma)\phi_0\rangle & \text{if } \widetilde{X} \in \widetilde{\Gamma}[(4\delta\mathfrak{o})^{-1}, 4\delta\mathfrak{o}] \\ 0 & \text{otherwise.} \end{cases}$$

  Then by ([HI13], §3 and §6), E^K is an idempotent in $\widetilde{\mathcal{H}}$.

- By Prop. 4.6 in [HI13], there is a unique (up to scalars) element $h \in \sigma$ fixed by the idempotent E^K. Its restriction to $\widetilde{K} = \widetilde{\mathrm{SL}_2(\mathfrak{o})}$ is given by (cf. [HI13], Prop. 4.4 and p. 1982)

$$h(\widetilde{X}) = \langle\omega_\psi(\widetilde{X}w(2))\phi_0, \phi_0\rangle,$$

  where $w(2) = \begin{pmatrix} & -2^{-1} \\ 2 & \end{pmatrix}$. Put

$$h^{(2)} = \sigma(m(2))h,$$

  where $m(2) = \begin{pmatrix} 2 & \\ & 2^{-1} \end{pmatrix}$.

The goal of this section is to prove the following.

Proposition 6.2.2. *For $g^{(\delta)}, h^{(2)}$ and ϕ as above, we have*

$$\langle g^{(\delta)}, g^{(\delta)}\rangle^{-1}\langle h^{(2)}, h^{(2)}\rangle^{-1}\langle \phi,\phi\rangle^{-1}\int_{\mathrm{SL}_2(F)}\langle \tau(X)g^{(\delta)}, g^{(\delta)}\rangle\overline{\langle \sigma(X)h^{(2)}, h^{(2)}\rangle\langle \omega_\psi(X)\phi,\phi\rangle}dX$$

$$= 2^{-d}\xi_F(2)\frac{L(1/2,\mathrm{Sym}^2\,\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}.$$

As in the $v \nmid 2$ case, we may and will assume that $\delta = 1$. Now, because of the isomorphism between the Kohnen plus space of half-integral weight modular forms and the space of Jacobi forms ([HI13]), we can write the integral $\int_{\mathrm{SL}_2(F)}\langle \tau(X)g, g\rangle\overline{\langle \sigma(X)h^{(2)}, h^{(2)}\rangle\langle \omega_\psi(X)\phi,\phi\rangle}dX$ as

$$\int_{\mathrm{SL}_2(F)}\langle \tau(X)g, g\rangle\overline{\langle \sigma\otimes\omega_\psi(X)\Phi,\Phi\rangle}dX$$

for some $\mathrm{J}(\mathfrak{o})$-fixed vector Φ. For the purposes of computation we will construct Φ explicitly. In fact, this construction has already been done in the classical language of modular forms in ([HI13],§13); we will recast that construction in the language of automorphic forms.

For $\lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}$, put $\phi_\lambda = \mathbb{1}_{\lambda+\mathfrak{o}} \in \mathcal{S}(F)$. Moreover, for any $\lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}$, let $h_\lambda \in \widetilde{I}_\psi(s)$ be such that

$$h_\lambda(\widetilde{X}) = \langle \omega_\psi(\widetilde{X})\phi_\lambda, \phi_0\rangle \qquad (\widetilde{X}\in\widetilde{K}).$$

Consider the subspace of $\mathcal{S}(F)$ spanned by the orthonormal vectors $\{\phi_\lambda | \lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}\}$. By Proposititon 3.3 in [HI13], the action of $\widetilde{K}$ by ω_ψ on this space gives an irreducible representation of $\widetilde{K}$ which we denote by ω_ψ^K. It follows that the subspace of $\widetilde{I}_\psi(s)$ spanned by $\{h_\lambda | \lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}\}$ is also invariant under the action of $\widetilde{K}$ by σ. We denote this representation by σ^K. Noting that $|\frac{1}{2}\mathfrak{o}/\mathfrak{o}| = 2^d$, it follows from Schur's orthogonality that $\{2^{d/2}h_\lambda | \lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}\}$ is an orthonormal basis of σ^K. Let $\iota : \omega_\psi^K \longrightarrow \sigma^K$ be the homomorphism with $\imath(\phi_\lambda) = 2^{d/2}h_\lambda$ for all $\lambda \in \frac{1}{2}\mathfrak{o}/\mathfrak{o}$. Then it is clear that $\imath$ is an isomorphism of representations of $\widetilde{K}$. Moreover, it is an isometry.

Lemma 6.2.3. *1. We have,*

$$2^{d/2}\gamma_\psi(1)h^{(2)} = \sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} h_\lambda.$$

2. We have,

$$\int_K \sigma(k)h^{(2)} \otimes \omega_\psi(k)\phi dk = 2^{-d/2}\gamma_\psi(1)^{-1} \sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} h_\lambda \otimes \phi_\lambda.$$

3. The element

$$\sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} h_\lambda \otimes \phi_\lambda \in \sigma \otimes \omega_\psi$$

is $\mathrm{J}(\mathfrak{o})$*-fixed.*

Proof. The first assertion follows from the observations,

$$\omega_\psi(w(1))\phi_0 = 2^{-d/2}\gamma_\psi(1)^{-1} \sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} \phi_\lambda \quad \text{and} \quad \sigma(w(1))h_0 = h^{(2)}.$$

For the second assertion, we note that

$$\phi = \sum_\lambda \phi_\lambda.$$

Moreover, the representations σ^K and $\overline{\omega_\psi^K}$ are dual to each other and $\{2^{d/2}h_\lambda\}$ and $\{\phi_\lambda\}$ form a dual basis. Together with Schur's orthogonality relation, we find that

$$\left\langle \int_K \sigma(k)h^{(2)} \otimes \omega_\psi(k)\phi dk, h_\lambda \otimes \phi_\mu \right\rangle = \begin{cases} 2^{-d} & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

The same relation holds for the right hand side of the assertion as well. This proves (2).

By (2), $\sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} h_\lambda \otimes \phi_\lambda$ is K-invariant. It is $H(\mathfrak{o})$-invariant by the the formulae (3.6) of the Schrödinger-Weil representation. This completes the proof of the lemma. □

The representation $\sigma \otimes \omega_\psi$ is isomorphic to the principle series representation ([BS98], Thm. 5.4.2)

$$\mathrm{Ind}_{B_{\mathrm{J}}(F)}^{\mathrm{J}(F)} \psi|\cdot|^s = \{f : \mathrm{J}(F) \longrightarrow \mathbb{C} \mid f(m(a)n(b)(0,\mu,\xi)g) = \psi(\xi)|a|^{s+3/2} f(g),\ g \in \mathrm{J}(F)\},$$

where the isomorphism is given by

$$h' \otimes \phi \mapsto (gu \mapsto h'(g)\omega_\psi(gu)\phi(0)); \quad h' \in \sigma, \phi \in \mathcal{S}(F), g \in \mathrm{SL}_2(F), u \in \mathrm{H}(F). \tag{6.9}$$

We denote this representation by ρ and the underlying space of functions by $I(s,\psi)$. For a suitable choice of inner product on ρ, $\rho \simeq \sigma \otimes \omega_\psi$ is an isometry. We will often identify ρ with $\sigma \otimes \omega_\psi$.

