ENRICHED ENUMERATIVE GEOMETRY

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Abstract. Enumerative geometry is the subfield of algebraic geometry concerned with counting the number of solutions to geometric questions. Classically one restricts to working over algebraically closed fields to get invariant answers to such problems. Enriched enumerative geometry uses tools from motivic homotopy theory (i.e., homotopy theory for schemes) to obtain invariant results over more general fields. In this paper, we give an overview of some standard techniques from enumerative geometry, both classical and enriched. We then restrict our attention to the problem of counting the number of lines of the complete intersection of two degree $n-2$ hypersurfaces in $\mathbb{P}^n_k$ when n is even.

Contents

1. Introduction

It is a classical result in algebraic geometry that there are 27 lines on a smooth cubic surface over an algebraically closed field k. In particular, the number of lines does not depend on the choice of smooth cubic. However, when k is not algebraically closed, this is no longer the case. For example, when $k = \mathbb{R}$, a smooth cubic can have $3, 7, 15$, or 27 lines. While a naive count of the number of lines on smooth cubic surfaces over arbitrary fields does not always give a fixed number, we hope that there is an invariant "enriched count", independent of the choice of smooth cubic, of the number of lines.

The general strategy for solving counting problems in algebraic geometry is to express the solution set as the zero locus of a section s of a vector bundle E over some moduli space X. In the smooth cubic case, X is chosen to be the Grassmannian $\mathrm{Gr}_k(2,4)$, which parameterizes the two-dimensional subspaces of $k^{\oplus 4}$. Let S be the canonical subplane bundle over $\mathrm{Gr}_k(2,4)$ and let f be the cubic polynomial that cuts out a smooth cubic $\Sigma \hookrightarrow \mathbb{P}^3_k$. One can associate a section σ of $\mathrm{Sym}^3(S^\vee)$ to f. Doing an appropriate "weighted" sum of the zeroes of σ yields a meaningful invariant.

To motivate the weighted sum proposed by Kass and Wickelgren, we examine a more classical case. In topology, an oriented (real) vector bundle over a manifold X is a vector bundle E and a trivialization of $\det(E)$. Let r be the rank of E. Then E has an element $e(E) \in H^r(X;\mathbb{Z})$ called the Euler class associated to it. Let s be a section of E and z an isolated zero of s. Assume that X is an oriented manifold of dimension r. Then s induces an isomorphism $ds : T_z(X) \longrightarrow T_z(X) \times E_z \longrightarrow E_z$. If this isomorphism preserves orientation (respectively, reverses orientation), then we define the sign $\mathrm{sgn}_z(s)$ of s at z to be 1 (respectively, -1). The Euler number can then be computed as

$$\sum_{z \in Z(s)} \mathrm{sgn}_z(s).$$

There is a homotopical interpretation of computing the sign. Note that since $ds : T_z(X) \longrightarrow E_z$ is proper, it induces an isomorphism

$$\widetilde{ds} : \mathbb{S}^r \longrightarrow \mathbb{S}^r.$$

of the one point compactification. Passing $\widetilde{ds}$ to the 0th stable homotopy group $\pi_0^s \cong \mathbb{Z}$ gives us either 1 or -1, which corresponds to $\mathrm{sgn}_z(s)$.

We now shift our attention back to the algebro-geometric case. Let $E \longrightarrow X$ be a vector bundle of rank r over a scheme X of dimension r and s be a section of E with isolated zeroes. Under nice situations, one can locally use s to compute an algebro-geometric analog of the Euler number. However, instead of the integers, this version of the Euler number is an element of the Grothendieck-Witt ring $GW(k)$, reflecting the fact that the 0th stable motivic homotopy group over k is $GW(k)$.

Let l be a line on a smooth cubic $\Sigma \longrightarrow \mathbb{P}^3_k$ defined by a homogeneous cubic polynomial f. Then l corresponds to a closed point p in $\mathrm{Gr}_k(2,4)$. We call the residue field L at this point the field of definition of l. We also get that f defines a section σ_f of $\mathrm{Sym}^3(S^\vee)$ where S is the canonical subplane bundle over $\mathrm{Gr}_k(2,4)$. Moreover, the closed points in the zero locus of σ_f are precisely the lines that lie on Σ. In [KW21], Kass and Wickelgren compute the Euler class $\tilde{e}(\mathrm{Sym}^3(S^\vee))$ locally using the section σ_f. They define the type $\alpha \in L^*/(L^*)^2$ of a line l and show that the lines on Σ satisfy

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{L/k}(\langle \alpha \rangle) = 15\langle 1 \rangle + 12\langle -1 \rangle.$$

We wish to use the ideas from [KW21] to get an equation similar to theirs for the number of lines on the complete intersection Σ of two degree $n-2$ hypersurfaces in $\mathbb{P}^n_k$ when n is even. This will generalize the result of [Dar22] on the number of lines on a degree 4 del Pezzo surface (equivalently, a complete intersection of two degree 2 hypersurfaces in $\mathbb{P}^4_k$). Most of our arguments are very straightforward generalizations of those in [Dar22]. For the case considered in [Dar22] (i.e., when $n = 4$), one component of the argument relies on the fact that for algebraically closed k, the number of lines on Σ is known. However, when n is an arbitrary even integer, this number is not known. To get around this problem, we use the splitting principle of Chern classes and some basic computational results from Schubert calculus.

