

ON THE ARITHMETIC GAN-GROSS-PRASAD CONJECTURE FOR $U(2) \times U(3)$

by

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We respectfully acknowledge the University of Arizona is on the land and territories of Indigenous peoples. Today, Arizona is home to 22 federally recognized tribes, with Tucson being home to the O'odham and the Yaqui. Committed to diversity and inclusion, the University strives to build sustainable relationships with sovereign Native Nations and Indigenous communities through education offerings, partnerships, and community service.

DEDICATION

For Nala

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ABSTRACT

In its broadest sense, the arithmetic Gan-Gross-Prasad (GGP) Conjecture relates the central value of the derivative of a certain L -function to a height pairing of certain algebraic cycles on Shimura varieties. Xue proved the arithmetic GGP conjecture for $U(2) \times U(3)$ operating under the assumption that the underlying Shimura varieties are projective as well as some hypotheses on the representations to which the L -function in question is attached. Assuming some working hypotheses about arithmetic theta lifting, we extend Xue's work to toroidal compactifications of Shimura varieties through the use of Hecke correspondences to annihilate contributions from the boundary thereby reducing much of the problem to the setting where many of Xue's results apply.

1 Introduction

The Gan-Gross-Prasad (GGP) Conjectures are a broad collection of open problems that propose deep connections between representation theory and number theory. Assuming some working hypotheses on the non-vanishing of arithmetic theta lifts, the main result of this paper is the proof of an unrefined version of the (arithmetic) GGP Conjecture for $U(2) \times U(3)$ when the underlying real field is \mathbb{Q} and the relevant automorphic representations have some prescribed properties. We will of course will make this more precise in the pages to come, but the basic idea is we will generalize the work of H. Xue in [Xue19] who used arithmetic theta lifting to give a precise formula relating the heights of certain algebraic cycles on the product of unitary Shimura varieties to the central derivative of a specific tensor product L -function. Xue's proof relies on the assumption that the underlying unitary Shimura varieties X_K are proper in addition to some hypotheses on the automorphic representations used to define the aforementioned L -function. In this paper, we address the situation when the Shimura varieties are only quasi-projective and extend Xue's techniques to a smooth canonical compactification \widetilde{X}_K of X_K .

We will first look at some of the history of the GGP conjectures and recount some of the key events and mathematical results that led to their formulation. In particular, we will discuss B. Gross and D. Zagier's seminal result from [GZ86], which many point to as the origins of the (arithmetic) GGP conjectures. From there, we will discuss the work of J.L. Waldspurger [Wal85], which itself is one of the first examples of a GGP-type result. Moreover, Waldspurger's methods were of critical importance to Gross when he formulated the "first" GGP Conjectures that he presented during a series of lectures at MSRI in 2001 [Gro22]. X. Yuan, S. Zhang, and W. Zhang would go on to prove these conjectures [YZZ13] and so we will recount their results as well. We note that we follow the exposition in Sections 1.1 and 1.4 of [YZZ13] to present the results of Gross-Zagier and Waldspurger, respectively. From there, we provide a general exposition of the GGP conjectures, subsequent developments, and their status currently. Finally, we will dive into arithmetic GGP conjecture for $U(n) \times U(n+1)$, Xue's main result from [Xue19], and state our main result.

1.1 Gross-Zagier Formula and Waldspurger's Formula

B. Gross and D. Zagier published their momentous paper [GZ86] in 1986. Though their publication included numerous substantial results, the so called *Gross-Zagier Formula*, which relates the Neron-Tate height of Heegner points on modular curves to the critical value of the derivative of an L -function attached to a modular form, is of particular significance to the GGP conjectures. In order to be precise, we fix the following notation:

- N is a positive integer;
- $f \in S_2(\Gamma_0(N))$ is a newform of weight 2;
- $E \subseteq \mathbb{C}$ is an imaginary quadratic extension of \mathbb{Q} whose discriminant D is both odd and relatively prime to N ;
- \mathcal{O}_E is the ring of integers in E ;
- H is the Hilbert class field of E ;
- χ is a character of $\text{Pic}(\mathcal{O}_E)$;
- $L(f, \chi, s)$ is the Rankin-Selberg convolution of the L -series $L(f, s)$ and the L -series $L(\chi, s)$.

Gross and Zagier showed that $L(f, \chi, s)$ has a holomorphic continuation to the entire complex plane and satisfies a functional equation relating s to $2 - s$ [GZ86, Proposition 5.5]. If we impose that E satisfies the ***Heegner condition***, which asserts that every prime factor of N is split in E , then the sign of the functional equation of $L(f, \chi, s)$ is odd from which it follows that $L(f, \chi, 1) = 0$. Naturally, this leads to the consideration of the value of $L'(f, \chi, 1)$.

Now let $X_0(N)$ be the modular curve over \mathbb{Q} whose \mathbb{C} -points parametrize those isogenies of elliptic curves over \mathbb{C} with kernel isomorphic to $\mathbb{Z}/N\mathbb{Z}$. As a consequence of the Heegner condition, there exists an ideal \mathcal{N} of \mathcal{O}_E such that $\mathcal{O}_E/\mathcal{N} \simeq \mathbb{Z}/N\mathbb{Z}$. For each nonzero ideal \mathcal{J} , we denote by $P_{\mathcal{J}}$ the point on $X_0(N)(\mathbb{C})$ representing the isogeny $\mathbb{C}/\mathcal{J} \rightarrow \mathbb{C}/\mathcal{J}\mathcal{N}^{-1}$; the point $P_{\mathcal{J}}$ is defined over H and depends only on the class of \mathcal{J} in $\text{Pic}(\mathcal{O}_E)$. Lastly, let P_{χ} denote

the following point in the Jacobian $J_0(N)$ of $X_0(N)$ using the cusp ∞ by

$$P_\chi := \sum_{[J] \in \text{Pic}(\mathcal{O}_E)} [P_J - \infty] \otimes \chi([J]) \in J_0(N)(H) \otimes_{\mathbb{Z}} \mathbb{C}$$

and set $P_\chi(f)$ to be the f -isotypical component of P_χ in $J_0(N)(H) \otimes_{\mathbb{Z}} \mathbb{C}$ under the action of the Hecke operators. Then the primary result of Gross and Zagier [GZ86, Theorem 6.3] is

$$\langle P_\chi(f), P_\chi(f) \rangle_{\text{NT}} = \frac{hu^2\sqrt{D}}{8\pi^2(f, f)_{\text{Pet}}} \cdot L'(f, \chi, 1) \quad (1.1.1)$$

where $\langle -, - \rangle_{\text{NT}}$ is the Neron-Tate height over H , h is the class number of E , $u := \frac{1}{2}(\#\mathcal{O}_E^\times)$, and $(-, -)_{\text{Pet}}$ is the Petersson inner product.

The most significant constituents of Equation (1.1.1) are the critical value of the L -derivative and the height pairing of some “special cycles.” The remainder of the terms in Equation (1.1.1) give some nonzero constant and so a slightly weaker statement would be

$$\langle P_\chi(f), P_\chi(f) \rangle_{\text{NT}} \neq 0 \iff L'(f, \chi, 1) \neq 0. \quad (1.1.2)$$

Statements of this type, i.e. relating the critical value of an L -derivative to a height pairing (such as the Neron-Tate height for curves or the conjectural Beilinson-Bloch pairing for higher dimensional varieties), are the essence of the (unrefined) *arithmetic* GGP. We will expand on this more in the subsequent sections.

Gross mentions in [Gro22] that he and Zagier had finished the proof of Equation (1.1.1) by the Fall of 1982, which serves as an appropriate starting point for the history of the GGP conjectures considering the Gross-Zagier formula is often identified as the first GGP-type result, certainly of the arithmetic flavor. Indeed Gross points to a conference held at the University of Durham in July 1983 where he and Zagier detailed their proof of Equation (1.1.1) as the beginning of a long journey that eventually led to the development of GGP conjectures as we know them today. Funnily enough, while Zagier was scheduled to speak at this conference in Durham, Gross initially was not. It was only after he and Zagier lobbied the organizers—B. Birch and R. Rankin—did he manage to get a spot on the speaking schedule. Both he and Zagier spoke back-to-back on the first day of the conference and, at the conclusion of

their talks, Birch, Rankin, along with J.P. Serre convinced Gross and Zagier to lecture every day for the remainder of the conference and to fill in every detail of their argument [Gro22].

After a week of lecturing, Gross finally caught a break on the last day of the conference, or so he thought. He intended to skip out on the final afternoon of talks, but M.F. Vignéras persuaded him to stay and attend the talk of one of her recent doctoral students, J.L. Waldspurger. Waldspurger had been working on a track parallel to Gross and Zagier and presented the main results of his celebrated paper [Wal85], in which he related the central value of an L -functions attached to an automorphic representation to toric periods of modular forms. Gross identifies Waldspurger's talk as his entry point to the subject and he singles out how Waldspurger's interpretation of the central value of the L -function using representation theory and automorphic forms was completely new to him [Gro22].