Put

$$\Phi = \sum_{\lambda\in\frac{1}{2}\mathfrak{o}/\mathfrak{o}} h_\lambda \otimes \phi_\lambda \in I(s,\psi).$$

Then Φ is $\mathrm{J}(\mathfrak{o})$-fixed by the lemma above. We have,

Lemma 6.2.4.

$$\int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \sigma(X)h^{(2)}, h^{(2)}\rangle \langle \omega_\psi(X)\phi, \phi\rangle} dX = 2^{-d} \int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \rho(X)\Phi, \Phi\rangle} dX.$$

Proof. Let $X = k_1 \operatorname{diag}[\varpi^t, \varpi^{-t}]k_2$ where $k_1, k_2 \in K$ and t is a nonnegative integer be the Cartan decomposition of $X \in \mathrm{SL}_2(F)$. We will denote $\operatorname{diag}[\varpi^t, \varpi^{-t}]$ by t. By Lemma 6.2.3, we have

$$\begin{aligned}
&\int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \sigma(X)h^{(2)}, h^{(2)}\rangle \langle \omega_\psi(X)\phi, \phi\rangle} dX \\
&= \int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \rho(X)(h^{(2)} \otimes \phi), h^{(2)} \otimes \phi\rangle} dX \\
&= \int_t \langle \tau(t)g, g\rangle \left(\int_K \int_K \overline{\langle \rho(tk_2)(h^{(2)} \otimes \phi), \rho(k_1^{-1})(h^{(2)} \otimes \phi)\rangle} dk_1 dk_2 \right) dt, \qquad (g \text{ is } K\text{-fixed}) \\
&= 2^{-d/2} \int_t \langle \tau(t)g, g\rangle \left(\int_K \overline{\langle \rho(tk_2)(h^{(2)} \otimes \phi), \Phi\rangle} dk_2 \right) dt, \qquad (\text{Lemma 6.2.3 (ii)}) \\
&= 2^{-d} \int_t \langle \tau(t)g, g\rangle \overline{\langle \rho(t)\Phi, \Phi\rangle} dt \\
&= 2^{-d} \int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \rho(X)\Phi, \Phi\rangle} dX \qquad (\mathrm{Vol}(K) = 1 \text{ and } \Phi \text{ is also } K\text{-fixed})
\end{aligned}$$

□

We will now compute

$$\int_{\mathrm{SL}_2(F)} \langle \tau(X)g, g\rangle \overline{\langle \rho(X)\Phi, \Phi\rangle} dX. \tag{6.10}$$

The computation is similar to that done in §6 of [Xue18]. Let

$$\eta = (1, 0, 0) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \in \mathrm{J}(F).$$

Let

$$T_{s', s, \psi} = \int_{\mathrm{SL}_2(F)} g(X) \overline{\Phi(\eta X)} dX.$$

Then by ([Xue17], §4), we have

$$(6.10) = \frac{\xi_F(2)}{\xi_F(1)^2} T_{s',s,\psi} T_{-s',-s,\psi^{-1}} \tag{6.11}$$

Therefore, it is enough to calculate $T_{s',s,\psi}$.

Lemma 6.2.5. *The support of Φ is contained in $B_R(F)R(\mathfrak{o})$.*

Proof. Substitute F for $\mathbb{Q}_2$ and $\mathfrak{o}$ for $\mathbb{Z}_2$ in the proof of ([Xue18], Lemma 6.4). □

Let $X = m(a)n(x)k$ be the usual Iwasawa decomposition for $X \in \mathrm{SL}_2(F)$. Then $dX = d^\times a dx dk$, where $d^\times a$ is the measure on $F^\times$ such that *vol* $\mathfrak{o}^\times = 1$. Then $g(X) = |a|^{s'+1}$. Moreover,

$$\Phi(\eta m(a)n(x)) = \begin{cases} |a|^{-s-\frac{3}{2}} \mathbb{1}_{\mathfrak{o}}(a^{-1}) & |x| \leqslant 1 \\ |ax|^{-s-\frac{3}{2}} \mathbb{1}_{\mathfrak{o}}(a^{-1}x^{-1})\psi(a^{-2}x^{-1}) & |x| > 1. \end{cases}$$

The case $|x| \leqslant 1$ is straightforward. The case $|x| > 1$ relies on the following decomposition of matrices,

$$\eta m(a)n(x) = (1,0,0)m(a^{-1})w(1)n(x) = (1,0,0)m(a^{-1})m(x^{-1})n(x) \begin{pmatrix} 1 & \\ x^{-1} & 1 \end{pmatrix}.$$

Therefore,

$$T_{s',s,\psi} = \int_{|x|\leqslant 1} \int_{|a|\geqslant 1} |a|^{-\frac{1}{2}-s+s'} d^\times a dx + \int_{|x|>1} \int_{|ax|\geqslant 1} |a|^{-\frac{1}{2}-s+s'} |x|^{-\frac{3}{2}-s} \psi(a^{-2}x^{-1}) dx d^\times a.$$

We have

$$\int_{|x|\leqslant 1} \int_{|a|\geqslant 1} |a|^{-\frac{1}{2}-s+s'} d^\times a dx = \sum_{n=0}^{n=\infty} q^{n(-\frac{1}{2}-s+s')} = \frac{1}{1-q^{-\frac{1}{2}-s+s'}},$$

and

$$\int_{|x|>1}\int_{|ax|\geqslant 1}|a|^{-\frac{1}{2}-s+s'}|x|^{-\frac{3}{2}-s}\psi(a^{-2}x^{-1})dxd^{\times}a$$

$$=\sum_{m=1}^{\infty}\sum_{n=-m}^{\infty}q^{n(-\frac{1}{2}-s+s')+m(-\frac{3}{2}-s)}\int_{|x|=q^m}\int_{|a|=q^n}\psi(a^{-2}x^{-1})dxd^{\times}a.$$

Since

$$\int_{x\in\mathfrak{o}^{\times}}\psi(\xi x)dx=\begin{cases}1-q^{-1} & \xi\in\mathfrak{o}\\ -q^{-1} & \mathrm{ord}(\xi)=-1\\ 0 & \text{otherwise,}\end{cases}$$

we see that

$$\int_{|x|=q^m}\int_{|a|=q^n}\psi(a^{-2}x^{-1})dxd^{\times}a=\begin{cases}q^{m-1}(q-1) & 2n+m\geqslant 0\\ -q^{m-1} & 2n+m=-1\\ 0 & \textit{otherwise}.\end{cases}$$

Therefore,

$$\int_{|x|>1}\int_{|ax|\geqslant 1}|a|^{-\frac{1}{2}-s+s'}|x|^{-\frac{3}{2}-s}\psi(a^{-2}x^{-1})dxd^{\times}a$$

$$=q^{-1}(q-1)\sum_{m\geqslant 1}\sum_{n\geqslant -\lceil\frac{m}{2}\rceil}q^{n(-\frac{1}{2}-s+s')+m(-\frac{1}{2}-s)}-q^{-1}\sum_{m\geqslant 1, m\text{ odd}}q^{\frac{m+1}{2}(\frac{1}{2}+s-s')+m(-\frac{1}{2}-s)}$$

$$=\frac{-1+q-q^{\frac{1}{2}+s}+q^{1+s'}}{q^{\frac{3}{2}+s+s'}(1-q^{-\frac{1}{2}-s-s'})(1-q^{-\frac{1}{2}-s+s'})}.$$