2. Background

In this section, we give a brief overview of the terminology and some basic results. k is always assumed to be a perfect field of characteristic not equal to 2. Since we are primarily interested in counting lines on closed subschemes of $\mathbb{P}^n_k$, our first order of business is to define what a line in $\mathbb{P}^n_k$ even is. For algebraically closed k, we can think of a line in $\mathbb{P}^n_k$ as a linear embedding $l : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^n_k$. We say two linear embeddings $l_1, l_2 : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^n_k$ define the same line if there is automorphism $\varphi : \mathbb{P}^1_k \hookrightarrow \mathbb{P}^1_k$ over $\mathbb{P}^n_k$.

For non-algebraically closed k, the definition of a line on $\mathbb{P}^n_k$ is more subtle. We wish for lines to also include linear embeddings $l : \mathbb{P}^1_{k'} \hookrightarrow \mathbb{P}^n_{k'}$ where k' is a finite extension of k. We also want two embeddings $l_1 : \mathbb{P}^1_{k_1} \longrightarrow \mathbb{P}^n_{k_1}$ and $l_2 : \mathbb{P}^1_{k_2} \longrightarrow \mathbb{P}^n_{k_2}$ to define the same line if their pullbacks to the composite field k_1k_2 are isomorphic over $\mathbb{P}^n_k$.

Definition 2.1. A line l in $\mathbb{P}^n_k$ is a closed point in the Grassmannian $\mathrm{Gr}_k(2, n+1)$. The residue field of l is called the **field of definition** of l.

Note that a point l in $\mathrm{Gr}_k(2, n+1)$ with residue field L determines linear embeddings $\mathbb{A}^2_L \longrightarrow \mathbb{A}^{n+1}_L$ and $\mathbb{P}^1_L \longrightarrow \mathbb{P}^n_L$. To see this, let $V \subseteq L^{\oplus(n+1)}$ be the vector space associated with the pullback of the canonical subplane bundle under the morphism $\mathrm{Spec}(L) \hookrightarrow \mathrm{Gr}_k(2, n+1)$. We then consider the dual $(L^{\oplus(n+1)})^\vee \longrightarrow V^\vee$ and apply $\mathit{Spec}\,(\mathrm{Sym}^\bullet(-))$ to it to get

$$\mathbb{A}^2_L \longrightarrow \mathbb{A}^{n+1}_L.$$

Note that $(L^{\oplus(n+1)})^\vee \longrightarrow V^\vee$ is surjective. Thus, by Lemma 27.11.13 in [Sta23, Tag 01MX], we can apply $\mathit{Proj}\,(\mathrm{Sym}^\bullet(-))$ to get the morphism

$$\mathbb{P}^1_L \longrightarrow \mathbb{P}^n_L.$$

Definition 2.2. Given a closed subscheme $\Sigma \hookrightarrow \mathbb{P}^k_n$ and a line l in $\mathbb{P}^n_k$, we say ***l* lies on Σ** if the composition

$$\mathbb{P}^1_L \xrightarrow{l} \mathbb{P}^n_L \to \mathbb{P}^n_k$$

has image in Σ.

Let $\Sigma \hookrightarrow \mathbb{P}^n_k$ be the hypersurface defined by a homogeneous degree d polynomial f. Let X be the Grassmannian $\mathrm{Gr}_k(2, n+1)$ and let S be the canonical subplane bundle over X. Let l be a closed point of $\mathrm{Gr}_k(2, n+1)$. Then l defines a linear embedding $\mathbb{A}^2_L \longrightarrow \mathbb{A}^{n+1}_L$. Under this embedding, f pulls back to a homogeneous polynomial of degree d in two variables. This polynomial can be identified with an element of $\mathrm{Sym}^d(S_l^\vee)$. In this way f determines a section σ_f of $\mathrm{Sym}^d(S^\vee)$.

Definition 2.3. We define the group $GW(k)$ as the group completion of the monoid of isomorphism classes of symmetric non-degenerate bilinear forms over k (where the group operation is given by the direct sum). The multiplicative structure on $GW(k)$ is given by the tensor product.

Since our enriched count will be done in the Grothedieck-Witt ring $GW(k)$, it is better to consider a more explicit description of $GW(k)$.

Proposition 2.4. *Let k be a field of characteristic not equal to 2. Then $GW(k)$ can be described as the abelian group generated by the symbols $\langle a \rangle$ for $a \in k^\times$ subject to the following relations*

- $\langle ab^2\rangle = \langle a\rangle$
- $\langle a\rangle + \langle -a\rangle = \langle 1\rangle + \langle -1\rangle$
- $\langle a\rangle + \langle b\rangle = \langle a+b\rangle + \langle (a+b)ab\rangle$ *when* $a \neq -b$

Moreover, the ring structure is given by $\langle a\rangle\langle b\rangle = \langle ab\rangle$*. The element* $\langle a\rangle$ *corresponds to (the equivalence class of) the bilinear form* $k \otimes_k k \longrightarrow k$ *given by* $x \otimes y \mapsto axy$*.*

Recall that when computing the Euler class of an oriented real vector bundle, the orientation allows one to "push forward" the local contributions at different zeroes of the section to $\pi_0^s \cong \mathbb{Z}$. In the algebro-geometric case, we want a similar notion of orientation to push forward to $GW(k)$ (the motivic 0th stable homotopy group).

Definition 2.5. We say a vector bundle E over a smooth k-scheme X is said to be **relatively orientable** if there exists a line bundle $\mathcal{L}$ and an isomorphism $\rho : \det(E) \otimes \omega_{X/k} \longrightarrow L^{\otimes 2}$. The isomorphism ρ is called a **relative orientation** of E and the pair (E, ρ) is called a **relatively oriented vector bundle**.

