KISIN VARIETIES ASSOCIATED TO REDUCIBLE GALOIS REPRESENTATIONS OF DIMENSION 2

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Contents

Abstract

We consider a semi-module decomposition of the Kisin variety associated to a 2-dimensional mod p reducible Galois representation $\overline{\rho}$ of a p-adic field K and a cocharacter μ. Upon associating a tuple of matrices b to the representation $\overline{\rho}$, there is a group theoretic description of the geometric points of the Kisin variety associated to $\overline{\rho}$ and μ, which we use to study these semi-modules. In order to do this, we note that b is determined up to a conjugation action and choose a representative b which allows us to compute the semi-modules and identify a finite set of cocharacters which correspond to the potentially nonempty semi-modules. We will also see that, for good choices of b and μ, all nonempty semi-modules are isomorphic to affine spaces.

1 Introduction

Fix a prime $p > 2$ and let K be a finite extension of $\mathbb{Q}_p$ with residue field k having degree f over $\mathbb{F}_p$. Fix an algebraic closure $\overline{K}$ of K and denote the absolute Galois group of K by G_K. Additionally, fix an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$. Let $\overline{\rho} : G_K \to \mathrm{Aut}(V_{\mathbb{F}})$ be a representation of G_K where $V_{\mathbb{F}}$ is an n-dimensional vector space over some finite extension $\mathbb{F}$ of $\mathbb{F}_p$. Given such a representation, one might ask in what ways it can be lifted to characteristic zero. More specifically, given a p-adic field F with residue field $\mathbb{F}$, what are the representations $\rho : G_K \to \mathrm{GL}_n(F)$ that complete the following diagram?

$$\begin{array}{ccc} G_K & \xrightarrow{\rho} & \mathrm{GL}_n(F) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow {\scriptstyle \mathrm{mod}\ p} \\ & & \mathrm{GL}_n(\mathbb{F}) \end{array}$$

Fix such a field F, denote the ring of integers of F by $\mathcal{O}_F$, and fix a uniformizer ϖ of $\mathcal{O}_F$. We define $\widehat{\mathcal{C}}_{\mathcal{O}}$ to be the category of complete local Noetherian $\mathcal{O}_F$-algebras with residue field $\mathbb{F}$ and consider the lifts of $\overline{\rho}$ to representations with coefficients A for any object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$. Specifically, we consider the flat deformations of $\overline{\rho}$ to A, roughly meaning the representations $\overline{\rho}_A : G_K \to \mathrm{Aut}(M)$ where M is a free A-module of rank n, $M \otimes_A \mathbb{F} \simeq V_{\mathbb{F}}$ as $\mathbb{F}[G_K]$-modules, and there exists a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ for which the G_K-representation given by $\mathcal{G}(\overline{K})$ is isomorphic to $\overline{\rho}_A$.

There is a functor $D_{\overline{\rho}}^{\mathrm{fl},\square} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ which sends an $\mathcal{O}_F$-module A in $\widehat{\mathcal{C}}_{\mathcal{O}}$ to the set of isomorphism classes of flat deformations of $\overline{\rho}$ to A. $D_{\overline{\rho}}^{\mathrm{fl},\square}$ is (pro)-representable by an element $R_{\overline{\rho}}^{\mathrm{fl},\square}$ of $\widehat{\mathcal{C}}_{\mathcal{O}}$, called the *universal flat deformation ring*. Moreover, for an embedding $K \to F$, there is a quotient $R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ of $R_{\overline{\rho}}^{\mathrm{fl},\square}$ which parametrizes the deformations of $\overline{\rho}$ satisfying a determinant condition given by μ.

In the case that K is unramified, the flat deformation ring $R_{\overline{\rho}}^{\mathrm{fl},\square}$ can be studied using Fontaine-Laffaille theory. More generally, in [Kis09], Kisin constructed a projective $R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$-scheme $X_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ which parameterizes the p-power torsion finite flat group schemes over $\mathcal{O}_K$ with generic fiber $\overline{\rho}$ satisfying a determinant condition given by μ. Moreover, the structure morphism $X_{\overline{\rho}}^{\mathrm{fl},\square,\mu} \to \operatorname{Spec} R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ becomes an isomorphism once p is inverted. As a result, $X_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ can be thought of as a (partial) resolution of $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$, which is in general quite singular.

The fiber of $X_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ over the closed point of $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$, which we denote by $C_\mu(\overline{\rho})$, is of particular interest because the connected components of $X_{\overline{\rho}}^{\mathrm{fl},\square,\mu} \otimes_{\mathcal{O}_F} F$ can often be related to those of $C_\mu(\overline{\rho})$ and because the finite flat models of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ are in bijection with $C_\mu(\overline{\rho})(\mathbb{F}')$ for any finite extension $\mathbb{F}'/\mathbb{F}$. In this paper, we will study $C_\mu(\overline{\rho})$, which was termed a Kisin variety by Pappas and Rapoport in [PR08], in the case where $n = 2$ and $\overline{\rho}$ is not irreducible.

Kisin conjectured the connected components of $C_\mu(b)$ to be given by open and closed subschemes on which the rank of the maximal multiplicative subobject and the maximal étale quotient are fixed. When the ramification index e is less than $p-1$, it is a special case of a theorem of Raynaud, [Con97, Theorem 1.6], that $C_\mu(\overline{\rho})$ consists of a single point. In the case where $n = 2$, this conjecture has been proven by Gee [Gee09], Hellman [Hel11], Imai [Ima10], and Kisin [Kis09]. In [CN20] Chen and Nie showed that, if $\overline{\rho}$ is absolutely irreducible and $\overline{\rho}$ is of a certain form or if $n = 3$, then $C_\mu(\overline{\rho})$ is connected. They also showed that Kisin's conjecture does not hold in general. The key tool they use is a semi-module decomposition of the Kisin variety into Iwahori strata. In this paper, we study the semi-modules of Kisin varieties in the case of a reducible 2-dimensional representations.

Fix a uniformizer π in K along with a consistent system (π_n) of p^nth roots of π. Define K_∞ to be the extension of K created by adjoining the roots π_n. Then the absolute Galois group G_{K_∞} of K_∞ is canonically isomorphic to the absolute Galois group of $k(\!(u)\!)$ [FW79]. Using this association and a result of Fontaine [Fon07, Proposition 1.2.6] the Tate twist of the restriction of $\overline{\rho}$ to a representation of G_{K_∞} can be associated to an étale φ-module with $\mathbb{F}$-coefficients (M, Φ) which is isomorphic as an étale φ-module to $((k \otimes_{\mathbb{F}_p} \mathbb{F}(\!(u)\!))^n, b\varphi)$ for some $b \in \operatorname{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)(\mathbb{F}(\!(u)\!))$ where φ is the $(k \otimes_{\mathbb{F}_p} \mathbb{F}(\!(u)\!))^n$- endomorphism which acts the identity on $\mathbb{F}$ and as the p-power map on $k(\!(u)\!)$. Since restriction from G_K to G_{K_∞} is fully faithful in the flat case [Bre02, Theorem 3.4.3] and this category of étale φ-modules is isomorphic to the category of G_K-representations with coefficients in $\mathbb{F}$, the representation $\overline{\rho}$ is identified by this element b of $\mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_n)(\mathbb{F}((u)))$. With this in mind, we change notation for the Kisin variety, $C_\mu(b) := C_\mu(\overline{\rho})$, and define the *Kisin variety associated to* b *and* μ, called the closed Kisin variety in [PR08], to be the reduced projective $\mathbb{F}$-variety $C_\mu(b) \subset \mathcal{G}rass_G \otimes_{\mathbb{F}_p} \mathbb{F}$ with geometric points given by

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O}_L) | g^{-1} b \sigma(g) \in \bigsqcup_{\substack{\nu \in Y \\ \nu \leq \mu}} G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L)\}.$$

Associating $\overline{\rho}$ to b provides a group theoretic description of the geometric points of $C_\mu(b)$. Let $L = \overline{\mathbb{F}}_p((u))$ and let $\varphi : L \to L$ be the homomorphism that acts as the p-power map on u and as the identity on $\overline{\mathbb{F}}_p$. Let $G = \mathrm{Res}_{k|\mathbb{F}_p} \mathrm{GL}_n$ and denote by σ the map induced by φ on $G(L)$. Additionally, let $Y \simeq \bigoplus_{i=1}^f \mathbb{Z}^n$ denote the cocharacter group of a maximal torus in G and fix a Bruhat ordering $\leq$. Then

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O}_L) | g^{-1} b \sigma(g) \in \bigsqcup_{\substack{\nu \in Y \\ \nu \leq \mu}} G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L)\}.$$

To compute with this, we will regard elements of $G(\mathcal{O}_L)$ and $G(L)$, as well as b, as f-tuples of matrices. Let I denote the Iwahori subgroup of $G(\mathcal{O}_L)$ given by the preimage of $B(\overline{\mathbb{F}}_p)$ under the natural map $G(\mathcal{O}_L) \to G(\overline{\mathbb{F}}_p)$ which sends u to 0, where B is some fixed Borel subgroup of G. $C_\mu(b)(\overline{\mathbb{F}}_p)$ can be written as a disjoint union

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \bigsqcup_{\lambda \in Y} C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$$

where each $C_\mu^\lambda(b)$ is a locally closed subscheme of $C_\mu(b) \times_{\mathbb{F}} \overline{\mathbb{F}}_p$ with geometric points

$$C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = I u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) \cap C_\mu(b)(\overline{\mathbb{F}}_p).$$

In the case where $\overline{\rho}$ is absolutely irreducible and b is of a certain form, Chen and Nie have classified which Iwahori strata $C_\mu^\lambda(b)$ are nonempty, showed that the nonempty strata are connected, and, when μ is minuscule, have determined the dimension of the Iwahori strata. This can be found in [CN20, Proposition 2.2].

In this paper, we consider the case where $\overline{\rho}$ is reducible representation on a 2-dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$. Recall that b comes from the Frobenius $b\varphi$ of the étale φ-module $((k \otimes_{\mathbb{F}_p} \mathbb{F}(\!(u)\!))^2, b\varphi)$ associated to $\overline{\rho}$. As such, selecting an f-tuple of 2×2 matrices to represent b requires a choice of basis for $(k \otimes_{\mathbb{F}_p} \mathbb{F}(\!(u)\!))^2$, so this representative b is not unique. In particular, a change of basis for $(k \otimes_{\mathbb{F}_p} \mathbb{F}(\!(u)\!))^2$ amounts to changing b by σ-conjugation. With this in mind, the isomorphism class of $C_\mu(b)$ does not depend on b, but on its σ-conjugacy class as we will see in Lemma 5.2. In particular, when $\overline{\rho}$ is reducible, we will see that there is an element $b \in \operatorname{Res}_{k|\mathbb{F}_p} \operatorname{GL}_n(\mathbb{F}(\!(u)\!))$ of the form

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}$$

to which $\overline{\rho}$ may be associated. However, while the isomorphism class of $C_\mu(b)$ does not depend on the specific choice of b, the semi-module decomposition of $C_\mu(b)$ does.

We will see that each such b is σ-conjugate to an element $\tilde{b} \in G(\mathbb{F}(\!(u)\!))$ of the same form where r_i is negative, s_i is positive and $f_i(u)$ is a polynomial divisible by u^{s_i+1}. For such $\tilde{b}$, the Iwahori composition is more straightforward to compute and we will be able to identify a finite set of cocharacters $S^{\natural(\tau)}$ which contains all cocharacters λ for which $C_\mu^\lambda(\tilde{b})$ is nonempty. Specifically, given b as above, let $\tau = ((r_i, s_i))_{i=1}^f$ and, for a cocharacter $\lambda = ((c_i, d_i))_{i=1}^f$, define $\lambda^{\natural(\tau)}$ to be the dominant conjugate of $-\lambda + \tau + \sigma(\lambda)$ (cf. [CN20]). We will prove the following, which combines Proposition 5.6, Theorem 6.1, and Theorem 6.5.

Theorem 1.1. *Any Kisin variety $C_\mu(b)$ associated to a reducible 2-dimensional representation $\overline{\rho} : G_K \to \operatorname{Aut}(V_{\mathbb{F}})$ is isomorphic to a Kisin variety $C_{\tilde{\mu}}(\tilde{b})$ where $\tilde{\mu} = ((m_i, 0))_{i=1}^f$ for some nonnegative integers m_i and $\tilde{b}$ is of the form*

$$\tilde{b} = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}$$

with $\alpha_i, \beta_i \in \mathbb{F}$, $r_i < 0$, $s_i > 0$, *and* $f_i(u) \in u^{s_i+1}\mathbb{F}[[u]]$. *If* $\lambda^{\natural(\tau)} \not\leq \tilde{\mu}$, *then* $C_{\tilde{\mu}}^{\lambda}(\tilde{b}) = \emptyset$. *If* $\lambda^{\natural(\tau)} \leq \tilde{\mu}$, *then* $C_{\tilde{\mu}}^{\lambda}(\tilde{b})$ *is either empty or isomorphic to* $\mathbb{A}_{\mathbb{F}}^{N_\lambda}$ *for some nonnegative integer* N_λ.

Further, for b as above, define

$$b^{ss} = \left(\begin{pmatrix} \alpha_i u^{r_i} & 0 \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}.$$

If b and μ satisfy the technical conditions stated in Theorem 1.1, we will see that that $C_\mu^\lambda(b) \neq \emptyset$ if and only if $\lambda^{\natural(\tau)} \leq \mu$, which has previously been proven for different choices of b and μ in [CN20, Proposition 2.2]. Thus we can conclude that the set of λ's for which $C_\mu^\lambda(b)$ is nonempty is a subset of those for which $C_\mu^\lambda(b^{\mathrm{ss}})$ is nonempty.

The proof uses the semi-module decomposition $C_\mu(b)(\overline{\mathbb{F}}_p) = \bigsqcup_{\lambda \in Y} C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ to compute $C_\mu(b)(\overline{\mathbb{F}}_p)$ by computing the quotients $(Iu^\lambda G(\mathcal{O}_L))/G(\mathcal{O}_L)$ and considering their intersections with $C_\mu(b)(\overline{\mathbb{F}}_p)$. In particular, we will compute the form of an element g of $(Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ and consider whether it is possible for $g^{-1}b\sigma(g)$ to belong to $G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)$ for some dominant cocharacter ν satisfying $\nu \leq \mu$. In the case $f = 1$, we have $G = \mathrm{GL}_2$, so $g \in Iu^\lambda GL_2(L)/\mathrm{GL}_2(\mathcal{O}_L)$ has a representative in $\mathrm{GL}_2(L)$ which is either upper or lower triangular depending on the difference of the two components in λ. If $f > 1$, then g instead belongs to $\mathrm{Res}_{W(k)|\mathbb{Z}_p} \mathrm{GL}_2(L) \simeq \prod_{i=1}^{f} \mathrm{GL}_2(L)$. Regarding g as an f-tuple of matrices, each of the matrices from which g is comprised will be either upper or lower triangular and the f-tuple itself may contain a combination of both. This, along with the shift in components involved in σ, complicates the computation of $g^{-1}b\sigma(g)$ in the $f > 1$ case. A key observation is that our choice of b will ensure that all nonempty Iwahori strata involve only upper triangular matrices, simplifying the computations significantly.

This paper is divided in to six sections, the first being this introduction. In the second section we establish some notation that will be used throughout the paper. Then, before we move on to the results, we will review some relevant background material in Section 3. First, in Subsection 3.1, we recall the construction of the universal framed deformation ring. Then, in order to be able to construct Kisin's resolution of the universal framed deformation ring as well as the Kisin variety in Subsections 3.6 and 3.7, we recall some results from p-adic Hodge theory in Subsections 3.2, 3.3, and 3.4. Meanwhile, in Subsection 3.5, we will recall the construction of the affine Grassmannian, specifically the affine Grassmannian associated to GL_n, because the Kisin variety is a subscheme of an affine Grassmannian. Following the background material, in Section 4, we describe the geometric points of the Iwahori strata of a Kisin variety. Afterward, in Section 5, we associate the representation $\overline{\rho}$ to a tuple of matrices b, which is unique up to a conjugation action, then find a nice representative of the conjugacy class of b. Finally, in Section 6 we prove the main results regarding which strata are empty as well as the form of the nonempty strata.

2 Notation

Let K be a finite extension of $\mathbb{Q}_p$ for some odd prime p. Denote the ring of integers of K by $\mathcal{O}_K$ and fix a uniformizer π of $\mathcal{O}_K$. Let k denote the residue field of K and define $f = [k : \mathbb{F}_p]$. Fix an algebraic closures $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ and $\overline{K}$ of K. Denote the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ of K by G_K. Likewise, for any field κ, G_κ will denote the absolute Galois group $\mathrm{Gal}(\kappa^{\mathrm{sep}}/\kappa)$ where κ^{sep} is the maximal separable subextension of $\overline{\kappa}/\kappa$ for some algebraic closure $\overline{\kappa}$ of κ. Fix a p-adic field F with residue field F, let $\mathcal{O}_F$ denote its ring of integers, and fix a uniformizer ϖ in $\mathcal{O}_F$.

Denote by B_{GL_2} the Borel subgroup of GL_2 consisting of upper triangular matrices and and by T_{GL_2} the maximal torus of GL_2 consisting of diagonal matrices. Denote by G the Weil restriction of GL_2, $G = \mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_2)$, which is characterized by the existence of bijections

$$\mathrm{Res}_{k|\mathbb{F}_p}(\mathrm{GL}_2)(X) \longrightarrow \mathrm{GL}_2(X \times_{\mathbb{F}_p} k)$$

x for any $\mathbb{F}_p$-scheme X. Let B and T denote the Weil restrictions $B = \mathrm{Res}_{k|\mathbb{F}_p} B_{\mathrm{GL}_2}$ and

$T = \mathrm{Res}_{k|\mathbb{F}_p} T_{\mathrm{GL}_2}$, respectively. If $\mathbb{F}$ is sufficiently large, there is a natural isomorphism

$$G_{\mathbb{F}} \longrightarrow \prod_{i=1}^{f} \mathrm{GL}_{2,\mathbb{F}}$$

and analogous isomorphisms for aforementioned subgroups of $G_{\mathbb{F}}$:

$$B_{\mathbb{F}} \longrightarrow \prod_{i=1}^{f} B_{\mathrm{GL}_{2,\mathbb{F}}} \quad \text{and} \quad T_{\mathbb{F}} \xrightarrow{\sim} \prod_{i=1}^{f} T_{\mathrm{GL}_{2,\mathbb{F}}}.$$

Let $R_{\mathrm{GL}_2} = (X_{\mathrm{GL}_2}, \Phi_{\mathrm{GL}_2}, Y_{\mathrm{GL}_2}, \Phi^{\vee}_{\mathrm{GL}_2})$ be the root datum associated to $T_{\mathrm{GL}_2} \subset B_{\mathrm{GL}_2} \subset \mathrm{GL}_2$ where $X_{\mathrm{GL}_2} = X^*(T_{\mathrm{GL}_2})$ is the character group, $Y_{\mathrm{GL}_2} = X_*(T_{\mathrm{GL}_2})$ is the the cocharacter group, $\Phi_{\mathrm{GL}_2} = \{e_1 - e_2, e_2 - e_1\}$ is the set of roots, and $\Phi^{\vee}_{\mathrm{GL}_2}$ is the set of coroots. Fix a set of positive roots $\Phi^+ = \{e_1 - e_2\}$. We identify the cocharacter group Y_{GL_2} with $\mathbb{Z}^2$ by associating the cocharacter $x \mapsto \mathrm{diag}(x^a, x^b)$ with the 2-tuple (a, b). Let $\leq$ denote the Bruhat order on Y_{GL_2} given by $(c, d) \leq (c', d')$ exactly when $(c' - c, d' - d) = (n, -n)$ for some positive integer n.

There is a corresponding root datum $R = (X, \Phi, Y, \Phi^{\vee})$ for $T_{\mathbb{F}} \subset B_{\mathbb{F}} \subset G_{\mathbb{F}}$ with identifications $Y = \oplus_{i=1}^{f} \mathbb{Z}^2$, $\Phi = \bigsqcup_{i=1}^{f} \Phi_{\mathrm{GL}_2}$, and $\Phi^+ = \bigsqcup_{i=1}^{f} \Phi^+_{\mathrm{GL}_2}$. The Bruhat order $\leq$ on Y_{GL_2} extends to Y with $(c_i, d_i)_{i=1}^{f} \leq (c'_i, d'_i)_{i=1}^{f}$ exactly when $(c'_i - c_i, d'_i - d_i)_{i=1}^{f} = (n_i, -n_i)_{i=1}^{f}$ for some positive integers n_i.

Denote by LG (resp. L^+G) the loop group of G (resp. the positive loop group of G), i.e. the ind-group scheme representing the functor which sends an $\mathbb{F}$-algebra R to $G(R(\!(u)\!))$ (resp. $G(R[[u]])$). Let $\mathcal{G}rass_G$ denote the affine Grassmian for G, which is the sheafification of the quotient of $\mathbb{F}$-spaces LG/L^+G. $\mathcal{G}rass_G$ is a ind-$\mathbb{F}$-scheme which contains the projective $\mathbb{F}$-scheme $C_\mu(b)$. We will discuss $\mathcal{G}rass_G$ in more detail in Section 3.5.

Let $L = \overline{\mathbb{F}}_p(\!(u)\!)$ and $\mathcal{O}_L = \overline{\mathbb{F}}_p[[u]]$. Let $\varphi : L \to L$ denote the endomorphism that acts as the identity on $\overline{\mathbb{F}}_p$ and as the p-power map on u. Then φ induces a $G(L)$-endomorphism σ given by

$$\sigma(g_1, g_2, \ldots, g_f) = (\varphi(g_2), \ldots, \varphi(g_f), \varphi(g_1))$$

as well as a Y-endomorphism given by $\sigma(\lambda) = \sigma \circ \lambda$. In particular, if $\lambda = ((c_i, d_i))_{i=1}^f$, then

$$\sigma(((c_1, d_1), (c_2, d_2), \ldots, (c_f, d_f))) = ((pc_2, pd_2), \ldots, (pc_f, pd_f), (pc_1, pd_1)).$$

Let I denote the Iwahori subgroup of $G(\mathcal{O}_L)$ consisting of f-tuples of matrices which are lower triangular modulo u and fix a cocharacter μ in Y along with an element b of $G(\mathbb{F}(\!(u)\!))$.

3 Background Material

In this section, we will recall the construction of the Kisin variety and, along the way, we will go over the definitions of framed deformation rings, étale φ-modules, Kisin modules, and affine Grassmannians to establish the through line between the representation $\overline{\rho}$ and the Kisin variety $C_\mu(b)$ by illustrating how the functor of points of the Kisin variety relates to the deformations of $\overline{\rho}$.

More specifically, let A be a $\mathbb{Z}_p$-algebra with finitely many elements; for example, we will make use of the case $A = \mathcal{O}_F/(\varpi^n)$. From a result of Fontaine [Fon07, Proposition 1.2.6], there is an equivalence of categories between the category of continuous G_K-representations with coefficients in A and étale φ-modules with coefficients in A, which allows us to identify a deformation of $\overline{\rho}$ to $\overline{\rho}_A$, having A coefficients, to semi-linear algebraic object called an étale φ-module with A-coefficients. Then, inside of that étale φ-module is a lattice with some additional structure akin to that of the étale φ-module, which is called a Kisin module with coefficients in A. In Section 3.5 we will discuss how such a lattice can be regarded as an A-point on the affine Grassmannian associated to G, $\mathcal{G}rass_G$. With these identifications, we will be able to construct a projective Spec $R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ scheme $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}$ which is a subscheme of $\mathcal{G}rass_G$ that parameterizes these Kisin modules and, therefore, the deformations of $\overline{\rho}$.

3.1 The framed deformation ring

Given a representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$, one natural question is which representations $\rho : G_K \to \mathrm{GL}_n(F)$, where F is a p-adic field with residue field $\mathbb{F}$, are lifts of $\overline{\rho}$. When we say that ρ is a lift of $\overline{\rho}$, we mean that it completes the following diagram.

$$\begin{array}{ccc} & & \mathrm{GL}_2(F) \\ & \overset{\rho}{\nearrow} & \downarrow \\ G_K & \xrightarrow{\overline{\rho}} & \mathrm{GL}_2(\mathbb{F}) \end{array}$$

Fix such a field F, denote the ring of integers of F by $\mathcal{O}_F$, and let ϖ be a uniformizer of $\mathcal{O}_F$. Define $\widehat{\mathcal{C}}_{\mathcal{O}}$ to be the category of complete local Noetherian $\mathcal{O}_F$-algebras with residue field $\mathbb{F}$.

Definition 3.1. *For an $\mathcal{O}_F$-module A in $\widehat{\mathcal{C}}_{\mathcal{O}}$, a* deformation *of $\overline{\rho}$ to A is a 4-tuple $(\overline{\rho}, A, \iota, \beta)$ where*

1. *M is a free A-module of rank n,*

2. *$\overline{\rho}_A : G_K \to \mathrm{Aut}_A(M)$ is a continuous representation of G_K with coefficients in M,*

3. *$\iota : M \otimes_A \mathbb{F} \to V_{\mathbb{F}}$ is an isomorphism of $\mathbb{F}$-vector spaces which respects the G_K actions on M and $V_{\mathbb{F}}$, and*

4. *β is an A-basis of M lifting some fixed basis of $V_{\mathbb{F}}$ under ι.*

Two such 4-tuples are said to be isomorphic if they are isomorphic in a way that respects the ι's. Additionally, we say that the deformation $(\overline{\rho}_A, M, \iota, \beta)$ is flat *if there exists a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ for which the G_K-representation $\mathcal{G}(\overline{K})$ has an A action and is isomorphic to ρ.*

Define the framed deformation functor $D^{\square}_{\overline{\rho}} : \widehat{\mathcal{C}}_{\mathcal{O}} \to \mathrm{Sets}$ by

$$D^{\square}_{\overline{\rho}}(A) = \{(\overline{\rho}_A, M, \iota, \beta)\}/\sim,$$

so that $D^{\square}_{\overline{\rho}}(A)$ is essentially the set of liftings of $\overline{\rho}$ to A. This functor is representable in $\widehat{\mathcal{C}}_{\mathcal{O}}$ by Schlessinger's criterion, meaning that there exists an object $R^{\mathrm{fl},\square}_{\overline{\rho}}$ for which

$$\operatorname{Spec} R^{\mathrm{fl},\square}_{\overline{\rho}}(A) = \operatorname{Hom}_{\mathcal{O}_F-\mathrm{alg}}(\operatorname{Spec} R^{\mathrm{fl},\square}_{\overline{\rho}}, A) = D^{\square}_{\overline{\rho}}(A)$$

for any object A in $\widehat{\mathcal{C}}_{\mathcal{O}}$.

3.2 Fontaine's theory of étale φ-modules

Using methods from p-adic Hodge theory, the representation $\overline{\rho}$ can be associated to a semi-linear algebra object called an étale φ-module. Our interests in reviewing the theory of étale φ-modules are two-fold. First, the group theoretic description of the geometric points of the Kisin variety,

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O})L \mid g^{-1}b\sigma(g) \in \bigsqcup_{\nu \leq \mu} G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)\},$$

uses of an element b of $\operatorname{Res}_{k|\mathbb{F}_p} \operatorname{GL}_2(\mathbb{F}(\!(u)\!))$ which classifies the étale φ-module associated to $\overline{\rho}$ and μ up to a choice of basis. Second, the theory of étale φ-modules is important to the construction of the Kisin variety.

Let E be a field of characteristic $p > 0$, not necessarily perfect. Fix an algebraic closure $\overline{E}$ of E and let E^{sep} denote the maximal separable sub-extension of $\overline{E}$. Let φ denote the p-power endomorphism on E^{sep} as well as its restriction to E. Denote by G_E the absolute Galois group $G_E = \operatorname{Gal}(E^{\mathrm{sep}}/E)$ endowed with the usual topology.

Definition 3.2. *A* linear representation *of a topological group G with coefficients in a field F is a finite-dimensional F-vector space V equipped with a linear action of G. If the field F has a topology and V is given the induced topology, such a representation V is called continuous if the action map $\rho : G \to \operatorname{Aut}_F(V)$ is continuous. Denote the category of continuous representations of G with coefficients in F by* $\operatorname{Rep}_F(G)$

Definition 3.3. *Let N be a finitely generated free $\mathbb{Z}_p$-module with the induced topology from $\mathbb{Z}_p$. We say that N is a* $\mathbb{Z}_p$-representation *of a topological group G if N is equipped with a G-action and the action map $\rho : G \to \operatorname{Aut}(N)$ is continuous. Denote the category of*

$\mathbb{Z}_p$-representations of G by $\mathrm{Rep}_{\mathbb{Z}_p}(G)$.

In this subsection, we will define categories of étale φ-modules which are equivalent to categories of representations of Galois groups of characteristic p fields with coefficients in $\mathbb{F}_p$, $\mathbb{Z}_p$, $\mathbb{Q}_p$. We begin by considering the étale φ-modules associated to $\mathrm{Rep}_{\mathbb{F}_p}(G_E)$.