It follows that

$$T_{s',s,\psi} = \frac{1}{1-q^{-\frac{1}{2}-s+s'}} + \frac{-1+q-q^{\frac{1}{2}+s}+q^{1+s'}}{q^{\frac{3}{2}+s+s'}(1-q^{-\frac{1}{2}-s-s'})(1-q^{-\frac{1}{2}-s+s'})}$$
$$= \frac{(1-q^{-1-s'})(1+q^{-\frac{1}{2}-s})}{(1-q^{-\frac{1}{2}-s+s'})(1-q^{-\frac{1}{2}-s-s'})}.$$

We thus end up with

$$(6.10) = \frac{\zeta_F(2)}{\zeta_F(1)^2}\frac{(1-q^{-1-s'})(1+q^{-\frac{1}{2}-s})(1-q^{-1+s'})(1+q^{-\frac{1}{2}+s})}{(1-q^{-\frac{1}{2}-s+s'})(1-q^{-\frac{1}{2}-s-s'})(1-q^{-\frac{1}{2}+s-s'})(1-q^{-\frac{1}{2}+s+s'})}$$
$$= \zeta_F(2)\frac{L(1/2,\mathrm{Sym}^2\,\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}.$$

Proof of Proposition 6.2.2. The above equation together with lemma 6.2.4 gives,

$$\int_{\mathrm{SL}_2(F)}\langle\tau(X)g,g\rangle\overline{\langle\sigma(X)h^{(2)},h^{(2)}\rangle}\langle\omega_\psi(X)\phi,\phi\rangle dX = 2^{-d}\zeta_F(2)\frac{L(1/2,\mathrm{Sym}^2\,\tau\times\pi)}{L(1,\pi,\mathrm{Ad})L(1,\tau,\mathrm{Ad})}.$$

We also have $\langle g,g\rangle = 1$, $\langle\phi,\phi\rangle = 2^d$ and by Lemma 6.2.3, $\langle h^{(2)},h^{(2)}\rangle = 2^{-d}$. □

Part IV

APPENDIX

A

SOME COMBINATORIAL IDENTITIES

Lemma A.0.1. *Let N be a nonnegative integer. We have the following identity.*

$$\sum_{i=0}^{n}(-1)^n \binom{N}{i} \frac{\Gamma(z+i)}{\Gamma(w+i)} = \frac{\Gamma(z)}{\Gamma(w-z)} \cdot \frac{\Gamma(w-z+N)}{\Gamma(w+N)}, \tag{A.1}$$

for every $z, w \in \mathbb{C}$.

Proof. Lemma 2.1 in [Ike98]. □

Lemma A.0.2. *For every $r, \kappa \in \mathbb{N}$, we have*

$$\sum_{0\leqslant l\leqslant \lfloor \frac{r}{2} \rfloor} \prod_{0\leqslant j\leqslant l-1} \frac{(r-2j)(r-2j-1)}{(2j+2)(2\kappa+2j+1)} = 2^r \frac{\binom{r+\kappa-1}{r}}{\binom{r+2\kappa-1}{r}}, \tag{A.2}$$

where we take the empty product corresponding to $i = 0$ to have value one.

Proof. We only sketch the proof here. Through some tedious but elementary calculations, one can verify that

$$(A.2) = \frac{\Gamma(\kappa+\frac{1}{2})}{\Gamma(\frac{\varepsilon-r}{2})} \sum_{0\leqslant l\leqslant \lfloor \frac{r}{2} \rfloor} (-1)^l \binom{\lfloor \frac{r}{2} \rfloor}{l} \frac{\Gamma(l+\frac{\varepsilon-r}{2})}{\Gamma(l+\kappa+\frac{1}{2})},$$

where $\varepsilon \equiv 1 + r (\mathrm{mod} 2)$. Now use Lemma A.0.1. □

Lemma A.0.3. *For every $r, \kappa' \in \mathbb{N}$, we have*

$$\sum_{0\leqslant i\leqslant \lfloor \frac{r}{2}\rfloor} \prod_{0\leqslant l\leqslant i-1} \frac{(r-2l)^2(r-2l-1)^2}{(2i-2l)(2i+2\kappa'-2l)(r-2i+2l+1)(r-2i+2l+2)} = 2^{-r}\frac{\binom{2\kappa'+2r}{r}}{\binom{\kappa'+r}{r}}, \tag{A.3}$$

where we take the empty product corresponding to $i=0$ to have value one.

Proof. We only sketch the proof here. Through some tedious but elementary calculations, one can verify that

$$(A.3) = \frac{\Gamma(\kappa'+1)}{\Gamma(\frac{\varepsilon-r}{2})} \sum_{0\leqslant l\leqslant \lfloor \frac{r}{2}\rfloor} (-1)^i \binom{\lfloor \frac{r}{2}\rfloor}{i} \frac{\Gamma(i+\frac{\varepsilon-r}{2})}{\Gamma(i+\kappa'+1)},$$

where $\varepsilon \equiv 1 + r(\mathrm{mod} 2)$. Now use Lemma A.0.1. □

B

DELIGNE'S CONJECTURE

B.1 STATEMENT OF THE GENERAL CONJECTURE

Let M be a motive over a number field k with coefficients in E, and let

$$L(s, M) = L_\infty(s, M)L_{\text{fin}}(s, M)$$

be its associated L-function. For the definitions of a motive and the associated L-function we refer the reader to [Del79].

Remark. *A word of caution: in [Del79], $L(s, M)$ is denoted as $\Lambda(s, M)$, and $L_{\text{fin}}(s, M)$ as $L(s, M)$.*

$L(s, M)$ satisfies a conjectural functional equation:

$$L(s, M) = \varepsilon(s, M)L(s, \check{M}),$$

where $\check{M}$ is the dual of M and $\varepsilon(s, M)$, considered as a function of s is the product of a constant and an exponential function.