Let (E, ρ) be a relatively oriented vector bundle over a smooth k-scheme X where $\mathrm{rk}E = \dim X = r$ and let s be a section of E with isolated zeroes. Let z be a zero of s and U open around z such that there is an isomorphism $u : U \xrightarrow{\sim} \tilde{U} \xhookrightarrow{open} \mathbb{A}_k^r$. Note that the standard coordinate vector fields of $\mathbb{A}_k^r$ determine a basis $\{\partial_{u_1}|_z, \ldots, \partial_{u_r}|_z\}$ of $(T_X)_z$. Suppose there is a trivialization of $E|_U$ by sections $\{e_1, \ldots, e_r\}$ such that $\det(T_X)_z \longrightarrow \det(E_z)$ given by

$$\partial_{u_1}|_z \wedge \cdots \wedge \partial_{u_r}|_z \longrightarrow e_1 \wedge \cdots \wedge e_r$$

is a square in $\mathrm{Hom}(\det(T_X)_z, \det E_z) \cong (\omega_{X/k})_z \otimes \det E_z \cong L_z^{\otimes 2}$. Then we make the following definition:

Definition 2.6. With notation as in the previous paragraph, assume also that the morphism $s_{u,e} : \widetilde{U} \longrightarrow \mathbb{A}_k^r$ is étale at z. Then $u : U \xrightarrow{\sim} \widetilde{U} \xhookrightarrow{open} \mathbb{A}_k^r$ along with the trivialization $\{e_1, \ldots, e_r\}$ is called a **good parametrization** near z.

Let $k(z)$ be the residue field at z. Taking the pullback of $s_{u,e}$ to $\mathrm{Spec}(k(z))$, we get a $k(z)$-morphism

$$s'_{u,e} : \widetilde{U}_{k(z)} \longrightarrow \mathbb{A}^r_{k(z)}.$$

Note that this morphism can be naturally identified by r global sections of $\widetilde{U}_{k(z)}$. Let $(s'_{u,e})_1, \ldots, (s'_{u,e})_r$ be these sections. We can then evaluate the Jacobian matric $\left(\frac{\partial (s'_{u,e})_i}{\partial u'_j}\right)$ at z. Since the morphism is smooth, the determinant of the Jacobian is not zero. Thus, we have an element $\left\langle \det\left(\frac{\partial (s'_{u,e})_i}{\partial u'_j}\right)_z \right\rangle$ of $GW(k(z))$.

However, we want an element of $GW(k)$. To get that, we define a morphism

$$\mathrm{Tr}_{k(z)/k} : GW(k(z)) \longrightarrow GW(k).$$

Let a be a non-zero element of $k(z)$. Then the element $\langle a\rangle$ corresponds to a morphism $k(z) \otimes_{k(z)} k(z) \longrightarrow k(z)$ given by $x \otimes y \mapsto axy$. Composing on the left by the natural map $k(z) \otimes_k k(z) \longrightarrow k(z) \otimes_{k(z)} k(z)$ and on the right by $\mathrm{Tr}_{k(z)/k} : k(z) \longrightarrow k$, we get a symmetric bilinear form over k, which we call $\mathrm{Tr}_{k(z)/k}\langle a\rangle$. Moreover, since k is perfect, the map $\mathrm{Tr}_{k(z)/k}$ is surjective. Thus, we get $\mathrm{Tr}_{k(z)/k}\langle a\rangle \in GW(k)$. Applying $\mathrm{Tr}_{k(z)/k}$ to $\left\langle \det\left(\frac{\partial (s'_{u,e})_i}{\partial u'_j}\right)_z \right\rangle$ gives us a rank

$[k(z):k]$ element of $GW(k)$. In [KW21], Kass and Wickelgren prove that

$$\mathrm{Tr}_{k(z)/k}\left\langle \det\left(\frac{\partial (s'_{u,e})_i}{\partial u'_j}\right)_z\right\rangle$$

is well-defined as long as a good parametrization is used.

Definition 2.7. Let (E,ρ) be a relatively oriented vector bundle over a smooth k-scheme X where $\mathrm{rk}E = \dim X = r$ and let s be a section of E whose zero locus consists of finitely many closed points. Assuming all zeroes of s have a good parametrization around them, we define the Jacobian form

$$\mathrm{Tr}_{k(z)/k}\langle \mathrm{Jac}_z(s,\rho)\rangle$$

at z to be

$$\mathrm{Tr}_{k(z)/k}\left\langle \det\left(\frac{\partial (s'_{u,e})_i}{\partial u'_j}\right)_z\right\rangle,$$

where the latter is computed using any good parametrization around z.

3. Statement of the main theorem

Let n be even and let $\Sigma \hookrightarrow \mathbb{P}^n_k$ be the complete intersection of two (general) degree $n-2$ hypersurfaces $Z_1 = Z(f_1)$ and $Z_2 = Z(f_2)$. Then f_1 and f_2 correspond to sections σ_1 and σ_2 of $\mathrm{Sym}^{n-2}(S^\vee)$ over $X = \mathrm{Gr}_k(2,n+1)$. The lines on Σ are precisely the zeroes of the section $\sigma_1\oplus\sigma_2$ of $E = \mathrm{Sym}^{n-2}(S^\vee)\oplus\mathrm{Sym}^{n-2}(S^\vee)$. Since f_1 and f_2 are general, $Z(\sigma_1\oplus\sigma_2)$ consists of finitely many closed points [DM98].