We again note that we will be following [YZZ13] for our exposition on Waldspurger's formula rather than the original source [Wal85] as the presentation of Waldspurger's results in [YZZ13] conforms more closely to the contemporary presentation of the GGP conjectures and is better suited to our purposes. With this in mind, we fix the following notation:

- F is a number field;
- B is a quaternion algebra over F ;
- E is a quadratic field extension of F with a fixed embedding $E \hookrightarrow B$ over F ;
- π is an irreducible cuspidal automorphic representation of $(B \otimes_F \mathbb{A}_F)^\times$ with central character $\omega_\pi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ and contragredient $\tilde{\pi}$;
- $\chi : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ is a character satisfying $\omega_\pi \cdot \chi|_{\mathbb{A}_F^\times} = 1$; and
- $\eta : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$ is the quadratic character associated to the extension E/F by class field theory.

Waldspurger's formula is truly a representation theoretic statement relating two vectors as scalar multiples of each other; we first describe the problem locally at a fixed place v of F just as Waldspurger did. The spaces we are interested in are $\mathcal{P}(\pi_v, \chi_v)$, $\mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$, and

$\mathcal{P}(\pi_v, \chi_v) \otimes \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$, where

$$\mathcal{P}(\pi_v, \chi_v) := \text{Hom}_{E_v^\times}(\pi_v \otimes \chi_v, \mathbb{C})$$

and $\mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$ is defined analogously. The work of J. Tunnel [Tun83] and H. Saito [Sai93] asserts that $\dim \mathcal{P}(\pi_v, \chi_v) \leq 1$ and is one-dimensional precisely when

$$\varepsilon\left(\frac{1}{2}, \pi_v, \chi_v\right) = \chi_v(-1)\eta_{E_v/F_v}(-1)\varepsilon_v(B_v),$$

where:

- $\varepsilon\left(\frac{1}{2}, \pi_v, \chi_v\right)$ is the local root number defined by

$$\varepsilon\left(\frac{1}{2}, \pi_v, \chi_v\right) = \varepsilon\left(\frac{1}{2}, \text{BC}(\sigma_v) \otimes \chi_v\right),$$

where σ_v is the Jacquet-Langlands correspondence of π_v on $\text{GL}_2(F_v)$ and $\text{BC}(\sigma_v)$ is the base change to E_v ; and

- $\varepsilon_v(B_v)$ is the *local Hasse invariant*, which is equal to $+1$ if $B_v \simeq M_2(F_v)$ and equal to -1 if B_v is isomorphic to the unique division algebra over F_v .

It also follows from the work of Tunnel and Saito that $\dim \mathcal{P}(\pi_v, \chi_v) = \dim \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$ and so the spaces $\mathcal{P}(\pi_v, \chi_v)$, $\mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$, and $\mathcal{P}(\pi_v, \chi_v) \otimes \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$ are either all zero or one-dimensional. Curiously, there is not a canonical choice in basis for either of the spaces $\mathcal{P}(\pi_v, \chi_v)$, $\mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$, but there is a canonical choice for the tensor product $\mathcal{P}(\pi_v, \chi_v) \otimes \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$ via the integration of local matrix coefficients.

In order to construct a generator for $\mathcal{P}(\pi_v, \chi_v) \otimes \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$, we let $B_v := B \otimes_F F_v$ and fix a B_v^\times -invariant pairing

$$\langle -, - \rangle_v : \pi_v \times \tilde{\pi}_v \longrightarrow \mathbb{C}$$

along with a Haar measure on E_v^\times/F_v^\times . We define the so-called *local canonical invariant*

form $\alpha_v : \pi_v \otimes \tilde{\pi}_v \rightarrow \mathbb{C}$ by

$$\alpha_v(f_1 \otimes f_2) := \frac{L(1, \eta_v)L(1, \pi_v, \text{ad})}{\zeta_{F_v}(2)L(\frac{1}{2}, \pi_v, \chi_v)} \int_{E_v^\times/F_v^\times} \langle \pi_v(t_v)f_1, f_2 \rangle_v \chi_v(t_v) dt_v, \quad f_1 \in \pi_v, f_2 \in \tilde{\pi}_v. \quad (1.1.3)$$

We note that $L(\frac{1}{2}, \pi_v, \chi_v) \neq 0$, which implies that the nonzero constant appearing before the integral in Equation (1.1.3) is well-defined. Also, we are using the embedding $E_v \hookrightarrow B_v$ in order to define the action of $\pi_v(t_v)$ on f_1 that appears in the first argument of the pairing in the integral in Equation (1.1.3). Further, we note that the integral in Equation (1.1.3) converges absolutely if both π_v and χ_v are unitary. With these conventions in place, we note that Waldspurger demonstrated [Wal85, Section III] that for $f_1 \in \pi$ and $f_2 \in \tilde{\pi}$, we have $\alpha_v(f_{1,v}, f_{2,v}) = 1$ for all but finitely many v and α_v generates $\mathcal{P}(\pi_v, \chi_v) \otimes \mathcal{P}(\tilde{\pi}_v, \chi_v^{-1})$ when the space does not vanish [Wal85, III.2, Lemme 10].

As we will explain in Section 1.2, we are primarily interested in a so-called *unrefined* GGP-type results. For this reason, the constant appearing before the integral in Equation (1.1.3) is not particularly significant for our purposes (but this constant is relevant for a so-called *refined* GGP-type results). Nonetheless, this constant is present solely by convention and is defined in such a way that, when the integral does not vanish, $\alpha_v(f_1 \otimes f_2) = 1$ in the following spherical situation:

- $B_v := M_2(F_v)$;
- E_v is an unramified extension of F_v ;
- π_v and χ_v are both unramified;
- dt_v is normalized so that $\text{vol}(\mathcal{O}_{E_v}^\times/\mathcal{O}_{F_v}^\times) = 1$; and
- $f_1 \in \pi_v^{\text{GL}_2(\mathcal{O}_F)}$ and $f_2 \in \tilde{\pi}_v^{\text{GL}_2(\mathcal{O}_F)}$ satisfy $(f_1, f_2)_v = 1$.

It is worth noting that any source on the GGP conjectures is littered with local results of the type above. We are mainly interested in the global GGP conjectures, but it is essentially impossible to discuss the global conjectures without the local conjectures interjecting in some fashion.

Waldspurger constructs a so called *toric period integral* $P_\chi : \pi \rightarrow \mathbb{C}$ by the formula

$$P_\chi(f) := \int_{E^\times \backslash \mathbb{A}_E^\times / \mathbb{A}_F^\times} f(t) \chi(t) dt,$$

where dt is a suitably normalized Haar measure; $P_{\chi^{-1}} : \tilde{\pi} \rightarrow \mathbb{C}$ is defined analogously. We denote by $L(s, \pi, \chi)$ the L -function

$$L(s, \pi, \chi) := L(s, \text{BC}(\sigma) \otimes \chi),$$

where σ is the Jacquet-Langlands lifting of π and $\text{BC}(\sigma)$ is the base change of π to E . If $f_1 \in \pi$ and $f_2 \in \tilde{\pi}$, then Waldspurger's formula may be stated as

$$P_\chi(f_1) \cdot P_{\chi^{-1}}(f_2) = \frac{\zeta_F(2)L(\frac{1}{2}, \pi, \chi)}{8L(1, \eta)^2 L(1, \pi, \text{ad})} \alpha(f_1 \otimes f_2), \quad (1.1.4)$$

where $\alpha : \pi \times \tilde{\pi} \rightarrow \mathbb{C}$ is the global bilinear form defined by $\alpha := \otimes \alpha_v$. We also note that Waldspurger normalized the functional equation of $L(s, \pi, \chi)$ to be symmetric about $s = \frac{1}{2}$, so $L(\frac{1}{2}, \pi, \chi)$ is the critical value.

The quantity

$$\frac{\zeta_F(2)L(\frac{1}{2}, \pi, \chi)}{8L(1, \eta)^2 L(1, \pi, \text{ad})}$$

is simply some nonzero constant, which makes $P_\chi(f_1) \cdot P_{\chi^{-1}}(f_2)$, $L(\frac{1}{2}, \pi, \chi)$, and $\alpha(f_1 \otimes f_2)$ the most interesting components of Waldspurger's formula because they control the vanishing. However, one can say more beyond

$$P_\chi(f_1) \cdot P_{\chi^{-1}}(f_2) \neq 0 \iff L\left(\frac{1}{2}, \pi, \chi\right) \neq 0 \text{ \underline{and} } \alpha(f_1 \otimes f_2) \neq 0,$$

by relating these quantities to linear forms in much the same way we did in the local situation earlier. In particular, if we define $\mathcal{P}(\pi, \chi) := \text{Hom}_{\mathbb{A}_E^\times}(\pi \otimes \chi, \mathbb{C})$, then Equation (1.1.4) relates the bilinear forms α and $P_\chi(-) \otimes P_{\chi^{-1}}(-)$, which are both readily verified to be elements of $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1})$. If $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1}) \neq 0$, then α is a generator but it is possible that $P_\chi(-) \otimes P_{\chi^{-1}}(-)$ is identically zero. When $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1}) \neq 0$, Waldspurger's

formula asserts $P_\chi(-) \otimes P_{\chi^{-1}}(-)$ is not identically zero precisely when $L\left(\frac{1}{2}, \pi, \chi\right) \neq 0$. More succinctly, if $\mathcal{P}(\pi, \chi) \otimes \mathcal{P}(\tilde{\pi}, \chi^{-1}) \neq 0$, then Waldspurger's formula may be viewed as an expression relating one form, $P_\chi(-) \otimes P_{\chi^{-1}}(-)$, as a scalar multiple (that may be zero) of a generator, α , of a one-dimensional space. Relating linear forms via L -functions in this way is the crux of the (non-arithmetic) GGP conjectures.