Definition B.1.1. *An integer n is a critical point for M if neither $L_\infty(s, M)$ nor $L_\infty(1 - s, \check{M})$ has a pole at $s = n$.*

Conjecture. *([Del79], Conjecture 2.8) Let n be a critical point for M. There exists a complex number, $c^+(M(n))$, such that for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\left(\frac{L_{\mathrm{fin}}(n, M)}{c^+(M(n))}\right)^\sigma = \frac{L_{\mathrm{fin}}(n, M^\sigma)}{c^+(M^\sigma(n))}.$$

Remark. *The complex number $c^+(M(n))$ is known as Deligne's period and is defined by a comparison of rational structures on the de Rham and Betti realizations of the motive M(cf. [Del79], §2).*

Remark. *Let $R_{F/\mathbb{Q}}(M)$ denote the motive over $\mathbb{Q}$ with coefficients over E obtained from M by the restriction of scalars. Then by ([Yos94], Eqn. (2.4)), Deligne's conjecture can also be written as*

$$\left(\frac{L_{\mathrm{fin}}(n, M)}{c^+(R_{F/\mathbb{Q}}(M(n)))}\right)^\sigma = \frac{L_{\mathrm{fin}}(n, M^\sigma)}{c^+(R_{F/\mathbb{Q}}(M^\sigma(n)))},$$

for some complex number $c^+(R_{F/\mathbb{Q}}(M(n)))$ and for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

B.2 DELIGNE'S CONJECTURE FOR $L(s, \mathrm{Sym}^2 g \times f)$

Let $\mathbb{Q}(f)$ resp. $\mathbb{Q}(g)$ be the number fields generated by the Fourier coefficients of f resp. g, and $\widetilde{F}$ denote the Galois closure of F. Put $E = \mathbb{Q}(f) \vee \mathbb{Q}(g) \vee \widetilde{F}$, the composite of $\mathbb{Q}(f)$, $\mathbb{Q}(g)$ and $\widetilde{F}$. Let $M_1 = M(f)$ and $M_2 = \mathrm{Sym}^2 M(g)$ be the (conjectured) motives over F with coefficients in E associated to f and $\mathrm{Sym}^2(g)$ respectively. M_1 is a motive of rank 2 and weight $2\kappa_0 - 1$; M_2 is a motive of rank 3 and weight $2\kappa_0'$. Put

$$M = M_2 \otimes M_1.$$

Then M is a motive over F with coefficients in E of rank 6 and weight $2\kappa_0 + 2\kappa_0' - 1$. Moreover, $\check{M} = M(2\kappa_0' + 2\kappa_0 - 1)$. In this section we will specialize Deligne's conjecture to the motivic L-function $L(s, M)$.

Remark. *Note that $\mathbb{Q}(f)$ and $\mathbb{Q}(g)$ are totally real fields since we are assuming that the "nebentypus" of f and g are trivial.*

Remark. *Note that $L(s, M) = L(s, \mathrm{Sym}^2 g \times f)$.*

Let m be a critical point for the motivic L-function $L(s, M)$. By definition, m is an integer such that neither $L(s, M)$ nor $L(1-s, \check{M}) = L(1 - s + 2\kappa_0' + 2\kappa_0 - 1, M)$ has a pole at $s = m$. It is not hard to find all such integers.

Put

$$t^0 = \min_{v \in \Sigma_{ub}} \{r_v\}, \quad a^0 = \min_{v \in \Sigma_b, 2\kappa_v > \kappa_v'} \{r_v\}, \quad b^0 = \min_{v \in \Sigma_b, \kappa' \geqslant 2\kappa} \{\kappa_v - 1\}.$$

Proposition B.2.1. *The set of critical points for $L(s, \mathrm{Sym}^2 g \times f)$ is*

$$\left\{ m \in \mathbb{Z} \mid \kappa_0 + \kappa_0' - \min\{a^0, b^0, t^0\} \leqslant m \leqslant \min\{a^0, b^0, t^0\} + \kappa_0 + \kappa_0' \right\}.$$

In particular, the center of $L(s, M)$, $s = \kappa_0 + \kappa_0'$, is a critical point.

Proof. Follows from Lemma 5.3.1 and Prop. 5.3.2 in §5.3. □

Then Deligne's conjecture (see the second remark after Conjecture B.1) predicts that there exists a complex number $c^+(R_{F/\mathbb{Q}}(M)(m))$ such that for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\left(\frac{L_{\mathrm{fin}}(m, M)}{c^+(R_{F/\mathbb{Q}}(M(m)))} \right)^\sigma = \frac{L_{\mathrm{fin}}(m, M^\sigma)}{c^+(R_{F/\mathbb{Q}}(M^\sigma(m)))}. \tag{B.1}$$

We will now express $c^+(R_{F/\mathbb{Q}}(M)(m))$ in terms of *Shimura's periods*. Our computations will closely follow [Yos94] and [Del79].

By ([Yos94], Prop. 2.2),

$$c^+(R_{F/\mathbb{Q}}(M(m))) = D_F^{d^+(M(m))/2} \prod_{v \in \Sigma_{\infty,F}} c_v^+(M(m)), \tag{B.2}$$

where $c_v^+(M(m)) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^\times = \prod_{\sigma \in \Sigma_{\infty,E}} (E \otimes_{E,\sigma} \mathbb{C})^\times$ are the *local periods*, $d^+(M(m))$ is the rank of the free $E \otimes_{\mathbb{Q}} F$-module $H^+_{DR}(M(m))$, and D_F is the discriminant of F. The right-hand side of the above equality is an element of $(E \otimes_{\mathbb{Q}} \mathbb{C})^\times$ and is well-defined up to multiplication by elements of the $E^\times$ viewed as a subgroup of $(E \otimes_{\mathbb{Q}} \mathbb{C})^\times$. Moreover, by ([Yos94], Eqn. (3.9)), for any $v \in \Sigma_{\infty,F}$,

$$c_v^+(M(m)) = \begin{cases} (2\pi i)^{m d_v^+(M)} c_v^+(M) & \text{if } m \text{ is even} \\ (2\pi i)^{m d_v^-(M)} c_v^-(M) & \text{if } m \text{ is odd} \end{cases} \tag{B.3}$$

where $d_v^+(M) = \dim_E H^+_{v,B} M$. Therefore, the problem of computing $c^+(R_{F/\mathbb{Q}}(M(m)))$ reduces to the problem of computing $d^+(M(m))$, $d_v^\pm(M)$ and $c_v^\pm(M)$, where $M = M_2 \otimes M_1$ and M_1 and M_2 are motives over the totally real field F with coefficients in a totally real field E.

By ([Yos94], Eqn. (3.2)),

$$d_v^+(M) = d_v^+(M_2) d_v^+(M_1) + d_v^-(M_2) d_v^-(M_1)$$

and

$$d_v^-(M) = d_v^+(M_2) d_v^-(M_1) + d_v^-(M_2) d_v^+(M_1).$$

Also, by ([Del79], proof of Prop. 7.7),

$$d_v^\pm(M_1) = 1, \quad d_v^+(M_2) = 2 \quad \text{and} \quad d_v^-(M_2) = 1.$$

Therefore,

$$d_v^{\pm}(M) = 3. \tag{B.4}$$

Hence also,

$$d^{+}(M(m)) = 3. \tag{B.5}$$

$c_v^{\pm}(M_1 \otimes M_2)$ can be computed by the formula given in ([Bha15], Thm. 3.2). These computations have already been done in ([CC19], Appendix). We state the results here:

$$c_v^{\pm}(M) = \begin{cases} c_v^{\pm}(M_1)\delta_v(M_1)(c_v^{+}(M_2)c_v^{-}(M_2)) & \text{if } v \in \Sigma_b \\ c_v^{\pm}(M_1)\delta_v(M_2)(c_v^{+}(M_1)c_v^{-}(M_1)) & \text{if } v \in \Sigma_{ub}. \end{cases} \tag{B.6}$$

Here, for any motive M, $\delta_v(M) = \det(I_v) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$ where I_v is the canonical isomorphism

$$I_v : H_{v,B}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong H_{DR}(M) \otimes_{F,v} \mathbb{C}.$$