Proposition 3.1. *$Z(\sigma_1\oplus\sigma_2)$ is étale over k.*

Proof. Since the residue field at each point is a finite extension of k and k is perfect, we get that $Z(\sigma_1\oplus\sigma_2)$ is étale over k. □

Proposition 3.2. *[Sta23, Tag 089R] $Gr_k(2,n+1)$ can be covered by open sets isomorphic to $\mathbb{A}^{2(n-1)}_k$.*

We now wish to use the results from the previous section to get an invariant on the number of lines on Σ. However, we run into a problem: E is not relatively orientable. To see this, Consider the Plücker embedding $X \hookrightarrow \mathbb{P}_k^{\frac{n(n+1)}{2}-1}$. Then the Picard group of X is generated by $\mathcal{O}_X(1)$, the pullback of $\mathcal{O}_{\mathbb{P}_k^{\frac{n(n+1)}{2}-1}}(1)$. We now show $\det(E)\otimes\omega_{X/k}$ is not a square in the Picard group. First, we compute $\omega_{X/k}$. Consider the tautological short exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}_X^{n+1} \longrightarrow Q \longrightarrow 0$$

This implies $\det(S)\otimes\det(Q)\cong\mathcal{O}_X$. Since the pullback of the tautological bundle via the Plücker embedding is $\det(S)$, we get $\det(S)\cong\mathcal{O}(-1)$ and $\det(Q)\cong\mathcal{O}(1)$. Using the isomorphism $\Omega_{X/k}\cong S\otimes Q^\vee$, we get $\omega_{X/k}\cong\mathcal{O}(-1)^{\otimes(n-1)}\otimes\mathcal{O}(-1)^{\otimes 2}\cong\mathcal{O}(-n-1))$. We note that $\omega_{X/k}$ is not a square as n is even. Since E is the direct sum of a vector bundle with itself, its determinant must be a square. Thus, we conclude E is not relatively orientable.

We can compute $\det(E)$ explicitly. First, note that since S is of rank 2, we have

$$\begin{aligned}\det(\mathrm{Sym}^{n-2}(S^\vee)) &\cong (det(S^\vee))^{\binom{2+(n-2)-1}{2}} \\ &\cong (det(S^\vee))^{\otimes\frac{(n-1)(n-2)}{2}} \\ &\cong \mathcal{O}\left(\frac{(n-1)(n-2)}{2}\right).\end{aligned}$$

Using this, we compute

$$\det(E) \cong \mathcal{O}((n-1)(n-2)).$$

Thus, we have

$$\det(E)\otimes\omega_{X/k} \cong \mathcal{O}((n-1)(n-2)-(n+1)).$$

To fix the issue of E not being relatively orientable, we define a different vector bundle $\widetilde{E}$ such that we can "carry over" the local calculations on it. We define $\widetilde{E}$ to be the vector bundle

$$\mathcal{O}(1)\otimes\mathrm{Sym}^{n-2}(S^\vee)\oplus\mathcal{O}(2)\otimes\mathrm{Sym}^{n-2}(S^\vee).$$

Proposition 3.3. *$\widetilde{E}$ is relatively orientable.*

Proof. Using the computations in the previous paragraph, we have

$$\begin{aligned}\det(\widetilde{E}) &= \det(\mathcal{O}(1)\otimes\mathrm{Sym}^{n-2}(S^\vee)\oplus\mathcal{O}(2)\otimes\mathrm{Sym}^{n-2}(S^\vee)) \\ &= \mathcal{O}\left(n-1\right)\otimes\mathcal{O}\left((n-1)(n-2)\right)\otimes\mathcal{O}\left(2(n-1)\right)\otimes\mathcal{O}\left((n-1)(n-2)\right) \\ &= \mathcal{O}\left((n-1)+2(n-1)^2\right).\end{aligned}$$

Note that $\det(\widetilde{E})$ is not a square as n is even. Since $\omega_{X/k}$ is not a square either, their tensor product is a square. Chasing through the isomorphisms in our computations, we get a canonical relative orientation ρ for $\widetilde{E}$. □

Let s be a section of $O(1)$ such that its zeroes are distinct from the zeroes of $\sigma_1\oplus\sigma_2$. We call s a **one-form non-degenerate on the lines on $\boldsymbol{\Sigma}$**. Let $U = X - Z(s)$. Then the morphism $E|_U \longrightarrow \widetilde{E}|_U$ given by $\alpha\oplus\beta\mapsto s\otimes\alpha\oplus s^{\otimes 2}\otimes\beta$ is an isomorphism. Of course, there is still the issue of proving the existence of a one-form non-degenerate on the lines on Σ.

Proposition 3.4. *If k is an infinite field, then there is a one-form non-degenerate on the lines on Σ.*

To prove this proposition, we make use of the following lemma.

Lemma 3.5. *Let k be a field and c a positive integer such that $|k| > c$. Let $p_1,\ldots,p_c$ be points in $\mathbb{P}^q_k$ for some q. Then there exists a section s of $\mathcal{O}_{\mathbb{P}^q_k}(1)$ such that $Z(s)$ does not contain any of the points $p_1,\ldots,p_c$.*

Proposition 3.5 follows by viewing $X\hookrightarrow\mathbb{P}_k^{\frac{n(n+1)}{2}-1}$ via the Plücker embedding, noting $Z(\sigma_1\oplus\sigma_2)$ is finite, using Lemma 3.5 to find a section $s\in\mathcal{O}_{\mathbb{P}_k^{\frac{n(n+1)}{2}-1}}(1)$ disjoint from $Z(\sigma_1\oplus\sigma_2)$, and taking its pullback. We will generalize Proposition 3.5 to sufficiently large finite fields later. For now, note that Proposition 3.5 guarantees the existence of a one-form non-degenerate on the lines on Σ when k is algebraically closed.