Definition 3.4. *A* φ-module over *E is a pair (M, φ_M) where M is an E-vector space and $\varphi_M : M \to M$ is a φ-semilinear endomorphism of M. We say that a φ-module (M, φ_M) is étale if M is finite dimensional as an E-vector space and the linearization*

$$\Phi_M : E \otimes_{\varphi_E, E} M \longrightarrow M$$

of φ_M is an isomorphism. The category of such objects is denoted by $\Phi M_E^{ét}$. The morphisms in this category are linear maps of vector spaces that commute with the semilinear maps φ_M.

Consider the condition $\Phi_M : E \otimes_{\varphi_E, E} M \xrightarrow{\sim} M$. Since M is finite dimensional as an E-vector space, Φ_M is an isomorphism if and only if Φ_M is injective or surjective. Moreover, if we fix an E-basis $\{e_1, \dots, e_d\}$ of M and define $a_{i,j}$ to be the elements of E for which

$$\varphi_M(e_j) = \sum_{i=1}^{d} a_{i,j} e_i,$$

then Φ_M is an isomorphism if and only if $(a_{i,j}) \in \mathrm{GL}_d(E)$, where $d = \dim_E M$.

The category $\Phi M_E^{ét}$ has natural notions of tensor products and dual objects. We take tensor products in the naive way; in particular, if (M, φ_M) and (M', φ'_M) are in $\Phi M_E^{ét}$, then $(M \otimes M', \varphi_M \otimes \varphi_{M'})$ is also in $\Phi M_E^{ét}$. On the other hand, let $M^\vee$ be the usual E-vector space dual of M. To construct the requisite semilinear endomorphism $\varphi_{M^\vee} : M^\vee \to M^\vee$, define the E-linear pullback functional $\varphi_E^* : M^\vee \to (E \otimes_{\varphi_E, E} M)^\vee$ so that $\ell \in M^\vee$ is taken to $\varphi_E^*(\ell) : E \otimes_{\varphi_E, E} M \to E$, which is given on pure tensors by $c \otimes m \mapsto c \cdot \ell(m)^p$. Define $\varphi_{M^\vee}(\ell) = \varphi_E^*(\ell) \circ \Phi^{-1}$; then $(M^\vee, \varphi_{M^\vee}) \in \Phi M_E^{ét}$. With these conventions for tensor and dual, $\Phi M_E^{ét}$ is an abelian category.

To any (M, φ_M) in $\Phi M_E^{ét}$ we assign an $\mathbb{F}_p$-vector space

$$V_E(M) = (E^{\text{sep}} \otimes_E M)^{\varphi=1}$$

where the self-map φ on the tensor product $E^{\text{sep}} \otimes_E M$ is given by $\varphi \otimes \varphi_M$. The tensor product $E^{\text{sep}} \otimes_E M$ is endowed with a G_E-action generated by $g(c \otimes x) = g(c) \otimes x$ for any $g \in G_E$, $c \in E^{\text{sep}}$, and $x \in V$. This action commutes with φ, so the subspace of φ-invariants, $V_E(M)$, also has a G_E-action. This induces a rank-preserving functor $V_E : \Phi M_E^{ét} \to \text{Rep}_{\mathbb{F}_p}(G_E)$ given by sending an étale φ-module (M, φ_M) to the G_E-representation $(E^{\text{sep}} \otimes_E M)^{\varphi=1}$ and sending a map between étale φ-modules

$$f : (M, \varphi_M) \longrightarrow (N, \varphi_N)$$

to a map between representations

$$V_E(f) : (E^{\text{sep}} \otimes_E M)^{\varphi=1} \longrightarrow (E^{\text{sep}} \otimes_E N)^{\varphi=1},$$

given by $1 \otimes f$.

Now we associate an étale φ-module to a representation in $\text{Rep}_{\mathbb{F}_p}(G_E)$. Given such a representation V, regard $E^{\text{sep}} \otimes_{\mathbb{F}_p} V$ as an E-vector space via the left tensor factor and define

$$D_E(V) = (E^{\text{sep}} \otimes_{\mathbb{F}_p} V)^{G_E}$$

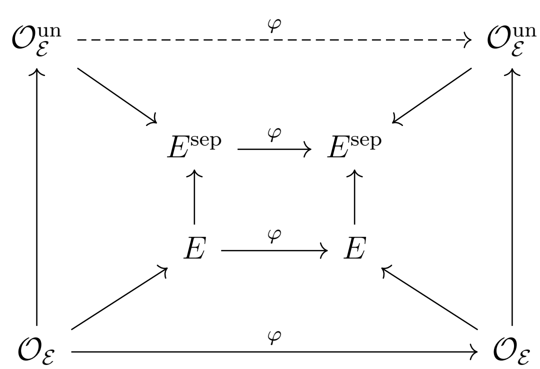
where the G_E-action on $E^{\text{sep}} \otimes_{\mathbb{F}_p} V$ is generated by $g(c \otimes x) = g(c) \otimes g(x)$ for $g \in G_E$, $c \in E^{\text{sep}}$, and $x \in V$. This association gives a rank-preserving functor from $\text{Rep}_{\mathbb{F}_p}(G_E)$ to $\Phi M_E^{ét}$.

Theorem 3.5. *(Fontaine) V_E and D_E are exact, rank-preserving quasi-inverse equivalences of categories between $\Phi M_E^{ét}$ to $\text{Rep}_{\mathbb{F}_p}(G_E)$ which are compatible with tensor products and duality.*

Now consider the category $\text{Rep}_{\mathbb{Z}_p}(G_E)$. The category of étale φ-modules that we will compare to $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ is defined very similarly to $\Phi M_E^{\acute{e}t}$, with the main difference being that the modules are not vector spaces over E, but rather finite type modules over a discrete valuation ring. Let $\mathcal{O}_\mathcal{E}$ be a complete discrete valuation ring with characteristic 0, uniformizer p, and residue field E. Fix an $\mathcal{O}_\mathcal{E}$ endomorphism φ lifting the map $\varphi : E \to E$. Let $\mathcal{E}$ denote the field of fractions of $\mathcal{O}_\mathcal{E}$, $\mathcal{E} = \mathcal{O}_\mathcal{E}[1/p]$. For example, if $E = k((u))$ for some finite field k, then we can define $\mathcal{O}_\mathcal{E}$ to be the p-adic completion of $W(k)[[u]]$,

$$\mathcal{O}_\mathcal{E} \simeq \left\{ \sum_{n\in\mathbb{Z}} a_n u^n | a_n \in W(k) \text{ and } a_n \to 0 \text{ as } n \to -\infty \right\}.$$

Additionally, define $\mathcal{O}_\mathcal{E}^{\mathrm{un}}$ to be the maximal unramified extension of $\mathcal{O}_\mathcal{E}$ and $\widehat{\mathcal{O}_\mathcal{E}^{\mathrm{un}}}$ to be its closure. Likewise, let $\mathcal{E}^{\mathrm{un}} = \mathcal{O}_\mathcal{E}^{\mathrm{un}}[1/p]$ and $\widehat{\mathcal{E}^{\mathrm{un}}} = \widehat{\mathcal{O}_\mathcal{E}^{\mathrm{un}}}[1/p]$. Note that $\mathcal{O}_\mathcal{E}^{\mathrm{un}}$ is strictly Henselian, so the Frobenius $\varphi : \mathcal{O}_\mathcal{E} \to \mathcal{O}_\mathcal{E}$ extends uniquely to $\mathcal{O}_\mathcal{E}^{\mathrm{un}}$, and therefore to $\widehat{\mathcal{O}_\mathcal{E}^{\mathrm{un}}}$, via the universal property of the strict henselization.



Moreover, there is a classical identification $\mathrm{Aut}_{\mathcal{O}_\mathcal{E}}(\widehat{\mathcal{O}_\mathcal{E}^{\mathrm{un}}}) = \mathrm{Gal}(\mathcal{E}^{\mathrm{un}}/\mathcal{E}) \simeq G_E$, which gives a continuous G_E-action on $\mathcal{O}_\mathcal{E}$ which commutes with φ.

Definition 3.6. *A* φ-module over $\mathcal{O}_\mathcal{E}$ *consists of a pair* (M, φ_M) *where* M *is an* $\mathcal{O}_\mathcal{E}$*-module of finite type and* φ_M *is a* φ*-semilinear endomorphism of* M*.*

To any φ-module over $\mathcal{O}_\mathcal{E}$, (M, φ_M), we associate an additional $\mathcal{O}_\mathcal{E}$-module $\mathcal{O}_\mathcal{E} \otimes_{\varphi,\mathcal{O}_\mathcal{E}} M$ along with a map $\Phi_M : \mathcal{O}_\mathcal{E} \otimes_{\varphi,\mathcal{O}_\mathcal{E}} M \to M$ generated on elementary tensors by $(c \otimes m) \to \varphi(c)\varphi_M(m)$. We call Φ_M the *linearization* of φ_M.

Definition 3.7. *Let* (M, φ_M) *be a* φ*-module over* $\mathcal{O}_{\mathcal{E}}$*. If the linearization* Φ_M *of* φ_M *is an isomorphism of* $\mathcal{O}_{\mathcal{E}}$*-modules, then we say that* (M, φ_M) *is an* étale φ*-module over* $\mathcal{O}_{\mathcal{E}}$*. Denote the category of such modules by* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$*.*

The category $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$ has notions of tensor and dual, which are defined similarly to those of $\Phi M^{ét}_E$, and, like $\Phi M^{ét}_E$, it is an abelian category. Moreover, $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$ is equivalent to $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ via functors $V_{\mathcal{O}_{\mathcal{E}}}$ and $D_{\mathcal{O}_{\mathcal{E}}}$ which are defined similarly to V_E and D_E.

To any φ-module over $\mathcal{O}_{\mathcal{E}}$, we associate a $\mathbb{Z}_p$-module $V_{\mathcal{O}_{\mathcal{E}}}(M)$,

$$V_{\mathcal{O}_{\mathcal{E}}}(M) = (\widehat{\mathcal{O}^{\mathrm{un}}_{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M)^{\varphi=1},$$

where the endomorphism φ on $\widehat{\mathcal{O}^{\mathrm{un}}_{\mathcal{E}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M$ is defined to be $\varphi \otimes \varphi_M$. In addition, $V_{\mathcal{O}_{\mathcal{E}}}(M)$ has a G_E-action inherited from its left tensor factor, so to any $V \in \mathrm{Rep}_{\mathbb{Z}_p}(G_E)$, we can associate an $\mathcal{O}_{\mathcal{E}}$-module $D_{\mathcal{O}_{\mathcal{E}}}(V)$ defined by

$$D_{\mathcal{O}_{\mathcal{E}}}(V) = (\widehat{\mathcal{O}^{\mathrm{un}}_{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V)^{G_E}$$

where the G_E-action on $\widehat{\mathcal{O}^{\mathrm{un}}_{\mathcal{E}}} \otimes_{\mathbb{Z}_p} V$ is given by $g(c \otimes v) = g(c) \otimes g(v)$ for $g \in G_E$, $c \in \widehat{\mathcal{O}^{\mathrm{un}}_{\mathcal{E}}}$, and $d \in V$.

Theorem 3.8. *([Fon07, Proposition 1.2.6])* $V_{\mathcal{O}_{\mathcal{E}}}$ *and* $D_{\mathcal{O}_{\mathcal{E}}}$ *are quasi-inverse equivalences of abelian categories between* $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ *and* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$*. These functors preserve rank and invariant factors over* $\mathcal{O}_{\mathcal{E}}$ *and* $\mathbb{Z}_p$ *and are compatible with tensor products.*

Lastly, consider the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$. The proofs for the equivalences of categories given by V_E and D_E and by $V_{\mathcal{O}_{\mathcal{E}}}$ and $D_{\mathcal{O}_{\mathcal{E}}}$ are quite involved, and have thus been omitted. However, using the existence of the equivalences $V_{\mathcal{O}_{\mathcal{E}}}$ and $D_{\mathcal{O}_{\mathcal{E}}}$ to achieve something similar for $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ is rather quick. This is because all elements of $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ are localizations of representations of G_E with coefficients in $\mathbb{Z}_p$, which is demonstrated in the following lemma.

Lemma 3.9. *For* $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$*, there exits a* G_E*-stable* $\mathbb{Z}_p$*-lattice* $\Lambda \subset V$*. In particular,* Λ *is a finite, free* $\mathbb{Z}_p$*-submodule of* V *and* $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda \simeq V$*.*

Proof. Let $\rho : G_E \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ be the continuous action map and fix a $\mathbb{Z}_p$ lattice $\Lambda_0 \subset V$. Then $\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0) \subset \mathrm{Aut}_{\mathbb{Q}_p}(V)$ because $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda_0$ and $\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0)$ is an open subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(G_E)$. Define $G_0 = \rho^{-1}(\mathrm{Aut}_{\mathbb{Z}_p}(\Lambda_0))$. Since G_0 is an open subgroup in the compact group G_E, it has finite index. Denote a set of coset representatives of G/G_0 by $\{g_i\}_{i=1}^n$. Then $\Lambda = \sum_i \rho(g_i)\Lambda_0$ is a G-stable lattice in V. □

For any $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$, define an $\mathcal{E}$-vector space

$$D_{\mathcal{E}}(V) = (\widehat{\mathcal{E}^{\mathrm{un}}} \otimes_{\mathbb{Q}_p} V)^{G_E}$$

where the G_E-action on $\widehat{\mathcal{E}^{\mathrm{un}}} \otimes_{\mathbb{Q}_p} V$ is induced by that of $\widehat{\mathcal{E}^{\mathrm{un}}}$. This vector space is equipped with a φ-semilinear endomorphism $\varphi_{D_{\mathcal{E}}}$ induced by the Frobenius endomorphism on the left tensor factor.

Proposition 3.10. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$, $D_{\mathcal{E}}(V)$ has finite $\mathcal{E}$ dimension equal to the $\mathbb{Q}_p$-dimension of V and the $\mathcal{E}$-linearization $\Phi : \widehat{\mathcal{E}^{un}} \otimes_{\varphi_{\mathcal{E}},\mathcal{E}} D_{\mathcal{E}}(V) \to D_{\mathcal{E}}(V)$ of $\varphi_{D_{\mathcal{E}}}$ is an isomorphism. Moreover, $D_{\mathcal{E}}(V)$ admits a $\varphi_{D_{\mathcal{E}}(V)}$-stable lattice L satisfying $(L, \varphi_{D_{\mathcal{E}}(V)}|_L) \in \Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$.*

Proof. By Lemma 3.9 we have $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Lambda$ for some $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ that is finite free as a $\mathbb{Z}_p$-module. Thus, from the definition,

$$D_{\mathcal{E}}(V) = (\widehat{\mathcal{E}^{\mathrm{un}}} \otimes_{\mathbb{Q}_p} V)^{G_E} = D_E(\Lambda)[1/p],$$

which is isomorphic to $\mathcal{E} \otimes_{\mathcal{O}_{\mathcal{E}}} D_{\mathcal{O}_{\mathcal{E}}}(\Lambda)$ as $\mathcal{E}$-vector spaces endowed with a $\varphi_{\mathcal{E}}$-semilinear endomorphism. Since Λ is a an object of $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ with $\mathrm{rank}_{\mathbb{Z}_p}(\Lambda) = \dim_{\mathbb{Q}_p}(V)$ and $D_{\mathcal{O}_{\mathcal{E}}}$ is rank preserving, $\dim_{\mathcal{E}}(D_{\mathcal{E}}(V)) = \dim_{\mathbb{Q}_p}(V)$. Take $L = D_{\mathcal{O}_{\mathcal{E}}}(\Lambda)$. □

This proposition motivates the following definition.

Definition 3.11. *An* étale φ-module *over $\mathcal{E}$ is a finite dimensional $\mathcal{E}$-vector space M along with a $\varphi_{\mathcal{E}}$-semilinear endomorphism $\varphi_M : M \to M$ whose* linearization $\Phi_M : \widehat{\mathcal{E}^{un}} \otimes_{\varphi_{\mathcal{E}},\mathcal{E}} M \to$ *M is an isomorphism and which admits a φ_M-stable lattice $L \subset M$ such that $(L, \varphi_M|_L) \in \Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}$. The category of such objects is denoted by $\Phi M^{\acute{e}t}_{\mathcal{E}}$.*

There is an evident functor $\Phi M^{\acute{e}t}_{\mathcal{O}_\mathcal{E}} \to \Phi M^{\acute{e}t}_{\mathcal{E}}$ given by

$$L \rightsquigarrow L[1/p] = \mathcal{E} \otimes_{\mathcal{O}_\mathcal{E}} L$$

and

$$\mathrm{Hom}_{\Phi M^{\acute{e}t}_{\mathcal{O}_\mathcal{E}}}(L, L')[1/p] = \mathrm{Hom}_{\Phi M^{\acute{e}t}_{\mathcal{E}}}(L[1/p], L[1/p]),$$

so $\Phi M^{\acute{e}t}_{\mathcal{E}}$ is identified with $\Phi M^{\acute{e}t}_{\mathcal{O}_\mathcal{E}}$. In particular, $\Phi M^{\acute{e}t}_{\mathcal{E}}$ is abelian.

Theorem 3.12. *The functors $D_\mathcal{E}(V) = (\widehat{\mathcal{E}^{un}} \otimes V)^{\varphi=1}$ and $V_\mathcal{E}(M) = (\widehat{\mathcal{E}^{un}} \otimes_\mathcal{E} M)^{G_E}$ are rank preserving, exact quasi-inverse equivalences of categories between $\mathrm{Rep}_{\mathbb{Q}_p}^{G_E}$ and $\Phi M^{\acute{e}t}_{\mathcal{E}}$.*

Proof. If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ and Λ is a G_E-stable $\mathbb{Z}_p$ lattice in V, then $D_\mathcal{E}(V) = D_{\mathcal{O}_\mathcal{E}}(\Lambda)[1/p]$. Analogously, if M is an étale φ-module over $\mathcal{E}$, then it has a Frobenius-stable $\mathcal{O}_\mathcal{E}$-lattice L and $V_\mathcal{E}(M) = V_\mathcal{E}(L)[1/p]$. Therefore, the equivalence of categories is immediately obtained by p-localization on the result comparing $\mathrm{Rep}_{\mathbb{Z}_p}(G_E)$ and $\Phi M^{\acute{e}t}_{\mathcal{O}_\mathcal{E}}$ restricted to the full subcategories objects with finite free module structures. □

3.3 Étale φ-modules with coefficients and Kisin modules

Using Theorem 3.5 from the previous subsection, we can associate an étale φ-module to $\overline{\rho}$ in the case where $\mathbb{F} = \mathbb{F}_p$. To do this, we begin by restricting $\overline{\rho}$ to G_{K_∞}. Then, since $G_{K_\infty} \simeq G_{k((u))}$ [FW79], we can regard it as a representation of $G_{k((u))}$, at which point we can apply the functor D_E to the Tate twist of $\overline{\rho}$. In this subsection we will extend to the case where $\mathbb{F}$ is not necessarily equal to $\mathbb{F}_p$ by introducing étale φ-modules with coefficients.

Set $K_0 = W(k)[1/p]$ and define $E(u)$ to be the minimal polynomial of π over K_0. Let $\mathfrak{S} = W(k)[[u]]$ and let $\mathcal{O}_\mathcal{E}$ be the p-adic completion of $\mathfrak{S}[1/u]$. We denote by φ the Frobenius endomorphism on $W(k)$ as well as its extension to $\mathfrak{S}$ such that $\varphi(u) = u^p$. Let φ also denote the unique continuous extension of the endomorphism $\varphi : \mathfrak{S} \to \mathfrak{S}$ to $\mathcal{O}_\mathcal{E}$. We have defined an étale φ-module over $\mathcal{O}_\mathcal{E}$, where $\mathcal{O}_\mathcal{E}$ is a complete discrete valuation ring with characteristic 0, uniformizer p, and to be a pair (M, φ_M) where M is a finite $\mathcal{O}_\mathcal{E}$-module and φ_M is a

φ-semilinear endomorphism such that the $\mathcal{O}_{\mathcal{E}}$-linear map $1 \otimes \varphi_M : \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} M \to M$ is an isomorphism. We now extend this definition to that of an étale φ-module with coefficients in A for any $\mathbb{Z}_p$-algebra A with finitely many elements. Such an object will have an action of $\mathcal{O}_{\mathcal{E}}$ and an action of A. We had previously noted that, in the case of $\mathrm{Rep}_{\mathbb{Z}_p}(G_{k((u))})$ we could choose $\mathcal{O}_{\mathcal{E}}$ to be the p-adic completion of $W((k))[[u]]$. With this in mind, we define $\mathfrak{S}_A = (W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$ and $\mathcal{O}_{\mathcal{E},A} = \mathfrak{S}_A[1/u]$.

Definition 3.13. *An* étale φ-module with coefficients in A *is an object of* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}}}$ *which is endowed with an* A*-action and is finite free as an* $\mathcal{O}_{\mathcal{E},A}$*-module. Denote the category of étale* φ*-modules with coefficients in* A *by* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}},A}$.

If V is a representation of G_K on a finite free A-module, then there's a composition of maps $\mathbb{Z}_p \to A \to V$ that establishes V as a finitely generated $\mathbb{Z}_p$-algebra, so V can be regarded as an element of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$. We denote the full subcategory of of $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ consisting of representations of G_K on finite free A-modules by $\mathrm{Rep}^{\mathrm{free}}_A(G_K)$.

Lemma 3.14. *The functor* $V_{\mathcal{O}_{\mathcal{E}}}$ *from Lemma 3.8 induces a rank-preserving equivalence of categories between* $\Phi M^{ét}_{\mathcal{O}_{\mathcal{E}},A}$ *and* $\mathrm{Rep}^{free}_A(G_{K_\infty})$.

$$V_{\mathcal{O}_{\mathcal{E}},A} : \Phi M^{ét}_{\mathcal{O}_{\mathcal{E}},A} \to \mathrm{Rep}^{\mathrm{free}}_A(G_{K_\infty})$$

Proof. See [Kis09] Lemma (1.2.7). □

It is with the functor, $V_{\mathcal{O}_{\mathcal{E}},\mathbb{F}}$ that we can attach an étale φ-module (M, φ_M) to $\overline{\rho} : G_K \to \mathrm{GL}_n(\mathbb{F})$, upon restriction to G_{K_∞}. In this case, M is a free $\mathcal{O}_{\mathcal{E},\mathbb{F}}$-module of rank n, so, as an $\mathbb{F}$-module,

$$M \simeq ((W(k) \otimes_{\mathbb{Z}_p} \mathbb{F})((u)))^n \simeq (k \otimes_{\mathbb{F}_p} \mathbb{F}((n)))^n,$$

and $\varphi_M = b\varphi$ for some $b \in \mathrm{Res}_{k|\mathbb{F}_p} \mathrm{GL}_2(\mathbb{F}((u)))$.

3.4 Stacks of étale φ-modules and Kisin modules

Kisin's original proof for the existence of his partial resolution of Spec $R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$ made use of various groupoids. However, as we recall this result, we have chosen to instead to use stacks parameterizing étale φ-modules and Kisin modules as constructed in [EG15], which we define in this subsection.

Let R be a complete local Noetherian $\mathcal{O}_F$-algebra, with maximal ideal m_R and residue field $\mathbb{F}$. For example, we will make use of the case where R is a quotient of $R_{\overline{\rho}}^{\mathrm{fl},\square}$. Fix a representation V_R of G_K on a finite free R-module of rank n. Then for each positive integer i, let R_i denote R/m_R^i and let V_{R_i} denote the representation $V_R \otimes R_i \in \mathrm{Rep}_{R_i}(G_K)$. Each V_{R_i} is a p^i-torsion representation of G_K which is free of rank n over R_i, so $D_{\mathcal{O}_\mathcal{E},R_i}(V_{R_i})$ is an étale φ-module with an R_i action which is free of rank n over $\mathcal{O}_{\mathcal{E},R_i}$. For each positive integer i, let $M_{R_i} = D_{\mathcal{O}_\mathcal{E},R_i}(V_{R_i})$ and define M_R to be the limit $M_R = \varprojlim M_{R_i}$.

For the rest of this subsection, fix a positive integer i. Let A be any $\mathcal{O}_F/\varpi^i$-algebra, no longer required to have finitely many elements. In particular, we will make use of the case where $A = R_i$. As before, define $\mathfrak{S}_A = (W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$ and $\mathcal{O}_{\mathcal{E},A} = \mathfrak{S}_A[1/u]$. It is easy to see that the Frobenius φ on $\mathfrak{S}$ is continuous with respect to the (p, u)-adic topology, so φ has a unique extension to an endomorphism of $\mathfrak{S}_A$ and, hence, of $\mathcal{O}_{\mathcal{E},A}$. Analogously to the situation in which A had finitely many elements, we define an étale φ-module with A-coefficients to be a finitely generated $\mathcal{O}_{\mathcal{E},A}$-module M_A along with a φ-semilinear endomorphism φ_{M_A} which induces an isomorphism of $\mathcal{O}_{\mathcal{E},A}$-modules $\Phi_{M_A} := 1 \otimes \varphi_{M_A} : \mathcal{O}_{\mathcal{E},A} \otimes_{\varphi,\mathcal{O}_{\mathcal{E},A}} M_A \to M_A$. Denote the category of étale φ-modules with A-coefficients which are projective of rank n over $\mathcal{O}_{\mathcal{E},A}$ by $\mathcal{R}_n^i(A)$.

Fix an étale φ-module $M_A \in \mathcal{R}_n^i(A)$ which is free over $\mathcal{O}_{\mathcal{E},A}$ of rank n. Additionally, fix an isomorphism $M_A \simeq \mathcal{O}_{\mathcal{E},A}^n$. For any A-algebra B, $\mathcal{O}_{\mathcal{E},B}$ is naturally an $\mathcal{O}_{\mathcal{E},A}$-algebra, and so we can consider the base change $M_B = \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$. The following lemma shows that M_B is an element of $\mathcal{R}_n^i(B)$.

Lemma 3.15. *For any free étale φ-module M_A with coefficients in A and any A-algebra B, $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$ is an étale φ-module with coefficients in B.*

Proof. To show that $M_B := \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$ is an étale φ-module with coefficients in B, we need to establish an isomorphism $\Phi_{M_B} : \varphi^* M_B \to M_B$. By definition M_A is equipped with an endomorphism φ_{M_A} which is semilinear with respect to the Frobenius φ on $\mathcal{O}_{\mathcal{E},A}$ and such that the linearization $\Phi_{M_A} : \varphi^* M_A \to M_A$ is an isomorphism, where $\varphi^* M_A := \mathcal{O}_{\mathcal{E},A} \otimes_{\varphi, \mathcal{O}_{\mathcal{E},A}} M_A$. Thus it will suffice to show that $\varphi^* M_B \simeq \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \varphi^* M_A$.

Since $M_A \simeq \mathcal{O}_{\mathcal{E},A}^n$ as an $\mathcal{O}_{\mathcal{E},A}$-module, we have the following isomorphism of $\mathcal{O}_{\mathcal{E},B}$-modules.

$$\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \varphi^* M_A \simeq \oplus_{j=1}^n \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A} \otimes_{\varphi, \mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A}.$$

But the Frobenius endomorphisms on $\mathcal{O}_{\mathcal{E},A}$ and $\mathcal{O}_{\mathcal{E},B}$ are compatible because the extensions of φ from $\mathcal{O}_{\mathcal{E}}$ to $\mathcal{O}_{\mathcal{E},A}$ and $\mathcal{O}_{\mathcal{E},B}$ are unique. In particular, the compositions $\mathcal{O}_{\mathcal{E},B} \to \mathcal{O}_{\mathcal{E},A} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{E},A}$ and $\mathcal{O}_{\mathcal{E},B} \xrightarrow{\varphi} \mathcal{O}_{\mathcal{E},B} \to \mathcal{O}_{\mathcal{E},A}$ are the same, so $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A} \otimes_{\varphi, \mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A}$ is naturally isomorphic to $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},B}, \varphi} \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A}$. Thus, there are natural isomorphisms

$$\begin{aligned} \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \varphi^* M_A &\simeq \oplus_{j=1}^n \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},B}, \varphi} \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} \mathcal{O}_{\mathcal{E},A} \\ &\simeq \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},B}, \varphi} \mathcal{O}_{\mathcal{E},B}^n \\ &\simeq \varphi^* M_B, \end{aligned}$$

which give the desired result.