Moreover, by ([Del79], 5.1.7, 5.1.8, Prop. 7.7),

$$\delta_v(M(f)) \sim (2\pi i)^{1-2\kappa_0}, \quad \delta_v(M(g)) \sim (2\pi i)^{-\kappa_0'}, \quad \delta_v(\mathrm{Sym}^2 M(g)) \sim (2\pi i)^{-3\kappa_0'},$$

and

$$c_v^{+}(\mathrm{Sym}^2 M(g))c_v^{-}(\mathrm{Sym}^2 M(g)) \sim (c_v^{+}(M(g))c_v^{-}(M(g)))^2\delta_v(M(g)),$$

where in $(E \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$, $\sim$ stands for equality up to elements of the subgroup $E^{\times}$. Thus, (B.6) can be rewritten as

$$c_v^{\pm}(\mathrm{Sym}^2 M(g) \otimes M(f)) \sim \begin{cases} (2\pi i)^{1-2\kappa_0-\kappa_0'}c_v^{\pm}(M(f))(c_v^{+}(M(g))c_v^{-}(M(g)))^2 & \text{if } v \in \Sigma_b \\ (2\pi i)^{-3\kappa_0'}c_v^{\pm}(M(f))(c_v^{+}(M(f))c_v^{-}(M(f))) & \text{if } v \in \Sigma_{ub}. \end{cases} \tag{B.7}$$

Hence, combining (B.7), (B.5), (B.4), (B.3) with (B.2), we get

$$\begin{aligned} c^+(R_{F/\mathbb{Q}}(M(m))) \sim (2\pi i)^* \cdot D_F^{3/2} \cdot \prod_{v \in \Sigma_{\infty,F}} c_v^+(M(f)) \\ \cdot \prod_{v \in \Sigma_{ub}} c_v^+(M(f)) c_v^-(M(f)) \cdot \prod_{v \in \Sigma_b} (c_v^+(M(g)) c_v^-(M(g)))^2 \end{aligned} \tag{B.8}$$

if m is even; and

$$\begin{aligned} c^+(R_{F/\mathbb{Q}}(M(m))) \sim (2\pi i)^* \cdot D_F^{3/2} \cdot \prod_{v \in \Sigma_{\infty,F}} c_v^-(M(f)) \\ \cdot \prod_{v \in \Sigma_{ub}} c_v^+(M(f)) c_v^-(M(f)) \cdot \prod_{v \in \Sigma_b} (c_v^+(M(g)) c_v^-(M(g)))^2 \end{aligned} \tag{B.9}$$

if m is odd. Here

$$* = |3m|_{\Sigma_\infty} - |3\kappa_0'|_{\Sigma_{ub}} + |1 - 2\kappa_0 - \kappa_0'|_{\Sigma_b}.$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\varepsilon = (\varepsilon_v)_{v \in \Sigma_{\infty,F}} \in \{0,1\}^{\Sigma_{\infty,F}}$ and $S \subset \Sigma_{\infty,F}$, put

$$U(f^\sigma, \varepsilon) = D_F^{1/2} \prod_{v \in \Sigma_{\infty,F}} c_v^{\varepsilon_v}(M(f)) \tag{B.10}$$

and

$$Q(f^\sigma, S) = \pi^{|2\kappa_0 - 2\kappa - 1|_S} \prod_{v \in S} c_v^+(M(f)) c_v^-(M(f)). \tag{B.11}$$

We can define $U(g^\sigma, \varepsilon)$ and $Q(g^\sigma, S)$ similarly. Here we understand that $c_v^0(M(f)) = c_v^+(M(f))$ and $c_v^1(M(f)) = c_v^-(M(f))$.

Take $\varepsilon_v = m \pmod 2$ for all v and let $\varepsilon = (\varepsilon_v)$. Then (B.8) and (B.9) can be together rewritten as:

$$c^+(R_{F/\mathbb{Q}}(M(m))) \sim (2\pi i)^* i^{|\Sigma_{ub}|} U(f, \varepsilon) Q(f, \Sigma_{ub}) Q(g, \Sigma_b)^2, \tag{B.12}$$

where

$$* = |3m - 2\kappa_0 - 3\kappa_0' + 1|_{\Sigma_\infty} + |2\kappa|_{\Sigma_{ub}} + |2\kappa' + 2|_{\Sigma_b}. \tag{B.13}$$

We can now rewrite Deligne's conjecture (B.1):

Conjecture. *If $m \in \mathbb{Z}$ is a critical point of the motivic L-function $L(s, \mathrm{Sym}^2 g \times f)$, then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\left(\frac{L_{\mathrm{fin}}(m, \mathrm{Sym}^2 g \times f)}{(2\pi i)^* i^{|\Sigma_{ub}|} U(f, \varepsilon) Q(f, \Sigma_{ub}) Q(g, \Sigma_b)^2} \right)^\sigma = \frac{L_{\mathrm{fin}}(m, \mathrm{Sym}^2 g^\sigma \times f^\sigma)}{(2\pi i)^* i^{|\Sigma_{ub}|} U(f^\sigma, \varepsilon) Q(f^\sigma, \Sigma_{ub}) Q(g^\sigma, \Sigma_b)^2}, \tag{B.14}$$

*where * is as in (B.13).*

Remark. *Note that in the above conjecture f and g are any Hilbert newforms of levels of the 'Γ_0-type'.*

C

A SURVEY OF SHIMURA'S INVARIANTS

Notation: For K a number field and $a, b \in \mathbb{C}^\times$, $a \sim_K b$ means that $a = bc$ for some $c \in K$.

C.1 THE u-INVARIANT

Let f be a Hilbert newform of weight $\kappa = (\kappa_1, \ldots, \kappa_n)$. Let

$$\kappa^0 = \min(\kappa_1, \ldots, \kappa_n) \quad \text{and} \quad \kappa_0 = \max(\kappa_1, \ldots, \kappa_n).$$

For a Hecke character χ of $\mathbb{A}_F^\times$ consider $L(s, f, \chi)$, the completed L-function attached to f twisted by χ; let $L_{\text{fin}}(s, f, \chi)$ denote its finite part. Then the set of integers m such that

$$(\kappa_0 - \kappa^0)/2 < m < (\kappa_0 + \kappa^0)/2$$

are all the critical points of $L(s, f, \chi)$. Shimura ([Shi78], Thm. 4.3) showed that there exists a family of complex numbers $\{u(\varepsilon, f^\sigma)\}_{\sigma,\varepsilon}$ indexed by $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\varepsilon \in \{0,1\}^n$ such that

(i) For a critical point m and any $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$L_{\mathrm{fin}}(m, f, \chi) \sim_{\mathbb{Q}(f,\chi)} (2\pi i)^{mn} \tau(\chi) u(\varepsilon, f)$$

and

$$\left(\frac{L_{\mathrm{fin}}(m, f, \chi)}{(2\pi i)^{mn} \tau(\chi) u(\varepsilon, f)} \right)^{\sigma} = \frac{L_{\mathrm{fin}}(m, f^{\sigma}, \chi^{\sigma})}{(2\pi i)^{mn} \tau(\chi^{\sigma}) u(\varepsilon, f^{\sigma})}, \tag{C.1}$$

where ε is prescribed by $\chi(a) = \mathrm{sign}[a^{\varepsilon} N(a)^m]$ for $a \in F_{\infty}^{\times}$ and $\tau(\chi)$ is the Gauss sum attached to χ.