Definition 3.6. Continuing the notation from this section, we write $\widetilde{\sigma}$ for $s \otimes \sigma_1 \oplus s^{\otimes 2} \otimes \sigma_2$ and define

$$\mathrm{Tr}_{k(z)/k} \left\langle \widetilde{\mathrm{Jac}}_z(f_1, f_2; s) \right\rangle := \mathrm{Tr}_{k(z)/k} \left\langle \mathrm{Jac}_z(\widetilde{\sigma}; \rho) \right\rangle .$$

We are now ready to state the main result.

Theorem 3.7. *Let Σ be the complete intersection of two degree $n-2$ hyperplanes $Z(f_1)$ and $Z(f_2)$ in $\mathbb{P}^n_k$ where n is even. Let s be a one-form non-degenerate on the lines on Σ. Then*

$$\sum_{\textit{lines } l \textit{ on } \Sigma} Tr_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle = m_n \left(\langle 1 \rangle + \langle -1 \rangle \right)$$

for some non-negative integer m_n.

The proof for this uses oriented intersection theory and follows from the exact same argument as the one in [Dar22].

4. Oriented Intersection Theory

All the work in this section is done to get an alternate expression for

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle ,$$

which we show is almost the Euler number of $\widetilde{E}$, differing only by the local contributions at the zeroes of s. We then show that both the Euler number and the local contributions are multiples of H, proving Theorem 3.7.

Let $p : X \longrightarrow \mathrm{Spec}(k)$ be smooth and proper of dimension r and E a rank r vector bundle over X. We define $e(E)$ as the top Chern class of E. Let $z : X \longrightarrow E$ be the zero section and $s : X \longrightarrow E$ a section such that $Z(s) \longrightarrow X$ is a regular embedding. Consider the Cartesian square

$$\begin{array}{ccc} Z(s) & \xrightarrow{j} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{z} & E \end{array}$$

Here i and j are both the inclusion morphism, but we label them differently to avoid a notational nightmare involving pullbacks and pushforwards. Since z is a regular embedding of codimension r, there is a refined Gypsin morphism

$$z^! : CH^*(X) \longrightarrow CH^{*+r-\mathrm{codim}(Z(s))}(Z(s))$$

is defined by $z^![V] = [Z(s) \cdot V]$. Composing $z^!$ with the pushforward

$$i_* : CH^*(Z(s)) \longrightarrow CH^{*+\mathrm{codim}(Z(s))}(X)$$

gives us a morphism

$$\phi : CH^*(X) \longrightarrow CH^{*+r}(X)$$

We write down an explicit description of ϕ. Since j is regular, it consists of regularly embedded clopen components $j_m : Z_m \longrightarrow X$. Let $\mathcal{N}_{X/Z_m}$ be the normal bundle of j_m (this is indeed a bundle as j_m is regular). Then there is a natural injective morphism $\mathcal{N}_{X/Z_m} \longrightarrow j_m^* \mathcal{N}_{E/X}$ where we embed X in E via the zero section, in which case $\mathcal{N}_{E/X} = E$. We define the excess bundle $\mathcal{E}_m$ as the quotient $j_m^* E / \mathcal{N}_{X/Z_m}$.

Given $\alpha \in CH^*(X)$, the excess intersection formula implies

$$\phi(\alpha) = \sum_m i_*(e(\mathcal{E}_m)(j_m^*(\alpha))).$$

Defining

$$\mathrm{ind}_{z_m}(s) := p_*(i_*(j_m^*([X]))) \in CH^0(k) = \mathbb{Z},$$

we get

$$p_*(e(E)([X])) = \sum_m \mathrm{ind}_{z_m}(s).$$

Barge-Morel in [BM00] and Fasel in [Fas08] define the Chow-Witt groups $\widetilde{CH}^*(X;\mathcal{L})$ of X twisted by a line bundle $\mathcal{L}$. They also define the Euler class

$$\tilde{e}(E) : \widetilde{CH}^*(X) \longrightarrow \widetilde{CH}^{*+r}(X; \det(E^\vee)).$$

These groups have natural morphisms to the ordinary Chow such that the following diagram commutes

$$\begin{array}{ccc} \widetilde{CH}^*(X) & \xrightarrow{\tilde{e}(E)} & \widetilde{CH}^{*+r}(X;\det(E^\vee)) \\ \downarrow & & \downarrow \\ CH^*(X) & \xrightarrow{e(E)} & CH^{*+r}(X) \end{array}$$

We wish to compute $\tilde{e}(E)$ as a sum of the local contributions using something analogous to the excess intersection formula. We note that Chow-Witt groups pull back under regular embeddings and push forward under proper morphisms. Namely, given a regular embedding $i : X \longrightarrow Y$, a proper morphism $f : X \longrightarrow Y$ of relative dimension d, and a line bundle $\mathcal{L}$ on Y, we have induced morphisms

$$\widetilde{CH}^*(Y;\mathcal{L}) \xrightarrow{i^*} \widetilde{CH}^*(X; i^*\mathcal{L}) \text{ and } \widetilde{CH}^*\left(X; f^*\mathcal{L}\otimes\omega_{X/k}\right) \xrightarrow{f_*} \widetilde{CH}^{*-d}\left(Y;\mathcal{L}\otimes\omega_{X/k}\right).$$