1.2 Gan-Gross-Prasad Conjectures: MSRI 2001 and Beyond

Following Waldspurger's talk at Durham in July of 1983, Gross sought to create a framework that would accommodate both his and Zagier's formula and Waldspurger's formula simultaneously. The biggest commonality between these two results is the use of critical values of the Rankin L -function.¹ However, the two results are basically treating separate cases: Gross and Zagier were studying when the sign of the functional equation was -1 (and so the order of vanishing is odd) whereas Waldspurger was studying when the sign of the functional equation was $+1$ (and so the order of vanishing was even).

In order to create this framework, Gross ostensibly worked backwards by starting with the L -function [Gro04, Section 3]. This led to a reformulation in terms adelic quaternion algebras that produced a refinement of Waldspurger's Formula [Gro04, Section 16] as well as a conjecture relating the height pairing of special cycles on quaternionic Shimura curves to the critical value of the derivative of a specific L -function [Gro04, Section 24]. The latter conjecture would eventually be resolved in the broadest sense in [YZZ13], though partial results and progress had been achieved earlier [Zha01]. Notably, this result extends the original Gross-Zagier formula and does away with many hypotheses, such as the Heegner condition.

To start, we let F be a totally real number field and E a quadratic extension of F . Assume that σ is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with finite order central character ω . We consider the torus $T := E^\times$ regarded as an algebraic group over F and let χ be a

¹It is not a fundamental difference, but Gross and Zagier had normalized the L -function so that the functional equation was symmetric about $s = 1$ and Waldspurger normalized so that the functional equation was symmetric about $s = \frac{1}{2}$.

Hecke character of $T(\mathbb{A}_F) = \mathbb{A}_E^\times$ satisfying

$$\chi|_{\mathbb{A}_F^\times} \cdot \omega = 1.$$

The Rankin-Selberg L -function $L(s, \sigma, \chi)$ (i.e. the L -function of the base change of σ to E twisted by χ) has a functional equation of the form

$$L(s, \sigma, \chi) = \varepsilon(s, \sigma, \chi)L(1 - s, \sigma, \chi).$$

Let Σ denote the set of places of F , let $\eta : F^\times \setminus \mathbb{A}_F^\times \rightarrow \{\pm 1\}$ denote the quadratic character associated to the extension E/F via class field theory and define $\mathbf{S}_\sigma \subseteq \Sigma$ to be the following subset of places of F :

$$\mathbf{S}_\sigma := \{v \in \Sigma : \varepsilon(\sigma_v, \chi_v) \neq \chi_v(-1)\varepsilon(-1)\}.$$

Then $\#\mathbf{S}_\sigma < \infty$ and the global root number, $\varepsilon(\frac{1}{2}, \sigma, \chi)$, is given by

$$\varepsilon\left(\frac{1}{2}, \sigma, \chi\right) = \prod_v \varepsilon(\sigma_v, \chi_v) = (-1)^{\#\mathbf{S}_\sigma}. \quad (1.2.1)$$

Now for each $v \in \Sigma$, we define a quaternion algebra B_v over F_v that is isomorphic to the unique division algebra over F_v if $v \in \mathbf{S}_\sigma$ and is isomorphic to $M_2(F_v)$ if $v \notin \mathbf{S}_\sigma$. Then we may embed $E_v \hookrightarrow B_v$ by the aforementioned work of Tunnel [Tun83] and Saito [Sai93] and σ_v has a nonzero Jacquet-Langlands correspondence π_v on B_v^\times such that $\dim \text{Hom}_{T_v}(\pi_v, \chi_v) = 1$. We may then define a quaternion algebra \mathbb{B} over \mathbb{A}_F by

$$\mathbb{B} := \bigotimes'_v B_v;$$

\mathbb{B} is the unique (up to isomorphism) \mathbb{A}_F -algebra that is free of rank 4 as an \mathbb{A}_F -module and whose localization $\mathbb{B}_v := \mathbb{B} \otimes_{\mathbb{A}_F} F_v$ is isomorphic to B_v at each place $v \in \Sigma$. Moreover, we have a representation $\pi := \otimes \pi_v$ of \mathbb{B}^\times satisfying $\text{Hom}_{T(\mathbb{A}_F)}(\pi, \chi) \neq 0$ due to our choices in B_v .

Everything up to this point is the common ground shared by Gross-Zagier and Wald-

spuriger. However, there is a significant change in approach when we consider the parity of $\#\mathbf{S}_\sigma$, in particular because of its impact on the global root number vis-a-vis Equation (1.2.1). When $\#\mathbf{S}_\sigma$ is even, there exists a unique (up to isomorphism) quaternion algebra B over F such that $B \otimes_F \mathbb{A}_F \simeq \mathbb{B}$; such an algebra \mathbb{B} is often called ***coherent*** due to terminology initiated by S. Kudla, e.g. [KR94] and [Kud97]. In this case, π is in fact a cuspidal automorphic representation and we recover Waldspurger's formula and may present it in a slightly different manner. If we let $f_1 \in \pi$, $f_2 \in \tilde{\pi}$, and let \mathbf{S}_π denote the places where the representation $\pi \otimes \chi$ is ramified, then the following are equivalent [YZZ13, Section 1.4.2]:

- (i) $L(\frac{1}{2}, \pi, \chi) \cdot \alpha(f_1 \otimes f_2) \neq 0$;
- (ii) $P_\chi(f_1) \cdot P_{\chi^{-1}}(f_2) \neq 0$;
- (iii) $\mathbf{S}_\sigma = \mathbf{S}_\pi$.

When $\#\mathbf{S}_\sigma$ is odd, \mathbb{B} is *not* the base change of any quaternion algebra over F , that is $\mathbb{B} \not\simeq B \otimes_F \mathbb{A}_F$ for all quaternion algebras B over F . Again, following Kudla's terminology, such an \mathbb{A}_F -algebra is called ***incoherent***. In this situation, we must impose \mathbf{S} contains all archimedean places of F (cf. [Gro04, Section 14]), which implies that E is a CM extension of F . Further, since $\#\mathbf{S}_\sigma$ is odd, $\varepsilon(\frac{1}{2}, \sigma, \chi) = -1$ and so the L -function, $L(s, \pi, \chi)$, vanishes at $s = \frac{1}{2}$. Thus all of the equivalent statements above are uninteresting and so it is natural to instead consider the critical value of the L -derivative, $L'(s, \pi, \chi)$.

The incoherent quaternion algebra \mathbb{B} determines a (compactified) Shimura curve $\mathrm{Sh}_K(\mathbb{B})$ over F for each compact open subgroup $K \subseteq \mathbb{B}_{\mathrm{fin}}^\times := (\mathbb{B} \otimes_{\mathbb{A}_F} \mathbb{A}_{F,\mathrm{fin}})^\times$. To be precise, let $\iota : F \hookrightarrow \mathbb{C}$ be any archimedean place of F . Then $\mathbf{S} \setminus \{\iota\}$ is even and there exists a quaternion algebra $B(\iota)$ over F such that $B(\iota) \otimes_F F_v$ is isomorphic to the unique division algebra over F_v for each $v \in \mathbf{S} \setminus \{\iota\}$ and isomorphic to $M_2(F_v)$ at each place $v \notin \mathbf{S} \setminus \{\iota\}$; we call $B(\iota)$ the ***nearby quaternion algebra***. Then the ι -adic uniformization of $\mathrm{Sh}_K(\mathbb{B})$ is

$$\mathrm{Sh}_{K,\iota}(\mathbb{B})(\mathbb{C}) \simeq B(\iota)^\times \setminus \mathcal{H}^\pm \times \mathbb{B}_{\mathrm{fin}}^\times / K \sqcup \{\mathrm{cusps}\}. \quad (1.2.2)$$

The notation appearing in Equation (1.2.2) is defined as follows:

- $\mathrm{Sh}_{K,\iota}(\mathbb{B})(\mathbb{C})$ denotes the complex analytification of $\mathrm{Sh}_K(\mathbb{B})$ at the place ι .

- $\mathcal{H}^\pm := \mathbb{C} \setminus \mathbb{R}$ is the union of the complex upper and lower half-plane.
- The group $B(\iota)$ acts on the left of $\mathcal{H}^\pm \times \mathbb{B}_{\text{fin}}^\times$. Specifically, $B(\iota)$ acts on the first factor \mathcal{H}^\pm by the usual Möbius transformations via an isomorphism $B(\iota) \otimes_F F_v \simeq M_2(\mathbb{R})$ and acts on the second factor $\mathbb{B}_{\text{fin}}^\times$ by left-multiplication via an isomorphism $\mathbb{B}_{\text{fin}} \simeq B(\iota) \otimes_F \mathbb{A}_{F, \text{fin}}$ of $\mathbb{A}_{F, \text{fin}}$ -algebras.
- The subgroup K acts on the second factor, $\mathbb{B}_{\text{fin}}^\times$, via right multiplication.