(ii) For any $\varepsilon,\ \varepsilon' \in \{0,1\}^n$ such that $\varepsilon + \varepsilon' = 1 \pmod 2$ and any $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$u(\varepsilon, f) u(\varepsilon', f) \sim_{\mathbb{Q}(f)} (2\pi i)^{n(1-\kappa_0)} \pi^{|\kappa|_{\Sigma_\infty}} \langle f, f \rangle$$

and

$$\left(\frac{u(\varepsilon, f) u(\varepsilon', f)}{(2\pi i)^{n(1-\kappa_0)} \pi^{|\kappa|_{\Sigma_\infty}} \langle f, f \rangle} \right)^{\sigma} = \frac{u(\varepsilon, f^{\sigma}) u(\varepsilon', f^{\sigma})}{(2\pi i)^{n(1-\kappa_0)} \pi^{|\kappa|_{\Sigma_\infty}} \langle f^{\sigma}, f^{\sigma} \rangle}.$$

Evidently, every member of such a family of complex numbers is uniquely determined up to an element of $\mathbb{Q}(f)^{\times}$. Shimura showed that one can take

$$u(\varepsilon, f^{\sigma}) = (2\pi i)^{|1-\kappa_0+m'|_{\Sigma_\infty}} \pi^{|\kappa|_{\Sigma_\infty}} \tau(\eta)^{-1} \frac{\langle f^{\sigma}, f^{\sigma} \rangle}{L_{\mathrm{fin}}(m', f^{\sigma}, \eta)}, \tag{C.2}$$

where $m' = (\kappa_0 + \kappa^0 - 2)/2$ is the right-most critical point of f and η is a Hecke character of order two such that $\eta(a) = \mathrm{sign}[a^{\varepsilon} N(a)^{m'+1}]$ for $a \in F_{\infty}^{\times}$.

Some more properties of the u-invariant.

(iii) From (C.2) it follows that,

$$u(\varepsilon, \overline{f}) = \overline{u(\varepsilon, f)}.$$

(iv) Let χ be a Hecke character of $\mathbb{A}_F^\times$ such that

$$\chi(a) = \mathrm{sign}(a^{\varepsilon_1}), \quad a \in F_\infty^\times.$$

Then for every $\varepsilon \in \{0,1\}^n$,

$$u(\varepsilon, f \otimes \chi) = u(\varepsilon + \varepsilon_1, f).$$

This property follows from property (i).

(v) For any $\varepsilon,\ \varepsilon',\ \eta,\ \eta' \in \{0,1\}^n$ such that $\varepsilon + \varepsilon' = 1 \pmod 2$ and $\eta + \eta' = 1 \pmod 2$,

$$u(\varepsilon, f)u(\varepsilon', f) \sim_{\mathbb{Q}(f)} u(\eta, f)u(\eta', f).$$

Immediately follows from property (ii).

(vi) If $\{\varepsilon_v, \varepsilon'_v\} = \{\eta_v, \eta'_v\}$ for every $v \in \Sigma_\infty$, then (cf. [Yos95])

$$u(\varepsilon, f)u(\varepsilon', f) \sim_{\overline{\mathbb{Q}}} u(\eta, f)u(\eta', f).$$

C.2 THE Q-INVARIANT

Let us now introduce Shimura's Q-invariant, which arises in the critical values of Rankin-Selberg convolution of two Hilbert modular forms. As before, $F/\mathbb{Q}$ is a totally real number field of degree n. Let $S,\ S'$ be subsets of Σ_∞ such that $S \cup S' = \Sigma_\infty$. Let B be a quaternion algebra over F such that B splits resp. ramifies at the archimedean places $v \in S$ resp. $v \in S'$. We call such a B a quaternion algebra of signature (S, S'). Now, let f be a Hilbert newform of weight $\kappa = (\kappa_1, \dots, \kappa_n)$. Assume that f corresponds to a nonzero automorphic form f_B on B by the Jacquet-Langlands correspondence (cf. §3.5.1.2). Define

$$Q(f,S) = \langle f_B, f_B \rangle. \tag{C.3}$$

We set $Q(f,\emptyset) = 1$. By the following results of Yoshida ([Yos94],§6), $Q(f,S)$ is well-defined up to a nonzero algebraic number.

- $\langle f_B, f_B \rangle$ mod $\overline{\mathbb{Q}}^{\times}$ is independent of the choice of f_B.
- If B' is another quaternion algebra over F of signature (S,S') and $f_{B'}$ a nonzero automorphic form on B' corresponding to f via Jacquet-Langlands correspondence, then
$$\langle f_B, f_B \rangle \sim_{\overline{\mathbb{Q}}} \langle f'_B, f'_B \rangle.$$

It is also possible to define $Q(f,S)$ when there is no Jacquet-Langlands correspondence of f on any quaternion algebra of signature (S,S'). Let F_1 be a totally real quadratic extension of of F. Let f_1 be the base change of f to F_1. Let S_1 be the set of archimedean places of F_1 over S. Then it is a fact that, when $Q(f,S)$ can be defined by (C.3) (that is, when f_B exists),

$$Q(f_1,S_1) = Q(f,S)^2.$$

Hence, one can set

$$Q(f,S) = Q(f_1,S_1)^{1/2}.$$

This defines $Q(f,S)$ independent of the existence of a Jacquet-Langlands lift of f to a quaternion algebra of signature $(S,\Sigma_\infty - S)$.

Now, let f resp. g be a Hilbert newform of weight $\kappa = (\kappa_1,\dots,\kappa_n)$ resp. $\kappa' = (\kappa'_1,\dots,\kappa'_n)$ such that $\kappa_i - \kappa'_i$ is independent of i. Let S and S' be two disjoint subsets of archimedean places whose union is Σ_∞ such that $\kappa_v > \kappa'_v$ for $v \in S$ and $\kappa'_v > \kappa_v$ for

$v \in S'$. Let $L_{\text{fin}}(s, f \times g)$ be the finite part of the standard Rankin-Selberg convolution of f and g. Shimura showed that ([Shi83], Thm 5.3)

$$L_{\text{fin}}(m, f \times g) \sim_{\overline{\mathbb{Q}}} \pi^{|\kappa|_S + |\kappa'|_{S'}} Q(f, S) Q(g, S'),$$

for any critical point m of $L_{\text{fin}}(s, f \times g)$.

Some properties of the Q-invariant (cf. [Yos95]):

(i) $Q(f, \Sigma_\infty) = \langle f, f \rangle$.

(ii) $Q(f, S_1) Q(f, S_2) \sim_{\overline{\mathbb{Q}}} Q(f, S_1 \cup S_2)$ if $S_1 \cap S_2 = \emptyset$ and $\kappa_v \geqslant 3$ for all $v \in S_1 \cup S_2$.

(iii) $Q(\overline{f}, S) = \overline{Q(f, S)}$.