Let $[X] \in \widetilde{CH}^0(X)$ be the fundamental class. Since j_m^* is regular, we have an element $j_m^*[X] \in \widetilde{CH}^0(Z_m)$. Applying $\tilde{e}(\mathcal{E}_m)$ gives us

$$\tilde{e}(\mathcal{E}_m)(j_m^*[X]) \in \widetilde{CH}^{\mathrm{rk}(\mathcal{E}_m)}(Z_m;\det(\mathcal{E}_m^\vee)) = \widetilde{CH}^{r-\mathrm{codim}(Z_m)}(Z_m;\det(\mathcal{E}_m^\vee)).$$

Using the short exact sequences

$$0 \longrightarrow \mathcal{N}_{X/Z_m} \longrightarrow j_m^*E \longrightarrow \mathcal{E}_m \longrightarrow 0$$

and

$$0 \longrightarrow T_{Z_m/k} \longrightarrow T_{X/k} \longrightarrow \mathcal{N}_{X/Z_m} \longrightarrow 0,$$

we get

$$\det(\mathcal{E}_m^\vee) \cong j_m^*\det(E^\vee)\otimes\det(\mathcal{N}_{X/Z_m}) \cong j_m^*\det(E^\vee)\otimes j_m^*\omega_{X/k}^\vee\otimes\omega_{Z_m/k}.$$

Thus, we may think of

$$\tilde{e}(\mathcal{E}_m)(j_m^*[X]) \in \widetilde{CH}^{r-\mathrm{codim}(Z_m)}\left(X; j_m^*(\det(E^\vee)\otimes\omega_{X/k}^\vee)\otimes\omega_{Z_m/k}\right).$$

Applying the pushforward i_* and noting $i|_{Z_m} = j_m$, we get

$$i_*(\tilde{e}(\mathcal{E}_m)(j_m^*[X])) \in \widetilde{CH}^r(X;\det(E^\vee)).$$

In [Fas09], Fasel proves an oriented version of the excess intersection formula:

$$\tilde{e}(E)([x]) = \sum_m i^*(\tilde{e}(\mathcal{E}_m)(j_m^*([X])))$$

Similar to the classical case, we would like to push this forward through $p : X \longrightarrow \mathrm{Spec}(k)$ to an element $\widetilde{CH}^0(\mathrm{Spec}(k)) = GW(k)$. However, we run into a problem. The only line bundle on $\mathrm{Spec}(k)$ is the trivial one and so we cannot push forward elements of $\widetilde{CH}^d(X; \det(E^\vee))$ using the pushforward mentioned earlier unless $\det(E^\vee) = p^*\mathcal{O}_{\mathrm{Spec}(k)} \otimes \omega_{X/k} = \omega_{X/k}$. This is where relative orientability helps us out.

First, we note the following fact: given line bundles $\mathcal{L}, \mathcal{L}'$, and $\mathcal{L}''$ over X and an isomorphism $\psi : \mathcal{L}' \longrightarrow \mathcal{L}'' \otimes \mathcal{L}^{\otimes 2}$, there is an induced isomorphism

$$\psi : \widetilde{CH}^*(X; \mathcal{L}') \longrightarrow \widetilde{CH}^*(X; \mathcal{L}'').$$

Thus, if (E, ρ) is a relative orientation of E, we have an isomorphism

$$\widetilde{CH}^*(X; \det(E^\vee)) \xrightarrow{\rho} \widetilde{CH}^*(X; \omega_{X/k}).$$

We define the augmented pushforward $p_*^\rho : \widetilde{CH}^*(X; \det(E^\vee)) \longrightarrow \widetilde{CH}^{*-r}(\mathrm{Spec}(k))$ as the composition

$$\widetilde{CH}^*(X; \det(E^\vee)) \xrightarrow{\rho} \widetilde{CH}^*(X; \omega_{X/k}) \xrightarrow{p_*} \widetilde{CH}^{*-r}(\mathrm{Spec}(k)).$$

Definition 4.1. Let (E, ρ) be a rank r oriented vector bundle over $p : X \longrightarrow \mathrm{Spec}(k)$ smooth and proper of dimension r. Let s be a section of E whose $i : Z(s) \longrightarrow X$ is regularly embedded. Let z be an isolated zero of s and $j_z : \{z\} \longrightarrow X$ be the inclusion. Then we define the local oriented index at z to be

$$\mathrm{ind}_z^{\mathrm{or}}(s; \rho) = p_*^\rho(i^*(\tilde{e}(\mathcal{E}_z)(j_z^*([X])))).$$

By Proposition 2.31 in [BW23], the local oriented index at z agrees with the Jacobian form at z when E is relatively oriented. We note this in the following proposition.