Furthermore, the incoherent quaternion algebra \mathbb{B} canonically² determines the curve $\text{Sh}_K(\mathbb{B})$ in the sense that it is independent of the choice of infinite place $\iota \in \Sigma$. Also the CM-extension E of F appears as a subfield of $B(\iota)$ for any infinite place ι of F and is the reflex field of the Shimura curve $\text{Sh}_K(\mathbb{B})$. Varying the compact open subgroup $K \subseteq \mathbb{B}_{\text{fin}}^\times$ gives rise to a projective system of Shimura curves defined over F attached to \mathbb{B} ; we let $\text{Sh}(\mathbb{B})$ denote this system.

At a fixed level $K \subseteq \mathbb{B}_{\text{fin}}^\times$, we let J_K denote the Jacobian variety of $\text{Sh}_K(\mathbb{B})$. We note that there is a canonical isomorphism $J_K \rightarrow J_K^\vee$ and, in this way, J_K may be regarded as either the Albanese variety or as the Picard variety of $\text{Sh}_K(\mathbb{B})$ of line bundles of degree 0 by canonical duality. We note the convention in [YZZ13] of identifying J_K with the Albanese variety and J_K^\vee with the Picard variety. While this may seem like an unnecessary distinction at a fixed level $K \subseteq \mathbb{B}_{\text{fin}}^\times$, [YZZ13, Section 3.1.2] warns that this “makes an essential difference” when varying K . Indeed, varying the level $K \subseteq \mathbb{B}_{\text{fin}}^\times$ is the source of this pedantry and is the starting point for a whole host of painstakingly crafted conventions so that the main result of [YZZ13] is compatible with the various underlying projective and direct systems of algebro-geometric objects so as to be valid at the infinite level.

Starting at the finite level with a pair of compact open subgroups $K, K' \subseteq \mathbb{B}_{\text{fin}}^\times$ such that $K' \subseteq K$, we have induced algebraic homomorphisms $(\pi_{K', K})_* : J_{K'} \rightarrow J_K$ given by the push-forward of divisors and $(\pi_{K', K})^* : J_K^\vee \rightarrow J_{K'}^\vee$ given by the pull-back of line bundles. This gives rise to a projective system $J := \{J_K\}_K$ and a direct system $J^\vee := \{J_K^\vee\}_K$ both indexed by compact open subgroups $K \subseteq \mathbb{B}_{\text{fin}}^\times$. Further, for a ring $R \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ and any

²We will discuss how an incoherent Hermitian space canonically determines a unitary Shimura variety in Section 4.2. The same logic applies here.

field extension F' of F , we define

$$\begin{aligned} J(F')_R &:= \varprojlim_K J_K(F') \otimes_{\mathbb{Z}} R \\ J^\vee(F')_R &:= \varinjlim_K J_K^\vee(F') \otimes_{\mathbb{Z}} R. \end{aligned}$$

We also have a direct system $\{\mathrm{Hom}(J_K, J_K^\vee)\}_K$, which in turn allows us to define

$$\mathrm{Hom}(J, J^\vee) := \varinjlim_K \mathrm{Hom}(J_K, J_K^\vee).$$

In the same spirit, we may define

$$\mathrm{Hom}^0(J, J^\vee) := \varinjlim_K \mathrm{Hom}(J_K, J_K^\vee) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which is the group of homomorphisms up to isogeny. We also define $\mathrm{Hom}^0(J, J^\vee)_{\mathbb{C}}$ to be the base change of $\mathrm{Hom}^0(J, J^\vee)$ to \mathbb{C} .

The extension E/F gives rise to a distinguished point on $\mathrm{Sh}_K(\mathbb{B})$ that plays a similar role to the cusp ∞ in the original Gross-Zagier formula. Specifically, \mathbb{A}_E^\times acts on $\mathrm{Sh}(\mathbb{B})$ by right multiplication; we let $\mathrm{Sh}^{E^\times}(\mathbb{B})$ denote the subscheme (which is also defined over F) of $\mathrm{Sh}(\mathbb{B})$ fixed by the action of E^\times viewed as a diagonally embedded subgroup of \mathbb{A}_E^\times . We fix a point $P \in \mathrm{Sh}^{E^\times}(\mathbb{B})(E^{\mathrm{ab}})$, which induces a point $P_K \in \mathrm{Sh}_K(\mathbb{B})(E^{\mathrm{ab}})$ for each compact open subgroup $K \subseteq \mathbb{B}_{\mathrm{fin}}^\times$. As explained in [YZZ13, Section 3.1.2], we may normalize in such a way that P_K is represented by the double coset $[z_0, 1]_K$ in the complex uniformization

$$\mathrm{Sh}_{K,\iota}(\mathbb{B})(\mathbb{C}) \simeq B(\iota)^\times \setminus \mathcal{H}^\pm \times \mathbb{B}_{\mathrm{fin}}^\times / K \sqcup \{\mathrm{cusps}\},$$

where z_0 is the unique point in \mathcal{H} fixed by E^\times via the action induced by the embedding $E \hookrightarrow B(\iota)$.

We may then define the χ -eigencomponent of P by

$$P_\chi := \frac{1}{\mathrm{vol}(T(F) \setminus T(\mathbb{A}_F)/Z(\mathbb{A}_F))} \int_{T(F) \setminus T(\mathbb{A}_F)/Z(\mathbb{A}_F)} \mathrm{T}_t(P - \xi_P) \chi(t) dt \in J(\overline{F})_{\mathbb{C}}.$$

In the expression above, T_t is the Hecke operator defined by “right multiplication by t ,” ξ_P is a certain bundle called the *Hodge bundle* normalized by P in some sense [YZZ13, Section 3.1.6], and dt is a suitably normalized measure. Further, the Hecke operator T_t is acting as a push-forward on the divisor $P - \xi_P$.

As described in [YZZ13, Section 3.3.1] there is a Hecke correspondence

$$T : \pi \otimes \tilde{\pi} \longrightarrow \mathrm{Hom}^0(J, J^\vee)_{\mathbb{C}}.$$

The precise definition of this action is not important our considerations at this moment but, for the purposes of illustrating the broader themes of the arithmetic GGP conjectures, we wish to emphasize this is a *Hecke action* arising from the representation π . For an automorphic form $f_1 \otimes f_2 \in \pi \otimes \tilde{\pi}$,

$$T(f_1 \otimes f_2) \in \mathrm{Hom}^0(J, J^\vee)_{\mathbb{C}} \subseteq \mathrm{Hom}^0(J(\overline{F})_{\mathbb{C}}, J^\vee(\overline{F})_{\mathbb{C}}),$$

and so $T(f_1 \otimes f_2)P_\chi \in J^\vee(\overline{F})_{\mathbb{C}}$. We may then consider $\langle T(f_1 \otimes f_2)P_\chi, P_{\chi^{-1}} \rangle_{\mathrm{NT}}$, where

$$\langle -, - \rangle_{\mathrm{NT}} : J(\overline{F})_{\mathbb{C}} \times J^\vee(\overline{F})_{\mathbb{C}} \longrightarrow \mathbb{C},$$

is obtained as the limit of the usual Neron-Tate height [YZZ13, Section 3.1.6]. Then the main theorem³ of [YZZ13] may be stated as follows:

Theorem 1.2.1 ([YZZ13], 3.15). For any $f_1 \otimes f_2 \in \pi \otimes \tilde{\pi}$, we have

$$\langle T(f_1 \otimes f_2)P_\chi, P_{\chi^{-1}} \rangle_{\mathrm{NT}} = \frac{\zeta_F(2)L'(\frac{1}{2}, \pi, \chi)}{4L(1, \eta)^2 L(1, \pi, \mathrm{ad})} \alpha(f_1 \otimes f_2).$$

We hope the pattern is unmistakable at this point and it will not come as a shock that, in the theorem above, both $\alpha(-)$ and $\langle T(-)P_\chi, P_{\chi^{-1}} \rangle_{\mathrm{NT}}$ are linear forms in the space $\mathrm{Hom}_{T(\mathbb{A}_E)}(\pi \otimes \chi, \mathbb{C}) \otimes \mathrm{Hom}_{T(\mathbb{A}_E)}(\tilde{\pi} \otimes \chi^{-1}, \mathbb{C})$ and that this space is at most one-dimensional. Hence, provided nothing vanishes, the main theorem of [YZZ13] relates these expressions via

³The authors of [YZZ13] offer multiple, equivalent formulations of their “main theorem.” See, for example, Theorem 1.2 and Theorem 3.13 in addition to Theorem 3.15

special values of derivatives of L -functions.

Since 2001, the field of study has undergone a significant expansion, encompassing a multitude of refinements, reformulations, generalizations, and, in certain cases, notable advancements. The sweeping set of conjectures as we know them today formally appeared in Gan, Gross, and Prasad's comprehensive paper [GGP12]. It is worth mentioning that we have focused on the *global* conjectures; there is an enormous volume of research dedicated to local analogues. It is an impossible task to capture the enormity of this subject in this space, but the work of Gross-Zagier and Waldspurger distills the fundamental spirit of the arithmetic and representation theoretic GGP conjectures, respectively, and captures their core ideas in their simplest cases.