(iii) If $\varepsilon + \varepsilon' = 1 \pmod 2$, then

$$u(\varepsilon, f) u(\varepsilon', f) \sim_{\overline{\mathbb{Q}}} \pi^{|1 - \kappa_0 + \kappa|_{\Sigma_\infty}} Q(f, \Sigma_\infty).$$

C.3 COMPARISON WITH DELIGNE'S PERIODS

Recall that in §B.2, we defined Deligne's U and Q-periods: for $\sigma \in \text{Aut}(\mathbb{C})$, $\varepsilon = (\varepsilon_v)_{v \in \Sigma_{\infty,F}} \in \{0, 1\}^{\Sigma_{\infty,F}}$ and $S \subset \Sigma_{\infty,F}$,

$$U(f^\sigma, \varepsilon) = D_F^{1/2} \prod_{v \in \Sigma_\infty} c_v^{\varepsilon_v}(M(f)) \tag{C.4}$$

and

$$Q(f^\sigma, S) = \pi^{|\kappa_0 - \kappa - 1|_S} \prod_{v \in S} c_v^+(M(f)) c_v^-(M(f)). \tag{C.5}$$

By comparing the known results on rationality of critical values of L-functions with Deligne's conjectures, one expects Shimura's u-invariant and Q-invariant to be essentially the same as Deligne's U and Q-periods, respectively. In particular, one can hope for a factorization of Shimura's invariants similar to Deligne's periods above. Yoshida ([Yos95], Main Theorem) showed that such a factorization indeed exists. Assume that $\kappa_v > 2$ for all v (and κ_v mod 2 is independent of v). Then for all $v \in \Sigma_\infty$, there exist nonzero complex numbers $c_v^{\pm}(f)$ determined uniquely modulo $\overline{\mathbb{Q}}^\times$ such that

$$u(\varepsilon, f) \sim_{\overline{\mathbb{Q}}} \prod_{v \in \Sigma_\infty} c_v^{\varepsilon_v}(f) \tag{C.6}$$

and

$$Q(f, S) \sim_{\overline{\mathbb{Q}}} \pi^{-|1-\kappa_0+\kappa|_S} \prod_{v \in S} c_v^{+}(f) c_v^{-}(f). \tag{C.7}$$

Here we understand that $c_v^+ = c_v^0$ and $c_v^- = c_v^1$.

We can now also define another of Shimura's invariants, the P-invariant, which is more basic than the u and Q-invariants. Put

$$P(f, S, \varepsilon) = \pi^{\frac{1}{2}|\kappa_0-\kappa-2|_S} \prod_{v \in S} c_v^{\varepsilon_v}(f). \tag{C.8}$$

Some properties of the P-invariant:

(i)

$$P(f, \Sigma_\infty, \varepsilon) \sim_{\overline{\mathbb{Q}}} \pi^{\frac{1}{2}|\kappa_0-\kappa-2|_{\Sigma_\infty}} u(\varepsilon, f).$$

Follows immediately from (C.6) and (C.8).

(ii) By ([Yos95], Eq. 4.9),

$$P(\overline{f}, S, \varepsilon) = \overline{P(f, S, \varepsilon)}.$$

(ii) If $\varepsilon + \varepsilon' = 1$ (mod 2), then

$$\pi^{|S|} P(f, S, \varepsilon) P(f, S, \varepsilon') \sim_{\overline{\mathbb{Q}}} Q(f, S).$$

Follows immediately from (C.7) and (C.8).

(iii) When f is of CM-type,

$$P(f,S,\varepsilon) \sim_{\overline{\mathbb{Q}}} p_K(\xi,\eta),$$

where p_K stands for the symbol of CM-periods introduced in [Shi80]. This is proved in [Yos95].

C.4 THE INVARIANTS COHOMOLOGICALLY

Shimura's u-invariant and Q-invariant have also been realized cohomologically, in [RT11] and [Har90] respectively.

Let π be an irreducible cuspidal automorphic representation of weight $\kappa = (\kappa_v)_{v\in\Sigma_\infty}$ and let $\varepsilon = (\varepsilon_v)_{v\in\Sigma_\infty} \in (\mathbb{Z}/2\mathbb{Z})^n$. In [RT11], a period $p^\varepsilon(\pi)$ has been defined by comparing rational structures on two different models for the finite part π_{fin} of π: one is a Whittaker model and the other is a cohomological realization of π_{fin}. Furthermore, it has been shown that ([RT11],Cor. 1.3)

Proposition. *Let π be an irreducible cuspidal automorphic representation of weight $\kappa = (\kappa_v)_{v\in\Sigma_\infty}$. Assume that $s = m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ is a critical point for the standard L-function attached to π. For any finite order character χ of $F^\times\backslash\mathbb{A}^\times$, and for any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\left(\frac{L_{\text{fin}}(m+\frac{1}{2},\pi\otimes\chi)}{(2\pi i)^{|\frac{\kappa}{2}+m|_{\Sigma_\infty}}\tau(\chi)p^{((-1)^m\varepsilon_\chi)}(\pi)}\right)^\sigma = \frac{L_{\text{fin}}(m+\frac{1}{2},\pi^\sigma\otimes\chi^\sigma)}{(2\pi i)^{|\frac{\kappa}{2}+m|_{\Sigma_\infty}}\tau(\chi^\sigma)p^{((-1)^m\varepsilon_{\chi^\sigma})}(\pi^\sigma)}, \tag{C.9}$$

where $\varepsilon_\chi = (\chi_v(-1))_{v\in\Sigma_\infty}$; and $\tau(\chi)$ is the Gauss sum of χ.

Comparing (C.9) with (C.1), we can find a relationship between the period $p^\varepsilon(\pi)$ and the complex number $u(\varepsilon,f)$, where f is the newform generating π. Indeed, noting that the critical point $s = \frac{1}{2}$ of $L(s,\pi\otimes\chi)$ corresponds to the critical point $s = \frac{\kappa_0}{2}$ of $L(s,f,\chi)$, we get that

Lemma C.4.1. *For all $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\left(\frac{p^{\varepsilon_\chi}(\pi)}{(2\pi i)^{|\frac{\kappa_0-\kappa}{2}|}u(\varepsilon', f)}\right)^\sigma = \left(\frac{p^{\varepsilon_{\chi^\sigma}}(\pi^\sigma)}{(2\pi i)^{|\frac{\kappa_0-\kappa}{2}|}u(\varepsilon', f^\sigma)}\right),$$

where $\varepsilon'_v = \varepsilon_{\chi,v} + \frac{\kappa_0}{2}$ (mod 2) for all $v \in \Sigma_\infty$.

Let us now give a cohomological interpretation of Shimura's Q-invariant. Let π be an irreducible cuspidal automorphic representation of weight $\kappa = (\kappa_v)_{v\in\Sigma_\infty}$ and let $S \subset \Sigma_\infty$. In [Har90], a period $\nu^S(\pi)$ has been defined by comparing rational structures on two different models for π_{fin}: one is a Whittaker model and the other is a cohomological realization of π_{fin}. The complex number $\nu^S(\pi)$ is well-defined up to an element of $\mathbb{Q}(\pi, S)$ which is the composite of $\mathbb{Q}(\pi)$ and a subfield of the Galois closure of F.