Proposition 4.2. *Let $p : X \longrightarrow Spec(k)$ be smooth and proper of dimension r and (E, ρ) a relatively oriented vector bundle of rank r over X. Let s be a section of E and z an isolated zero of s admitting a good parameterization. Then*

$$Tr_{k(z)/k}\langle Jac_z(s, \rho)\rangle = ind_z^{or}(s; \rho).$$

5. Proof of the main result

We are now ready to prove Theorem 3.7. Let Σ be the complete intersection of two degree $n-2$ hyperplanes $Z(f_1)$ and $Z(f_2)$ in $\mathbb{P}_k^n$ where n is even. Let E and $\widetilde{E}$ be as defined in Section 3. Let s be a one-form non-degenerate on the lines on Σ. We show that

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle$$

is almost the Euler number of $\widetilde{E}$, differing only by the local contributions at the zeroes of s. First, note that

$$Z(\widetilde{\sigma}) = Z(s) \coprod Z(\sigma_1 \oplus \sigma_2)$$

where $\widetilde{\sigma} = s \otimes \sigma_1 \oplus s^{\otimes 2} \otimes \sigma_2$. To verify $Z(\widetilde{\sigma}) \hookrightarrow X$ is regular, it suffices to show $Z(s) \hookrightarrow X$ and $Z(\sigma_1 \oplus \sigma_2) \hookrightarrow X$ are regular. The latter being regular follows from $Z(\sigma_1 \oplus \sigma_2)$ and X being smooth over k. $Z(s) \hookrightarrow X$ being regular follows from the Grassmannian being integral and s not being the 0 section.

Let $Z_0 := Z(s)$ and let $Z_1, \ldots, Z_m$ be the closed points of $Z(\sigma_1 \oplus \sigma_2)$. Let $\mathcal{E}_k$ be the excess bundle of $Z_k \longrightarrow X, j_k : Z_m \longrightarrow X$ the inclusion of the clopen components, and $i : Z(\widetilde{\sigma}) \longrightarrow X$ the inclusion of the entire zero locus. Applying p_*^ρ to the oriented excess intersection formula, we have

$$p_*^\rho(\tilde{e}(\widetilde{E})([X])) = \sum_{k=0}^{m} p_*^\rho(i^*(\tilde{e}(\mathcal{E}_k)(j_k^*([X])))).$$

We can split the sum on the right side to get

$$p_*^\rho(\tilde{e}(\widetilde{E})([X])) = p_*^\rho(i^*(\tilde{e}(\mathcal{E}_0)(j_0^*([X])))) + \sum_{z \in Z(\sigma_1 \oplus \sigma_2)} \mathrm{ind}_z^{or}(\widetilde{\sigma}; \rho).$$

Note that since n is even, $\widetilde{E}$ has an odd-rank summand. Using Theorem 1.3 from [Ana20], we get $p_*^\rho(\tilde{e}(\widetilde{E}([X])))$ is a multiple of H. Since X has even dimension and $Z(s)$ is codimension 1, the excess bundle $\mathcal{E}_0$ has odd-rank. Using Theorem 1.3 from [Ana20] again, we get that $p_*^\rho(i^*(\tilde{e}(\mathcal{E}_0)(j_0^*([X]))))$ is also a multiple of H. Applying Proposition 4.2, we conclude

$$\sum_{z \in Z(\sigma_1 \oplus \sigma_2)} \mathrm{ind}_z^{or}(\widetilde{\sigma}; \rho) = \sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle$$

is a multiple of H.

Now that we know

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle = m_n \left(\langle 1 \rangle + \langle -1 \rangle\right)$$

for some non-negative integer m_n, we would like compute what m_n is explicitly. First, note that Theorem 13.3.1 of [Fas08] implies

$$\mathrm{rk}\left(\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle\right) = p_*(e(E)([X])).$$

Combining this result with the observation that $\mathrm{rk}\left((\langle 1 \rangle + \langle -1 \rangle)\right) = 2$, we get

$$m_n = \frac{p_*(e(E)([X]))}{2}.$$

We would like a description of $e(E)([X])$ in terms of the Chern classes of S. Using the splitting principle, we may treat our computation as if there exists an exact sequence

$$0 \longrightarrow L_1 \longrightarrow S \longrightarrow L_2 \longrightarrow 0$$

where L_1 and L_2 are line bundles. Let $c(L_1) = 1 + x_1$ and $c(L_2) = 1 + x_2$. Then $c(S) = 1 + s_1 + s_2$ where $s_1 = x_1 + x_2$ and $s_2 = x_1 \cdot x_2$. Using some basic properties of Chern classes, we have

$$c(\mathrm{Sym}^{n-2}(S^\vee)) = \prod_{i=0}^{n-2} (1 + (-ix_1 - (n-2-i)x_2))$$

We then get

$$
\begin{aligned}
e(E) &= \left(\prod_{i=0}^{n-2}(-ix_1-(n-2-i)x_2)\right)^2 \\
&= \left(\binom{n-2}{2} s_1 \prod_{i=0}^{\frac{n-2}{2}-1}(i^2+(n-2-i)^2)s_2 + i(n-2-i)(s_1^2-2s_2)\right)^2
\end{aligned}
$$

Now, note that the top Chow group $CH^{2(n-1)}(\mathrm{Gr}_k(2,n+1))$ is generated by the Schubert cycle $\sigma_{(n-1,n-1)}$. Under the pushforward of the structure morphism $\mathrm{Gr}_k(2,n+1) \longrightarrow \mathrm{Spec}(k)$, the cycle $\sigma_{(n-1,n-1)}$ gets mapped to 1. Thus, to compute $p_*(e(E)([X]))$, it suffices to write $e(E)([X])$ as a multiple of $\sigma_{(n-1,n-1)}$. We do this using some basic results of Schubert classes.