$\square$

Definition 3.16. *For an $\mathcal{O}_F/\varpi^i$-algebra A we define a* Kisin module with A-coefficients*, sometimes called a Breuil-Kisin module, to be a finitely generated, u-torsion free $\mathfrak{S}_A$-module $\mathfrak{M}$ along with a φ-semilinear endomorphism $\varphi_{\mathfrak{M}}$ such that the linearization $\Phi_{\mathfrak{M}} : 1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S}_A \otimes_{\varphi, \mathfrak{S}_A} \mathfrak{M} \to \mathfrak{M}$ is an injection of $\mathfrak{S}_A$-modules. We say that $\mathfrak{M}$ has* height ≤ 1 *if, moreover, the cokernel of $\Phi_{\mathfrak{M}}$ is annihilated by $E(u)$. Denote the category of Kisin modules with A-coefficients of height ≤ 1 which are furthermore projective of constant rank n over $\mathfrak{S}_A$ by $\mathcal{C}_n^{i, \leq 1}(A)$.*

Lemma 3.17. *If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module in $\mathcal{C}_n^{i, \leq 1}(A)$ then $\mathfrak{M}[1/u]$ is an étale φ-module in $\mathcal{R}_n^i(A)$ when considered along with the extension of $\varphi_{\mathfrak{M}}$ to $\mathfrak{M}[1/u]$.*

Proof. First we show that $\mathfrak{M}[1/u]$ is a projective rank n $\mathcal{O}_{\mathcal{E},A}$-module. Let $\mathfrak{p} \subset \mathcal{O}_{\mathcal{E},A}$ be a prime ideal and let $\mathfrak{q}$ be the pullback of $\mathfrak{p}$ to $\mathfrak{S}_A$. Since $\mathfrak{M}$ is projective of rank n over $\mathfrak{S}_A$, $\mathfrak{M}_\mathfrak{q} \simeq (\mathfrak{S}_A)_\mathfrak{q}^n$, so $\mathfrak{M}_\mathfrak{q}[1/u] \simeq ((\mathfrak{S}_A)_\mathfrak{q})^n[1/u]$ and $\mathfrak{M}[1/u]_\mathfrak{p} \simeq ((\mathcal{O}_{\mathcal{E},A})_\mathfrak{p})^n$.

Now consider the linearization of $\varphi_\mathfrak{M}$. Since $\mathfrak{M}$ is a Kisin module with A-coefficients, this linearization is an injection, so there is an exact sequence

$$0 \to \mathfrak{S}_A \otimes_{\varphi,\mathfrak{S}_A} \mathfrak{M} \xrightarrow{\Phi_\mathfrak{M}} \mathfrak{M} \to \operatorname{coker} \Phi_\mathfrak{M} \to 0.$$

When this sequence is localized at u, it yields

$$0 \to \mathcal{O}_{\mathcal{E},A} \otimes_{\varphi,\mathcal{O}_{\mathcal{E},A}} \mathfrak{M}[1/u] \to \mathfrak{M}[1/u] \to \operatorname{coker} \Phi_\mathfrak{M}[1/u] \to 0,$$

so to show that the linearization of the extension of $\varphi_\mathfrak{M}$ to $\mathfrak{M}[1/u]$ is an isomorphism, it suffices to show that $\operatorname{coker} \Phi_\mathfrak{M}[1/u] = 0$. By definition, $\operatorname{coker} \Phi_\mathfrak{M}$ is killed by $E(u)$, and $\operatorname{coker} \Phi_\mathfrak{M}[1/u] = 0$ if each element of $\operatorname{coker} \Phi_\mathfrak{M}$ is killed by u^N for some N, so it will be enough to show that for some positive integer powers N and M, $E(u)^M = u^N$ in $\mathfrak{S}_A$. Recall that $E(u)$ is the minimal polynomial of a uniformizer $\pi \in \mathcal{O}_K$ over K_0 where K/K_0 is a totally ramified extension, so $E(u)$ is an Eisenstein polynomial. In particular, $E(u) \equiv u^e \mod p$ where $e = [K : K_0]$. Let s be the smallest positive integer for which $p^s = 0$ in A. Then $E(u)^{p^s} \equiv u^{ep^s} \mod p^s$, so $E(u)^{p^s} = u^{ep^s} \in \mathfrak{S}_A$.

□

For any $\mathcal{O}_F/\varpi^i$-algebra A with a fixed free étale φ-module of rank n, M_A, Lemma 3.15 establishes a map $\operatorname{Spec} A \to \mathcal{R}_n^i$ and Lemma 3.17 establishes a map $\mathcal{C}_n^{i,\leq 1} \to \mathcal{R}_n^i$. Consider the case where $A = R_i$ and $M_A = M_{R_i} = D_{\mathcal{O}_E,R_i}(V_R \otimes_R R_i)$. In [EG15], Emerton and Gee establish that $\mathcal{C}_n^{i,\leq 1}$ and $\mathcal{R}_n^i$ are both stacks over $\operatorname{Spec} \mathcal{O}_F/\varpi^i$ and that these maps are indeed morphisms of stacks, so it makes sense to consider the 2-fibered product $\mathcal{C}_n^{i,\leq 1} \times_{\mathcal{R}_n^i} \operatorname{Spec} R_i$

in the category of stacks.

$$\begin{array}{ccc} \mathcal{C}_n^{i,\leq 1} \times_{\mathcal{R}_n^i} \operatorname{Spec} R_i & \longrightarrow & \operatorname{Spec} R_i \\ \downarrow & & \downarrow \\ \mathcal{C}_n^{i,\leq 1} & \longrightarrow & \mathcal{R}_n^i \end{array}$$

In subsection 3.6, we will identify this product with a closed, projective subscheme of the affine Grassmannian of GL_n over $\operatorname{Spec} R$. Before proceeding, however, let us consider the A-points of this 2-fibered product for an arbitrary $\mathcal{O}/\varpi^i$-algebra A. An A point of $\operatorname{Spec} R_i$ is an R_i-algebra structure on A; an A point of $\mathcal{C}_n^{i,\leq 1}$ is a Kisin module with A-coefficients that is projective or rank n over $\mathfrak{S}_A$; and an A point of $\mathcal{R}_n^i$ is an étale φ-module with A-coefficients that is projective of rank n over $\mathcal{O}_{\mathcal{E},A}$. Thus, an A point of $\mathcal{C}_n^{i,\leq 1} \times_{\mathcal{R}_n^i} \operatorname{Spec} R_i$ is a triple $(\mathfrak{M}_A, y, f)$ where $\mathfrak{M}_A \in \mathcal{C}_n^{i,\leq 1}(A)$, y establishes an R_i-algebra structure on A, and f is an isomorphism of étale φ-modules $\mathfrak{M}_A[1/u] \xrightarrow{\sim} \mathcal{O}_{\mathcal{E},A} \otimes_{\mathcal{O}_{\mathcal{E},R_i}} M_{R_i}$ in $\mathcal{R}_n^i(A)$.

3.5 The affine Grassmannian of $\mathrm{Res}_{k|\mathbb{F}_p} \mathrm{GL}_n$

In subsection 3.7, we will construct Kisin's partial resolution of $\operatorname{Spec} R_{\overline{\rho}}^{\mathrm{fl},\square,\mu}$, which is a subscheme of a Weil restriction of the affine Grassmannian associated to GL_n. Before we review the results of Kisin that construct this variety, it will be prudent to include some exposition on affine Grassmannians because it will help to underscore that the Kisin variety is indeed a variety and not merely a set of finite flat group schemes over $\mathcal{O}_K$. We have already seen that certain representations can be associated to étale φ-modules which contain lattices called Kisin modules. In this section, we will see that, if the representation has coefficients in A, the Kisin module can be realized as an A-point of a Weil restriction of the affine Grassmannian of GL_n.

To begin defining the affine Grassmannian, we first recall the definition of an ind-scheme.

Definition 3.18. *(1.) Let κ be a field. A κ-space is a functor $F : (\mathrm{Sch}_\kappa)^{\mathrm{opp}} \to (\mathrm{Sets})$ which is a sheaf for the fpqc topology.*

(2.) An ind-scheme is a κ-space which can be written as the inductive limit of of an inductive system of schemes in the category of κ-spaces where the index set is $\mathbb{N}$ and all of the transition maps are closed immersions.

Let G be a connected, reductive group over a field κ. Define LG to be the functor $(\mathrm{Sch}_\kappa)^{\mathrm{opp}} \to (\mathrm{Sets})$ that sends an affine κ-scheme $\operatorname{Spec} A$ to the $A(\!(u)\!)$-points of G.

$$LG(A) = G(A(\!(u)\!))$$

LG is called the *loop group* of G and is an ind-scheme, but, in general, is not a scheme. Define L^+G to be the functor $(\mathrm{Sch}_\kappa)^{\mathrm{opp}} \to (\mathrm{Sets})$ that sends an affine κ-scheme $\operatorname{Spec} A$ to the $A[[u]]$-points of G.

$$L^+G(A) = G(A[[u]]).$$

$L + G$ is called the *positive loop group* of G and is a scheme, albeit infinite-dimensional. To see this, consider the example where $G = \mathrm{GL}_n$. In this case, we can identify L^+G with an affine subspace of $\prod_{i=1}(\mathbb{A}^{n^2} \times \mathbb{A}^{n^2})$. For a κ-algebra A, $L^+\,\mathrm{GL}_n(A) = \mathrm{GL}_n(A[[u]])$, which we can associate with a subset of $\mathrm{Mat}_{n\times n}(A[[u]]) \times \mathrm{Mat}_{n\times n}(A[[u]])$ via the closed embedding

$$\mathrm{GL}_n \longhookrightarrow \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times n}$$

given by $M \mapsto (M, M^{-1})$. Moreover, $\mathrm{Mat}_{n\times n}(A[[u]]) \times \mathrm{Mat}_{n\times n}(A[[u]])$ can be identified with $\prod_{i=1}^{\infty}(\mathbb{A}^{n^2}(A) \times \mathbb{A}^{n^2}(A))$. Then $L^+G(A)$ can be cut out of $\prod_{i=1}^{\infty}(\mathbb{A}^{n^2}(A) \times \mathbb{A}^{n^2}(A))$ by splitting the matrix equality $M_1M_2 = 1$ into equations for each u-component and we can recognize L^+G as a closed subscheme of $\prod_{i=1}^{\infty}(\mathbb{A}^{n^2} \times \mathbb{A}^{n^2})$. While a similar process will show that

$$LG \hookrightarrow \prod_{i=-\infty}^{\infty} (\mathbb{A}^{n^2} \times \mathbb{A}^{n^2}),$$

we are unable to express condition on negative degree terms for Laurent series with polynomial equations, and LG cannot likewise be recognized as a closed subscheme of $\prod_{i=-\infty}^{\infty}(\mathbb{A}^{n^2} \times \mathbb{A}^{n^2})$.

Definition 3.19. *The affine Grassmannian associated to G is the sheafification of the quotient, LG/L^+G. We denote it by $\mathcal{G}rass_G$.*

With this definition in mind, let us revisit the description of the geometric points of $C_\mu(b)$ from the introduction.

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in \bigsqcup_{\substack{\nu \in Y^+ \\ \nu \leq \mu}} G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)\}$$

In this case, $G = \operatorname{Res}_{k|\mathbb{F}_p} \mathrm{GL}_2$ so any point $g \in C_\mu(b)(\overline{\mathbb{F}}_p)$ comes from

$$G(L)/G(\mathcal{O}_L) = \operatorname{Res}_{k|\mathbb{F}_p} GL_2(\overline{\mathbb{F}}_p((u)))/\operatorname{Res}_{k|\mathbb{F}_p} GL_2(\overline{\mathbb{F}}_p[[u]]) = LG(\overline{\mathbb{F}}_p)/L^+G(\overline{\mathbb{F}}_p),$$

so the $\overline{\mathbb{F}}_p$-points of $C_\mu(b)$ certainly lie in $\operatorname{Res}_{k|\mathbb{F}_p} \mathcal{G}rass_{\mathrm{GL}_2}$. To realize the Kisin variety as a subspace of $\operatorname{Res}_{k|\mathbb{F}_p} \mathcal{G}rass_{\mathrm{GL}_n}$, however, it will be valuable to have another description of the affine Grassmannian associated to GL_n. Specifically, for a κ-algebra A, the A-points of $\mathcal{G}rass_{\mathrm{GL}_n}$ can be regarded as lattices in $A((u))^n$.

Definition 3.20. *Let A be a κ-algebra. A* lattice in $A((u))^n$ *is a finitely generated, projective $A[[u]]$-submodule $\mathscr{L}$ of $A((u))^n$ such that $\mathscr{L} \otimes_{A[[u]]} A((u)) = A((u))^n$.*

To recognize the A-points of $\mathcal{G}rass_{\mathrm{GL}_n}$ as the set of lattices in $A((u))^n$, we will assign to any such lattice an element of $\mathrm{GL}_n(A((u)))/\mathrm{GL}_n(A[[u]])$. If a lattice $\mathscr{L}$ was additionally free as an $A[[u]]$-submodule, then a choice of basis for $\mathscr{L}$ over $A[[u]]$ would correspond to an element of $\mathrm{GL}_n(A((u)))$. Further, a change of basis would be equivalent to multiplication by an element of $\mathrm{GL}_n(A[[u]])$, so $\mathscr{L}$ is identified uniquely by an element of $\mathcal{G}rass_{\mathrm{GL}_n}(A) = \mathrm{GL}_n(A((u)))/\mathrm{GL}_n(A[[u]])$.

In the following lemma, we see that $\mathscr{L}$ is fpqc-locally free on A so, there is a faithfully flat ring homomorphism $A \to A'$ for which $\mathscr{L} \otimes_{A[[u]]} A'[[u]]$ is indeed free as an $A'[[u]]$-module and we can associate $\mathscr{L} \otimes_{A[[u]]} A'[[u]]$ to an element of $\mathcal{G}rass_{\mathrm{GL}_n}(A')$. Since $\mathcal{G}rass_{\mathrm{GL}_n}$ is the sheafification of $L\,\mathrm{GL}_n/L^+\,\mathrm{GL}_n$, it is sufficient to show that $\mathcal{G}rass_{\mathrm{GL}_n}(A)$ can be identified with the set of lattices in $A((u))^n$.

Lemma 3.21. *Let $\mathscr{L} \subset A(\!(u)\!)^n$ be an $A[[u]]$-submodule. The following are equivalent:*

(1.) The submodule $\mathscr{L}$ is a lattice.

(2.) There exists a nonnegative integer N with $u^N A[[u]]^n \subseteq \mathscr{L} \subseteq u^{-N} A[[u]]^n$ and the quotient $u^{-N} A[[u]]^n / \mathscr{L}$ is locally free of finite rank over A.

(3.) Zariski-locally on A, $\mathscr{L}$ is a free $A[[u]]$-module of rank n (i.e. there exist $f_1, \dots, f_r \in A$ such that $(f_1, \dots, f_r) = 1$ and for all i, $\mathscr{L} \otimes_{A[[u]]} A_{f_i}[[u]]$ is a free $A_{f_i}[[u]]$-module of rank n) and $\mathscr{L} \otimes_{A[[u]]} A(\!(u)\!) = A(\!(u)\!)^n$.

(4.) fpqc-locally on A, $\mathscr{L}$ is a free $A[[u]]$-module of rank n (i.e. there exists a faithfully flat ring homomorphism $A \to A'$ such that $\mathscr{L} \otimes_{A[[u]]} A'[[u]]$ is a free $A'[[u]]$-module of rank n) and $\mathscr{L} \otimes_{A[[u]]} A(\!(u)\!) = A(\!(u)\!)^n$.

Proof. [GÖ0, Lemma 2.11] □

The preceding discussion and the equivalence of (1.) and (4.) in Lemma 3.21 give us the following proposition.

Proposition 3.22. *The affine Grassmannian for GL_n is the κ-space that assigns to any κ-algebra A the set of lattices in $A(\!(u)\!)^n$.*

Using this description of $\mathcal{G}rass_{\mathrm{GL}_n}$, we will now see that it is ind-projective (i.e. that it can be written as the inductive limit of an inductive system of projective schemes where the index set is $\mathbb{N}$ and all of the transition maps are closed immersions). First note that by the equivalence of (1.) and (2.) in Lemma 3.21, for any lattice $\mathscr{L}$ in $A(\!(u)\!)^n$, we can fix a nonnegative integer N for which

$$u^N A[[u]] \subseteq \mathscr{L} \subseteq u^{-N} A[[u]]$$

and the quotient $u^{-N} A[[u]]^n / \mathscr{L}$ is locally free and of finite rank over A. With this in mind, define $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ to be the subspace of $\mathcal{G}rass_{\mathrm{GL}_n}$ given by

$$\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}(A) = \{\mathscr{L} \in \mathcal{G}rass_{\mathrm{GL}_n}(A) \mid u^N A[[u]] \subseteq \mathscr{L} \subseteq u^{-N} A[[u]]\},$$

so that $\mathcal{G}rass_{\mathrm{GL}_n} = \lim_{N\to\infty} \mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$.

Theorem 3.23. *$\mathcal{G}rass_{\mathrm{GL}_n}$ is represented by an ind-projective scheme.*

Proof. Since $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n} = \lim_{N\to\infty} \mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ it suffices to show that each $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ is represented by a projective scheme, which we do by relating the A-points of $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ to those of a finite union of Grassmannians. For positive integers m and ℓ let $\mathrm{Gr}(m,\ell)$ denote the usual Grassmannian having the following functor of points.

$$\mathrm{Gr}(m,\ell)(A) = \left\{ \begin{array}{c} \text{locally free rank } \ell \text{ quotients of the rank } m \\ \text{free sheaf } \mathcal{O}_A^{\oplus m} \text{ up to isomorphism} \end{array} \right\}$$

Fix a nonnegative integer N. Using the description of lattices in Lemma 3.21 (2.), a lattice $\mathscr{L} \in \mathcal{G}rass_{\mathrm{GL}_n}(A)$ is defined by the existence of an inclusion of of A-modules

$$\mathscr{L}/u^N A[[u]]^n \hookrightarrow u^{-N}A[[u]]^n/u^N A[[u]]^n \simeq A^{2Nn},$$

and a surjection

$$A^{2Nn} \longrightarrow \frac{u^{-N}A[[u]]^n/u^N A[[u]]^n}{\mathscr{L}/u^N A[[u]]^n} \simeq u^{-N}A[[u]]^n/\mathscr{L},$$

where $u^{-N}A[[u]]^n/\mathscr{L}$ is locally free of finite rank over A. Thus, $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ can be identified with a subscheme of the finite union $\bigsqcup_{\ell=1}^{2Nn} \mathrm{Gr}(2Nn,\ell)$, which is a projective scheme.

□

Now that we have recognized $\mathcal{G}rass_{\mathrm{GL}_n}$ as an ind-projective scheme parameterizing lattices, we conclude this subsection by considering $\mathrm{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}$. In the following section, we'll see that a Kisin module with coefficients in A can be regarded as an A-point on $\mathrm{Res}_{W(k)|\mathbb{Z}_p} Gr_{\mathrm{GL}_n} \times_{\mathbb{Z}_p} \mathrm{Spec}\, \mathcal{O}_F$. Recall that the Weil restriction $\mathrm{Res}_{W(k)/\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_d}$ is characterized by the existence of a bijection

$$\mathrm{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}(\mathrm{Spec}\, A) \to \mathcal{G}rass_{\mathrm{GL}_d}(\mathrm{Spec}\, A \otimes_{\mathbb{Z}_p} W(k))$$

for any $\mathbb{Z}_p$-algebra A. Accordingly, for any $\mathbb{Z}_p$-algebra A, $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}(A)$ is the set of all finitely generated, projective $(W(k) \otimes_{\mathbb{Z}_p} A)[[u]]$-submodules $\mathscr{L}$ of $(W(k) \otimes_{\mathbb{Z}_p} A)((u))^n$ such that $\mathscr{L} \otimes_{(W(k)\otimes_{\mathbb{Z}_p} A)[[u]]} (W(k) \otimes_{\mathbb{Z}_p} A)((u)) = (W(k) \otimes_{\mathbb{Z}_p} A)((u))^n$. In the notation we've been using, $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}(A)$ is exactly the set of all finitely generated, projective $\mathfrak{S}_A$-submodules $\mathscr{L}$ of $\mathcal{O}^n_{\mathcal{E},A}$ such that $\mathscr{L} \otimes_{\mathfrak{S}_A} \mathcal{O}_{\mathcal{E},A} = \mathcal{O}^n_{\mathcal{E},A}$. Observe that this is very nearly the set of Kisin modules with coefficients in A, only the data of the Frobenius endomorphism is missing. In the following lemma we see that, like $\mathcal{G}rass_{\mathrm{GL}_n}$, $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \times_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F$ is indeed an ind-scheme.

Lemma 3.24. *Assume F contains a copy of K_0. The Weil restriction $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \times_{\mathbb{Z}_p} \mathcal{O}_F$ exists and is an ind-projective scheme.*

Proof. By definition, the functor of points of $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}$ is given by

$$\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}(\operatorname{Spec} A) = \operatorname{Hom}_{W(k)}(\operatorname{Spec} A \otimes_{\mathbb{Z}_p} W(k), \mathcal{G}rass_{\mathrm{GL}_d})$$

for any $\mathbb{Z}_p$-algebra A, so the functor of points for the base change $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \times_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F$ given by

$$\begin{aligned}((\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}) &\otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F)(\operatorname{Spec} A) \\ &= \operatorname{Hom}_{\mathcal{O}_F \otimes_{\mathbb{Z}_p} W(k)}(\operatorname{Spec} A \otimes_{\mathbb{Z}_p} W(k), \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F)\end{aligned}$$

for any $\mathcal{O}_F$-algebra A.

Since $\mathcal{O}_F$ contains a copy of $W(k)$ and $W(k)$ is an étale extension of $\mathbb{Z}_p$, $W(k) \otimes_{Z_p} \mathcal{O}_F \simeq \bigoplus_{i=1}^{f} \mathcal{O}_F$, where f is the number of embeddings $W(k) \hookrightarrow \mathcal{O}_F$. As a result, $\operatorname{Spec} A \otimes_{\mathbb{Z}_p} W(k) \simeq$

$\bigsqcup_{i=1}^{f} \operatorname{Spec} A$ and the functor of points can be rewritten as

$$\begin{aligned}(\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \otimes_{\mathbb{Z}_p} \mathcal{O}_F)(A) &= \operatorname{Hom}_{\mathcal{O}_F}(\operatorname{Spec} A \otimes_{\mathbb{Z}_p} W(k), \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F) \\ &= \operatorname{Hom}_{\mathcal{O}_F}(\bigsqcup_{i=1}^{f} \operatorname{Spec} A, \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F) \\ &= \prod_{i=1}^{f} \operatorname{Hom}_{\mathcal{O}_F}(\operatorname{Spec} A, \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F) \\ &= \operatorname{Hom}_{\mathcal{O}_F}(\operatorname{Spec} A, \prod_{i=1}^{f} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F).\end{aligned}$$

Thus $\operatorname{Res}_{W(k)/\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ exists and is isomorphic to $\prod_{i=1}^{f} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F$.

It remains to see that $\prod_{i=1}^{f} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \operatorname{Spec} \mathcal{O}_F$ is ind-projective. Fix projective $W(k)$-schemes X_m so that $\mathcal{G}rass_{\mathrm{GL}_d} = \varinjlim X_m$. Then $\prod_{i=1}^{f} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \mathcal{O}_F = \varinjlim \prod_{i=1}^{f} X_m \otimes_{\mathbb{Z}_p} \mathcal{O}_F$, so $\prod_{i=1}^{f} \mathcal{G}rass_{\mathrm{GL}_d} \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ is an ind-projective $\mathcal{O}_F$-scheme because each $\prod_{i=1}^{f} X_m \otimes_{\mathbb{Z}_p} \mathcal{O}_F$ is a projective $\mathcal{O}_F$-scheme.

□

3.6 Locating Kisin modules in $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n}$

Let A be a $\mathbb{Z}_p$-module and fix a free étale φ-module M_A of rank n with coefficients in A. For any A-algebra B, set $M_B := \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$. Fix an isomorphism $M_A \simeq \mathcal{O}_{\mathcal{E},A}^n$ and let $\mathfrak{N}_A$ be the preimage of $\mathfrak{S}_A^n$ under this map. For any $\mathcal{O}/\varpi^i$-algebra B, let $\mathcal{G}rass_{M_A}(B)$ denote the set of $\mathfrak{S}_B$-lattices Λ of $\mathcal{O}_{\mathcal{E},B}^n$ which are projective of rank n and satisfy $\Lambda \otimes_{\mathfrak{S}_B} \mathcal{O}_{\mathcal{E},B} \simeq \mathcal{O}_{\mathcal{E},B}^n$. Using the isomorphism $M_A \simeq \mathcal{O}_{\mathcal{E},A}^n$, $\mathcal{G}rass_{M_A}(B)$ can be identified with the set of $\mathfrak{S}_B \simeq \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{N}_A$-lattices Λ of M_B which are projective of rank n and satisfy $\Lambda \otimes_{\mathfrak{S}_B} \mathcal{O}_{\mathcal{E},B} \simeq M_B$. In particular, using this isomorphism, $\mathcal{G}rass_{M_A}(B)$ can be identified with $(\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \otimes_{\mathbb{Z}_p} A)(B)$. Let $\mathcal{G}rass_{M_A}$ denote $\operatorname{Res}_{W(k)|\mathbb{Z}_p} \mathcal{G}rass_{\mathrm{GL}_n} \otimes_{\mathbb{Z}_p} A$.

Now suppose that for some A-algebra B, Λ is an element of $\mathcal{G}rass_{M_A}(B)$. Then Λ is a subset of $M_B = \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$. Since M_A is an étale φ-module with coefficients in A, it is equipped with a Frobenius endomorphism φ_{M_A} and M_B is an étale φ-module with coefficients in B by Lemma 3.15. In particular, it has a Frobenius endomorphism $\varphi_{M_B} = \varphi \otimes \varphi_{M_A}$ which we use to define a closed subspace $\mathcal{G}rass_{M_A}^{\leq 1} \subset \mathcal{G}rass_{M_A}$ with conditions on the functor of points that mimic those used to define $\mathcal{C}_n^{i,\leq 1}$.

Define $\mathcal{G}rass_{M_A}^{\leq 1}(B)$ to be the subset of $\mathcal{G}rass_{M_A}(B)$ consisting of those lattices Λ which are invariant under the Frobenius of M_B and such that the cokernel of that Frobenius restricted to Λ is killed by $E(u)$. We say that such lattices have *height* ≤ 1. In this section we will show that $\mathcal{G}rass_{M_A}^{\leq 1}$ is a closed, projective subscheme of $\mathcal{G}rass_{M_A}$ and then that $\mathcal{G}rass_{M_A}^{\leq 1}$ can be naturally identified with the aforementioned 2-fiber product $\mathcal{C}_n^{i,\leq 1} \times_{\mathcal{R}_n^i} \operatorname{Spec} A$.

In order to show that $\mathcal{G}rass_{M_R}^{\leq 1}$ is a closed subspace of $\mathcal{G}rass_{M_A}$, we need to show that, for any $\mathcal{O}_F$-algebra B and any $y \in \mathcal{G}rass_{M_A}(B)$, that the induced map $\mathcal{G}rass_{M_A}^{\leq 1} \times_{\mathcal{G}rass_{M_A}} \operatorname{Spec} B \to B$ is a closed immersion.

$$\begin{array}{ccc}
\mathcal{G}rass_{M_A}^{\leq 1} \times_{\mathcal{G}rass_{M_A}} \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} B \\
\big\downarrow & & \big\downarrow y \\
\mathcal{G}rass_{M_A}^{\leq 1} & \longrightarrow & \mathcal{G}rass_{M_A}
\end{array}$$

Such a B-point y represents a $\mathfrak{S}_B$-lattice $\mathfrak{M}_B$ in $\mathcal{O}_{\mathcal{E},B}^n$. We want to show that it suffices to check this in the case where $\mathfrak{M}_B$ is a free $\mathfrak{S}_B$-module. When that is the case, we will be able to fix an isomorphism $\mathfrak{S}_B^n \simeq \mathfrak{M}_B$ and, therefore, a basis for $\mathfrak{M}_B$ so that we can regard the Frobenius endomorphism as a $n \times n$ matrix, which will make it more straightforward to show that the height condition is in fact a closed condition. The following two lemmas, which along with their proofs are from [EG15], will allow us to make this reduction.

Lemma 3.25. *[EG15, Proposition 5.1.8.] Let S be a commutative $\mathbb{Z}_p$-module and let $\mathfrak{M}$ be an $S[[u]]$-algebra. The following are equivalent.*

1. *$\mathfrak{M}$ is a finitely generated projective $S[[u]]$-module.*

2. *$\mathfrak{M}$ is a u-torsion free and u-adically complete and separated $S[[u]]$-module and $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective S-module.*

If $\mathfrak{M}/u\mathfrak{M}$ is furthermore a free S-module, then $\mathfrak{M}$ is a free S-module.

Proof. First, suppose $\mathfrak{M}$ is a finitely generated projective $S[[u]]$-module. Then $\mathfrak{M}$ is a direct summand of a finite free $S[[u]]$-module, and so certainly is a u-torsion free and u-adically complete and separated $S[[u]]$-module. Of course $\mathfrak{M}/u\mathfrak{M}$ is additionally a finitely generated projective S-module.

Now suppose $\mathfrak{M}$ satisfies the second condition. Since $\mathfrak{M}$ is u-torsion free, for any positive integers n and m there is an exact sequence of A-modules

$$0 \longrightarrow \mathfrak{M}/u^n\mathfrak{M} \xrightarrow{u^m} \mathfrak{M}/u^{n+m}\mathfrak{M} \longrightarrow \mathfrak{M}/u^m\mathfrak{M} \longrightarrow 0.$$

By assumption $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective S module, so by [Sta21, Tag 00NX] $\mathfrak{M}/u\mathfrak{M}$ is a finitely presented and flat S-module. Then, using the exact sequence above, we can show inductively that each $\mathfrak{M}/u^n\mathfrak{M}$ is finitely presented as an S-module. Indeed, suppose this is the case for $\mathfrak{M}/u^n\mathfrak{M}$; then the following sequence is exact, so $\mathfrak{M}/u^{n+1}\mathfrak{M}$ is finitely presented as well.

$$0 \longrightarrow \mathfrak{M}/u\mathfrak{M} \xrightarrow{u^m} \mathfrak{M}/u^{n+1}\mathfrak{M} \longrightarrow \mathfrak{M}/u^n\mathfrak{M} \longrightarrow 0$$

Therefore, by [Sta21, Tag 0561], each $\mathfrak{M}/u^n\mathfrak{M}$ is finitely presented as an $S[[u]]/u^n$-module so by [Gro60a, Prop 0.7.2.10(ii)], $\mathfrak{M}$ is a finitely generated projective $S[[u]]$-module.