Theorem ([Har94], Thm. 1)**.** *Let f denote the newform corresponding to π. Then,*

$$\nu^S(\pi) \sim_{\overline{\mathbb{Q}}} Q(f, S).$$

Remark. *The period $\nu^S(\pi)$ appears in critical values of Rankin-Selberg convolutions and certain triple product L-functions (cf. [Har90]) and has the advantage over $Q(f, S)$ of expressing results on rationality of these values that behave well under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

REFERENCES

[BCdS+03] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla, *An introduction to the Langlands program*, Birkhäuser Boston, Inc., Boston, MA, 2003, Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001, Edited by Joseph Bernstein and Stephen Gelbart. MR 1990371

[Bha15] Chandrasheel Bhagwat, *On Deligne's periods for tensor product motives*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 3, 191–195. MR 3306483

[BR93] Don Blasius and Jonathan D. Rogawski, *Motives for Hilbert modular forms*, Invent. Math. **114** (1993), no. 1, 55–87. MR 1235020

[BS98] Rolf Berndt and Ralf Schmidt, *Elements of the representation theory of the Jacobi group*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1998, [2011 reprint of the 1998 original] [MR1634977]. MR 3013729

[Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR 1431508

[CC19] Shih-Yu Chen and Yao Cheng, *On Deligne's conjecture for certain automorphic L-functions for* $GL(3) \times GL(2)$ *and* $GL(4)$, Doc. Math. **24** (2019), 2241–2297. MR 4033818

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, With an appendix by N. Koblitz and A. Ogus, pp. 313–346. MR 546622

[GJ79] Stephen Gelbart and Hervé Jacquet, *Forms of* $\mathrm{GL}(2)$ *from the analytic point of view*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 213–251. MR 546600

[GR00] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, sixth ed., Academic Press, Inc., San Diego, CA, 2000, Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. MR 1773820

[Har90] Michael Harris, *Period invariants of Hilbert modular forms. I. Trilinear differential operators and L-functions*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Lecture Notes in Math., vol. 1447, Springer, Berlin, 1990, pp. 155–202. MR 1082965

[Har94] ———, *Period invariants of Hilbert modular forms. II*, Compositio Math. **94** (1994), no. 2, 201–226. MR 1302316

[HI13] Kaoru Hiraga and Tamotsu Ikeda, *On the Kohnen plus space for Hilbert modular forms of half-integral weight I*, Compos. Math. **149** (2013), no. 12, 1963–2010. MR 3143703

[Hid16] Haruzo Hida, *Arithmetic of adjoint L-values*, p-adic aspects of modular forms, World Sci. Publ., Hackensack, NJ, 2016, pp. 185–236. MR 3587958

[HK91] Michael Harris and Stephen S. Kudla, *The central critical value of a triple product L-function*, Ann. of Math. (2) **133** (1991), no. 3, 605–672. MR 1109355

[Ich05] Atsushi Ichino, *Pullbacks of Saito-Kurokawa lifts*, Invent. Math. **162** (2005), no. 3, 551–647. MR 2198222

[Ike98] Tamotsu Ikeda, *On the gamma factor of the triple L-function. II*, J. Reine Angew. Math. **499** (1998), 199–223. MR 1631136

[IP21] Atsushi Ichino and Kartik Prasanna, *Periods of quaternionic Shimura varieties. I*, Contemporary Mathematics, vol. 762, American Mathematical Society, [Providence], RI, [2021] ©2021. MR 4228250

[Iwa69a] Kenkichi Iwasawa, *Analogies between number fields and function fields*, Some Recent Advances in the Basic Sciences, Vol. 2 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1965-1966), Yeshiva Univ., Belfer Graduate School of Science, New York, 1969, pp. 203–208. MR 0255510

[Iwa69b] ______, *On p-adic L-functions*, Ann. of Math. (2) **89** (1969), 198–205. MR 269627

[Jac09] Hervé Jacquet, *Archimedean Rankin-Selberg integrals*, Automorphic forms and L-functions II. Local aspects, Contemp. Math., vol. 489, Amer. Math. Soc., Providence, RI, 2009, pp. 57–172. MR 2533003

[JL70] H. Jacquet and R. P. Langlands, *Automorphic forms on* $\mathrm{GL}(2)$, Lecture Notes in Mathematics, Vol. 114, Springer-Verlag, Berlin-New York, 1970. MR 0401654

[Lap03] Erez M. Lapid, *On the nonnegativity of Rankin-Selberg L-functions at the center of symmetry*, Int. Math. Res. Not. (2003), no. 2, 65–75. MR 1936579

[LL79] J.-P. Labesse and R. P. Langlands, *L-indistinguishability for* $SL(2)$, Canadian J. Math. **31** (1979), no. 4, 726–785. MR 540902

[Qiu14] Yannan Qiu, *Periods of Saito-Kurokawa representations*, Int. Math. Res. Not. IMRN (2014), no. 24, 6698–6755. MR 3291638

[RT11] A. Raghuram and Naomi Tanabe, *Notes on the arithmetic of Hilbert modular forms*, J. Ramanujan Math. Soc. **26** (2011), no. 3, 261–319. MR 2865819

[Shi73] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481. MR 332663

[Shi76] ______, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804. MR 434962

[Shi78] ______, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), no. 3, 637–679. MR 507462

[Shi80] ______, *The arithmetic of certain zeta functions and automorphic forms on orthogonal groups*, Ann. of Math. (2) **111** (1980), no. 2, 313–375. MR 569074

[Shi83] ______, *Algebraic relations between critical values of zeta functions and inner products*, Amer. J. Math. **105** (1983), no. 1, 253–285. MR 692113

[Shi87] ______, *On Hilbert modular forms of half-integral weight*, Duke Math. J. **55** (1987), no. 4, 765–838. MR 916119

[Shi88] ______, *On the critical values of certain Dirichlet series and the periods of automorphic forms*, Invent. Math. **94** (1988), no. 2, 245–305. MR 958833

[Wal80] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. (9) **59** (1980), no. 1, 1–132. MR 577010

[Wal91] Jean-Loup Waldspurger, *Correspondances de Shimura et quaternions*, Forum Math. **3** (1991), no. 3, 219–307. MR 1103429

[Xue17] Hang Xue, *Refined global Gan-Gross-Prasad conjecture for Fourier-Jacobi periods on symplectic groups*, Compos. Math. **153** (2017), no. 1, 68–131. MR 3622873

[Xue18] ______, *Fourier-Jacobi periods of classical Saito-Kurokawa lifts*, Ramanujan J. **45** (2018), no. 1, 111–139. MR 3745067

[Xue19] ______, *Central values of degree six L-functions*, J. Number Theory **203** (2019), 350–359. MR 3991408

[Yos94] Hiroyuki Yoshida, *On the zeta functions of Shimura varieties and periods of Hilbert modular forms*, Duke Math. J. **75** (1994), no. 1, 121–191. MR 1284818

[Yos95] ______, *On a conjecture of Shimura concerning periods of Hilbert modular forms*, Amer. J. Math. **117** (1995), no. 4, 1019–1038. MR 1342839