We conclude this section with a coarse statement of the representation theoretic GGP conjecture for unitary groups. We restrict to this case for the sake of simplicity as well as its relevance to this paper; we will address the arithmetic GGP conjecture for unitary groups in the section that follows. Even though we only discuss unitary groups, many of the statements transfer to orthogonal groups in an expected manner.

With that said, let F be a totally real number field, E a CM-extension of F , and let $\tau : E \rightarrow E$ denote the nontrivial involution in $\text{Gal}(E/F)$. Let $(V, \langle -, - \rangle)$ be an Hermitian space over E of dimension $n \geq 2$. Let $W \subseteq V$ be a nondegenerate subspace of V with $V = W \oplus W^\perp$ such that $\dim_E W^\perp = 1$. We define groups $G_W := \text{U}(W)$, $G_V := \text{U}(V)$, and $G := G_V \times G_W$. The inclusion $W \hookrightarrow V$ induces an embedding $G_W \hookrightarrow G_V$ with G_W acting trivially on W^\perp . This gives rise to a diagonal embedding of $G_W \hookrightarrow G$; we let $H := \Delta G_W$ denote the image of G_W in G under the embedding Δ . All of the groups in question are defined over F .

If we let $\mathcal{A}(G)$ denote the space of cuspidal automorphic forms of $G(\mathbb{A}_F)$, then there is a natural $H(\mathbb{A}_F)$ -invariant linear functional $\mathcal{P}_H : \mathcal{A}(G) \rightarrow \mathbb{C}$ defined by

$$\mathcal{P}_H(f) := \int_{H(F) \backslash H(\mathbb{A}_F)} f(h) dh,$$

where dh is a suitably normalized Haar measure on $H(F) \backslash H(\mathbb{A}_F)$. If $\pi := \pi_1 \boxtimes \pi_2 \subseteq \mathcal{A}(G)$ is a cuspidal representation of $G(\mathbb{A}_F)$, then the restriction of \mathcal{P}_H to π defines an element of

$\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$. Then the GGP conjecture asserts the following are equivalent:

- (i) \mathcal{P}_H does not vanish on π , i.e. there exists some $\phi \in \pi$ such that $\mathcal{P}_H(\phi) \neq 0$;
- (ii) $\text{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$ for all places v of F and $L(\frac{1}{2}, \pi) \neq 0$, where $L(s, \pi)$ is a Rankin-Selberg L -function attached to $\pi = \pi_1 \boxtimes \pi_2$.

1.3 The Arithmetic GGP Conjecture for $\text{U}(n) \times \text{U}(n+1)$

As we have alluded to in the earlier sections, the arithmetic GGP conjectures predict an equality relating the critical value of the derivative of some L -function and the highly conjectural Beilinson-Bloch pairing on the space of cohomologically trivial cycles much in the same way that the Gross-Zagier formula relates the Neron-Tate height and critical value of the derivative of an L -function attached to a modular form. In order to proceed, we assume the following hypothesis about the Beilinson-Bloch pairing:

Hypothesis 1.3.1. For any smooth projective variety S over a number field F , we assume the following hypotheses of Beilinson and Bloch:

- (BB1) The Beilinson and Bloch pairing $\langle -, - \rangle_{\text{BB}} : \text{Ch}^i(S)_0 \times \text{Ch}_{i-1}(S)_0 \rightarrow \mathbb{C}$ is well-defined
- (BB2) If δ is a correspondence on S and δ_* acts trivially on $\text{H}^{2r-1}(S)$, then it acts trivially on $\text{Ch}^r(S)_0$.

We will state the arithmetic GGP conjecture for $\text{U}(n) \times \text{U}(n+1)$ when $F = \mathbb{Q}$, though the statement in its full generality (which may be found in Section 1.1 of [Xue19]) requires minimal effort beyond what we present here. To this end, let E be an imaginary quadratic extension of \mathbb{Q} and let $W \subseteq V$ be Hermitian spaces of dimensions n and $n+1$ and of signature $(n-1, 1)$ and $(n, 1)$ at the single archimedean place, respectively. Unlike the representation theoretic GGP conjectures, the imposition of these hypotheses on the signatures is necessary because the arithmetic theory is far less developed and, at the present time, it is not even known what the conjecture *should* be.

We let $G_W := \text{U}(W)$, $G_V := \text{U}(V)$, and $G := \text{U}(W) \times \text{U}(V)$, which are all reductive algebraic groups over \mathbb{Q} . Let $K \subseteq G_W(\mathbb{A}_{\text{fin}})$ and $K'' \subseteq G_V(\mathbb{A}_{\text{fin}})$ be compact open subgroups

and denote by K' the compact open subgroup $K' := K \times K''$ of $G(\mathbb{A}_{\text{fin}})$. As we explain in Section 4.2 and Section 4.3, we may attach compactified Shimura varieties Y_K , $X_{K''}$, and $M_{K'}$ of dimensions $n-1$, n , and $2n-1$ attached to the groups G_W , G_V , and G , respectively. We obtain projective systems of Shimura varieties $\{Y_K\}_K$, $\{X_{K''}\}_{K''}$ and $\{M_{K'}\}_{K'}$ as K and K'' vary, which we denote by Y , X , and M , respectively.

For fixed levels $K \subseteq G_W(\mathbb{A}_{\text{fin}})$ and $K'' \subseteq G_V(\mathbb{A}_{\text{fin}})$, the inclusion of $W \hookrightarrow V$ induces an embedding of $G_W \hookrightarrow G_V$, which induces a diagonal embedding $G_W \hookrightarrow G$. The embedding $G_W \hookrightarrow G$ then induces an embedding $Y_K \hookrightarrow M_{K'}$ of Shimura varieties; the image of Y_K in $M_{K'}$ under this embedding defines an element of $\text{Ch}^n(M_{K'})$, which we denote by y_K . We let $\text{cl} : \text{Ch}^n(M_{K'}) \rightarrow \text{H}^{2n}(M_{K'})$ denote the cycle class map and set $\text{Ch}^n(M_{K'})_0 := \ker \text{cl}$, which is the space of cohomologically trivial cycles. It is expected that there exists a Hecke equivariant projection $\text{Ch}^n(M_{K'}) \rightarrow \text{Ch}^n(M_{K'})_0$. This will be true for the case of $n=2$ (see Section 5), i.e. $M_{K'}$ is a product of a curve and a surface, but we henceforth assume such a projection exists for all n in order to state the conjecture in its full generality. We let $y_{K,0}$ denote the image of y_K in $\text{Ch}^n(M_{K'})_0$.

If we set \mathcal{A} to be the set of irreducible admissible representations of $G(\mathbb{A}_{\text{fin}})$ which appear in $\text{H}^{2n-1}(M_{K'})$, then we have a surjective map

$$C_c^\infty(G(\mathbb{A}_{\text{fin}})) \longrightarrow \bigoplus_{\pi \in \mathcal{A}} \pi \otimes \bar{\pi}. \quad (1.3.1)$$

Fix $\pi \in \mathcal{A}$ and, for each $\varphi, \varphi' \in \pi$, let $t_{\varphi, \varphi'} \in C_c^\infty(G(\mathbb{A}_{\text{fin}}))$ be a function that maps to $\varphi \otimes \bar{\varphi}'$ via the map in Equation (1.3.1). As we explain in Section 5, $t_{\varphi, \varphi'}$ acts on $M_{K'}$ via a Hecke correspondence; let $T(t_{\varphi, \varphi'})$ denote the Hecke correspondence on $M_{K'}$ given by $t_{\varphi, \varphi'}$. The *unrefined* arithmetic GGP conjecture states the following:

Conjecture 1.3.2. Assume $\text{Hom}_{G_W(\mathbb{A}_{\text{fin}})}(\pi, \mathbb{C}) \neq 0$. Then the following are equivalent:

(i) The map

$$\begin{aligned} \pi \otimes \bar{\pi} &\longrightarrow \mathbb{C} \\ \varphi \otimes \bar{\varphi}' &\longmapsto \langle T(t_{\varphi, \varphi'})_* y_{K,0}, y_{K,0} \rangle_{\text{BB}} \end{aligned}$$

is not identically zero;

(ii) $L'(\frac{1}{2}, \pi) \neq 0$.

Remark 1.3.3. When $n = 1$, this is essentially [YZZ13] (cf. [Zha12, Remark 9]).

The setting of this paper is when $n = 2$, i.e. we are in the situation where Y_K is a unitary Shimura curve and $X_{K''}$ is a Picard modular surface. For tempered cuspidal $\pi \in \mathcal{A}$, we may write $\pi = \pi_2 \boxtimes \pi_3$ where π_2 is a tempered irreducible cuspidal automorphic representation of $G_W(\mathbb{A}_{\text{fin}})$ and π_3 is a tempered irreducible cuspidal automorphic representation of $G_V(\mathbb{A}_{\text{fin}})$. We also assume $\varphi = \varphi_2 \otimes \varphi_3$ and $\varphi' = \varphi'_2 \otimes \varphi'_3$ for some $\varphi_2, \varphi'_2 \in \pi_2$ and $\varphi_3, \varphi'_3 \in \pi_3$. Our main theorem is the following:

Theorem 5.3.1. Keeping with the notation above and assuming some working hypotheses about arithmetic theta lifting, if there are irreducible cuspidal automorphic representations σ_2 and σ_3 so that π_2 and π_3 are theta lifts of σ_2 and σ_3 , respectively, then Conjecture 1.3.2 holds.