Lastly, we need to show that if $\mathfrak{M}/u\mathfrak{M}$ is furthermore free as an S-module, then $\mathfrak{M}$ is free as an $S[[u]]$-module. Since $\mathfrak{M}/u\mathfrak{M}$ is projective as an A-module, we can choose an A-linear section of quotient map $\mathfrak{M} \to \mathfrak{M}/u\mathfrak{M}$.

$$\begin{array}{ccc} & & \mathfrak{M} \\ & \nearrow & \downarrow \\ \mathfrak{M}/u\mathfrak{M} & = & \mathfrak{M}/u\mathfrak{M} \end{array}$$

This lift provides us with a map of projective $S[u]$-modules $\mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]] \to \mathfrak{M}$ which, modulo u, is the identity map on $\mathfrak{M}/u\mathfrak{M}$. Therefore, by the sixth statement of Nakayama's Lemma in [Sta21, Tag 00DV], $\mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]] \to \mathfrak{M}$ is surjective. To show that this map is an isomorphism, consider the exact sequence of $A[[u]]$-modules

$$0 \to K \to \mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]] \to \mathfrak{M} \to 0.$$

Since both $\mathfrak{M}$ and $\mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]]$ are projective over $A[[u]]$ this sequence splits and $\mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]] \simeq K \oplus \mathfrak{M}$, so K is a finitely generated, projective $A[[u]]$-module and, thus, is u-adically separated. However, considering the isomorphism $\mathfrak{M}/u\mathfrak{M} \otimes_A A[[u]] \simeq K \oplus \mathfrak{M}$ modulo u, we see that $\mathfrak{M}/u\mathfrak{M} \simeq K \oplus \mathfrak{M}/u\mathfrak{M}$, so $K/uK = 0$. Since K is u-adically separated, this implies $K = 0$. □

Lemma 3.26. *[EG15, Proposition 5.1.9. (1)] Let S be any $\mathbb{Z}_p$-algebra. If $\mathfrak{M}$ is a finitely generated projective $S[[u]]$-module, then $\mathfrak{M}$ is Zariski locally on* $\operatorname{Spec} S$ *free of finite rank as an $S[[u]]$-module. In other words, there exists a Zariski open cover $\bigcup \operatorname{Spec} S_i$ of $\operatorname{Spec} S$ such that the base changes $\mathfrak{M} \otimes_{S[[u]]} S_i[[u]]$ are locally free of finite rank.*

Proof. If $\mathfrak{M}$ is a finitely generated $S[[u]]$-module, then $\mathfrak{M}/u\mathfrak{M}$ is a finitely generated projective S-module by Lemma 3.25 and is therefore Zariski locally finite free over $\operatorname{Spec} S$. Fix an affine open cover $\bigcup \operatorname{Spec} S_i$ of $\operatorname{Spec} S$ such that for each $\operatorname{Spec} S_i$, $\mathfrak{M}/u\mathfrak{M} \otimes_S S_i$ is finite free over S_i. In the proof of the preceding lemma, we showed that there exists an isomorphism $(\mathfrak{M}/u\mathfrak{M}) \otimes_S S[[u]] \simeq \mathfrak{M}$, so $\mathfrak{M} \otimes_{S[[u]]} S_i[[u]] \simeq (\mathfrak{M}/u\mathfrak{M}) \otimes_S S_i[[u]]$, which is free of finite rank over $S_i[[u]]$.

□

Proposition 3.27. *$\mathcal{G}rass^{\leq 1}_{M_A}$ is a closed, projective sub-scheme of $\mathcal{G}rass_{M_A}$.*

Proof. First we want to show that $\mathcal{G}rass^{\leq 1}_{M_A}$ is a closed subspace of $\mathcal{G}rass_{M_A}$. For this we must show that for any $\mathcal{O}_F/\varpi^i$-algebra B and any B-valued point y of $\mathcal{G}rass_{M_A}$, that the induced map $\mathcal{G}rass^{\leq 1}_{M_A} \otimes_{\mathcal{O}_F} B \to \operatorname{Spec} B$ is a closed immersion, which can be checked Zariski locally.

$$\begin{array}{ccc} \mathcal{G}rass^{\leq 1}_{M_A} \times_{\mathcal{G}rass_{M_A}} B & \longrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \mathcal{G}rass^{\leq 1}_{M_A} & \longrightarrow & \mathcal{G}rass_{M_A} \end{array} \tag{1}$$

Note that the inclusion $\mathcal{G}rass^{\leq 1}_{M_A} \to \mathcal{G}rass_{M_A}$ is a monomorphism, so its base change $\mathcal{G}rass^{\leq 1}_{M_A} \otimes_{\mathcal{O}_F} B \to \operatorname{Spec} B$ is as well. Thus, to show that this map is a closed immersion, we need to show that its image is closed in $\operatorname{Spec} B$. In particular, we need to show that there is an ideal I in B such that the image of $\mathcal{G}rass^{\leq 1}_{M_A} \otimes_{\mathcal{O}_F} B \to \operatorname{Spec} B$ is exactly $\operatorname{Spec} B/I$. To do this, we construct an ideal I such that, for $\mathfrak{M}_B \in \mathcal{G}rass_{M_A}(B)$, we have $\mathfrak{M}_B \otimes_{\mathfrak{S}_B} \mathfrak{S}_{B/I} \in \mathcal{G}rass^{\leq 1}_{M_A}(B/I)$ if and only if $\mathfrak{M}_B \in \mathcal{G}rass^{\leq 1}_{M_A}(B)$.

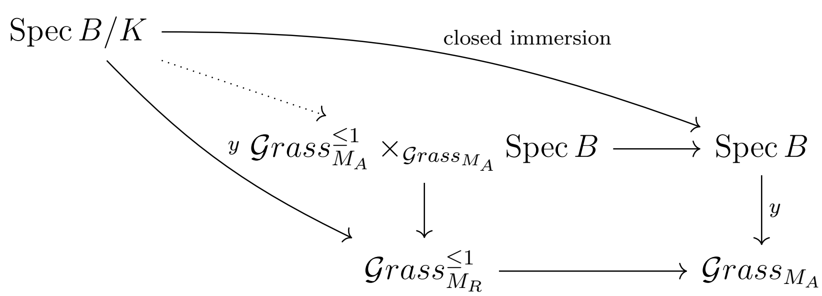
Fix such an $\mathfrak{M}_B$. Then $\mathfrak{M}_B$ is defined to be a finitely generated projective $W(k) \otimes_{\mathbb{Z}_p} B[[u]]$-module but, using the isomorphism below, $\mathfrak{M}_B$ may be regarded instead as a finitely generated projective $B[[u]]^f$-module where f is the number of embeddings of $W(k)$ into $\mathcal{O}_F$.

$$W(k) \otimes_{\mathbb{Z}_p} B \simeq (W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_F) \otimes_{\mathcal{O}_F} B \simeq (\oplus_{i=1}^{f} \mathcal{O}_F) \otimes_{\mathcal{O}_F} B,$$

Denote the decomposition of $\mathfrak{M}_B$ using idempotents by $\mathfrak{M}_B = \oplus \mathfrak{M}_B^{(i)}$. Note that each factor $\mathfrak{M}_B^{(i)}$ is finite and projective projective over $B[[u]]$. By Lemma 3.26, for each of the $\mathfrak{M}_B^{(i)}$ there exists a Zariski open cover of $\operatorname{Spec} B$ on which $\mathfrak{M}_B^{(i)}$ is finite free over $B[[u]]$ and, therefore, over $W(k) \otimes_{\mathbb{Z}_p} B[[u]]$ so we can reduce to the case where $\mathfrak{M}_B$ is finite free over $\mathfrak{S}_B$.

Set $M_B := \mathfrak{M}_B[1/u]$ and let φ_{M_B} denote the Frobenius on M_B coming from $\mathfrak{M}_B$. Fix a basis for $\mathfrak{M}_B$ over $\mathfrak{S}_B$. When regarded as a basis for $\varphi^* M_B$ over $\mathcal{O}_{\mathcal{E},B}$, this basis can be used to represent the linearization of φ_{M_B} with a matrix. Denote this matrix by $A_{\varphi_{M_B}} \in \operatorname{Mat}_{d\times d}(\mathcal{O}_{\mathcal{E},B})$ and denote its (i,j)th entry by $(A_{\varphi_{M_B}})_{i,j} = \sum_{\ell=N_{i,j}}^{\infty} a_{i,j,\ell} u^\ell$. If $N_{i,j}$ is strictly negative, $(A_{\varphi_{M_B}})_{i,j}$ is not integral in $\mathcal{O}_{\mathcal{E},B}$, but we can construct an ideal $I_{i,j} \subset B$ for which $(A_{\varphi_{M_B}})_{i,j}$ is integral in $\mathcal{O}_{\mathcal{E},B/I_{i,j}}$. Recall that $W(k) \otimes_{\mathbb{Z}_p} B \simeq B^{\oplus f}$ where f is the number of embeddings $W(k) \hookrightarrow \mathcal{O}_F$, so $\mathfrak{S}_B \simeq \oplus_{i=1}^{f} B[[u]]$ and we may regard the $a_{i,j,k}$ as elements of B^f. For $N_{i,j} \leq k \leq -1$ denote the entries of $a_{i,j,k}$ by $(b_{i,j,k,1}, b_{i,j,k,2}, \ldots, b_{i,j,k,f})$ and define $I_{i,j}$ to be the ideal of B generated by $\{b_{i,j,k,\ell} : 1 \leq k \leq N_{i,j},\ 1 \leq \ell \leq f\}$ so that $I_{i,j}$ is the minimal ideal of B for which $(A_{\varphi_{M_B}})_{i,j} \in \mathfrak{S}_{B/I_{i,j}} \subset \mathcal{O}_{\mathcal{E},B/I_{i,j}}$ and $I = \sum_{i,j} I_{i,j}$ is the minimal ideal in B for which all of the $(A_{\varphi_{M_B}})_{i,j}$ are integral in $\mathcal{O}_{\mathcal{E},B/I}$.

Analogously, consider the conditions under which the kernel of φ_{M_B} restricted to $\mathfrak{M}_B$ is killed by $E(u)$. This is the case if and only if for each $x \in \mathfrak{M}_B$, there exists some $y \in \mathfrak{M}_B$ for which $E(u)x = A_{\varphi_{M_B}} y$. Since M_A is an étale φ-module, the linearization of φ_{M_A} is an isomorphism and $A^{-1}_{\varphi_{M_B}}$ exists. Therefore, kernel of φ_{M_B} restricted to $\mathfrak{M}_B$ is killed by $E(u)$ if and only if each entry of $E(u)A^{-1}_{\varphi_{M_B}}$ is an element of $\mathfrak{S}_B$. Using the same process as above, one can construct an ideal $J \subset B$ which is the minimal ideal of B such that each entry of $E(u)A^{-1}_{\varphi_{M_B}}$ is integral in $\mathcal{O}_{\mathcal{E},B/J}$. Then, letting $K = I + J$, K is the minimal ideal of B for which $\mathfrak{M}_B \otimes_{\mathfrak{S}_B} \mathfrak{S}_{B/K} \in \mathcal{G}rass^{\leq 1}_{M_A}(B/K)$. In other words, the map $y : \operatorname{Spec} B \to \mathcal{G}rass_{M_A}$ to which $\mathfrak{M}_B$ is associated has image in $\mathcal{G}rass^{\leq 1}_{M_A}$ when restricted to $\operatorname{Spec} B/K$, so we have the following diagram wherein the image of $\mathcal{G}rass^{\leq 1}_{M_A} \times_{\mathcal{G}rass_{M_A}} \operatorname{Spec} B \to \operatorname{Spec} B$ coincides with the image of the closed immersion $\operatorname{Spec} B/K \to \operatorname{Spec} B$.



Having seen that $\mathcal{G}rass^{\leq 1}_{M_A}$ is closed inside of $\mathcal{G}rass_{M_A}$, to show that $\mathcal{G}rass^{\leq 1}_{M_A}$ is a projective scheme, it will suffice to show that $\mathcal{G}rass^{\leq 1}_{M_A}$ is contained in one of the projective schemes from which $\mathcal{G}rass_{M_A}$ is constructed. Recall that $\mathcal{G}rass_{\mathrm{GL}_n} = \varinjlim \mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ where the the $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ are projective subschemes characterizing bounded lattices. In particular, for a $\mathbb{Z}_p$-algebra B, $\mathcal{G}rass^{(N)}_{\mathrm{GL}_n}$ is the set of all finitely generated projective submodules $\Lambda \subset B((u))^n$ such that $\Lambda \otimes_{B[[u]]} B((u)) = B((u))^n$ and $u^N B[[u]]^n \subset \Lambda \subset u^{-N} B[[u]]^n$.

Fix an isomorphism $M_A \simeq \mathcal{O}^n_{\mathcal{E},A}$ and let $\mathfrak{N}_A$ be the preimage of $\mathfrak{S}^n_A$ under this map. Then, by construction, $\mathcal{G}rass_{M_A}$ can be written as the colimit $\mathcal{G}rass_{M_A} = \varinjlim \mathcal{G}rass^{(N)}_{M_A}$ where, for any $\mathcal{O}_F$-algebra B, $\mathcal{G}rass^{(N)}_{M_A}(B)$ is the subset of $\mathcal{G}rass_{M_A}(B)$ consisting of those lattices Λ

in $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A \simeq \mathcal{O}_{\mathcal{E},B}^n$ such that $u^N \mathfrak{N}_B \subset \Lambda \subset u^{-N}\mathfrak{N}_B$ where $\mathfrak{N}_B = \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{N}_A$. It will thus suffice to show that for all $\mathcal{O}_F$-algebras B there exists some positive integer N, not depending on B, for which any $\mathfrak{M}_B \in \mathcal{G}rass_{M_A}^{\leq 1}(B)$ satisfies $u^N\mathfrak{N}_B \subset \mathfrak{M}_B \subset u^{-N}\mathfrak{N}_B$.

Since $\mathfrak{N}_A \subset M_A$, it is equipped with a restriction of φ_{M_A} which may not have image in $\mathfrak{N}_A$, so we regard it as a map $\varphi_{M_A} : \mathfrak{N}_A \to M_A$. Let r be the smallest integer such that $u^r\mathfrak{N}_A \subset (1 \otimes \varphi_{\mathfrak{N}_A})(\varphi^*\mathfrak{N}_A) \subset u^{-r}\mathfrak{N}_A$, where $\varphi^*\mathfrak{N}_A = \mathfrak{S}_A \otimes_{\mathfrak{S}_A} \mathfrak{N}_A$ as usual and $1 \otimes \varphi_{\mathfrak{N}_A}$ is regarded as a map $\varphi^*\mathfrak{N}_A \to M_A$. Then, for any $\mathcal{O}_F/\varpi^i$-algebra B with a point $\mathfrak{M}_B \in \mathcal{G}rass_{M_R}^{\leq 1}(B)$, define $\mathfrak{N}_B$ by $\mathfrak{N}_B = \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{N}_A$ Then r is additionally the smallest integer such that $u^r\mathfrak{N}_B \subset (1 \otimes \varphi_{M_B})(\varphi^*\mathfrak{N}_B) \subset u^{-r}\mathfrak{N}_B$.

Now let s be the smallest integer such that $\mathfrak{N}_B \subset u^{-s}\mathfrak{M}_B$. As before, we can use 3.26 to reduce to the case where $\mathfrak{M}_B$ is free as a $\mathfrak{S}_B$-module. Fix a $\mathfrak{S}_B$-basis $\{\beta_1, \dots, \beta_n\}$ for $\mathfrak{N}_B$. Then each β_i is of the form $u^{-s}m_i$ where $m_i \in \mathfrak{M}_B$, so $\varphi_{M_B}(\beta_i) = u^{-sp}\varphi(m_i) \in u^{-sp}\mathfrak{M}_B$. Accordingly, the smallest integer j such that $(1 \otimes \varphi)\varphi^*\mathfrak{N}_B \subset u^{-j}(1 \otimes \varphi)\varphi^*\mathfrak{M}_B$ is sp.

Now let k be the smallest integer such that $p^k = 0$ in A. Since B is an A-algebra, $p^k = 0$ in B as well and so $p^k = 0$ in $\mathfrak{S}_B$. Since E is an Eisenstein polynomial, $E(u) \equiv u^e$ modulo p as an element of $\mathcal{O}_{K_0}[u]$, so $E(u)^k = u^{ek}$ in $\mathfrak{S}_B$. Therefore,

$$\begin{aligned}
(1 \otimes \varphi)(\varphi^*\mathfrak{N}_B) &\subset u^{-r}\mathfrak{N}_B \\
&\subset u^{-s-r}\mathfrak{M}_B \\
&= E(u)^{-1}u^{-s-r}(E(u)\mathfrak{M}_B) \\
&\subset u^{-s-r-ek}(1 \otimes \varphi)(\varphi^*\mathfrak{M}_B),
\end{aligned}$$

so $sp \leq ek + s + r$ and s can be bounded above by $(ek + r)/(p-1)$, which does not depend on B or $\mathfrak{M}_B$. Similarly, if t is the least integer such that $\mathfrak{M}_B \subset u^{-t}\mathfrak{N}_B$, then

$$(1 \otimes \varphi)(\varphi^*\mathfrak{M}_B) \subset \mathfrak{M}_B \subset u^{-t}\mathfrak{N}_B \subset u^{-t-r}(1 \otimes \varphi)(\varphi^*\mathfrak{N}_B),$$

so $t + r \leq sp$, and $t \leq r/(p-1) \leq s$. Thus, letting $N = (ek + r)/(p-1)$, we can conclude that $\mathcal{G}rass_{M_A}^{\leq 1} \subset \mathcal{G}rass_{M_A}^{(N)}$, so $\mathcal{G}rass_{M_A}^{\leq 1}$ is projective.

□

Lemma 3.28. *$\mathcal{G}rass^{\leq 1}_{M_A}$ can be naturally identified with the 2-fiber product $\mathcal{C}^{i,\leq 1}_n \times_{\mathcal{R}^i_n} \operatorname{Spec} A$.*

Proof. Suppose $(\mathfrak{M}, \eta, f) \in (\mathcal{C}^{i,\leq 1}_n \times_{\mathcal{R}^i_n} \operatorname{Spec} A)(B)$. By definition, $\mathfrak{M}$ is a finitely generated, projective $\mathfrak{S}_B$-module of constant rank n and f is an isomorphism $f : \mathfrak{M}[1/u] \to \mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A$. But $M_A \simeq \mathcal{O}^n_{\mathcal{E},A}$ as an $\mathcal{O}_{\mathcal{E},A}$-module, so this map can be regarded as an isomorphism $g : \mathfrak{M}[1/u] \to \mathcal{O}^n_{\mathcal{E},B}$ which establishes $\mathfrak{M}$ as a $\mathfrak{S}_B$-submodule of $\mathcal{O}^n_{\mathcal{E},B}$ such that $\mathfrak{M} \otimes_{\mathfrak{S}_B} \mathcal{O}_{\mathcal{E},B} = \mathcal{O}^n_{\mathcal{E},B}$, so $\mathfrak{M} \in \mathcal{G}rass_{M_A}(B)$.

To see that $\mathfrak{M}$ is furthermore an element of $\mathcal{G}rass^{\leq 1}_{M_A}(B)$, we need to see that $\mathfrak{M}$ is invariant under the Frobenius φ_{M_B} of M_B and that the restriction of φ_{M_B} to $\mathfrak{M}$ has cokernel killed by $E(u)$. This follows from the fact that f is indeed a morphism of étale φ-modules and so commutes with φ_{M_B} on $\mathfrak{M}[1/u]$ and φ on M_B.

For the other inclusion, if $\Lambda \in \mathcal{G}rass^{\leq 1}_{M_A}(B)$, then the definition of $\mathcal{G}rass^{\leq 1}_{M_A}$ gives us $\Lambda \in \mathcal{C}^{i,\leq 1}_n(B)$. Moreover, an A-algebra structure on B satisfying $\mathcal{O}_{\mathcal{E},B} \otimes_{\mathcal{O}_{\mathcal{E},A}} M_A \simeq \mathfrak{M}[1/u]$ is inherent in the definition of Λ, so $\Lambda \in \mathcal{C}^{i,\leq 1}_n \times_{\mathcal{R}^i_n} \operatorname{Spec} A$. □

Now for any positive integer i, we a have constructed a projective morphism of schemes $\mathcal{G}rass^{\leq 1}_{M_{R_i}} \to \operatorname{Spec} R_i$. These maps form a compatible system, so taking an inductive limit, we get a formal scheme $\mathcal{G}rass^{\leq 1}_{M_R} = \varinjlim_i \mathcal{G}rass^{\leq 1}_{M_{R_i}}$ equipped with a morphism $\mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spf} R$. In [Fal03, p.42-43] Faltings constructs an ample line bundle over the affine grassmannian whose restriction to any closed subscheme of finite type is very ample. These line bundles pull back to line bundles $\mathcal{L}_i$ on $\mathcal{G}rass^{\leq 1}_{M_{R_i}}$ such that $\mathcal{L}_i \otimes_{R_i} R_{i+1} = \mathcal{L}_{i+1}$. Therefore, it follows from formal GAGA that $\mathcal{G}rass^{\leq 1}_{M_R}$ is a projective scheme equipped with a projective morphism $\mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spec} R$ which, modulo m^i_R, recovers $\mathcal{G}rass^{\leq 1}_{M_{R_i}} \to \operatorname{Spec} R_i$

3.7 Constructing the Kisin variety

As before, let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field k. Define $K_0 = W(k)[1/p]$. Fix a positive integer n and, for each $\mathbb{Q}_p$-algebra embedding $\psi : K \to K_0^{\text{sep}}$, choose a cocharacter ν_ψ of the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0) \in \mathbb{Z}^n$. Set $\nu = (\nu_\psi)_\psi$. Then, for each $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, fix a lifting $\tilde{\sigma}$ of σ to K_0^{sep}. For such σ, we denote the embeddings $\psi : K \to K_0^{\mathrm{sep}}$ satisfying $\psi_0 = \sigma$ by $(\psi_{\sigma,1}, \ldots, \psi_{\sigma,r})$. Define $\nu_\sigma = (\nu_{\psi_{\sigma,j}})$ and let F be a finite, Galois extension of $\mathbb{Q}_p$ containing a copy of K_0 and the reflex field of ν_σ for every $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$. As before, let $\mathcal{O}_F$ denote the ring of integers of F.

Let R be a complete local Noetherian $\mathcal{O}_F$ algebra with maximal ideal m_R and residue field $\mathbb{F}$. Fix a representation V_R of G_K on a free R-module of rank n. For each positive integer i, let R_i denote the quotient R/m_R^i. Assume that, for any such i, the G_K-representation $V_{R_i} := V_R \otimes_R R/m_R^i$ is isomorphic to $\mathcal{G}_i(\overline{K})$ for some finite flat group scheme $\mathcal{G}_i$ over $\mathcal{O}_K$.

Suppose we are given a finite dimensional F-vector space D_F, and a filtration of $D_{F,K} = D_F \otimes_{\mathbb{Q}_p} K$ by $F \otimes_{\mathbb{Q}_p} K$-submodules such that the associated grading is concentrated in degrees $[0, h]$. Set $\nu = \{D_F, \ \mathrm{Fil}^i D_{F,K}, \ i = 0, \ldots, h\}$. If B is a finite F-algebra and V_B is a finite free B-module, equipped with a continuous action of G_K which makes V_B a de Rham representation, then we say that V_B is of *p-adic Hodge type ν* if all of the Hodge-Tate weights of V_B are between 0 and h and, for $i = 0, 1, \ldots, h$ there is an isomorphism of $B \otimes_{\mathbb{Q}_p} K$-modules

$$\mathrm{gr}^i \operatorname{Hom}_{B[G_K]}(V_B, B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} B) \xrightarrow{\sim} \mathrm{gr}^i D_{F,K} \otimes_F B,$$

where B_{dR} denotes the de Rham period ring. In our setting, where the representations are flat, the only jump in the filtration is between 0 and 1, so the grading is concentrated in degrees 0 and 1.

For any finite extension E/F, we say that an $\mathcal{O}_F$-algebra homomorphism $y : R \to \mathcal{O}_E$ has *p-adic Hodge type ν* if the induced representation $V_R \otimes_R E$ has p-adic Hodge type ν. In [Kis09], Kisin uses the p-adic analytic space attached to the formal spectrum of R to argue that there is a subset of the set of connected components of $\operatorname{Spec} R[1/p]$ such that $y : \mathcal{O}_E \to R$ has p-adic Hodge type ν if and only if the image of y lies on one of those connected components. Define R^ν to be the quotient of R corresponding to the closure of those connected components in $\operatorname{Spec} R$. Then $\operatorname{Spec} R^\nu[1/p]$ is the union of those connected components.

Having carved out a closed subscheme of $\operatorname{Spec} R$ corresponding to objects with p-adic

Hodge type ν, we proceed to do the same for $\mathcal{G}rass^{\leq 1}_{M_R}$. Let B be an $\mathcal{O}_F$-algebra and fix $\mathfrak{M}_B \in \mathcal{G}rass^{\leq 1}_{M_R}(B)$. Denote the image of the linearization $\Phi_{\mathfrak{M}_B} : \varphi^*\mathfrak{M}_B \to \mathfrak{M}_B$ by $\mathfrak{N}_B$. We say that $\mathfrak{M}_B$ has *p-adic Hodge type* ν if it satisfies the condition

$$\det_B(B \mid \mathfrak{N}_B/E(u)\mathfrak{M}_B) = \prod_\psi \psi(a)^{v_\psi}$$

for all $b \in \mathcal{O}_K$. This definition is clearly well-defined in the case where $\mathfrak{N}_B/E(u)\mathfrak{N}_B$ is a finite free B-module, which is not necessarily the case. However, $\mathfrak{N}_B/E(u)\mathfrak{M}_B$ is a finite projective B-module by [Kis09, Lemma 1.2.2], so we can define the determinant Zariski locally.

Define $\mathcal{G}rass^{\leq 1,\nu}_{M_R}$ to be the closed subscheme of $\mathcal{G}rass^{\leq 1}_{M_R}$ so that, for any $\mathcal{O}_F$-algebra B, $\mathcal{G}rass^{\leq 1,\nu}_{M_R}(B)$ is the set of all elements of $\mathcal{G}rass^{\leq 1}_{M_R}(B)$ whose corresponding Kisin module has p-adic Hodge type ν. Let $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ be the closed subscheme of $\mathcal{G}rass^{\leq 1,\nu}_{M_R}$ corresponding to the ideal sheaf of p-power torsion sections. We will show that, when restricted to $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$, the proper $\mathcal{O}_F$-scheme morphism $\mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spec} R$ factors through the closed subscheme $\operatorname{Spec} R^\nu \subset \operatorname{Spec} R$ by looking at the generic fiber. Specifically, we will consider the map on generic fibers $\mathcal{G}rass^{\leq 1,\nu}_{M_R} \times_{O_F} \operatorname{Spec} F \to \operatorname{Spec} R \otimes_{\mathcal{O}_F} F$, which is sufficient because $\mathcal{G}rass^{\leq 1,\nu}_{M_R}$ and $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ have isomorphic generic fibers. This is demonstrated in the following lemma.

Lemma 3.29. *Let S be an $\mathcal{O}_F$-scheme and define S^{loc} to be the closed subscheme of S corresponding to the ideal sheaf of p-power torsion sections. Then $S^{\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ can be naturally identified with $S \otimes_{\mathcal{O}_F} F$*

Proof. Let i denote the closed immersion $i : S^{\mathrm{loc}} \hookrightarrow S$ and let $i \otimes F$ denote the induced map on generic fibers. Then $S \otimes_{\mathcal{O}_F} F$ has a basis of affine opens of the form $\operatorname{Spec} A \otimes_{\mathcal{O}_F} F \simeq \operatorname{Spec} A[1/p]$ where $\operatorname{Spec} A$ is an open subset of S. The preimage of such a set under $i \otimes 1$ is exactly $\operatorname{Spec} A/A_{p^\infty} \otimes_{\mathcal{O}_F} F$, where $A_{p^\infty} \subset A$ is the ideal consisting of p-power torsion elements. But $A/A_{p^\infty} \otimes_{\mathcal{O}_F} F \simeq A[1/p]$, so $i \otimes F$ is an isomorphism onto its image and it remains only to see that $i \otimes F$ is surjective. To see this, note that any $x \in S \otimes_{\mathcal{O}_F} F$ belongs to an affine open of the form $\operatorname{Spec} A \otimes_{\mathcal{O}_F} F$, and so x corresponds to a prime ideal of $A[1/p]$. Such a prime ideal cannot have p-power torsion, so x corresponds to an ideal of

$\operatorname{Spec} A/A_{p^\infty}[1/p]$. ☐

To prove that the restriction of $\mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spec} R$ factors through $\operatorname{Spec} R^\nu$ when restricted to $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$, we will also need a result of Kisin that establishes an equivalence between p-divisible groups over $\mathcal{O}_K$ with certain $\mathfrak{S}$-modules. For any $\mathbb{Z}_p$-algebra A, let $\mathcal{C}^{\leq 1}(A)$ denote the category of finite projective $\mathfrak{S} \otimes_{\mathbb{Z}_p} A$-modules $\mathfrak{M}$ equipped with an endomorphism $\varphi_{\mathfrak{M}}$ that is semilinear with respect to the Frobenius on $\mathfrak{S}$ and has cokernel killed by $E(u)$. Note that, for any $\mathcal{O}_F$-algebra A and any compatible system of Kisin modules $\mathfrak{M}_{A/\varpi^i A} \in \mathcal{C}^{i,\leq 1}_n(A/\varpi^i A)$, the limit $\varprojlim_n \mathfrak{M}_{A/\varpi^n A}$ lies in $\mathcal{C}^{\leq 1}(A)$, but not every object of $\mathcal{C}^{\leq 1}(A)$ is of this form.