First, note that $s_1 = -\sigma_1$ and $s_2 = \sigma_{1,1}$. We can expand our expression for the Euler class to express it as a sum of terms of the form $\sigma_1^a\sigma_{1,1}^b$ where $a+2b = 2(n-1)$. Our computation now reduces to finding the coefficient when $\sigma_1^a\sigma_{1,1}^b$ is expressed as an integer multiple of $\sigma_{n-1,n-1}$ Using Proposition 4.11 from Chapter 4 of [EH16], for any Schubert cycle $\sigma_{p,q}$, we have

$$\sigma_{1,1}\cdot\sigma_{p,q} = \sigma_{p+1,q+1} \text{ and } \sigma_1\cdot\sigma_{p,q} = \sigma_{p+1,q}+\sigma_{p,q+1},$$

interpreting $\sigma_{r,s} = 0$ if $r < s$ or $r > n-1$. Repeatedly applying these identities gives us an expression of $\sigma_1^a\sigma_{1,1}^b$ as an integer multiple of $\sigma_{n-1,n-1}$. In this way, we get an algorithm to determine m_n for any n. This computation also shows that m_n does not depend on the choice of field k.

As an example, we compute m_4. We have

$$
\begin{aligned}
e(E) &= \left(\prod_{i=0}^{n-2}(-ix_1-(n-2-i)x_2)\right)^2 \\
&= 16\left(\sigma_1^2\cdot\sigma_{1,1}^2\right) \\
&= 16\left(\sigma_1^2\cdot\sigma_{2,2}\right) \\
&= 16\left(\sigma_1\cdot\sigma_{3,2}\right) \\
&= 16\left(\sigma_{3,3}\right)
\end{aligned}
$$

We get $m_4 = 8$ and deduce the equality

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k}\left\langle \widetilde{Jac}_z(f_1,f_2;s)\right\rangle = 8\left(\langle 1\rangle + \langle -1\rangle\right)$$

of Theorem 1.2 of [Dar22]. We can compute the rank for larger n using the following SageMath script.

```
import sage.libs.lrcalc.lrcalc as lrcalc
def rank(n):
    S=[]
    for p in range(n):
        for q in range(p+1):
            S.append([p,q])
    def m(s,M): #represents multiplication of s, a simple cycle, by an element M of the Chow ring, viewed as a lower traingular n-1 x n-1 matrix
```

```
        N=matrix(n)
        for k in S:
            d=0
            for l in S:
                d=d+M[l[0],l[1]]*lrcalc.lrcoef(k,s,l)
            N[k[0],k[1]]=d
        return N
    def f(a,b): #represents s_1^a*s_2^b as a multiple of sigma_n-1,n-1
        N=matrix(n)
        N[0,0]=1
        for i in range(a):
            N=m([1,0],N)
        for j in range(b):
            N=m([1,1],N)
        cf=N[n-1,n-1]
        return (-1)^a*cf
    R.<x,y> = PolynomialRing(ZZ, 'x,y')
    h=(n-2)*x/2
    for i in range((n-2)/2):
        h=h*((i^2+(n-2-i)^2)*y+i*(n-2-i)*(x^2-2*y))
    g=h^2
    r=0
    for a in range(2*(n-1)+1):
        if a % 2 == 0:
            r=r+f(a,(n-1)-a/2)*g.coefficient({x:a,y:(n-1)-a/2})
    return r
```

We note the rank for small values of n.

n	rank(n)
4	16
6	111616
8	4782986496
10	735546407124992

6. Extension of the main result

Let Σ be the complete intersection of two degree $n-2$ hypersurfaces of $\mathbb{P}^n_k$ where n is even. The proof of Theorem 3.7 relies on the existence of a one-form s non-degenerate on the lines of Σ. As noted in Proposition 3.4, such an s always exists when k is an infinite field. In this section, we extend Proposition 3.4 (and consequently Theorem 3.7) to the case of sufficiently large finite fields. First, note that if there is a one-form s non-degenerate on the lines of Σ, then $|Z(\sigma_1 \oplus \sigma_2)|$ is bounded above by the rank of

$$\sum_{\text{lines } l \text{ on } \Sigma} \mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle$$

To see this, we apply Theorem 3.7 and note that

$$\mathrm{Tr}_{k(l)/k} \left\langle \widetilde{Jac}_z(f_1, f_2; s) \right\rangle$$

has positive rank as it is represented by a non-degenerate bilinear form $k(z) \otimes_k k(z) \longrightarrow k$.

Proposition 6.1. *Let k be a field such that $|k| > 2m_n$. Then there is a one-form non-degenerate on the lines on Σ.*

Proof. Let $\overline{\sigma}_1, \overline{\sigma}_2$, and $\overline{\Sigma}$ and be the pullbacks of σ_1, σ_2, and Σ to the algebraic closure of k. Then there exists a one-form s non-degenerate on the lines on $\overline{\Sigma}$. By our observation in the previous paragraph, we have

$$|Z(\overline{\sigma}_1 \oplus \overline{\sigma}_2)| \leq \mathrm{rk}\left(\sum_{l \in Z(\overline{\sigma}_1 \oplus \overline{\sigma}_2)} \mathrm{Tr}_{k(l)/k}\left\langle \widetilde{Jac}_z(f_1, f_2; s)\right\rangle\right) = 2m_n < |k|.$$

Thus, we get $|Z(\sigma_1 \oplus \sigma_2)| \leq |Z(\overline{\sigma}_1 \oplus \overline{\sigma}_2)| < |k|$. Lemma 3.6 implies the existence of a one-form s non-degenerate on the lines of Σ and extends our results to sufficiently large finite fields. □

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