The assumptions that π_3 and π_2 are theta lifts of σ_3 and σ_2 , respectively, are quite strong but not hopelessly restrictive. Moreover, the assumption on π_2 is considerably mild because if π_2 is not a theta lift, then we can modify π_2 by sufficiently many twists in order to produce a representation that is a theta lift. We do nevertheless rely on these assumptions quite heavily because they imply the L -function in question, $L(s, \pi)$, factors into a product of two well-understood L -functions (cf. Section 5.3). Philosophically, our situation is somewhat analogous to assuming an elliptic curve (defined over a number field) is equipped with complex multiplication rather than just an arbitrary elliptic curve because Deuring's theorem implies that the L -function attached to a CM elliptic curve factors into a product of two (simpler) L -functions attached to Hecke characters. Our result may be viewed as an edge case of the arithmetic GGP conjecture for $U(2) \times U(3)$ that will (hopefully) shed light on proving the conjecture in its full generality.

The case when $F \neq \mathbb{Q}$ implies that that the Shimura varieties in question are all projective (cf. Section 4.2) and Hang Xue proved Theorem 5.3.1 using the theory of arithmetic theta lifting in [Xue19]. In fact, Xue proved a so called *refined* version of Theorem 5.3.1 by proving

an equality of the form

$$\langle T(t_{\varphi, \varphi'})_* y_{K,0}, y_{K,0} \rangle_{\text{BB}} = (*) L' \left(\frac{1}{2}, \pi \right), \quad (1.3.2)$$

where $(*)$ is an explicit nonzero constant. Unfortunately, it is not the case that all of the Shimura varieties are projective when $F = \mathbb{Q}$. While there are canonical choices for smooth compactifications of the Shimura varieties, which are necessary for computing $\langle -, - \rangle_{\text{BB}}$, the theory is far less developed and not as well understood as the $F \neq \mathbb{Q}$ case. This makes it quite difficult, if not impossible, to replicate the arguments in [Xue19] step-by-step at the present time. We will address these challenges as they arise (e.g. Section 4.5 and Section 5.2), but the general idea is there are some incredibly challenging problems involving contributions from boundary components and its broader implications on modularity.

We sidestep these obstacles through the use of Hecke operators to annihilate (unknown) contributions from the boundary. However, our use of Hecke operators settles Theorem 5.3.1 through qualitative arguments, which prevents certain explicit calculations. In particular, our argument is not sufficient for proving a refined version of Theorem 5.3.1 along the lines of Equation (1.3.2). Nonetheless, the theorem in its present form is sufficient for many applications, e.g. Euler systems.

2 Notation and Conventions

- We denote the ring of finite adeles by \mathbb{A}_{fin} , i.e. $\mathbb{A}_{\text{fin}} := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Similarly, $\mathbb{A} := \mathbb{R} \times \mathbb{A}_{\text{fin}}$ is the ring of adeles. Moreover, For a number field F , we define $\mathbb{A}_F := \mathbb{A} \otimes_{\mathbb{Q}} F$, $\mathbb{A}_{F,\text{fin}} := \mathbb{A}_{\text{fin}} \otimes_{\mathbb{Q}} F$, and $F_{\infty} := \mathbb{R} \otimes_{\mathbb{Q}} K \simeq \prod_{v|\infty} F_v$.
- If F is a number field, Σ is the set of places of F , and

$$A = \bigotimes'_{v \in \Sigma} A_v$$

is an adelic object (e.g. a group over \mathbb{A}_F , a module over \mathbb{A}_F , or an element of an adelic

group/module), then for any $\mathbf{S} \subseteq \Sigma$ we define

$$A_{\mathbf{S}} := \bigotimes'_{v \in \mathbf{S}} A_v \quad \text{and} \quad A^{\mathbf{S}} := \bigotimes'_{v \notin \mathbf{S}} A_v.$$

We will often say $A_{\mathbf{S}}$ is the object A *at* \mathbf{S} and $A^{\mathbf{S}}$ is the object A *away from* \mathbf{S} .

- For an \mathbb{A}_F -algebra \mathbb{V} , we set $\mathbb{V}_{\text{fin}} := \mathbb{V} \otimes \mathbb{A}_{F,\text{fin}}$.
- If F is a number field, A is an F -algebra or module, and v is a place of F , then we define $A_v := A \otimes_F \mathbb{F}_v$.
- For a ring R and positive integers m, n , we denote by $\text{Mat}_{m,n}(R)$ the ring of $m \times n$ matrices with entries in R . We also set $\text{Mat}_n(R) := \text{Mat}_{n,n}(R)$. We also set $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) to be the $n \times n$ identity matrix (resp. $n \times n$ zero matrix). Lastly, for $g \in \text{Mat}_{m,n}(R)$, we define ${}^t g$ to be the transpose of the matrix g .
- For a commutative ring R , a commutative étale algebra S over R of rank 2 with nontrivial R -automorphism of S $\tau : s \mapsto s^\tau$ for $s \in S$, and a positive integer n , a **Hermitian space** (resp. **skew-Hermitian space**) of rank n over S with respect to τ is a free S -module V of rank n equipped with a map

$$\langle -, - \rangle : V \times V \rightarrow S$$

such that, for all $s, t \in S$ and $x, y, z \in V$, $\langle -, - \rangle$ satisfies the following properties:

- (i) $\langle -, - \rangle$ is S -linear in the first variable, i.e. $\langle sx + ty, z \rangle = s\langle x, z \rangle + t\langle y, z \rangle$;
- (ii) $\langle -, - \rangle$ is τ -linear in the second variable, i.e. $\langle x, sy + tz \rangle = s^\tau \langle x, y \rangle + t^\tau \langle x, z \rangle$;
- (iii) $\langle -, - \rangle$ is τ -symmetric (resp. τ -antisymmetric), i.e. $\langle x, y \rangle = \langle y, x \rangle^\tau$ (resp. $\langle x, y \rangle = -\langle y, x \rangle^\tau$).

We say a Hermitian or skew-Hermitian space V over S of rank n is **nondegenerate** if there exists an S -basis $\{v_1, \dots, v_n\}$ for V such that the matrix $(\langle v_i, v_j \rangle)_{1 \leq i, j \leq n}$ has determinant in S^\times . All Hermitian and skew-Hermitian spaces will be assumed to be nondegenerate, unless explicitly stated otherwise.

- If E/F is a quadratic extension of number fields, we define a Hermitian space \mathbb{V} over \mathbb{A}_E to be a restricted tensor product

$$\mathbb{V} = \bigotimes'_{v \in \Sigma} \mathbb{V}_v,$$

where each \mathbb{V}_v is a Hermitian space over E_v . Following the conventions of coherent and incoherent quadratic spaces studied by S. Kudla in the orthogonal case in [KR94] and [Kud97], we say that \mathbb{V} is **coherent** if there exists a Hermitian space V over E such that $\mathbb{V} \simeq V \otimes_E \mathbb{A}_E$; we say that \mathbb{V} is **incoherent** if no such V exists.

- If G is an algebraic group over a global field F , then we define the **automorphic quotient** of G to be the quotient topological space

$$[G] := G(F) \backslash G(\mathbb{A}_F).$$

- For a smooth projective variety X of dimension n and defined over a number field F , we let $\text{Ch}^i(X)$ (resp. $\text{Ch}_i(X)$) denote the \mathbb{C} -vector space of codimension i (resp. dimension i) cycles on X . There is an intersection pairing between $\text{Ch}^i(X)$ and $\text{Ch}_i(X)$, which we denote by $\alpha \cdot \beta$ for $\alpha \in \text{Ch}^i(X)$ and $\beta \in \text{Ch}_i(X)$.
- For an algebraic variety X over a field F , we let $H^*(X)$ denote the (Betti) cohomology of $X(\mathbb{C})$ (for some embedding $F \hookrightarrow \mathbb{C}$). Unless stated otherwise, we assume $H^*(X)$ has coefficients in \mathbb{C} and so there is a well-defined notion of complex conjugation on these groups.

3 The Weil Representation and Theta Lifts

Let F be a totally real number field and E a totally imaginary quadratic extension of F . We let τ denote the nontrivial element in $\text{Gal}(E/F)$ and set $\eta_{E/F} : F^\times \backslash \mathbb{A}_F^\times \rightarrow \{\pm 1\}$ to be the quadratic character associated to E by class field theory. For the remainder of the paper, we will let Σ (resp. Σ_{fin} , Σ_∞) denote the set of all (resp. finite, infinite) places of F and let Σ° , $\Sigma_{\text{fin}}^\circ$, and Σ_∞° denote the corresponding places over E . Additionally, we fix a nontrivial

additive character, $\psi : F \setminus \mathbb{A}_F \rightarrow \mathbb{C}^\times$ such that $\psi_v(x) = \exp(2\pi i x)$ for each archimedean place. Finally, set $\psi_E(x) := \psi\left(\frac{1}{2} \operatorname{Tr}_{E/F} x\right)$.

Though our end goal of proving an unrefined result is somewhat insensitive to measures, we nonetheless define the following measures for the remainder of the paper following the exposition in [Xue19, Section 1.5].