Proposition 3.30. *The category of p-divisible groups over $\mathcal{O}_K$ is equivalent to the category $\mathcal{C}^{\leq 1}(\mathbb{Z}_p)$.*

Proof. See [Kis09, Cor. 2.2.22]. ☐

Additionally, we will require a result of Tate which, in our case, provides that a p-divisible group over $\mathcal{O}_K$ is uniquely determined by its Tate module. In the case where a representation V of G_K is isomorphic to the Tate module $T_p(\mathcal{G}) := \varprojlim \mathcal{G}_i(\overline{K})$ of some p-divisible group $\mathcal{G} = (\mathcal{G}_i)$ over $\mathcal{O}_K$, then $\mathcal{G}$ is the unique such p-divisible group by the following theorem.

Theorem 3.31. *(Tate) Let S be an integrally closed Noetherian integral domain whose field of fractions E is of characteristic* 0. *Let $\mathcal{G}$ and $\mathcal{H}$ be p-divisible groups over S. A homomorphism $f : \mathcal{G}\otimes_S E \to \mathcal{H}\otimes_S E$ of the generic fibers extends uniquely to a homomorphism $\mathcal{G} \to \mathcal{H}$. In particular, where $G_E := \mathrm{Gal}(\overline{E}/E)$, the map $\mathrm{Hom}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_{G_E}(T_p(\mathcal{G}), T_p(\mathcal{H}))$ is bijective.*

Proof. See [Tat67, Theorem 4]. ☐

Lemma 3.32. *When restricted to $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$, the proper $\mathcal{O}_F$-scheme morphism $\mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spec} R$ factors through the closed subscheme $\operatorname{Spec} R^\nu \subset \operatorname{Spec} R$.*

Proof. Let θ denote the composition $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R} \hookrightarrow \mathcal{G}rass^{\leq 1}_{M_R} \to \operatorname{Spec} R$. First we show that, in order to check that θ factors through $\operatorname{Spec} R^\nu$, it suffices to check that the induced map on the generic fibers factors through $\operatorname{Spec} R^\nu \otimes_{\mathcal{O}_F} F$. To see this, suppose that the induced map on generic fibers does indeed factor through $\operatorname{Spec} R^\nu \otimes_{\mathcal{O}_F} F$ and denote the maps as below.

$$\begin{array}{ccc} \mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} \operatorname{Spec} F & \xrightarrow{\theta\times 1} & \operatorname{Spec} R \otimes_{\mathcal{O}_F} F \\ & \searrow & \uparrow{\scriptstyle i\times 1} \\ & & \operatorname{Spec} R^\nu \otimes_{\mathcal{O}_F} F \end{array}$$

Consider the induced maps on global sections.

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}) \otimes_{\mathcal{O}_F} F & \xleftarrow{(\theta)^\# \otimes 1} & R \otimes_{\mathcal{O}_F} F \\ & \nwarrow & \downarrow{\scriptstyle i^\# \otimes 1} \\ & & R^\nu \otimes_{\mathcal{O}_F} F \end{array}$$

Let $x \in I$ be arbitrary. Since the diagram above commutes, $\theta^\#(I) \otimes F = 0$, where I is the ideal of R for which $R^\nu = R/I$. In particular, $\theta^\#(x) \otimes 1 = 0$ in $\mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}) \otimes_{\mathcal{O}_F} F$, so $\theta^\#(x) = 0$ as an element of $\mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}})[1/p]$. This means that $\theta^\#(x)$ is p^n-torsion for some positive integer n as an element of $\mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}})$. But $\mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}})$ has no p-power torsion, so $\theta^\#(x) = 0$ in $\mathcal{O}_{\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}}(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}})$. Therefore, $\theta^\#$ factors through R^ν and θ factors through $\operatorname{Spec} R^\nu$.

To show that the map on generic fibers factors through $\operatorname{Spec} R^\nu[1/p] \otimes_{\mathcal{O}_F} F$, we first show that for a finite extension E/F, the map on E-points factors through $(\operatorname{Spec} R^\nu[1/p])(E)$. Let y be an E point of $\mathcal{G}rass_{M_R}^{\leq 1} \otimes_{\mathcal{O}_F} F$. When composed with $\mathcal{G}rass_{M_R}^{\leq 1} \otimes_{\mathcal{O}_F} F \to \operatorname{Spec} R[1/p]$, y induces a map $R[1/p] \to E$, which restricts to a map of $\mathcal{O}_F$-algebras $\tilde{y} : R \to \mathcal{O}_E$. Using $\tilde{y}$, we associate a G_K-representation $V_{\mathcal{O}_E} := V_R \otimes_R \mathcal{O}_E$ to y. Additionally, using the valuative criterion for properness, $\tilde{y}$ associates an $\mathcal{O}_E$-point of $\mathcal{G}rass_{M_R}^{\leq 1}$ to y which corresponds to an element $\mathfrak{M}_{\mathcal{O}_E}$ of $\mathcal{C}^{\leq 1}(\mathcal{O}_E)$.

$$\begin{array}{ccc} \operatorname{Spec} E & \longrightarrow & \mathcal{G}rass_{M_R}^{\leq 1} \\ \downarrow & \nearrow & \downarrow \\ \operatorname{Spec} \mathcal{O}_E & \xrightarrow{\tilde{y}} & \operatorname{Spec} R \end{array}$$

For each positive integer i, let $V_{\mathcal{O}_E,i}$ denote the torsion representation $V_{\mathcal{O}_E} \otimes_{R_i} V_{R_i}$. Since V_{R_i} is flat, $V_{\mathcal{O}_E,i}$ is flat as a result of [Con97, Theorem 2.3]. For each positive integer i, fix a finite flat group scheme $\mathcal{H}_i$ over $\mathcal{O}_K$ for which $V_{\mathcal{O}_E,i} \simeq \mathcal{H}_i(\overline{K})$. We can choose the $\mathcal{H}_i$ to fit in a compatible system for which $\mathcal{H} := (\mathcal{H}_i)$ is a p-divisible group over $\mathcal{O}_K$ by a result of Raynaud [Ray74, Proposition 2.3.1]. Then $\mathcal{H}$ satisfies $V_{\mathcal{O}_E} \simeq T_p(\mathcal{H})$ by definition. Further, $\mathcal{H}$ is the unique p-divisible group associated to $\mathfrak{M}_{\mathcal{O}_E}$ by the functor of Proposition 3.30.

The quotient R^ν is defined in such a way that the image of y lies in $\operatorname{Spec} R^\nu[1/p]$ if and only if $V_{\mathcal{O}_E}$ has p-adic Hodge type ν, so the map $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F \to \operatorname{Spec} R[1/p]$ factors through $\operatorname{Spec} R^\nu[1/p]$ if, whenever $\mathfrak{M}_{\mathcal{O}_E}$ has p-adic Hodge type ν, $V_{\mathcal{O}_E}$ does as well. This is the case by various p-adic Hodge theory results of Breuil which are referenced in the proof of [Kis09, Proposition 2.4.8]

We have established that, on E points, the restriction θ factors through $\operatorname{Spec} R^\nu[1/p]$. Note that all closed points of $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}$ and $\operatorname{Spec} R^\nu[1/p]$ are of this form. Additionally, $R^\nu[1/p]$ and $R[1/p]$ is are Jacobson rings, so the closed points of $\operatorname{Spec} R^\nu[1/p]$ are dense. Moreover, since θ is a projective map, $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ is Jacobsen as well, so its closed points are dense. From this we can conclude that $\theta \otimes_{\mathcal{O}_F} F : \mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F \to \operatorname{Spec} R[1/p]$ factors through $\operatorname{Spec} R^\nu[1/p]$.

To see this, let $U = (\operatorname{Spec} R[1/p]) \setminus (\operatorname{Spec} R^\nu[1/p])$. Since U is open in $\operatorname{Spec} R[1/p]$, if the preimage of U is nonempty, then it is open in $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$, so it contains a closed point. But the image of that closed point must be in $\operatorname{Spec} R^\nu[1/p]$, which is a contradiction.

□

Now that we have a map $\theta : \mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \to \operatorname{Spec} R^\nu$, in order to show that it induces an isomorphism on generic fibers, we will need to refine the definition of R. Until now, R has been any complete local Noetherian $\mathcal{O}_F$-algebra, but in in the proof of the following proposition we will require $R[1/p]$ to be a regular ring and, to ensure that, we will assume that R is a deformation ring.

Using the same definition for F, $\mathcal{O}_F$, and $\mathbb{F}$ as before, let $V_{\mathbb{F}}$ be an $\mathbb{F}$-vector space of dimension n endowed with a continuous G_K action. Fix a basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$ and let $\overline{\rho} : G_K \to$

$\mathrm{GL}_n(V_{\mathbb{F}})$ be the homomorphism corresponding to the basis $\beta_{\mathbb{F}}$. Let $\widehat{\mathcal{C}}_{\mathcal{O}_F}$ denote the category of complete local Noetherian $\mathcal{O}_F$-algebras with residue field $\mathbb{F}$. Recall the flat deformation functor $D^{\square}_{\overline{\rho}} : \widehat{\mathcal{C}}_{\mathcal{O}_F} \to \mathrm{Sets}$ given by

$$D^{\square}_{\overline{\rho}}(A) = \{(\overline{\rho}_A, M, \iota, \beta)\}/\sim$$

where M is a free A-module of rank n, $\rho : G_K \to \mathrm{GL}_A(M)$ is a continuous representation, $\iota : \rho \otimes_A \mathbb{F} \xrightarrow{\sim} \overline{\rho}$ is an isomorphism, and two such 4-tuples are equivalent if there exists an isomorphism between the representations which respects the ι's. $D^{\square}_{\overline{\rho}}$ is represented by a ring $R^{\square}_{\overline{\rho}}$ in $\widehat{\mathcal{C}}_{\mathcal{O}_F}$. For the rest of the section, we assume that $R = R^{\mathrm{fl},\square}_{\overline{\rho}}$ for some representation $\overline{\rho}$ on a n-dimensional $\mathbb{F}$-vector space. In this case, we have the following results from Kisin.

Proposition 3.33. *1. $R^{\nu}[1/p]$ is a regular ring.*

2. $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ is normal and its special fiber is reduced.

Proof. For (1), see [Kis09, Corollary 2.3.11] and, for (2), see [Kis09, Proposition 2.4.6] □

Proposition 3.34. *The map $\theta : \mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R} \to \mathrm{Spec}\, R^{\nu}$ becomes an isomorphism after base change to F.*

Proof. We begin by showing that, for any finite extension E/F, this map induces a bijection on $\mathcal{O}_E$-valued points. Let y be an $\mathcal{O}_E$-point of $\mathrm{Spec}\, R^{\nu}$. Then y represents an $\mathcal{O}_F$ algebra map $y : R^{\nu} \to \mathcal{O}_E$, which defines an R-algebra structure on $\mathcal{O}_E$, and so y corresponds to a G_K-representation $V_{\mathcal{O}_E} := \mathcal{O}_E \otimes_R V_R$. By definition, $V_{\mathcal{O}_E}/p^n V_{\mathcal{O}_E}$, which is flat by [Con97, Theorem 2.3] Then, by [Ray74, Prop. 2.3.1], $V_{\mathcal{O}_E}$ corresponds to a unique p-divisible group $\mathcal{G}_{\mathcal{O}_E}$ over $\mathcal{O}_K$ and, therefore, by [Tat67, Thm. 4], to a unique p-divisible group $\mathcal{G}_{\mathcal{O}_E}$ having Tate module $V_{\mathcal{O}_E}$. It follows by Proposition 3.30 that $\mathcal{G}_{\mathcal{O}_E}$ corresponds to a unique Kisin module in $\mathcal{C}^{\leq 1}(\mathbb{Z}_p)$, which is projective of rank n and is equipped with an $\mathcal{O}_E$-action by functoriality. Denote this Kisin module by $\mathfrak{M}_{\mathcal{O}_E}$.

Since $\mathfrak{M}_{\mathcal{O}_E}$ is a finitely generated projective $\mathfrak{S}$-module, $\mathfrak{M}_{\mathcal{O}_E}/u\mathfrak{M}_{\mathcal{O}_E}$ is a finitely generated projective $\mathfrak{S}/u\mathfrak{S}$-module, so $\mathfrak{M}_{\mathcal{O}_E}/u\mathfrak{M}_{\mathcal{O}_E}$ is finitely generated and projective over

$\mathfrak{S}_{\mathcal{O}_E}/u\mathfrak{S}_{\mathcal{O}_E}$.This implies that $\mathfrak{M}_{\mathcal{O}_E}$ is finitely generated and projective over $\mathfrak{S}_{\mathcal{O}_E}$ by Lemma 3.25, so $\mathfrak{M}_{\mathcal{O}_E}$ is an object in $\mathcal{C}^{\leq 1}(\mathcal{O}_E)$ which is of constant rank n over $\mathfrak{S}_{\mathcal{O}_E}$.

Now, to show injectivity, suppose $\mathfrak{M}'_{\mathcal{O}_E}$ is another point of $\mathcal{G}rass^{\leq 1,\nu}_{M_R}(\mathcal{O}_E)$ which is taken to y. Then $\mathfrak{M}'_{\mathcal{O}_E}$ corresponds to a p-divisible group $\mathcal{G}_{\mathcal{O}_\mathcal{E}}{}'$ over K. But, since $\mathfrak{M}'_{\mathcal{O}_E}$ corresponds to $V_{\mathcal{O}_E}$, the Tate modules of $\mathcal{G}_{\mathcal{O}_\mathcal{E}}$ and $\mathcal{G}'_{\mathcal{O}_\mathcal{E}}$ are isomorphic. By Theorem 3.31, this implies that $\mathcal{G}_{\mathcal{O}_\mathcal{E}} \simeq \mathcal{G}'_{\mathcal{O}_\mathcal{E}}$, so $\mathfrak{M}'_{\mathcal{O}_E} \simeq \mathfrak{M}_{\mathcal{O}_E}$. Thus the map induces a bijection on $\mathcal{O}_E$-valued points. Moreover, since the $\mathcal{O}_E$-points of $\operatorname{Spec} R$ can be identified with the E-points of $\operatorname{Spec} R$ and the map $\mathcal{G}rass^{\leq 1,\nu}_{M_R} \to \operatorname{Spec} R^\nu$ is proper, this induces a bijection on E-points $\mathcal{G}rass^{\leq 1,\nu}_{M_R}(E) \leftrightarrow \operatorname{Spec} R^\nu(E)$. Further, since E is equipped with an F-algebra structure, this gives a bijection on the E-valued points of the generic fibers $(\operatorname{Spec} R \otimes_{\mathcal{O}_F} F)(E) \leftrightarrow (\mathcal{G}rass^{\leq 1,\nu}_{M_R} \otimes_{\mathcal{O}_F} F)(E)$ which, as previously noted, is a bijection on the closed points of the generic fibers.

Next, we will show that θ is a finite map. Since θ is proper, it suffices to prove that it is quasi-finite. In particular, we need to show that each point $x \in \mathcal{G}rass^{\leq 1,\nu}_{M_R} \otimes_{\mathcal{O}_F} F$ is isolated in its fiber. We call the set of points of $\mathcal{G}rass^{\leq 1,\nu}_{M_R} \otimes_{\mathcal{O}_F} F$ which are isolated in their fiber the quasi-finite locus. Because θ is a finite type morphism of Noetherian schemes, the set of points which are isolated in their fibers is an open set. But, if θ had 0-dimensional fibers over the closed points of $\mathcal{G}rass^{\leq 1,\nu}_{M_R} \otimes_{\mathcal{O}_F} F$, then the quasi-finite locus would all closed points of $\mathcal{G}rass^{\leq 1,\nu}_{M_R} \otimes_{\mathcal{O}_F} F$, so the quasi-finite locus would be the whole space and θ would be quasi-finite.

To see that θ is quasi-finite, we then let $y \in \operatorname{Spec} R^\nu[1/p]$ be a closed point defined over a finite extension E_y of F. The fiber $\theta^{-1}(y)$ is a finite type E_y-scheme. For any finite extension E'/E_y, the points $\theta^{-1}(y)(E')$ are the E'-points x of $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R} \otimes_{\mathcal{O}_F} F$ such that $\theta(x) = y \otimes E'$. We have seen that, for each finite extension E'/F, there is exactly one point of $(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R} \otimes_{\mathcal{O}_F} F)(E')$ mapping to y, so there is exactly one point of $\theta^{-1}(y)$ over $\overline{E}_y$ and so $\theta^{-1}(y)$ is 0-dimensional.

Finally, to show that θ is an isomorphism, we check that it is an isomorphism on open affine subsets of connected components. Since $\operatorname{Spec} R^\nu[1/p]$ is regular by Proposition 3.33, its connected components coincide with its irreducible components which are additionally integral. Let X be a connected component of $\operatorname{Spec} R^\nu[1/p]$. Note that, since θ is projective, it is closed, so $\theta^{-1}(X)$ must be connected as well. Let $\operatorname{Spec} B \subset X$ be an affine open subset and fix $\operatorname{Spec} A = \theta^{-1}(\operatorname{Spec} B)$. Recall from Proposition 3.33 that $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}$ is normal, so $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ is as well. Since $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ is additionally Noetherian, this implies that $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ is a finite disjoint union of normal, integral schemes. In particular, A and B are both integral domains so the induced ring map $B \to A$ gives a map on the fraction fields as long as it is injective.

Let I denote the kernel of this map. Then $\operatorname{Spec} A \to \operatorname{Spec} B$ factors through the closed subscheme $\operatorname{Spec} B/I$. But θ induces a bijection on closed points, so $\operatorname{Spec} B/I$ is a closed subset of $\operatorname{Spec} B$ containing all of the closed points of $\operatorname{Spec} B$. Further $\operatorname{Spec} R^\nu[1/p]$ is Jacobson, so the closed points of $\operatorname{Spec} R^\nu[1/p]$ are dense and, therefore, the closed points of $\operatorname{Spec} B$ are dense in $\operatorname{Spec} B$, so $\operatorname{Spec} B/I = \operatorname{Spec} B$ and I is the zero ideal.

It remains to see that $B \to A$ is surjective. Since this map is injective, it induces map on fraction fields $\operatorname{Frac} B \to \operatorname{Frac} A$, which must be an isomorphism by [Sta21, Tag 02JX]. Let $a \in A$ be arbitrary and fix $u \in \operatorname{Frac} B$ that is sent to a. Since $B \to A$ is finite, it is integral and there is a monic polynomial $p(x) \in B[x]$ that is satisfied by a, and therefore satisfied by u. Thus u is integral over B, but B is integrally closed in its fraction field, so u is an element of B.

□

Theorem 3.35. *For a topological space X, we denote the set of connected components of X by $H_0(X)$. Let $\mathcal{G}rass_{M_R,0}^{\leq 1,\nu,\mathrm{loc}}$ denote the fiber over the closed point of R^ν. There is a bijection $H_0(\mathcal{G}rass_{M_R,0}^{\leq 1,\nu,\mathrm{loc}}) \simeq H_0(\operatorname{Spec} R^\nu[1/p])$.*

Proof. We denote the global sections of a scheme S by $\mathcal{O}_S$. Fix a connected component of $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} F$ and let e be the idempotent of $\mathcal{O}_{(\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}})} \otimes_{\mathcal{O}_F} F$ that is 1 on this connected component and vanishes everywhere else. Pick m to be the smallest nonnegative integer for which $\varpi^m e$ extends to a section of $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}}$. Then $(\varpi^m e)^2 = \varpi^m(\varpi^m e)$, so if $m > 0$, ϖ^{me} induces a nonzero-nilpotent function on the scheme $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} \mathbb{F}$. But by [Kis09, Prop. 2.4.6(2)], $\mathcal{G}rass_{M_R}^{\leq 1,\nu,\mathrm{loc}} \otimes_{\mathcal{O}_F} \mathbb{F}$ is reduced, so $m = 0$ and e extends to a section of $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$. This induces a bijection $H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R} \otimes_{\mathcal{O}_F} F) \simeq H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R})$. By the isomorphism of Proposition 3.34, this induces a bijection $H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}) \simeq H_0(\operatorname{Spec} R^\nu[1/p])$.

Let $\widehat{\mathcal{G}rass}^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ denote completion of $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ along the maximal ideal of R. Then $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R,0}$ and $\widehat{\mathcal{G}rass}^{\leq 1,\nu,\mathrm{loc}}_{M_R}$ have the same underlying topological space so $H_0(\widehat{\mathcal{G}rass}^{\leq 1,\nu,\mathrm{loc}}_{M_R}) \xrightarrow{\sim} H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R,0})$. Additionally, there is an isomorphism $H_0(\widehat{\mathcal{G}rass}^{\leq 1,\nu,\mathrm{loc}}_{M_R}) \to H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R})$ which follows from [Gro60b, Theorem 4.1.5], so $H_0(\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R,0}) \simeq H_0(\operatorname{Spec} R^\nu[1/p])$. □

In this paper, when we refer to a Kisin variety, morally, we are referring to the fiber over the closed point of R^ν, $\mathcal{G}rass^{\leq 1,\nu,\mathrm{loc}}_{M_R,0}$. This term was coined in Pappas and Rapoport's paper [PR08] wherein they globalize this construction by formally forgetting about the Galois representations and finite flat group schemes and considering instead the Kisin modules themselves. Albeit, what we call a Kisin variety here, they refer to as a closed Kisin variety. Recall that $\nu = (\nu_\psi)_\psi$ where ψ runs over the embeddings $K \to K_0^{\mathrm{sep}}$ and each ν_ψ is a cocharacter of the form $(1,1,\ldots,1,0,0,\ldots,0) \in \mathbb{Z}^n$. Fix an embedding $\sigma : k \to \mathbb{F}_p^{\mathrm{sep}}$ and let S_σ denote the set of all embeddings $K \to K_0^{\mathrm{sep}}$ which lift σ. Define

$$\mu = \sum_{\psi \in S_\sigma} \nu_\psi.$$

Specifically, Pappas and Rapoport showed that there is reduced, projective $\mathbb{F}$-variety $C_\mu(b) \subset \operatorname{Res}_{k|\mathbb{F}_p} \mathcal{G}rass_{\mathrm{GL}_n} \otimes_{\mathbb{F}_p} \mathbb{F}$ whose geometric points are given by

$$C_\mu(b)(\overline{\mathbb{F}}_p) = \{g \in G(L)/G(\mathcal{O}_L) | g^{-1} b\sigma(b) \in \bigsqcup_{\substack{\nu \in Y^+ \\ \nu \leq \mu}} G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L)\}.$$

Definition 3.36. *We define the* Kisin variety *associated to b and μ to be this reduced, projective variety $C_\mu(b)$.*

4 Describing elements of $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$

Fix an element $b \in \mathrm{Res}_{k|\mathbb{F}_p} \mathrm{GL}_n(\mathbb{F}(\!(u)\!))$ and a cocharacter $\mu \in Y$. The geometric points of the Kisin variety $C_\mu(b)$ associated to b and μ can be written as a disjoint union indexed by the set of cocharacters Y. For any $\lambda \in Y$, we define the *Iwahori stratum associated to* λ, which we denote by $C_\mu^\lambda(b)$, to be the reduced locally closed subscheme of $C_\mu(b) \times_{\mathbb{F}} \overline{\mathbb{F}}_p$ with $\overline{\mathbb{F}}_p$-points

$$C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = C_\mu(b)(\overline{\mathbb{F}}_p) \cap Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L).$$

Then $C_\mu(b)(\overline{\mathbb{F}}_p) = \bigsqcup_{\lambda \in Y} C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$. To determine which Iwahori strata, $C_\mu^\lambda(b)$, could be nonempty for b of the form

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i \end{pmatrix} \right)_{i=1}^{f},$$

satisfying some technical conditions and $\mu = ((m_i, 0))_{i=1}^f$ for nonnegative integers m_i, we will directly compute the elements of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ and then consider whether they could belong to $C_\mu(b)(\overline{\mathbb{F}}_p)$, in particular, whether σ-conjugating b by some $g \in Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ could yield an element of $G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)$ for some $\nu \leq \mu$.

For a dominant cocharacter $\mu = ((m_{1,i}, m_{2,i}))_{i=1}^f$ define a subfunctor $L^{\leq\mu}G \subset LG$ by defining $L^{\leq\mu}G(R)$ to be the subset of f-tuples of matrices g in $LG(R)$ defined by two conditions:

1. $\det g \in \prod_{i=1}^f u^{m_{1,i}+m_{2,i}} R[[u]]^\times$ and
2. $u^{-m_{2,i}} g \in \prod_{i=1}^f Mat_{2\times 2}(R[[u]])$.

Lemma 4.1. $L^{\leq\mu}G(\overline{\mathbb{F}}_p) = \bigsqcup_{\substack{\nu \in Y^+ \\ \nu \leq \mu}} G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)$.

Proof. First, suppose $g \in G(\mathcal{O}_L)u^\nu G(\mathcal{O}_L)$ for some dominant cocharacter $\nu = ((x_i, y_i))_{i=1}^f$ with $\nu \leq \mu$. It is clear that $\det g \in \prod_{i=1}^f u^{m_{1,i}+m_{2,i}} \mathcal{O}_L^\times$. Fix $h_1, h_2 \in G(\mathcal{O}_L)$ for which $g = h_1 u^\nu h_2$. Then $u^{-m_{2,i}} g = h_1 u^{(x_i - m_{2,i}, y_i - m_{2,i})} h_2$ and both $x_i - m_{2_i}$ and $y_i - m_{2_i}$ are positive since ν and μ are both dominant and $\nu \leq \mu$, so $g \in LG^{\leq\mu}(\overline{\mathbb{F}}_p)$.

Now suppose $g \in L^{\leq\mu}G(\overline{\mathbb{F}}_p)$. Then, using the Cartan decomposition, there exist $h_1, h_2 \in G(\mathcal{O}_L)$ and $((x_i, y_i))_{i=1}^f \in Y^+$ for which $g = h_1 u^{(x_i, y_i)} h_2$. From the determinant condition on elements of $L^{\leq\mu}G(\overline{\mathbb{F}}_p)$ we get that $x_i + y_i = m_{1,i} + m_{2,i}$, so to see that $((x_i, y_i))_{i=1}^f \leq \mu$ it remains to see that $y_i - m_{2,i}$ is nonnegative. For this, note that $u^{-m_{2,i}} g = h_1 u^{(x_i - m_{2,i}, y_i - m_{2,i})} h_2 \in \prod_{i=1}^f \mathrm{Mat}_{2\times 2}(\mathcal{O}_L)$, so $y_i - m_{2,i}$ must be nonnegative.

□

This lemma allows us to define a scheme-theoretic version of the Kisin variety. Consider the map $LG \to LG$ given on R-points by $g \mapsto g^{-1} b\sigma(g)$. Since $L^{\leq\mu}G$ is stable under left and right multiplication by L^+G, the pullback of $L^{\leq\mu}G$ under this map descends to a closed subscheme of LG/L^+G whose geometric points coincide with those of $C_\mu(b)$ by Lemma 5.7. Denote this scheme by $X_\mu(b)$. Both $C_\mu(b)$ and $X_\mu(b)$ are finite type over $\mathbb{F}$ and hence Jacobson, which means their $\overline{\mathbb{F}}_p$-points are dense. Furthermore, since the geometric points of $C_\mu(b)$ and $X_\mu(b)$ coincide and since $C_\mu(b)$ is reduced, $C_\mu(b) \subset X_\mu(b)$ with the same topological space, hence the underlying reduced of $X_\mu(b)$ is $C_\mu(b)$. Since we are primarily interested in the topological information of the Kisin variety, it will generally suffice to study $X_\mu(b)$. Moreover, since $C_\mu(b)$ is reduced, the underlying reduced scheme of $X_\mu(b)$ is exactly $C_\mu(b)$.

One benefit of looking at $X_\mu(b)$ is that the methods we use to compute $X_\mu(b)(\overline{\mathbb{F}}_p) = C_\mu(b)(\overline{\mathbb{F}}_p)$ will be sufficient to compute $X_\mu(b)(R)$ for any $\mathbb{F}$-algebra R and, therefore, to compute the isomorphism class of $X_\mu(b)$. During most of this thesis, we work with specifically with the geometric points of $C_\mu(b)$, but in section 5.4, we will compute an example of a Kisin variety and this distinction is notable there.

In the following subsection, once we put technical conditions on b and μ, we will see that Lemma 4.1 also provides a useful computational tool in the form of necessary and sufficient conditions for an element g of $u^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ to additionally belong to $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$. Using this, we will be able to directly compute the intersection

$$C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = (Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)) \cap C_\mu(b)(\overline{\mathbb{F}}_p).$$

We first establish the following lemma, in which we see a natural isomorphism between

$(Iu^\lambda G(\mathcal{O}_L))/G(\mathcal{O}_L)$ and the easier to compute $(I/(u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I))u^\lambda$, allowing us to identify a helpful set of representatives which are used throughout the thesis.

Lemma 4.2. *Fix a cocharacter* $\lambda = ((c_i, d_i))_{i=1}^f \in Y$.

1. *There is a natural identification*

$$Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) \leftrightarrow \left(\frac{I}{u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I}\right) u^\lambda.$$

2. *Each element* g *of* $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ *has a representative where the* i*th component has the form:*

   - $\begin{pmatrix} u^{c_i} & u^{d_i}h_i(u) \\ 0 & u^{d_i} \end{pmatrix}$ *if* $c_i - d_i > 1$,
   - $\begin{pmatrix} u^{c_i} & 0 \\ u^{c_i}h_i(u) & u^{d_i} \end{pmatrix}$ *if* $c_i - d_i < 0$, *or*
   - $\begin{pmatrix} u^{c_i} & 0 \\ 0 & u^{d_i} \end{pmatrix}$ *if* $c_i - d_i = 0, 1$.

   In the first two cases, $h_i(u)$ *is a polynomial in* $\overline{\mathbb{F}}_p[u]$ *with degree at most* $|c_i - d_i| - 1$.

Proof. For any element $hu^\lambda g$ of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$, $hu^\lambda g \sim hu^\lambda gg^{-1} \sim hu^\lambda$, so we will use representatives for cosets of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ of this form. Consider a map from $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ to $(I/(u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I))u^\lambda$ given by

$$(hu^\lambda \mod G(\mathcal{O}_L)) \mapsto (h \mod u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I)u^\lambda.$$

To see that this map is well-defined, we need to see that, if $hu^\lambda \sim h'u^\lambda$ modulo $G(\mathcal{O}_L)$, then h and h' are equivalent modulo $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I$. If $hu^\lambda \sim h'u^\lambda$ modulo $G(\mathcal{O}_L)$, then for some $g \in G(\mathcal{O}_L)$, $hu^\lambda g = h'u^\lambda$, so

$$\begin{aligned} hu^\lambda g &= h'u^\lambda \\ u^\lambda g &= h^{-1}h'u^\lambda \\ u^\lambda g u^{-\lambda} &= h^{-1}h', \end{aligned}$$

which implies h and h' are equivalent modulo $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I$. Of course $u^\lambda g u^{-\lambda}$ is in $u^\lambda G(\mathcal{O}_L)u^{-\lambda}$ and, since it is equal to $h^{-1}h'$, it is in I as well, so this map is well defined.

Going the other direction, we define a map from $(I/(u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I))u^\lambda$ to $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ given by

$$(h \mod u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I)u^\lambda \mapsto (hu^\lambda \mod G(\mathcal{O}_L))$$

If elements h and h' of I are equivalent modulo $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I \subset G(\mathcal{O}_L)$, then they are equivalent modulo $G(\mathcal{O}_L)$, so this map is well defined. Moreover, since these two maps are inverses, they must be bijections.

For the second part of the lemma, let g be an arbitrary element of I.

$$g = \left(\begin{pmatrix} f_{1,1,i}(u) & uf_{1,2,i}(u) \\ f_{2,1,i}(u) & f_{2,2,i}(u) \end{pmatrix} \right)_{i=1}^{f}.$$

We will construct elements A, B, and C of $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I$ for which

$$(gABC)_j = \begin{cases} \begin{pmatrix} 1 & \sum_{i=1}^{c_j-d_j-1} a_{j,i}u^i \\ 0 & 1 \end{pmatrix} & c_j - d_j > 1 \\ \begin{pmatrix} 1 & 0 \\ \sum_{i=0}^{d_j-c_j-1} a_{j,i}u^i & 1 \end{pmatrix} & c_j - d_j < 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & c_j - d_j = 0, 1 \end{cases},$$

so $(gABC)u^\lambda$ has the desired form. An element of $u^\lambda G(\mathcal{O}_L)u^{-\lambda}$ has the form

$$\left(\begin{pmatrix} * & u^{c_i-d_i}* \\ u^{d_i-c_i}* & * \end{pmatrix} \right)_{i=1}^{f},$$

so the intersection $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I$ depends on the value of $c_i - d_i$ in each of the f entries. With this in mind, we will construct A, B, and C component by component.

First, fix an index $j \in \{1, \ldots, f\}$ for which $c_j - d_j > 1$. Then the jth component of an element of $u^\lambda G(\mathcal{O}_L)u^{-\lambda} \cap I$ has the form

$$\begin{pmatrix} f_{1,1,j}(u) & u^{c_j-d_j} f_{1,2,j}(u) \\ f_{2,1,j}(u) & f_{2,2,j}(u) \end{pmatrix}$$

for some $f_{i_1,i_2,j}(u) \in \overline{\mathbb{F}}_p[[u]]$.

Note that $\det g_j = f_{1,1,j}(u)f_{2,2,j}(u) - uf_{1,2,j}(u)f_{2,1,j}(u)$ belongs to $\mathcal{O}_L^\times$, which implies $f_{1,1,j}(u)f_{2,2,j}(u) \in \mathcal{O}_L^\times$. Therefore, the matrix

$$\begin{pmatrix} f_{2,2,j}(u) & -u^{c_j-d_j} f_{1,2,j}(u) \\ -f_{2,1,j}(u) & f_{1,1,j}(u) \end{pmatrix},$$

which is a slight variation of g_j^{-1}, has determinant $f_{1,1,j}(u)f_{2,2,j}(u) - u^{c_j-d_j}f_{1,2,j}(u)f_{2,1,j}(u) \in \mathcal{O}_L^\times$. Set

$$A_j = \begin{pmatrix} f_{2,2,j}(u) & -u^{c_j-d_j}f_{1,2,j}(u) \\ -f_{2,1,j}(u) & f_{1,1,j}(u) \end{pmatrix} \quad \text{and} \quad B_j = \begin{pmatrix} 1/\det g_j & 0 \\ 0 & 1/\det A_j \end{pmatrix}.$$

Then

$$g_jA_jB_j = \begin{pmatrix} 1 & \frac{f_{1,1,j}(u)f_{1,2,j}(u)}{\det A_j}(u - u^{c^j-d^j}) \\ 0 & 1 \end{pmatrix}.$$

Denote the coefficients of the power series $\frac{f_{1,1,j}(u)f_{1,2,j}(u)}{\det A_j}(u - u^{c^j-d^j})$ by

$$\frac{f_{1,1,j}(u)f_{1,2,j}(u)}{\det A_j}(u - u^{c^j-d^j}) = \sum_{i=1}^{\infty} a_{j,i}u^i$$

and define $h_j(u) = \sum_{i=1}^{c_j-d_j-1} a_{j,i}u^i$ and $\tilde{h}_j(u) = \sum_{i=c_j-d_j}^{\infty} a_{j,i}u^i$. Define C_j to be the upper triangular 2×2 matrix with 1's on the diagonal and upper right entry $-\tilde{h}_j(u)$ so that

$$g_jA_jB_jC_j = \begin{pmatrix} 1 & h_j(u) \\ 0 & 1 \end{pmatrix}$$

as desired.

Now fix and index $k \in \{1, \dots, f\}$ for which $c_k - d_k < 0$. Then the kth component of an element of $u^\lambda G(\mathcal{O}_L)^{-\lambda}$ takes the form

$$\begin{pmatrix} f_{1,1,k}(u) & uf_{1,2,k}(u) \\ u^{d_k-c_k}f_{2,1,k}(u) & f_{2,2,k}(u) \end{pmatrix}$$

for some $f_{i_1,i_2,k}(u) \in \overline{\mathbb{F}}_p[[u]]$. Similarly, to the pervious case, let

$$A_k = \begin{pmatrix} f_{2,2,k}(u) & -uf_{1,2,k}(u) \\ -u^{d_k-c_k}f_{2,1.k} & f_{1,1,k}(u) \end{pmatrix} \quad \text{and} \quad B_k = \begin{pmatrix} 1/\det A_k & 0 \\ 0 & 1/\det g_k \end{pmatrix}$$

so that

$$g_k A_k B_k = \begin{pmatrix} 1 & 0 \\ \frac{f_{2,1,k}(u) f_{2,2,k}(u)}{\det A_k}(1 - u^{d_k - c_k}) & 1 \end{pmatrix}.$$

Denote the coefficients of the power series $\frac{f_{2,1,k}(u) f_{2,2,k}(u)}{\det A_k}(1 - u^{d_k - c_k})$ by

$$\frac{f_{2,1,k}(u) f_{2,2,k}(u)}{\det A_k}(1 - u^{d_k - c_k}) = \sum_{i=0}^{\infty} a_{k,i} u^i$$

and define $h_k(u) = \sum_{i=1}^{d_k - c_k - 1} a_{k,i} u^i$ and $\tilde{h}_k(u) = \sum_{i=d_k-c_k}^{\infty} a_{k,i} u^i$. Define C_k to be the lower triangular 2×2 matrix with 1's on the diagonal and $-\tilde{h}_k(u)$ in the lower left entry so that

$$g_k A_k B_k C_k = \begin{pmatrix} 1 & h_k(u) \\ 0 & 1 \end{pmatrix}.$$

Finally, fix an index ℓ for which $c_i - d_i = 0, 1$. In this case, an element of $u^{\lambda} G(\mathcal{O}_L) u^{-\lambda} \cap I$ is of the form

$$\begin{pmatrix} f_{1,1,\ell}(u) & u f_{1,2,\ell}(u) \\ f_{2,1,\ell}(u) & f_{2,2,\ell}(u) \end{pmatrix}.$$

Set $A_\ell = g_\ell^{-1}$ and $B_\ell = C_\ell = \mathrm{id}$. Then $g_\ell A_\ell B_\ell C_\ell = \mathrm{id}$.

□

5 σ-Conjugacy Class Representative

5.1 Associating $\overline{\rho}$ to b

As before, let K be a finite extension of $\mathbb{Q}_p$ with residue field k and let $\mathbb{F}$ be a finite field of characteristic p for which $k \hookrightarrow \mathbb{F}$. Fix such an embedding σ. In this subsection, we see that whenever $\overline{\rho} : G_K \to \mathrm{Aut}(V_{\mathbb{F}})$ is a reducible representation on a 2-dimensional $\mathbb{F}$ vector space there is a representative $b \in \operatorname{Res}_{k|\mathbb{F}_p} \mathrm{GL}_n$ of $\overline{\rho}$ of the form

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}$$

where $\alpha_i, \beta_i \in \mathbb{F}^\times$ and $f_i(u) \in \mathbb{F}(\!(u)\!)$.

To see that there is a representative b of this form, note that such a representation $\overline{\rho}$ admits a 1-dimensional subrepresentation $\overline{\chi}_1 : G_K \to \operatorname{Aut}(W_{\mathbb{F}})$ and, accordingly, a 1-dimensional quotient $\overline{\chi}_2 : G_K \to \operatorname{Aut}(V_{\mathbb{F}}/W_{\mathbb{F}})$. The functor, $D_{\mathcal{O}_\mathcal{E},\mathbb{F}}$, which associates an étale φ-module to this representation preserves invariant factors, so the same can be said for the étale φ-module associated to $\overline{\rho}$. Denote this étale φ-module by (M, φ_M), its 1-dimensional subobject by (M_1, φ_{M_1}) and its 1-dimensional quotient by (M_2, φ_{M_2}).

For any étale φ-module with $\mathbb{F}$-coefficients (N, φ_N), recall that N is a $(k \otimes_{\mathbb{F}_p} \mathbb{F})(\!(u)\!)$-vector space. Let $S = \{\sigma_0, \sigma_1, \ldots, \sigma_{f-1}\}$ denote the set of embeddings of k into $\mathbb{F}$ where $\sigma_0 = \sigma$ and for $i = 1, \ldots, f-1$, $\sigma_i = \sigma_i \circ \varphi$ where φ is the p-power map on k. Then there are f orthogonal idempotents,

$$\varepsilon_i = -\sum_{x \in k^\times} x \otimes \sigma_i(x)^{-1},$$

in $(k \otimes_{\mathbb{F}_p} \mathbb{F})(\!(u)\!)$ which sum to 1 and satisfy $(\varphi \otimes 1)(\varepsilon_{\sigma_i}) = \varepsilon_{i+1}$ [Sav08]. Using these idempotents, we can then break up $(k \otimes_{\mathbb{F}_p} \mathbb{F})(\!(u)\!)$ into the direct sum

$$(k \otimes_{\mathbb{F}_p} \mathbb{F})(\!(u)\!) \simeq \bigoplus_{i=1}^{f} \mathbb{F}(\!(u)\!),$$

and, accordingly, decompose N as $N = \oplus_{i=0}^{f-1} N^{(i)}$, where $N^{(i)} = \varepsilon_i N$. Then $\varphi(N^{(i)}) \subseteq N^{(i+1)}$, so we can regard φ as an f-tuple of maps $\varphi^{(i)} : N^{(i)} \to N^{(i+1)}$.

$$\begin{array}{c} N_i^{(1)} \xrightarrow{\varphi^{(1)}} N_i^{(2)} \xrightarrow{\varphi^{(2)}} \cdots \xrightarrow{\varphi^{(f-2)}} N_i^{(f-1)} \xrightarrow{\varphi^{(f-1)}} N_i^{(f)} \\ \underbrace{\longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-}_{\varphi^{(f)}} \end{array}$$

and we can represent φ by an f-tuple of 2×2 matrices with entries in $\mathbb{F}(\!(u)\!)$.

In particular, we can do this process for (M, φ_M). For each $i = 1, \ldots, f$ we select an $\mathbb{F}((u))$-basis $\{e_1^{(i)}\}$ of $M_1^{(i)} \subset M$ and a basis $\{\overline{e}_2^{(i)}\}$ of $M_2^{(i)} \subset M/M_1$. Let $e_2^{(i)}$ be any lift of $\overline{e}_2^{(i)}$ to M. Then $\{e_1^{(i)}, e_2^{(i)}\}$ is an $\mathbb{F}((u))$-basis for $M^{(i)}$ and φ can be represented by $b = (b_i)_{i=1}^f$ where b_i is the matrix representing $\varphi^{(i)}$ with respect to the bases $\{e_1^{(i)}, e_2^{(i)}\}$ and $\{e_1^{(i+1)}, e_2^{(i+1)}\}$.

Lemma 5.1. *[GLS12, Lemma 6.2] Suppose $\overline{\mathfrak{M}}$ is a rank 1 Kisin module, then there exists non-negative integers $r_0, \ldots, r_{f-1}$ and $\alpha \in \mathbb{F}^\times$ such that $\overline{\mathfrak{M}}$ is isomorphic to $(\overline{\mathfrak{M}}(r_i;\alpha), \varphi_{\overline{\mathfrak{M}}(r_i;\alpha)})$, where $(\overline{\mathfrak{M}}(r_i;\alpha), \varphi_{\overline{\mathfrak{M}}(r_i;\alpha)})$ is the Kisin module with coefficients in $\mathbb{F}$ that has rank 1 over $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathbb{F}$ and satisfies*

- *$\overline{\mathfrak{M}}(r_i;\alpha)_i$ is generated by e_i, and*
- *$\varphi_{\overline{\mathfrak{M}}(r_i;\alpha)}(e_{i-1}) = \alpha_i u^{r_i} e_i$.*

Here $\alpha_i = \alpha$ if $i = 1$ and $\alpha_i = 1$ otherwise.

The 1-dimensional étale φ-modules (M_1, φ_{M_1}) and (M_2, φ_{M_2}) contain rank 1 Kisin modules $(\mathfrak{M}_1, \varphi_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, \varphi_{\mathfrak{M}_2})$ which are thus isomorphic to Kisin modules of this form. For each index i, select integers r_i and s_i as well as $\alpha_i, \beta_i \in \mathbb{F}^\times$ for which $(\mathfrak{M}_1, \varphi_{\mathfrak{M}_1}) \simeq (\overline{\mathfrak{M}}(r_i;\alpha), \varphi_{\overline{\mathfrak{M}}})$ and $(\mathfrak{M}_2, \varphi_{\mathfrak{M}_2}) \simeq (\overline{\mathfrak{M}}(s_i,;\beta), \varphi_{\overline{\mathfrak{M}}})$. Fix a generator $e_1^{(i)}$ for each $\mathfrak{M}_1^{(i)}$ and a generator $\overline{e}_2^{(i)}$ for each $\mathfrak{M}_2^{(i)}$ as in the lemma. Then $\{e_1^{(i)}\}$ and $\{\overline{e}_2^{(i)}\}$ form $\mathbb{F}((u))$-bases for $M_1^{(i)} \subset M^{(i)}$ and $M_2^{(i)} = M^{(i)}/M_1^{(i)}$, respectively. Choose a lift of each $\overline{e}_2^{(i)}$ to $M^{(i)}$ so that $\{e_1^{(i)}, e_2^{(i)}\}$ forms a basis for $M^{(i)}$. Then, since M_1 is φ-invariant,

$$\varphi(e_1^{(i)}) = \alpha_i u^{r_i} e_1^{(i+1)} + 0 e_2^{(i+1)} \text{ and}$$
$$\varphi(e_2^{(i)}) = f_i(u) e_1^{(i+1)} + \beta_i u^{s_i} e_2^{(i+1)},$$

for some $f_i(u) \in \mathbb{F}((u))$, and φ_M can be represented by an f-tuple of matrices of the desired form:

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^f .$$

5.2 σ-conjugation

In this subsection, we will see that the isomorphism class of $C_\mu(b)$ depends not on b, but on its σ-conjugacy class, which will allow us to find a representative $\tilde{b}$ of this σ-conjugacy class that satisfies certain technical conditions on the r_i, s_i, and $f_i(u)$, which will simplify computations moving forward. We say that $\tilde{b}$ is *σ-conjugate* to b if for some $h \in G(\mathcal{O}_L)$, $\tilde{b} = h^{-1}b\sigma(h)$. Recall that, in the definition of the Kisin variety, b is an element of $G(\mathbb{F}((u)))$ for which the étale φ-module associated to a given representation $\overline{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ is isomorphic to $(k \otimes_{\mathbb{F}_p} \mathbb{F}((u))^n, b\varphi)$, so using a matrix to represent b amounts to choosing a basis for $\mathbb{F}((u))^n$. Likewise, performing a σ-conjugation on b amounts to a change of basis. Therefore, replacing b by another element of its σ-conjugacy class does not change the isomorphism class of the Kisin variety, which we will demonstrate in the following lemma.

Lemma 5.2. *Let b be any element of $\mathrm{Res}_{k|F_p} GL_n(\mathbb{F}((u)))$ and fix a dominant cocharacter $\mu \in (\mathbb{Z}^n)^f$. For any central cocharacter $z \in (\mathbb{Z}^n)^f$, the geometric points of $C_\mu(b)$ are exactly those of $C_{\mu+z}(bu^z)$. Moreover, for any σ-conjugate $\tilde{b}$ of b, there is a natural bijection between $C_\mu(b)(\overline{\mathbb{F}}_p)$ and $C_\mu(\tilde{b})(\overline{\mathbb{F}}_p)$. Thus, for any σ-conjugate of $\widetilde{bu^z}$ of bu^z, $C_\mu(b)$ is isomorphic to $C_{\mu+z}(\widetilde{bu^z})$.*

Proof. Suppose $g \in C_\mu(b)(\overline{\mathbb{F}}_p)$. Then for some $h_1, h_2 \in G(\mathcal{O}_L)$ and some cocharacter ν satisfying $\nu \leq \mu$, $g^{-1}b\sigma(g) = h_1 u^\nu h_2$. Since z is central, multiplying both sides of this equation by u^z yields $g^{-1}(bu^z)\sigma g = h_1 u^{\nu+z} h_2$, so $g \in C_{\mu+z}(bu^z)$. The other inclusion is identical.

Now fix $h \in G(\mathcal{O}_L)$ for which $\tilde{b} = h^{-1}b\sigma(h)$. We claim that g belongs to $C_\mu(b)(\overline{\mathbb{F}}_p)$ if and only if gh is an element of $C_\mu(\tilde{b})$. Suppose $g \in C_\mu(b)(\overline{\mathbb{F}}_p)$. Then for some $h_1, h_2 \in G(\mathcal{O}_L)$ and some cocharacter ν satisfying $\nu \leq \mu$, $g^{-1}b\sigma(g) = h_1 u^\nu h_2$. Multiplying this equation by h^{-1} on the left and $\sigma(h)$ on the right yields

$$(gh)^{-1}b\sigma(gh) = (h^{-1}h_1)u^\nu(h_2\sigma(h)),$$

which illustrates that gh belongs to $C_\mu(\tilde{b})(\overline{\mathbb{F}}_p)$. Likewise, if $gh \in C_\mu(\tilde{b})$, then multiplying the equation $(gh)^{-1}b\sigma(gh)$ by h on the left and by $\sigma(h^{-1})$ on the right shows that $g \in C_\mu(b)(\overline{\mathbb{F}}_p)$.

$\square$

Fix an element $b \in \operatorname{Res}_{k|\mathbb{F}_p} \mathrm{GL}_2(\mathbb{F}(\!(u)\!))$ of the form

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix}\right)_{i=1}^{f}$$

and a cocharacter $\mu = ((m_i, 0))_{i=1}^f$ where each m_i is a nonnegative integer. Let τ be the cocharacter whose entries are the exponents on the diagonal of b, $\tau = ((r_i, s_i))_{i=1}^f$. For any cocharacter $\lambda \in Y$, we define $\lambda^{\natural(\tau)}$ to be the dominant conjugate of $\lambda + \tau - \sigma(\lambda)$ (cf. [CN20]). For $\lambda = ((c_i, d_i))_{i=1}^f$,

$$-\lambda + \tau + \sigma(\lambda) = ((pc_{i+1} - c_i + r_i, pd_{i+1} - d_i + s_i))_{i=1}^f,$$

so we define

$$\begin{aligned} c_i^{\natural(\tau)} &= \max(pc_{i+1} - c_i + r_i, pd_{i+1} - d_i + s_i) \text{ and} \\ d_i^{\natural(\tau)} &= \min(pc_{i+1} - c_i + r_i, pd_{i+1} - d_i + s_i), \end{aligned}$$

so that $\lambda^{\natural(\tau)} = ((c_i^{\natural(\tau)}, d_i^{\natural(\tau)}))_{i=1}^f$.

Lemma 5.3. *Suppose Y contains an element $\lambda = ((c_i, d_i))_{i=1}^f$ for which $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for each index i. Then there is a σ-conjugate*

$$\tilde{b} = \left(\begin{pmatrix} \alpha_i u^{\tilde{r}_i} & \tilde{f}_i(u) \\ 0 & \beta_i u^{\tilde{s}_i} \end{pmatrix}\right)_{i=1}^{f}$$

of b for which:

- *$\tilde{r}_i$ is negative,*
- *$\tilde{s}_i$ is positive,*

- *$\tilde{f}_i(u)$ is a power series which is divisible by $u^{\tilde{s}_i+1}$*

- *$\tilde{r}_i + \tilde{s}_i = m_i$ for all $i = 1, \ldots, f$.*

Proof. Fix a cocharacter $\lambda = (c_i, d_i)_{i=1}^f \in Y$ as in the statement of the lemma. We begin by conjugating b by $u^{(0,c_i+d_i)}$. In the ith component this yields,

$$\begin{pmatrix} 1 & 0 \\ 0 & u^{-c_i-d_i} \end{pmatrix} \begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{pc_{i+1}+pd_{i+1}} \end{pmatrix} = \begin{pmatrix} \alpha_i u^{r_i} & u^{pc_{i+1}+pd_{i+1}} f_i(u) \\ 0 & \beta_i u^{pc_{i+1}+pd_{i+1}-c_i-d_i+s_i} \end{pmatrix}.$$

For a positive integer N, conjugate the result by $u^{(-N,N)}$. In the ith component this yields,

$$\begin{pmatrix} u^N & 0 \\ 0 & u^{-N} \end{pmatrix} \begin{pmatrix} \alpha_i u^{r_i} & u^{pc_{i+1}+pd_{i+1}} f_i(u) \\ 0 & \beta_i u^{pc_{i+1}+pd_{i+1}-c_i-d_i+s_i} \end{pmatrix} \begin{pmatrix} u^{-pN} & 0 \\ 0 & u^{pN} \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_i u^{r_i-(p-1)N} & u^{pc_{i+1}+pd_{i+1}+(p+1)N} f_i(u) \\ 0 & \beta_i u^{pc_{i+1}+pd_{i+1}-c_i-d_i+s_i+(p-1)N} \end{pmatrix}.$$

Set $\tilde{r}_i = r_i - (p-1)N$, $\tilde{s}_i = pc_{i+1} + pd_{i+1} - c_i - d_i + s_i + (p-1)N$, and $\tilde{f}_i(u) = u^{pc_{i+1}+pd_{i+1}+(p+1)N} f_i(u)$. Note that increasing the value of N makes $\tilde{r}_i$ smaller, $\tilde{s}_i$ larger, and multiplies $\tilde{f}_i(u)$ by larger powers of u. Therefore, for sufficiently large N, conditions (1), (2), and (3) are satisfied. Condition (4) can be verified by a quick computation:

$$\begin{aligned} \tilde{r}_i + \tilde{s}_i &= r_i - (p-1)N + pc_{i+1} + pd_{i+1} - c_i - d_i + s_i + (p-1)N \\ &= c_i^{\natural(\tau)} + d_i^{\natural(\tau)} \\ &= m_i. \end{aligned}$$

□

Given a reducible 2-dimensional representation $\overline{\rho}$ associated to an element $b \in \mathrm{Res}_{k|\mathbb{F}_p}(F((u)))$ and a cocharacter μ, the discussion in the previous subsection and Lemma 5.3 allow us to select nice a representative $\tilde{b}$ and a desirable cocharacter $\tilde{\mu}$ for which $C_\mu(b) \simeq C_{\tilde{\mu}}(\tilde{b})$ as long as there exist integers c_i and d_i for which $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for each index i. However, such integers c_i and d_i do not exist for each choice of b and μ, which we see in the following example. Regardless, the lemma that follows demonstrates that, if no $\lambda = ((c_i, d_i))_{i=1}^f$ for which $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ exists, the $C_\mu(b) = \emptyset$.

Example 5.4. *Consider the example where $p = 11$, $\mu = (1, 0)$, and*

$$b = \begin{pmatrix} u^6 & f(u) \\ 0 & u^3 \end{pmatrix}$$

for some $f(u) \in \mathbb{F}(\!(u)\!)$. Then, for any $\lambda = (c, d) \in Y$, $c^{\natural(\tau)} + d^{\natural(\tau)} = 10(c + d) + 9$, so $c^{\natural(\tau)} + d^{\natural(\tau)} = m_i$ does not have an integer solution.

Lemma 5.5. *For $\lambda = ((c_i, d_i))_{i=1}^f \in Y$, if $c_i^{\natural(\tau)} + d_i^{\natural(\tau)}$ is not equal m_i for each index i, then $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = \emptyset$.*

Proof. Suppose for some $\lambda = ((c_i, d_i))_{i=1}^f \in Y$, that $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) \neq \emptyset$ and fix an element g of $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$. Then, since g belongs to $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$, $g = h_1 u^\lambda h_2$ for some $h_1, h_2 \in G(\mathcal{O}_L)$. Let $v = \det(h_1 h_2) \in \prod_{i=1}^f \mathcal{O}_L^\times$.

On the other hand, g is in $C_\mu(b)(\overline{\mathbb{F}}_p)$, so for some $\nu \in Y^+$ satisfying $\nu \leq \mu$ and some $h_3, h_4 \in G(\mathcal{O}_L)$, $g^{-1} b\sigma(g) = h_3 u^\nu h_4$. Let $v' = \det(h_3 h_4) \in \bigsqcup_{i=1}^f \mathcal{O}_L^\times$. Take the determinant of both sides of $g^{-1} b\sigma(g) = hu^\nu h$:

$$\begin{aligned} \det(g^{-1} b\sigma(g)) &= \det(hu^\nu h) \\ (u^{pc_{i+1} + pd_{i+1} - c_i - d_i + r_i + s_i})_{i=1}^f v^{-1}\sigma(v) &= (u^{m_i})_{i=1}^f v' \\ (u^{c_i^{\natural(\tau)} + d_i^{\natural(\tau)}})_{i=1}^f v^{-1}\sigma(v) &= (u^{m_i})_{i=1}^f v'. \end{aligned}$$

Since $(u^{c_i^{\natural(\tau)} + d_i^{\natural(\tau)}})_{i=1}^f$ and $(u^{m_i})_{i=1}^f$ differ by something in $\prod_{i=1}^f \mathcal{O}_L^\times$, their exponents must be equal, so $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for each index i. $\square$

Thus, whenever the Kisin variety is nonempty, we are able choose a cocharacter $\lambda = ((c_i, d_i))_{i=1}^f$ such that $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ and can apply in Lemma 5.3. The results of this subsection thus far are summarized in the following proposition.