For any $v \in \Sigma$, let V be a Hermitian space over E_v of dimension n . Let $\mathfrak{u}(V)$ denote the Lie algebra of $U(V)$ and let

$$\mathfrak{c}_v : \mathfrak{u}(V) \longrightarrow U(V) \quad X \longmapsto (1 + X)(1 - X)^{-1}$$

denote the Cayley transform. There is a self-dual measure of $\mathfrak{u}(V)$ and we denote by $d'h_v$ the unique measure on $U(V)$ that makes \mathfrak{c}_v measure preserving. If we suppose that $n = 2r$, then $d'h$ satisfies

$$\int_{\operatorname{Herm}_{2r}(F_v)} \left(\int_{V^{2r}} \phi(x) \psi(\operatorname{Tr} nQ(x)) dx \right) \psi(-\operatorname{Tr} nQ) dn = \gamma_V \int_{U(V)(F_v)} \phi(h_v^{-1}x_Q) d'h_v,$$

where Herm_{2r} is the space of $2r \times 2r$ Hermitian matrices, dn is the self-dual measure on Herm_{2r} , and x_Q is any fixed element in V^{2r} with $Q(x) = Q$, where $Q(x) := \frac{1}{2}(\langle x_i, x_j \rangle_v)_{1 \leq i, j \leq 2r}$ is the moment matrix of $x := {}^t(x_1 \cdots x_{2r}) \in V^{2r}$. We put

$$dh_v := L(1, \eta_v) \zeta_{F_v}(2) \cdots L(n, \eta_v^n) d'h_v$$

and call $d'h_v$ the **unnormalized local measure** and dh_v the **normalized local measure**.

In the global situation, let V an n -dimensional Hermitian space over E . Then the Tamagawa measure on $[U(V)]$ equals

$$(L(1, \eta) \cdots L(n, \eta^n))^{-1} \prod_{v \in \Sigma} dh_v.$$

In the adelic situation, for an incoherent Hermitian space \mathbb{V} of rank n over \mathbb{A}_E , we abuse

notation and also call the measure

$$(L(1, \eta)\zeta(2) \cdots L(n, \eta^n))^{-1} \prod_{v \in \Sigma} dh_v.$$

the Tamagawa measure on $U(\mathbb{V})(\mathbb{A}_F)$.

For a compact open subgroup $K \subseteq U(V)(\mathbb{A}_{F, \text{fin}})$, we denote by $\text{vol } K$ the volume of K with respect to the measure

$$(2L(1, \eta)\zeta(2) \cdots L(n, \eta^n))^{-1} \prod_{v \in \Sigma_{\text{fin}}} dh_v,$$

where dh_v is the normalized local measure at v ; we use $\text{vol}' K$ to denote the volume of K with respect to the unnormalized local measures $d'h_v$.

3.1 Symplectic Groups

Fix a positive integer r for the remainder of the paper and denote by W_r the standard skew-Hermitian space of dimension $2r$ over E with respect to the involution τ , which is endowed with a skew-Hermitian form $\langle -, - \rangle : W_r \times W_r \rightarrow E$. Moreover, there exists an E -basis $\{e_1, \dots, e_{2r}\}$ for W_r satisfying the following properties for $1 \leq i, j \leq r$:

- $\langle e_i, e_j \rangle = 0$;
- $\langle e_{r+i}, e_{r+j} \rangle = 0$;
- $\langle e_i, e_{r+j} \rangle = \delta_{ij}$.

Thus, the form $\langle -, - \rangle$ is represented by the matrix

$$\mathbf{w}_r := \begin{pmatrix} & \mathbf{1}_r \\ -\mathbf{1}_r & \end{pmatrix}$$

with respect to the basis $\{e_1, \dots, e_{2r}\}$.

We put $H_r := U(W_r)$ to be the quasi-split unitary group of W_r , which is a reductive group

over F . We also set

$$N_r(F) := \left\{ n(b) := \begin{pmatrix} \mathbf{1}_r & b \\ & \mathbf{1}_r \end{pmatrix} : b \in \text{Herm}_r(E) \right\};$$

$$M_r(F) := \left\{ m(a) := \begin{pmatrix} a & \\ & {}^t a^{\tau, -1} \end{pmatrix} : a \in \text{GL}_r(E) \right\}$$

Then the group $H_r(F)$ (resp. $H_r(F_v)$ for some place $v \in \Sigma$), is generated by the parabolic subgroup $P_r(F) := N_r(F)M_r(F)$ and the element \mathbf{w}_r (resp. $P_r(F_v) := N_r(F_v)M_r(F_v)$ and \mathbf{w}_r).

3.2 Classification of Hermitian Spaces

Fix an integer $m \geq 1$ and a place $v \in \Sigma$ of F and let E_v denote the quadratic F_v -algebra $E_v := E \otimes_F F_v$. Since F is a totally real number field and E is a totally imaginary extension of F , we have $F_v \simeq \mathbb{R}$ and $E_v \simeq \mathbb{C}$ for all $v \in \Sigma_\infty$. For $v \in \Sigma_{\text{fin}}$, it is the case that either E_v/F_v is a quadratic extension of local fields or $E_v \simeq F_v \oplus F_v$ (as an F_v -algebra). We say that E_v/F_v is **non-split** if E_v is a field and **split** if $E_v \simeq F_v \oplus F_v$; similarly, we say that E is non-split (resp. split) at $v \in \Sigma_v$ if E_v is non-split (resp. split).

For a Hermitian space $(V_v, \langle -, - \rangle_v)$ over E_v of rank m , we define

$$\eta_v(V_v) := \eta_{E_v/F_v} \left((-1)^{\frac{m(m-1)}{2}} \det V_v \right) \in \{\pm 1\}$$

In this local setting, there are three cases (modulo isometry):

- If $v \in \Sigma_{\text{fin}}$ and E is non-split at v then there are two unique Hermitian spaces V_v^\pm over E_v of dimension $m \geq 1$ determined by $\eta_v(V_v^\pm) = \pm 1$;
- If $v \in \Sigma_{\text{fin}}$ and E is split at v , then there is only one Hermitian space V_v^+ over E_v of dimension m and $\eta_v(V_v^+) = 1$;
- If $v \in \Sigma_\infty$, then there are $m + 1$ distinct Hermitian spaces E_v and are characterized by their signature. To be precise, they are the m -dimensional spaces of signature $(s, m - s)$ and $\eta_v(V_s) = (-1)^{\frac{m(m+1)}{2} - s}$, where $0 \leq s \leq m$

The last bullet point should come as no surprise from basic linear algebra, but we do wish to flesh out some of the details in the non-archimedean setting of the first two bullet points (cf. [Har11, Section 1.2] for additional details). First suppose that $E_v = F_v \oplus F_v$ is split. Then $V_v \simeq V_{v,1} \oplus V_{v,2}$, where $V_{v,1}, V_{v,2}$ are each F_v -vector spaces of rank m ; for each $x, y \in V_v$, write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, y_1 \in V_{v,1}$ and $x_2, y_2 \in V_{v,2}$. Then there exists non-degenerate bilinear pairings $h_i : V_{v,1} \times V_{v,2} \rightarrow F_v$ for $i = 1, 2$ such that the form $\langle -, - \rangle_v$ on V_v is given by

$$\langle x, y \rangle = (h_1(x_1, y_2), h_2(y_1, x_2)).$$

In this case, $U(V_v) \simeq \text{GL}(V_{v,1})$.

On the other hand, suppose that E_v/F_v is non-split. When $m = 1$, V_v can be identified with E_v and the Hermitian form $\langle -, - \rangle_v$ is given by

$$\langle e_1, e_2 \rangle_v := ae_1e_2^{\tau_v}$$

for some $a \in F_v^\times$. We denote this space by $E_v(a)$ and it is a fact that $E_v(a) \simeq E_v(b)$ if and only if $a/b \in \text{Nm}_{E_v/F_v}(E_v^\times)$. For $m = 1$, we have $V^+ \simeq E_v(a)$ for some $a \in \text{Nm}_{E_v/F_v}(E_v^\times)$ and $V^- \simeq E_v(b)$ for some $b \notin \text{Nm}_{E_v/F_v}(E_v^\times)$.

For $m = 2$, fix a basis $\{e_1, e_2\}$ for E_w^2 . Then V^+ is isometric to the **hyperbolic plane** U equipped with the form

$$\langle ae_1 + be_2, ce_1 + de_2 \rangle := a^{\tau_v}d - b^{\tau_v}c.$$

On the other hand, $V^- \simeq E_v(a) \oplus E_v(b)$ for $a, b \neq 0$ and $-a/b \notin \text{Nm}_{E_v/F_v}(E_v^\times)$; we denote such a space by $W_2(a, b)$, which is often called an **anisotropic plane**. In general, for $m = 2k$, we have

$$V^+ \simeq U^{\oplus k} \quad \text{and} \quad V^- \simeq U^{\oplus(k-1)} \oplus W_2$$

where U (resp. W_2) is the hyperbolic plane (resp. anisotropic plane) from above. For $m = 2k + 1$, then $V^\pm \simeq U^{\oplus m} \oplus W^\pm$ with $W^\pm \simeq E_v(a)$ according to whether $a \in \text{Nm}_{E_v/F_v}(E_v^\times)$ or

not.