Proposition 5.6. *If the Kisin variety associated to a reducible 2-dimensional representation $\overline{\rho}$ and a fixed dominant cocharacter is nonempty, then it is isomorphic to one of the form $C_\mu(b)$ where $\mu = ((m_i, 0))_{i=1}^f$ for some nonnegative integers m_i and b is of the form*

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^f$$

where $\alpha_i, \beta_i \in \mathbb{F}^\times$, $r_i < 0$, $s_i > 0$, $r_i + s_i = m_i$, and $f_i(u) \in u^{s_i+1}\mathbb{F}[[u]]$ for each index i.

Now suppose μ and b satisfy the conditions of Proposition 5.6. The following two lemmas showcase some of the benefits of working with μ and b of this form. First, we see a variant of Lemma 4.1 reformulated for this circumstance. In particular, we see that, when computing the intersection $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) \cap C_\mu(b)(\overline{\mathbb{F}}_p)$ for some $\lambda = ((c_i, d_i))_{i=1}^f$ with $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$, it will suffice to determine whether the entries of $g^{-1} b \sigma(g)$ for an arbitrary element g of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ belong to $\mathcal{O}_L$. After that, we will see how this pursuit is simplified by forcing r_i to be negative, s_i to be positive, and $m_i = r_i + s_i$.

Lemma 5.7. *Suppose b and μ satisfy the conditions of Proposition 5.6 and suppose $\lambda = ((c_i, d_i))_{i=1}^f$ satisfies $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for each index i. Then $g \in Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ also belongs to $C_\mu(b)(\overline{\mathbb{F}}_p)$ if and only if $g^{-1} b \sigma(g) \in \prod_{i=1}^f \mathrm{Mat}_{2\times 2}(\mathcal{O}_L)$.*

Proof. First suppose $g \in Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L) \cap C_\mu(b)(\overline{\mathbb{F}}_p)$. Then for some dominant cocharacter ν with $\nu \le \mu$, $g \in G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L)$. Denote the entries of ν by $\nu = ((x_i, y_i))$. The condition $\nu \le \mu$ is precisely

$$((m_i - x_i, -y_i))_{i=1}^f = ((n_i, -n_i))_{i=1}^f$$

for some nonnegative integers n_i. Since ν is dominant, this implies $x_i \ge y_i \ge 0$, so $u^\nu \in \prod_{i=1}^f \mathrm{Mat}_{2\times 2}(\mathcal{O}_L)$. Thus, $G(\mathcal{O}_L) u^\nu G(\mathcal{O}_L) \subset \mathrm{Mat}_{2\times 2}(\mathcal{O}_L)$.

Now suppose $g^{-1}b\sigma(g) \in \prod_{i=1}^{f} \mathrm{Mat}_{2\times 2}(\mathcal{O}_L)$. Since $g \in Iu^{\lambda}G(\mathcal{O}_L)/G(\mathcal{O}_L)$, $\det g = u^{c_i+d_i}v$ for some unit v in $\prod_{i=1}^{f} \mathcal{O}_L$, so

$$\det g^{-1}b\sigma(g) = u^{pc_{i+1}-c_i+r_i+pd_{i+1}-d_i+s_i}v\sigma(v) = u^{m_i}v\sigma(v) \in \prod_{i=1}^{f} u^{m_i}\mathcal{O}_L^{\times}.$$

By Lemma 4.1, it follows that $g^{-1}b\sigma(g) \in \bigsqcup_{\substack{\nu\in Y^+\\ \nu\leq\mu}} G(\mathcal{O}_L)u^{\nu}G(\mathcal{O}_L)$, so $g \in C_{\mu}(b)(\overline{\mathbb{F}}_p)$.

□

Lemma 5.8. *Suppose b is as in Lemma 5.3 and $\lambda = ((c_i, d_i))_{i=1}^{f} \in S^{\natural(\tau)}$. Then*

(1.) $c_i + d_i = 0$ for each index i,

(2.) $c_i > 0$ for each index i, and

(3.) each element of $Iu^{\lambda}G(\mathcal{O}_L)/G(\mathcal{O}_L)$ has a representative of the form

$$\left(\begin{pmatrix} u^{c_i} & u^{d_i}h_i(u) \\ 0 & u^{d_i} \end{pmatrix}\right)_{i=1}^{f}$$

where $h_i(u) \in \overline{\mathbb{F}}_p(u)$ has degree at most $c_i - d_i - 1$.

Proof. We begin by computing the value of $c_i + d_i$. Note that $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ because $\lambda^{\natural(\tau)} \leq \mu$, so the entries c_i and d_i of λ satisfy the system of f equations

$$-c_i + pc_{i+1} - d_i + pd_{i+1} = m_i - r_i - s_i = 0.$$

Let A denote the $f \times f$ matrix with -1's on the main diagonal, p's on the super diagonal, and a p in the bottom left entry.

$$A = \begin{pmatrix} -1 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & p & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & -1 & p & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & p \\ p & 0 & 0 & \cdots & 0 & 0 & -1 \end{pmatrix},$$

Then this system of equations corresponds to the block matrix $[\,A\ \ A\,]$, where the first f columns correspond to the c_i and the latter f columns correspond to the d_i. Note that A has a nonzero determinant and, therefore, can be row reduced to the $f \times f$ identity matrix. From this we can conclude that $c_i + d_i = 0$ for each index i.

Now consider the sign of c_i. Since $c_i^{\natural(\tau)}$ must be nonnegative, there exist f integers a_i, each between 0 and m_i, for which,

$$pc_{i+1} - c_i = m_i - a_i - r_i.$$

Let $x_i = m_i - a_i - r_i$ and consider the augmented matrix encoding the equations in the variables c_i, $[\,A\ \ x\,]$.

$$\begin{pmatrix} -1 & p & 0 & 0 & \cdots & 0 & 0 & x_1 \\ 0 & -1 & p & 0 & \cdots & 0 & 0 & x_2 \\ 0 & 0 & -1 & p & \cdots & 0 & 0 & x_3 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & p & x_{f-1} \\ p & 0 & 0 & 0 & \cdots & 0 & -1 & x_f \end{pmatrix}$$

Referring to the ith row of this matrix by ρ_i, we replace the first row by

$$\frac{1}{p^f-1}(\rho_1 + p\rho_2 + \cdots + p^{f-1}\rho_f),$$

which yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{p^f-1}\sum_{j=1}^{f} p^{j-1}x_j \\ 0 & -1 & p & 0 & \cdots & 0 & 0 & x_2 \\ 0 & 0 & -1 & p & \cdots & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & \cdots & -1 & p & x_{f-1} \\ p & 0 & 0 & 0 & \cdots & 0 & -1 & x_{f-1} \end{pmatrix}.$$

Now replace ρ_f by $p\rho_1 - \rho_f$ to get

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{p^f-1}\sum_{j=1}^{f} p^{j-1}x_j \\ 0 & -1 & p & 0 & \cdots & 0 & 0 & x_2 \\ 0 & 0 & -1 & p & \cdots & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & \cdots & -1 & p & x_{f-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \frac{1}{p^f-1}\sum_{j=1}^{f-1} p^{j}x_j - x_f \end{pmatrix}.$$

Then, for $i = 2, 3, \dots, f-1$, we replace the ith row by $-\rho_i + p\rho_{i+1}$ to see inductively that

$$c_i = \frac{1}{p^f-1}\left(\sum_{j=1}^{i-1} p^{j+f-i}(m_j - a_j - r_j) + \sum_{j=i}^{f} p^{j-i}(m_j - a_j - r_j)\right).$$

Therefore, since $r_j < 0$ for each index j, $m_j - a_j - r_j > 0$ for each index j, so c_i must be positive as well.

The third statement of the Lemma follows immediately from Lemma 4.2. □

Lemma 5.8 shows that our choice of b necessitates that any $\lambda \in S^{\natural(\tau)}$ is of the form $((c_i, d_i))_{i=1}^{f}$ where $c_i > 0$ and $d_i < 0$. In Proposition 6.1 we will see that this implies that any nonempty Iwahori strata is comprised only of elements consisting of upper triangular matrices. Negative values of c_i would likewise correspond to lower triangular components g_i.

However, this is not the case for all choices of b, as we see in the following example.

Example 5.9. *Let $p = 5$, $f = 1$, $\mu = (4, 0)$,*

$$b = \begin{pmatrix} u^4 & u^M \\ 0 & u^4 \end{pmatrix}, \quad \textit{and} \quad b^{ss} = \begin{pmatrix} u^4 & 0 \\ 0 & u^4 \end{pmatrix}$$

for some integer M. For $\lambda = (-1, 0)$, an element g of $(Iu^\lambda G(\mathcal{O}_L))/G(\mathcal{O}_L)$ is of the form

$$\begin{pmatrix} u^{-1} & 0 \\ au^{-1} & 1 \end{pmatrix}$$

for some $a \in \overline{\mathbb{F}}_p$. One computes

$$g^{-1}b\sigma(g) = \begin{pmatrix} 1 + au^{M-4} & u^{M+1} \\ -a^2u^{M-5} & -au^M + u^4 \end{pmatrix} \quad \textit{and} \quad g^{-1}b^{ss}\sigma(g) = \begin{pmatrix} 1 & 0 \\ 0 & u^4 \end{pmatrix}.$$

Since $g^{-1}b^{ss}\sigma(g)$ has entries in $\mathcal{O}_L$ for any choice of $a \in \overline{\mathbb{F}_p}$,

$$C^\lambda_\mu(b^{ss})(\overline{\mathbb{F}}_p) = \left\{ \begin{pmatrix} u^{-1} & 0 \\ au^{-1} & 1 \end{pmatrix} \mid a \in \overline{\mathbb{F}}_p \right\}.$$

In particular, $C^\lambda_\mu(b^{ss})(\overline{\mathbb{F}}_p)$ is nonempty, consists of lower triangular matrices, and can be identified with $\overline{\mathbb{F}}_p$. On the other hand, $C^\lambda_\mu(b)(\overline{\mathbb{F}}_p)$ depends on the value of M. If $M \geq 5$, then $C^\lambda_\mu(b)(\overline{\mathbb{F}}_p)$ can also be identified with $\overline{\mathbb{F}}_p$ as any value of a would make each entry in $g^{-1}b\sigma(g)$ integral. For M between -1 and 4, the only value of a that would place $g^{-1}b\sigma(g)$ in $C_\mu(b)(\overline{\mathbb{F}}_p)$ is $a = 0$, so $C^\lambda_\mu(b)(\overline{\mathbb{F}}_p)$ consists of a single point. If $M < -1$, the upper right entry of $g^{-1}b\sigma(b)$ does not belong to $\mathcal{O}_L$, so $C^\lambda_\mu(b)(\overline{\mathbb{F}}_p) = \emptyset$.

5.3 Computing $S^{\natural(\tau)}$

Fix a cocharacter $\mu = ((m_i, 0))_{i=1}^f \in Y$ and an element b of $G(\mathbb{F}((u)))$ of the form

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ 0 & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^f$$

where $r_i, s_i \in \mathbb{Z}$, $f_i(u) \in \mathbb{F}((u))$, and $\alpha_i, \beta_i \in \mathbb{F}^\times$. Given $\lambda = ((c_i, d_i))_{i=1}^f \in Y$, not assumed to be dominant, $\lambda \leq \mu$ if and only if $\mu - \lambda = ((n_i, -n_i))_{i=1}^f$ for some positive integers n_i. Since $\mu - \lambda = ((m_i - c_i, -d_i))_{i=1}^f$, this is equivalent to $d_i \geq 0$ and $c_i + d_i = m_i$. More explicitly,

$$\{\lambda \in Y^+ \mid \lambda \leq \mu\} = \{((m_i - a_i, a_i))_{i=1}^f \in Y \mid 0 \leq a_i \leq \lfloor \frac{m_i}{2} \rfloor\},$$

so $\lambda^{\natural(\tau)} \leq \mu$ if and only if $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ and $0 \leq c_i^{\natural(\tau)}, d_i^{\natural(\tau)} \leq m_i$ for each $i = 1, \ldots, f$.

Note that there is not necessarily a cocharacter λ with $\lambda^{\natural(\tau)} \leq \mu$ for each choice of μ and b. Explicitly, a cocharacter λ with $\lambda^{\natural(\tau)} \leq \mu$ exists if and only if the system of f linear equations in the $2f$ variables $c_1, c_2, \ldots, c_f, d_1, \ldots, d_f$ given by

$$pc_{i+1} - c_i + r_i + pd_{i+1} - d_i + s_i = m_i,$$

which can be abbreviated to $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$, has a solution for which all of the c_i and d_i are integers and both of $c_i^{\natural(\tau)}$ and $d_i^{\natural(\tau)}$ are positive. In the proof of Lemma 5.8 we saw that for any $\lambda = ((c_i, d_i))_{i=1}^f$ with $\lambda^{\natural(\tau)} \leq \mu$, in every index i,

$$c_i = \frac{1}{p^f - 1} \left(\sum_{j=1}^{i-1} p^{j+f-i}(m_i - a_i - r_i) + \sum_{j=i}^{f} p^{j-i}(m_i - a_i - r_i) \right)$$

for some a_i between 0 and m_i. An identical row reduction will show that

$$d_i = \frac{1}{p^f - 1} \left(\sum_{j=1}^{i-1} p^{j+f-i}(a_i - s_i) + \sum_{j=i}^{f} p^{j-i}(a_i - s_i) \right).$$

Thus $S^{\natural(\tau)}$ is a finite set containing no more than $\prod_{i=1}^{f}(m_i+1)$ elements. To be more specific, $S^{\natural(\tau)}$ is the set of all elements of $\bigoplus_{i=1}^{f}\mathbb{Z}^2$ of the form

$$\left(\frac{1}{p^f-1}\left(\sum_{j=1}^{i-1}p^{j+f-i}(m_i-a_i-r_i)+\sum_{j=i}^{f}p^{j-i}(m_i-a_i-r_i)\right),\frac{1}{p^f-1}\left(\sum_{j=1}^{i-1}p^{j+f-i}(a_i-s_i)+\sum_{j=i}^{f}p^{j-i}(a_i-s_i)\right)\right)_{i=1}^{f}$$

where $0\le a_i\le m_i$ for each index i. Of course, the denominator $\frac{1}{p^f-1}$ complicates the existence of integer solutions.

5.4 A Kisin variety computation in the case $f=1$

In this section, we will compute the isomorphism class of the Kisin variety $C_\mu(b)$ associated to the matrix

$$b=\begin{pmatrix}u^2 & xu\\ 0 & u\end{pmatrix}$$

for some $x\in\overline{\mathbb{F}}^{\times}$ and the cocharacter $\mu=(15,0)$ in two ways: first, using this representative b, and later by computing $C_\mu(\tilde{b})$ for a σ-conjugate $\tilde{b}$ of b as in Lemma 5.3. In both cases, we will do so by computing the Iwahori strata. The aim here is to demonstrate the methods of the proof of the main result and to showcase how the specific σ-conjugate we've chosen will simplify the form of the Iwahori strata.

Starting with $C_\mu(b)$, let a cocharacter $\lambda=(c,d)\in Y$, not necessarily dominant. Per Lemma 5.5, $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)=\emptyset$ unless $c^{\natural(\tau)}+d^{\natural(\tau)}=15$. Recall that $\lambda^\natural$ is the dominant conjugate of $-\lambda+\tau+\sigma(\lambda)$ and, here,

$$-\lambda+\tau+\sigma(\lambda)=(4c+2,4d+1),$$

so the condition $c^{\natural(\tau)}+d^{\natural(\tau)}=15$ amounts to $c+d=3$. We consider such characters three cases, particularly, the three cases from Lemma 4.2 (2): $c-d\ge 2$, $c-d\le -1$, and $c-d=0,1$

First, if $c-d \geq 2$, then an element g of $Iu^\lambda G(\mathcal{O}_L)$ has a representative of the form

$$g = \begin{pmatrix} u^c & u^d h(u) \\ 0 & u^d \end{pmatrix},$$

for some polynomial $h(u) \in \overline{\mathbb{F}}_p[u]$ of degree at most $c-d-1$, for which

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^{4c+2} & u^{6d-1}\varphi(h(u)) + u^{6d-2}h(u) + xu^{6d-2} \\ 0 & u^{4d+1} \end{pmatrix}.$$

Note that, in this case the lower right entry is exactly $u^{d^{\natural(\tau)}}$. In order for this entry to lie in $\mathcal{O}_L$, we must have $d^{\natural(\tau)} \geq 0$, which coincides with the condition $\lambda^{\natural(\tau)} \leq \mu$. In this case, that condition mandates $d \geq 0$, but, since $c+d=3$ and $c-d \geq 2$, it must be the case that $d \leq 0$. Therefore, the only possible λ with $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) \neq \emptyset$ in this case is $\lambda = (3,0)$.

Meanwhile, if $c-d \leq -1$, an element $g \in Iu^\lambda G(\mathcal{O}_L)$ is of the form

$$g = \begin{pmatrix} u^c & 0 \\ u^c h(u) & u^d \end{pmatrix},$$

for some polynomial $h(u) \in \overline{\mathbb{F}}_p[u]$ of degree at most d, so

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^{4c+2} + xu^{4c+1}\varphi(h(u)) & xu^{6d-2} \\ -u^{6c-1}h(u) + xu^{6c-2}h(u)\varphi(h(u)) + u^{6c-2}\varphi(h(u)) & u^{4d+1} - u^{4d+1}xh(u) \end{pmatrix}$$

Immediately, we notice that the product $g^{-1}b\sigma(g)$ is more complicated in the case where g is lower triangular than in the case where g is upper triangular. Regardless, in the upper left entry, note that the nonzero terms of $\varphi(h(u))$ have degree $5i$ for values of i between 0 and $d-c-1$. Therefore, the polynomial $xu^{4c+1}\varphi(h(u))$ does not have a nonzero term of degree u^{4c+2} and, in order for this entry to lie in $\mathcal{O}_L^\times$, we must have $4c+2 \geq 0$. Along with the other conditions of this case, this limits the value of c to $0 \leq c \leq 1$, giving us two more candidates λ for which $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ could be nonempty: $\lambda = (0,3)$ and $\lambda = (1,2)$.

The final case is $c - d = 0, 1$, which has only one option, $\lambda = (2, 1)$, which we can check directly. Here $Iu^{\lambda}G(\mathcal{O}_L)/G(\mathcal{O}_L)$ is a singleton set whose element has a representative $g = \mathrm{diag}(u^2, u)$, so

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^{10} & xu^4 \\ 0 & u^5 \end{pmatrix},$$

which has integral entries, so $C_\mu^{(2,1)}(b)(\overline{\mathbb{F}}_p)$ consists of a single point by Lemma 5.7

It is easily verified that these are exactly the cocharacters λ for which $\lambda^{\natural(\tau)} \leq \mu$. We now directly compute $C_\mu^{\lambda}(b)(\overline{\mathbb{F}}_p)$ for the three remaining λ's.

Consider $C_\mu^{(3,0)}(b)(\overline{\mathbb{F}}_p)$. An element $g \in Iu^{(3,0)}G(\mathcal{O}_L)/G(\mathcal{O}_L)$ has a representative of the form

$$g = \begin{pmatrix} u^3 & h(u) \\ 0 & 1 \end{pmatrix},$$

for some polynomial $h(u) \in u\overline{\mathbb{F}}_p[u]$ of degree at most 2. Denote the coefficients of $h(u)$ by $h(u) = au^2 + bu$. Then

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^{14} & xu^{-2} + u^{-1}\varphi(h(u)) - u^{-2}h(u) \\ 0 & u \end{pmatrix},$$

so in order for g to be an element of $C_\mu^{(3,0)}(b)(\overline{\mathbb{F}}_p)$ the upper left entry must belong to $\mathcal{O}_L$. The upper left entry is precisely

$$au^9 + bu^4 - a + -bu^{-1} + xu^{-2},$$

which does not lie in $\mathcal{O}_L$ regardless of the values of a and b due to the degree -2 term, xu^{-2}. Therefore, $C_\mu^{(3,0)}(b)(\overline{\mathbb{F}}_p) = \emptyset$

Now consider $C_\mu^{(1,2)}(b)(\overline{\mathbb{F}}_p)$. An element $g \in Iu^{(1,2)}G(\mathcal{O}_L)/G(\mathcal{O}_L)$ has a representative of the form

$$g = \begin{pmatrix} u & 0 \\ au & u^2 \end{pmatrix}$$

for some $a \in \overline{\mathbb{F}}_p$. Then

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^6 + axu^5 & xu^{10} \\ -au^5 - a^2xu^4 + au^4 & u^9 - axu^9 \end{pmatrix},$$

which has entries in $\mathcal{O}_L$ regardless of the value of a, so

$$C_\mu^{(1,2)}(b) \simeq \operatorname{Spec} \mathbb{F}[a] \simeq \mathbb{A}^1_{\mathbb{F}}.$$

Lastly, consider $C_\mu^{(0,3)}(b)(\overline{\mathbb{F}}_p)$. In this case, an element $g \in Iu^{(0,3)}$ has a representative of the form

$$g = \begin{pmatrix} 1 & 0 \\ h(u) & u^3 \end{pmatrix},$$

where $h(u) \in \overline{\mathbb{F}}_p[u]$ has degree at most 2. Denote the coefficients of $h(u)$ by $h(u) = au^2 + bu + c$. Then

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^2 + xu\varphi(h(u)) & xu^{16} \\ -u^{-1}h(u) + u^{-2}\varphi(h(u)) - xu^{-2}h(u)\varphi(h(u))0 & u^{13} - xu^{13}h(u) \end{pmatrix},$$

so the upper left, upper right, and lower right entries belong to $\mathcal{O}_L$ no matter the values of the coefficients of $h(u)$. Consider the lower left entry, which expands to

$$-a^2xu^{10} - abxu^9 + (a - acx)u^8 - abxu^5 - b^2xu^4 + (b - bcx)u^3 - au - acx - b + (-c - bcx)u^{-1} + (c - c^2x)u^{-2}.$$

Thus, this entry belongs to $\mathcal{O}_L$ if and only if $c(1 + bx) = c(1 - cx) = 0$, which is satisfied if $c = 0$ or if $b = -x^{-1}$ and $c = x^{-1}$, so

$$C_\mu^{(0,3)}(b) \simeq \operatorname{Spec}\left(\frac{\mathbb{F}[a,b,c]}{(c)}\right) \sqcup \operatorname{Spec}\left(\frac{\mathbb{F}[a,b,c]}{(b + x^{-1}), (c - x^{-1})}\right) \simeq \mathbb{A}^2_{\mathbb{F}} \sqcup \mathbb{A}^1_{\mathbb{F}}.$$

Thus, we can conclude that the Iwahori strata of the aforementioned cocharacters have the isomorphism classes in the following table and $C_\mu(b) = C_\mu^{(2,1)}(b) \cup C_\mu^{(1,2)}(b) \cup C_\mu^{(0,3)}(b) \simeq$

$\operatorname{Spec}\mathbb{F} \cup \mathbb{A}^1_{\mathbb{F}} \cup \mathbb{A}^1_{\mathbb{F}} \cup \mathbb{A}^2_{\mathbb{F}}$.

Cocharacter λ	Isomorphism class of $C^{\lambda}_{\mu}(b)$
$(3,0)$	$\emptyset$
$(1,2)$	$\mathbb{A}^1_{\mathbb{F}}$
$(2,1)$	$\operatorname{Spec}\mathbb{F}$
$(0,3)$	$\mathbb{A}^2_{\mathbb{F}} \sqcup \mathbb{A}^1_{\mathbb{F}}$
all others	$\emptyset$

Now let us find a σ-conjugate $\tilde{b}$ of b as in Lemma 5.3. To do this, we follow the proof of that lemma, first conjugating by $u^{(0,3)}$ and then by $u^{(-N,N)}$ for some positive integer N, which yields

$$\tilde{b} = \begin{pmatrix} u^{-4N+2} & xu^{6N+16} \\ 0 & u^{4N+13} \end{pmatrix}.$$

We want $-4N+2<0$, $4N+13>0$, and $6N+16>13+4N$, which is satisfied any $N \geq 1$. Selecting $N=1$ gives us

$$\tilde{b} = \begin{pmatrix} u^{-2} & xu^{22} \\ 0 & u^{17} \end{pmatrix}.$$

Lastly, we'll compute $C_{\mu}(\tilde{b})$. Per Lemma 5.5, for a cocharacter $\lambda = (c,d)$, if $c^{\natural(\tau)} + d^{\natural(\tau)} \neq 15$, then $C^{\lambda}_{\mu}(\tilde{b}) = \emptyset$. In this case, $c^{\natural(\tau)} + d^{\natural(\tau)} = 4(c+d)+15$, so $C^{\lambda}_{\mu}(b)(\overline{\mathbb{F}}_p) = \emptyset$ unless $c+d=0$. Fix $\lambda = (c,d)$ and assume $c+d=0$. An element g of $Iu^{\lambda}G(\mathcal{O}_L)/G(\mathcal{O}_L)$ has a representative of one of three possible forms by Lemma 4.2, but when $c+d=0$, we can restate the conditions. In particular, the upper triangular case occurs when $c>0$, the lower triangular case occurs when $c<0$, and the diagonal case occurs when $c=0$.

First, suppose $c<0$. Then, for $g \in Iu^{\lambda}G(\mathcal{O}_L)/G(\mathcal{O}_L)$,

$$g^{-1}b\sigma(g) = \begin{pmatrix} u^{4c-2} + xu^{4c+22}\varphi(h(u)) & xu^{5d-c+22} \\ -u^{5c-d-2}h(u) - xu^{5c-d+22}h(u)\varphi(h(u)) + u^{17-d+5c}\varphi(h(u)) & -xu^{4d+22}h(u) + u^{4d+17} \end{pmatrix},$$

for some polynomial $h(u) \in \overline{\mathbb{F}}_p[u]$ of degree at most $2|c|-1$. Note that the degree $4c-2$

term in the upper left entry is exactly u^{4c-2}. Since $4c-2<0$, this entry cannot lie in $\mathcal{O}_L$ regardless of the choice of $h(u)$, so $C_\mu^\lambda(\tilde{b})=\emptyset$.

Similarly, if $c=0$, $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ is a singleton set whose element has a representative $g=\operatorname{diag}(1,1)$, for which $g^{-1}b\sigma(g)=b$. As the upper left entry of b is not an element of $\mathcal{O}_L$, we can thus conclude that $C_\mu^\lambda(\tilde{b})=\emptyset$.

Finally, suppose $c>0$. Then, for $g\in Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$,

$$g^{-1}b\sigma(g)=\begin{pmatrix} u^{4c-2} & u^{-6c-2}\varphi(h(u))+xu^{-6c+22}-u^{17-6c}h(u) \\ 0 & u^{-4c-17} \end{pmatrix}.$$

Consider the lower right entry. This does not lie in $\mathcal{O}_L$ unless $c\leq 4$, leaving 4 cocharacters which could possibly have nonempty Iwahori strata: $(1,-1)$, $(2,-2)$, $(3,-3)$, and $(4,-4)$. It can be easily verified that this is precisely the cocharacters λ for which $\lambda^{\natural(\tau)}\leq\mu$.

Using the same methods as for $C_\mu^\lambda(b)$, one computes that the isomorphism classes of the $C_\mu^\lambda(\tilde{b})$ are of the following forms, so, just like $C_\mu(b)$, $C_\mu(\tilde{b})\simeq \operatorname{Spec}\mathbb{F}\cup\mathbb{A}^1_{\mathbb{F}}\cup\mathbb{A}^1_{\mathbb{F}}\cup\mathbb{A}^2_{\mathbb{F}}$.

Cocharacter λ	Isomorphism class of $C_\mu^\lambda(\tilde{b})$
$(-1,1)$	$\operatorname{Spec}\mathbb{F}$
$(-2,2)$	$\mathbb{A}^1_{\mathbb{F}}$
$(-3,3)$	$\mathbb{A}^2_{\mathbb{F}}$
$(-4,4)$	$\mathbb{A}^1_{\mathbb{F}}$
all others	$\emptyset$

While we see here that $C_\mu(b)\simeq C_\mu(\tilde{b})$, which must be the case by Lemma 5.2, it is worth noting that the σ-conjugation from b to $\tilde{b}$ did change the Iwahori strata; not only did it change which strata are non-empty, but it also changed the isomorphism classes of the nonempty strata. In particular, $C_\mu^{(0,3)}(b)\simeq\mathbb{A}^2_{\mathbb{F}}\sqcup\mathbb{A}^1_{\mathbb{F}}$, which is disconnected. After σ-conjugating to $\tilde{b}$, all of the nonempty strata are not only connected, but isomorphic to $\mathbb{A}^{N_\lambda}_{\mathbb{F}}$ for some non-negative integer N_λ. In Theorem 6.5 we will see that this is always the case for such μ and $\tilde{b}$, i.e. those satisfying the conditions of Proposition 5.6.

6 Computing the Iwahori Strata

6.1 Empty Iwahori strata

In the following proposition we will show that, if b and μ satisfy the conditions in Proposition 5.6, then $C_\mu^\lambda(b) = \emptyset$ whenever $\lambda^{\natural(\tau)} \not\leq \mu$. To do this, we will directly compute elements of g of $u^\lambda G(\mathcal{O}_L)u^{-\lambda}$ for λ with $\lambda^{\natural(\tau)} \not\leq \mu$ and show that they cannot belong to $C_\mu(b)(\overline{\mathbb{F}}_p)$ because $g^{-1}b\sigma(g)$ contains an entry that does not belong to $\mathcal{O}_L$.

Theorem 6.1. *Suppose b satisfies the conditions from Proposition 5.6. If $\lambda^{\natural(\tau)} \not\leq \mu$, then $C_\mu^\lambda(b) = \emptyset$.*

Proof. Fix a cocharacter $\lambda \in Y \setminus S^{\natural(\tau)}$. If $c_i^{\natural(\tau)} + d_i^{\natural(\tau)}$ is not equal to m_i for each index i, then $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = \emptyset$ by Lemma 5.5, so we assume $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for each index i. Since $\lambda^{\natural(\tau)} \not\leq \mu$, but $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for all indices i, there is some index i for which

$$c_i^{\natural(\tau)} = \max(pc_{i+1} - c_i + r_i, pd_{i+1} - d_i + s_i) > m_i \text{ and}$$
$$d_i^{\natural(\tau)} = \min(pc_{i+1} - c_i + r_i, pd_{i+1} - d_i + s_i) < 0.$$

Fix such an index i. To show that $C_\mu^\lambda(b) = \emptyset$, we will look at an arbitrary element g of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ and show some entry in the ith component of $g^{-1}b\sigma(g)$ lies in $L \setminus \mathcal{O}_L$, which implies that $g \notin C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ by Lemma 5.7.

Note that the ith component of $g^{-1}b\sigma(g)$ is $g_i^{-1}b_i\varphi(g_{i+1})$, so its form depends on the value of $c_i - d_i$ as well as that of $c_{i+1} - d_{i+1}$. This gives four cases for the computation of the ith component of $g^{-1}b\sigma(g)$:

(i) g_i and g_{i+1} are upper triangular,

(ii) g_i and g_{i+1} are lower triangular,

(iii) g_i is upper triangular and g_{i+1} is lower triangular, and

(iv) g_i is lower triangular and g_{i+1} is upper triangular.

The cases where $c_i - d_i$ or $c_{i+1} - d_{i+1}$ is equal to 0, forcing g_i or g_{i+1} to be diagonal, have been omitted as they correspond the polynomial $h_i(u)$ in the upper triangular cases being equal to 0 and thus follow from the four cases listed.

Case (i) g_i and g_{i+1} are upper triangular

In this case, the ith and $(i+1)$th components of an element g of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ have a unique representatives of the form

$$g_i = \begin{pmatrix} u^{c_i} & u^{d_i} h_i(u) \\ 0 & u^{d_i} \end{pmatrix} \quad \text{and} \quad g_{i+1} = \begin{pmatrix} u^{c_{i+1}} & u^{d_{i+1}} h_{i+1}(u) \\ 0 & u^{d_{i+1}} \end{pmatrix}$$

so

$$g_i^{-1} = \begin{pmatrix} u^{-c_i} & -u^{-c_i} h_i(u) \\ 0 & u^{-d_i} \end{pmatrix}, \quad \text{and} \quad \sigma(g)_i = \begin{pmatrix} u^{pc_{i+1}} & u^{pd_{i+1}} \varphi(h_{i+1}(u)) \\ 0 & u^{pd_{i+1}} \end{pmatrix}.$$

Computing the product $g^{-1} b\sigma(g)_i$ yields

$$(g^{-1} b\sigma(g))_i = \begin{pmatrix} \alpha_i u^{pc_{i+1} - c_i + r_i} & \alpha_i u^{pd_{i+1} - c_i + r_i} \varphi(h_{i+1}(u)) + u^{pd_{i+1} - c_i} f_i(u) - \beta_i u^{pd_{i+1} - c_i + s_i} h_i(u) \\ 0 & \beta_i u^{pd_{i+1} - d_i + s_i} \end{pmatrix}$$

Observe that exponents in the upper left and lower right entries of $(g^{-1} b\sigma(g))_i$ are $c_i^{\natural(\tau)}$ and $d_i^{\natural(\tau)}$, not necessarily in that order, with the former being negative. Furthermore, recall that α_i and β_i are nonzero, so $g^{-1} b\sigma(g)$ contains an entry that does not lie in $\mathcal{O}_L$ and g does not lie in $C_\mu(b)(\overline{\mathbb{F}}_p)$.

Case (ii) g_i and g_{i+1} are lower triangular

In this case, the ith and $i+1$th components of an element of $Iu^\lambda G(\mathcal{O}_L)$ have unique representatives of the form

$$g_i = \begin{pmatrix} u^{c_i} & 0 \\ u^{c_i} h_i(u) & u^{d_i} \end{pmatrix} \quad \text{and} \quad g_{i+1} = \begin{pmatrix} u^{c_{i+1}} & 0 \\ u^{c_{i+1}} h_{i+1}(u) & u^{d_{i+1}} \end{pmatrix}$$

so

$$(g^{-1})_i = \begin{pmatrix} u^{-c_i} & 0 \\ -u^{-d_i}h_i(u) & u^{-d_i} \end{pmatrix}, \text{ and } \sigma(g)_i = \begin{pmatrix} u^{pc_{i+1}} & 0 \\ u^{pc_{i+1}}\varphi(h_{i+1}(u)) & u^{pd_{i+1}} \end{pmatrix},$$

and the ith component of the product $g^{-1}b\sigma(g)$ has the following entries.

Upper left: $\alpha_i u^{pc_{i+1}-c_i+r_i} + u^{pc_{i+1}-c_i} f_i(u)\varphi(h_{i+1}(u))$

Upper right: $u^{pd_{i+1}-c_i} f_i(u)$

Lower left: $-\alpha_i u^{pc_{i+1}-d_i+r_i} h_i(u) - u^{pc_{i+1}-d_i} f_i(u)h_i(u)\varphi(h_{i+1}(u)) + \beta_i u^{pc_{i+1}-d_i+s_i}\varphi(h_{i+1}(u))$

Lower right: $-u^{pd_{i+1}-d_i} h_i(u)f_i(u) + \beta_i u^{pd_{i+1}-d_i+s_i}$

Consider the upper left entry. The exponent of the first term is either $c_i^{\natural(\tau)}$ or $d_i^{\natural(\tau)}$. If it is $d_i^{\natural(\tau)}$, then the degree $d_i^{\natural(\tau)}$ term of this entry is precisely $u^{d_i^{\natural(\tau)}}$ because $f_i(u)\varphi(h_{i+1}(u))$ is a polynomial and $r_i < 0$. Given that $d_i^{\natural(\tau)} < 0$, we can conclude that this entry does not lie in $\mathcal{O}_L$ if $pc_{i+1} - c_i + r_i = d_i^{\natural(\tau)}$.

On the other hand suppose that $d_i^{\natural(\tau)} = pd_{i+1} - d_i + s_i$ and consider the lower right entry of $(g^{-1}b\sigma(g))_i$:

$$u^{pd_{i+1}-d_i+s_i}(-u^{-s_i}h_i(u)f_i(u) + \beta_i)$$

We assumed $f_i(u)$ to be divisible by u^{s_i+1}, so the constant term of $-u^{-s_i}h_i(u)f_i(u)+\beta_i$ is β_i. Thus, the degree $pd_{i+1} - d_i + s_i$ term of this entry is $\beta_i u^{pd_{i+1}-d_i+s_i}$ and it follows that this entry is not in $\mathcal{O}_L$.

Case (iii) g_i is upper triangular and g_{i+1} is lower triangular

In this case, the ith component of an element g of $Iu^\lambda G(\mathcal{O}_L)$ has a unique representative of the form

$$g_i = \begin{pmatrix} u^{c_i} & u^{d_i}h_i(u) \\ 0 & u^{d_i} \end{pmatrix},$$

whereas the $i+1$th component has a unique representative of the form

$$g_{i+1} = \begin{pmatrix} u^{pc_{i+1}} & 0 \\ u^{pc_{i+1}}\varphi(h_{i+1}(u)) & u^{pd_{i+1}} \end{pmatrix}.$$

The ith component of $g^{-1}b\sigma(g)$ the following entries.

Upper left: $\alpha_i u^{pc_{i+1}-c_i+r_i} + u^{pc_{i+1}-c_i} f_i(u)\varphi(h_{i+1}(u)) - \beta_i u^{pc_{i+1}-c_i+s_i} h_i(u)\varphi(h_{i+1}(u))$

Upper right: $u^{pd_{i+1}-c_i} f_i(u) - \beta_i u^{pd_{i+1}-c_i+s_i} h_i(u)$

Lower left: $\beta_i u^{pc_{i+1}-d_i+s_i}\varphi(h_{i+1}(u))$

Lower right: $\beta_i u^{pd_{i+1}-d_i+s_i}$

Either $d_i^{\natural(\tau)} = pd_{i+1} - d_i + s_i$ or $d_i^{\natural(\tau)} = pc_{i+1} - c_i + r_i$. In the former case, the lower right entry is not in $\mathcal{O}_L$. In the latter case, consider the upper left entry,

$$u^{pc_{i+1}-c_i+r_i}(\alpha_i + u^{-r_i} f_i(u)\varphi(h_{i+1}(u)) - \beta_i u^{-r_i+s_i} h_i(u)\varphi(h_{i+1}(u))).$$

Since s_i is a positive integer and r_i is negative one, $u^{-r_i} f_i(u)\varphi(h_{i+1}(u)) - u^{s_i-r_i} h_i(u)\varphi(h_{i+1}(u))$ is a polynomial which is divisible by a positive power of u. Therefore, the degree $u^{pc_{i+1}-c_i+r_i}$ term of this entry is exactly $\alpha_i u^{pc_{i+1}-c_i+r_i}$. Since $pc_{i+1} - c_i + r_i < 0$, this entry is not in $\mathcal{O}_L$.

Case (iv) g_i is lower triangular and g_{i+1} is upper triangular

In this case, the ith component of an element g of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ has a unique representative of the form

$$g_i = \begin{pmatrix} u^{c_i} & 0 \\ u^{c_i} h_i(u) & u^{d_i} \end{pmatrix},$$

whereas the $i+1$th component has a unique representative of the form

$$g_{i+1} = \begin{pmatrix} u^{c_{i+1}} & u^{d_{i+1}} h_{i+1}(u) \\ 0 & u^{d_{i+1}} \end{pmatrix}.$$

The ith component of the product $g^{-1}b\sigma(g)$ has the following entries.

$$\begin{aligned}
\text{Upper left:}\quad & \alpha_i u^{pc_{i+1}-c_i+r_i} \\
\text{Upper right:}\quad & \alpha_i u^{pd_{i+1}-c_i+r_i}\varphi(h_{i+1}(u)) + u^{pd_{i+1}-c_i} f_i(u) \\
\text{Lower left:}\quad & -\alpha_i u^{pc_{i+1}-d_i+r_i} h_i(u) \\
\text{Lower right:}\quad & -\alpha_i u^{pd_{i+1}-d_i+r_i} h_i(u)\varphi(h_{i+1}(u)) - u^{pd_{i+1}-d_i} f_i(u)h_i(u) + \beta_i u^{pd_{i+1}-d_i+s_i}
\end{aligned}$$

Consider the upper left entry. If $pc_{i+1} - c_i + r_i = d_i^{\natural(\tau)} < 0$, then this entry is not in $\mathcal{O}_L$. Otherwise, $pc_{i+1} - c_i + r_i = c_i^{\natural(\tau)} > m_i$. In this case, there must be some other index $j \in \{1, \dots, \widehat{i}, \dots, f\}$ for which $(c_j^{\natural(\tau)}, d_j^{\natural(\tau)}) \not\leq (m_j, 0)$ and the jth component of $g^{-1}b\sigma(g)$ falls under one of the cases we have already covered. The existence of such an index j is shown below in Lemma 6.2, which completes the proof.

□

Lemma 6.2. *Suppose b satisfies the conditions from Lemma 5.3. Fix for a cocharacter $\lambda = ((c_i, d_i))_{i=1}^f \in Y$ for which $c_i^{\natural(\tau)} + d_i^{\natural(\tau)} = m_i$ for all i. Suppose that for some $i_0 \in \{1, \dots, f\}$, $(c_{i_0}^{\natural(\tau)}, d_{i_0}^{\natural(\tau)}) \not\leq (m_{i_0}, 0)$ and the following inequalities are satisfied*

1. $pc_{i_0+1} - c_{i_0} + r_{i_0} > m_{i_0}$,

2. $c_{i_0} < 0$, *and*

3. $c_{i_0+1} > 0$.

Then there is another index $j_0 \in \{1, \dots, f\}$ for which $(c_{j_0}^{\natural(\tau)}, d_{j_0}^{\natural(\tau)}) \not\leq (m_{j_0}, 0)$ and at least one of the inequalities above is not satisfied.

Proof. Let S denote the set of all indices in $\{1, \dots, f\}$ for which $(c_i^{\natural(\tau)}, d_i^{\natural(\tau)}) \not\leq (m_i, 0)$ and suppose that for all $i \in S$ the three inequalities in the statement of the lemma are satisfied. Reorder the c_i so that the index i_0 becomes the index 1, i.e. (c_1, d_1) satisfies the inequalities in the statement of the lemma and $1 \in S$.

For any $i \in S$, since $c_i < 0$ and $c_{i+1} > 0$, the indices $i \pm 1$ are not elements of S. In particular, $S \neq \{1, \dots, f\}$. For any index $j \in \{1, \dots, f\} \setminus S$, the following inequality is satisfied because $(c_j^{\natural(\tau)}, d_j^{\natural(\tau)}) \leq (m_j, 0)$,

$$0 \leq pc_{j+1} - c_j + r_j \leq m_j$$

so $-r_j \leq pc_{j+1} - c_j$. For $j \in \{1, \ldots, f\} \setminus S$, let E_j denote the inequality $-r_j \leq pc_{j+1} - c_j$.

For $i \in S$, we can rewrite the first inequality from the statement of the lemma as $s_i < pc_{i+1} - c_i$ since $m_i - r_i = s_i$. For $i \in S$, let E_i denote the inequality $s_i < pc_{i+1} - c_i$. Now consider the sum $\sum_{i=1}^{f} p^{i-1} E_i$:

$$\begin{aligned} \sum_{i \in S} p^{i-1} s_i - \sum_{j \notin S} p^{j-1} r_j &< \sum_{i=1}^{f} p^i c_{i+1} - \sum_{i=1}^{f} p^{i-1} c_i \\ \sum_{i \in S} p^{i-1} s_i - \sum_{j \notin S} p^{j-1} r_j &< (pc_2 + \cdots + p^{f-1} c_f + p^f c_1) - (c_1 + pc_2 + \cdots + p^{f-1} c_f) \\ \sum_{i \in S} p^{i-1} s_i - \sum_{j \notin S} p^{j-1} r_j &< (p^f - 1) c_1 \\ \frac{1}{p^f - 1} \left(\sum_{i \in S} p^{i-1} s_i - \sum_{j \notin S} p^{j-1} r_j \right) &< c_1 \end{aligned}$$

We assumed all s_i to be positive and all r_j to be negative, so the value on the left hand side of this inequality is positive. But the index 1 belongs to S, so $c_1 < 0$, which forces a contradiction. □

6.2 Nonempty Iwahori strata

Having seen that $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) = \emptyset$ whenever $\lambda \notin S^{\natural(\tau)}$, it is natural to ask whether $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ is nonempty if and only if $\lambda \in S^{\natural(\tau)}$. This is, however, not the case, as we see in the following example.

Example 6.3. *Let* $p = 3$, $\mu = (1, 0)$, *and*

$$b = \begin{pmatrix} u^{-40} & u^{42} \\ 0 & u^{41} \end{pmatrix}$$

so that $r = -40$, $s = 41$, *and* $f(u) = u^{s+1}$. *Note that this choice of* b *and* μ *satisfies the conditions of Proposition 5.6.*

In this example, $\{\lambda \in Y^+ \mid \lambda \leq \mu\} = \{(1,0)\}$. *Additionally, for any cocharacter* $\lambda = (c,d)$, $\lambda^{\natural(\tau)}$ *is the dominant conjugate of* $(2c-40, 2d+41)$, *so* $\lambda^{\natural(\tau)} \leq \mu$ *if and only if* $2c - 40 = 1$ *and* $2d + 41 = 0$ *or* $2c - 40 = 0$ *and* $2d + 41 = 1$. *Thus* $S^{\natural(\tau)}$ *is precisely the singleton set* $\{(20, -20)\}$. *Let*

$$g = \begin{pmatrix} u^{20} & u^{-20}h(u) \\ & u^{-20} \end{pmatrix}$$

be an arbitrary element of $Iu^{(20,-20)}G(\mathcal{O}_L)/G(\mathcal{O}_L)$. *Then* $\deg h(u) \leq 39$. *Denote the coefficients of* $h(u)$ *by* $h(u) = \sum_{i=1}^{39} a_i u^i$. *Then*

$$g^{-1}b\sigma(g) = \begin{pmatrix} 1 & u^{-120}\varphi(h(u)) - u^{-39}h(u) + u^{-38} \\ 0 & u \end{pmatrix}.$$

All entries other than the upper right entry belong to $\mathcal{O}_L$, *so we can focus on whether the upper right entry belongs to* $\mathcal{O}_L$ *in order to determine whether* $g \in C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$. *Consider the degree* -38 *term of this entry. Since* $u^{-120}\varphi(h(u)) = \sum_{i=1}^{39} a_i u^{3i-120}$ *and there is no integer* i *for which* $3i - 120 = -38$, *this term is precisely* $a_1 u^{-38} + u^{-38}$. *Thus, for* g *to belong to* $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$, a_1 *must be* -1. *To force a contradiction, consider the degree* -117 *term of this entry. Since* $h(u)$ *is a polynomial, this term is precisely the degree* -117 *term of* $u^{-120}\varphi(h(u))$, *which is* $a_1 u^{-117}$, *so for* g *to belong to* $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$, a_1 *must be* 0. *We can, therefore, conclude that* $C_\mu^{(20,-20)}(b)(\overline{\mathbb{F}}_p) = \emptyset$.

The preceding example demonstrated that, given μ and b which satisfy the conditions of Proposition 5.6, it is possible for $C_\mu^\lambda(b)$ to be empty when $\lambda^{\natural(\tau)} \leq \mu$. However, this is not the case for a slight variation of b. Given

$$b = \left(\begin{pmatrix} \alpha_i u^{r_i} & f_i(u) \\ & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}$$

we define

$$b^{\mathrm{ss}} := \left(\begin{pmatrix} \alpha_i u^{r_i} & \\ & \beta_i u^{s_i} \end{pmatrix} \right)_{i=1}^{f}.$$

The results in this paper are largely inspired by Proposition 2.2 of [CN20], which says that $C_\mu^\lambda(b^{\mathrm{ss}})(\overline{\mathbb{F}}_p) \neq \emptyset$ if and only if $\lambda^{\natural(\tau)} \leq \mu$ when the fixed point ν of the Y-endomorphism $u^\tau\sigma$ satisfies $0 < \langle \alpha, \nu \rangle < 1$ for each $\alpha \in \Phi^+$. While this condition is not, in general, satisfied by b and μ as in Proposition 5.6, we will be able to recover this result for such b and μ in Corollary 6.4 using the methods from the proof of Theorem 6.1.

In the meantime, to see that b and μ as in Proposition 5.6 do not satisfy the conditions of [CN20, Proposition 2.2], we will compute the fixed point ν of $u^\tau\sigma$ directly. For a cocharacter $\nu = ((c_i, d_i))_{i=1}^f$ in Y,

$$u^\tau\sigma(\nu) = ((pc_{i+1} + r_i, pd_{i+1} + s_i))_{i=1}^f,$$

so the unique fixed point of $u^\tau\sigma$ is the solution to the system of equations

$$pc_{i+1} - c_i = -r_i$$
$$pd_{i+1} - d_i = -s_i,$$

which is very similar to one that we solved in section 5.3 to compute c_i. Using the same row reductions, we get the unique solution $\nu = ((c_i, d_i))_{i=1}^f$ where

$$c_i = \frac{-1}{p^f - 1} \left(\sum_{j=1}^{i-1} p^{j+f-i} r_j + \sum_{j=i}^{f} p^{j-i} r_j \right) \text{ and } d_i = \frac{-1}{p^f - 1} \left(\sum_{j=1}^{i-1} p^{j+f-i} s_j + \sum_{j=i}^{f} p^{j-i} s_j \right).$$

Let α_i denote the element of Φ^+ that acts as $e_1 - e_2$ on the ith component. Then,

$$\langle \nu, \alpha_i \rangle = \frac{1}{p^f - 1} \left(\sum_{j=1}^{i-1} p^{j+f-i}(-r_j + s_j) + \sum_{j=i}^{f} p^{j-i}(-r_j + s_j) \right).$$

Consider the case where $f = 1$ and $p = 3$ and let $\alpha := e_1 - e_2$ denote the only element of

Φ^+ in this case. Then $\langle \nu, \alpha_i \rangle = \frac{-r+s}{2}$, so the inequality $0 < \langle \nu, \alpha \rangle < 1$ is not satisfied for any choices of negative r and positive s. On the other hand, if $f = 1$ and $p = 13$, this inequality becomes $0 < \frac{-r+s}{12} < 1$ which is satisfied by many choices of negative r and positive s. So in our case, b^{ss} satisfies the conditions of [CN20, Proposition 2.2] sometimes, but not always.

Corollary 6.4. *Suppose b is as in Lemma 5.3. Then $C_\mu^\lambda(b^{ss})(\overline{\mathbb{F}}_p) \neq \emptyset$ if and only if $\lambda^{\natural(\tau)} \leq \mu$. Moreover, the set of λ for which $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) \neq \emptyset$ is a subset of the set of λ for which $C_\mu^\lambda(b^{ss})(\overline{\mathbb{F}}_p) \neq \emptyset$.*

Proof. Suppose $\lambda = ((c_i, d_i))_{i=1}^f$ satisfies $\lambda^{\natural(\tau)} \leq \mu$. Then $c_i > 0$ in each index i by Lemma 5.8, so any element g of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ is of the form

$$g = \left(\begin{pmatrix} u^{c_i} & u^{d_i} h_i(u) \\ 0 & u^{d_i} \end{pmatrix} \right)_{i=1}^f .$$

Let g_0 denote the element of $Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$ for which $h_i(u) = 0$ in each index i. Then

$$g_0^{-1} b \sigma(g_0) = \left(\begin{pmatrix} \alpha_i u^{pc_{i+1} - c_i + r_i} & 0 \\ 0 & \beta_i u^{pd_{i+1} - d_i + s_i} \end{pmatrix} \right)_{i=1}^f ,$$

which is an element of $G(\mathcal{O}_L) u^{\lambda^{\natural(\tau)}} G(\mathcal{O}_L)$, so $g_0 \in C_\mu^\lambda(b^{\mathrm{ss}})(\overline{\mathbb{F}}_p)$. The rest of the theorem follows from Theorem 6.1. $\square$

Now we consider the nonempty Iwahori strata of a Kisin variety $C_\mu(b)$ with b and μ satisfying the conditions of Proposition 5.6. Suppose $\lambda = ((c_i, d_i))_{i-1}^f$ satisfies $\lambda_{\natural(\tau)} \leq \mu$. Then for each index i, $c_i > 0$, $d_i = -c_i$, and for an element $g \in Iu^\lambda G(\mathcal{O}_L)/G(\mathcal{O}_L)$, $g^{-1} b \sigma(g)$ has the following form.

$$(g^{-1} b \sigma(g))_i = \begin{pmatrix} \alpha_i u^{pc_{i+1} - c_i + r_i} & \alpha_i u^{pd_{i+1} - c_i + r_i} \varphi(h_{i+1}(u)) - \beta_i u^{pd_{i+1} - c_i + s_i} h_i(u) + u^{pd_{i+1} - c_i} f_i(u) \\ 0 & \beta_i u^{pd_{i+1} - d_i + s_i} \end{pmatrix}$$

Since the exponents in the upper left and lower right entries are $c_i^{\natural(\tau)}$ and $d_i^{\natural(\tau)}$, they must both be nonnegative because $\lambda^{\natural(\tau)} \leq \mu$. Since these entries therefore lie in $\mathcal{O}_L$, whether g belongs to $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ depends entirely upon whether the upper left entry belongs to $\mathcal{O}_L$. Denote the coefficients of the polynomial $h_i(u)$ by $h_i(u) = \sum_{j=1}^{2c_i-1} a_{i,j}u^j$ and the coefficients of $f_i(u)$ by $f_i(u) = \sum_{j=1}^{\infty} b_{i,j}u^j$ so that the upper left entry is

$$\alpha_i \sum_{j=1}^{2c_i-1} a_{i,j}u^{pj+pd_{i+1}-c_i+r_i} - \beta_i \sum_{j=1}^{2c_i-1} a_{i,j}u^{j+pd_{i+1}-c_i+s_i} + \sum_{j=1}^{\infty} b_{i,j}u^{j+pd_{i+1}-c_i}.$$

Let S denote the set of tuples (N,i) where N is a positive integer for which, treating the $a_{i,j}$ as variables, the upper right entry of $(g^{-1}b\sigma(g))_i$ has a nonzero degree term of degree $-N$.

Going from left to right through the summands, a nonzero degree $-N$ term of $\alpha_i \sum_{j=1}^{2c_i-1} a_{i,j}u^{pj+pd_{i+1}-c_i+r_i}$ would come from the degree $N_1 := \frac{1}{p}(-N - pd_{i+1} - c_i + r_i)$ term of $h_{i+1}(u)$ in the event that $\frac{1}{p}(N - pd_{i+1} - c_i + r_i)$ is an integer between 0 and $2c_{i+1} - 1$. Set $a'_{i+1,N_1} = a_{i+1,N_1}$ if this is the case and $a'_{i+1,N_1} = 0$ otherwise. Likewise, a degree $-N$ term of $\beta_i \sum_{j=1}^{2c_i-1} a_{i,j}u^{j+pd_{i+1}-c_i+s_i}$ would come from the degree $N_2 := -N - pd_{i+1} + c_i + s_i$ term of $h_i(u)$ in the event that $0 \leq N_2 \leq 2c_i - 1$. Set $a'_{i,N_2} = a_{i,N_2}$ if this is the case and $a'_{i,N_2} = 0$ otherwise. Lastly, a degree $-N$ term of $\sum_{j=1}^{\infty} b_{i,j}u^{j+pd_{i+1}-c_i}$ would come from the degree $N_3 = -N - pd_{i+1} + c$ term of $f_i(u)$ in the event that $f_i(u)$ has a nonzero degree term of degree N_3. Set $b'_{i,N_3} = b_{i,N_3}$ if this is the case and $b'_{i,N_3} = 0$ otherwise. Then the degree $-N$ term of the upper right entry of $(g^{-1}b\sigma(g))_i$ has the form $\alpha_i a'_{i+1,N_1} - \beta_i a'_{i,N_2} + b'_{N_3}$.

In order for g to belong to $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$, it is necessary and sufficient that, for each $(N,i) \in S$ the linear equation $\alpha_i a'_{i+1,N_1} - \beta_i a'_{i,N_2} + b'_{N_3} = 0$ is satisfied. Observe that, in the event $a'_{i+1,N_1} = a'_{i,N_2} = 0$ and b'_{N_3} is nonzero, this is an impossibility, so $C_\mu^\lambda(b) = \emptyset$. If for each such (N,i) this is not the case, let I denote the ideal of $\overline{\mathbb{F}}_p[a_{1,1}, \ldots, a_{f,2c_f-1}]$ generated by the set

$$\{\alpha_i a'_{i+1,N_1} - \beta_i a'_{i,N_2} - b'_{N_3} \mid (N,i) \in S\}.$$

Then $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ can be identified with $\overline{\mathbb{F}}_p[a_{1,1}, \ldots, a_{f,2c_f-1}]/I$. As such, $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p)$ is the solution set of a system of linear equations in $\sum_{i=1}^{f}(2c_f - 1)$ variables, so for some $N_\lambda \leq \sum_{i=1}^{f}(2c_f - 1)$, $C_\mu^\lambda(b)(\overline{\mathbb{F}}_p) \simeq \overline{\mathbb{F}}_p^{N_\lambda}$. Since $C_\mu^\lambda(b)$ is a reduced, Jacobson scheme, we can thus conclude that $C_\mu^\lambda(b) \simeq \mathbb{A}_\mathbb{F}^{N_\lambda}$. This gives us the following theorem.

Theorem 6.5. *Suppose b and μ satisfy the conditions of Proposition 5.6. If $\lambda^{\natural(\tau)} \leq \mu$, either $C_\mu^\lambda(b) = \emptyset$ or $C_\mu^\lambda(b) \simeq \mathbb{A}_\mathbb{F}^{N_\lambda}$ for some nonnegative integer N_λ.*

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