In the global setting, all Hermitian spaces V over E of dimension m are classified by their signatures at infinite places and $\det V \in F^\times / \text{Nm } E^\times$. In particular, V is determined by the local data $V_v := V \otimes_F F_v$ for all $v \in \Sigma$.

In the adelic setting, recall that a nondegenerate Hermitian space \mathbb{V} over \mathbb{A}_E of rank m is a restricted tensor product

$$\mathbb{V} = \bigotimes'_{v \in \Sigma} \mathbb{V}_v,$$

where each \mathbb{V}_v is a Hermitian space over E_v , and there exists an \mathbb{A}_E -basis B for \mathbb{V} such that the matrix representing the Hermitian form on \mathbb{V} with respect to B is invertible in $\text{GL}_m(\mathbb{A}_E)$. For any place $v \in \Sigma$, we set the following notation:

$$\begin{aligned} \mathbb{V}_v &:= \mathbb{V} \otimes_{\mathbb{A}_F} F_v \\ \mathbb{V}_{\text{fin}} &:= \mathbb{V} \otimes_{\mathbb{A}_F} \mathbb{A}_{F, \text{fin}} \\ \Sigma(\mathbb{V}) &:= \{v \in \Sigma : \eta_v(\mathbb{V}_v) = -1\} \\ \eta(\mathbb{V}) &:= \prod_{v \in \Sigma} \eta_v(\mathbb{V}_v). \end{aligned}$$

Recall from Section 2 that a nondegenerate Hermitian space \mathbb{V} over \mathbb{A}_E is coherent if and only if there exists a Hermitian space V over E such that $\mathbb{V} \simeq V \otimes_F \mathbb{A}_F$ and \mathbb{V} is incoherent if such a V does not exist. By the Hasse Principle, the following are equivalent:

- (i) \mathbb{V} is coherent (resp. incoherent);
- (ii) $\#\Sigma(\mathbb{V})$ is even (resp. odd);
- (iii) $\eta(\mathbb{V}) = 1$ (resp. $\eta(\mathbb{V}) = -1$).

3.3 The Weil Representation and Abstract Theta Lifts

Fix a place $v \in \Sigma$ of a F and let $(V_v, \langle -, - \rangle_v)$ be a Hermitian space over E_v of dimension m with corresponding unitary group $\text{U}(V_v)$. Further, fix a character $\chi_v : E_v^\times \rightarrow \mathbb{C}^\times$ that satisfies $\chi_v|_{F_v^\times} = \eta_{E_v/F_v}^m$. Then the Weil representation ω_{χ_v} , or ω_v for short when the character χ_v is clear from context, of $H_r(F_v) \times \text{U}(V_v)$ is realized on $\mathcal{S}(V_v^r)$, the space of

Schwartz functions on V_v^r . Explicitly, for $\phi \in \mathcal{S}(V_v^r)$ and $h \in \mathrm{U}(V_v)(F_v)$, ω_{χ_v} is defined by the following formulae:

- $(\omega_{\chi_v}(n(b))\phi)(x) := \psi_v(\mathrm{Tr} bQ(x))\phi(x),$
- $(\omega_{\chi_v}m(a))\phi(x) := |\det a|_{E_v}^{\frac{m}{2}}\chi_v(\det a)\phi(xa),$
- $(\omega_{\chi_v}(\mathbf{w}_r)\phi)(x) := \gamma_{V_v}\widehat{\phi}(x),$
- $(\omega_{\chi_v}(h)\phi)(x) := \phi(h^{-1}x)$

where

- $Q(x)$ is the moment matrix of $x \in V_v^r$, i.e.

$$Q(x) := \frac{1}{2}(\langle x_i, x_j \rangle_v)_{1 \leq i, j \leq r} \quad x := \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} \in V_v^r; \quad (3.3.1)$$

- γ_{V_v} is the Weil index associated to V_v ; and
- $\widehat{\phi}$ is the Fourier transform of ϕ , which is defined by the formula

$$\int_{V_v^r} \phi(y)\psi_E(\langle x, y \rangle_v) dy,$$

using the self-dual measure dy on V_v^r with respect to ψ .

For an irreducible admissible representation σ_v of $H_r(F_v)$, we say that $\theta_{\chi_v}(\sigma_v)$ is an **abstract theta lift** of σ_v to $\mathrm{U}(V_v)(F_v)$ if $\theta_{\chi_v}(\sigma_v)$ is an irreducible admissible representation of $\mathrm{U}(V_v)(F_v)$ such that $\sigma_v \boxtimes \theta_{\chi_v}(\sigma_v)$ is the maximal semisimple quotient of the σ_v -isotypic part of ω_{χ_v} ; abstract theta lifts of $\mathrm{U}(V_v)(F_v)$ to $H_r(F_v)$ are defined analogously. Howe [How89] proved the existence theta lifts in the archimedean case and Waldspurger [Wal90] handled the non-archimedean case (with odd residue characteristic). More recently, Gan and Takeda [GT16] proved the existence of theta lifts in full generality.

In the adelic setting,

$$\mathbb{V} := \bigotimes'_{v \in \Sigma} V_v$$

is a Hermitian module over \mathbb{A}_E and the Weil representation is defined by taking restricted tensor products. That is, if $\chi : E^\times \setminus \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ is the character defined by $\chi := \otimes \chi_v$, then the global Weil representation ω_χ of $H_r(\mathbb{A}_F) \times \mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ is defined on the space

$$\mathcal{S}(\mathbb{V}^r) := \bigotimes'_{v \in \Sigma} \mathcal{S}(V_v^r).$$

Abstract theta lifts in the adelic setting are defined by restricted tensor products as well. More precisely, if $\sigma := \otimes \sigma_v$ is an irreducible admissible representation of $H_r(\mathbb{A}_F)$, then an abstract theta lift of σ to $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ is defined to be an irreducible admissible representation $\pi := \otimes \pi_v$ of $\mathrm{U}(\mathbb{V})(\mathbb{A}_F)$ such that $\pi_v \simeq \theta_{\chi_v}(\sigma_v)$ for all $v \in \Sigma$. Lastly, assume \mathbb{V} is coherent, i.e. $\mathbb{V} = V \otimes_E \mathbb{A}_E$. Then for each $\phi \in \mathcal{S}(\mathbb{V}^r)$, we define a theta series by the formula

$$\theta(g, h; \phi) := \sum_{x \in V(E)^r} (\omega_\chi(g, h)\phi)(x), \quad g \in H_r(\mathbb{A}_F) \text{ and } h \in H(\mathbb{A}_F),$$

which is a smooth, slowly increasing function on $H_r(F) \setminus H_r(\mathbb{A}_F) \times H(F) \setminus H(\mathbb{A}_F)$.

4 Unitary Shimura Varieties

This section lays out all of the key ingredients for our main result. We begin with some common facts about arbitrary Shimura varieties before proceeding to a general construction of unitary Shimura varieties in Section 4.2 followed by their smooth toroidal compactifications in Section 4.3. We discuss a broad class of cycles on unitary Shimura varieties in Section 4.4 and introduce the vitally important generating series of special cycles in Section 4.5. The generating series is a central object in the so-called *Kudla program*. Initiated by Kudla in [Kud97] and developed further in [Kud04], the Kudla program is a broad collection of ideas and conjectures with considerable overlap with the arithmetic GGP conjectures. In particular, the Kudla program relates special cycles on orthogonal and unitary Shimura varieties to automorphic forms. For our purposes, the generating series acts as an integral

kernel function of sorts; we define an *arithmetic theta lift* in Section 5.2, which takes automorphic forms to cycles in the Chow group by integrating against the generating series. Arithmetic theta lifts are an indispensable tool in this area and they play a critical role in our argument.

4.1 Basics of Shimura Varieties

Though a rigorous presentation of the general theory of Shimura varieties is not completely necessary for our interests, we nonetheless present some basic definitions and facts here. We follow the model provided by J. Milne in [Mil05], which is based on P. Deligne's original axiomatic treatment of the subject in 1970s.

To this end, we take note of a few notations and definitions for algebraic groups. First, for our purposes, an ***algebraic group*** G over a field k is a group scheme of finite type over k .⁴ For any $g \in G(k)$, g acts on G by the inner automorphism $\text{ad}(g)(x) := gxg^{-1}$ and hence on $\text{Lie}(G)$ by an automorphism $\text{Ad}(g)$. We define the ***adjoint group*** of an algebraic group G to be the quotient of G by its center and denote it by G^{ad} . Further, we denote by G^+ (resp. G°) a connected component of the identity relative to a real topology (resp. Zariski topology). Finally, we denote by \mathbb{S} the ***Deligne torus***, which is defined to be the algebraic torus over \mathbb{R} obtained from \mathbb{G}_m over \mathbb{C} by restriction of scalars. Thus for any \mathbb{R} -algebra A , we have $\mathbb{S}(A) = \mathbb{G}_m(A \otimes_{\mathbb{R}} \mathbb{C})$; in particular, $\mathbb{S}(\mathbb{R}) \simeq \mathbb{C}^\times$, $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m$

Definition 4.1.1. A ***Shimura datum*** is a pair (G, X) consisting of a reductive algebraic group G over \mathbb{Q} and a $G(\mathbb{R})$ -conjugacy class X of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying the following conditions:

- (i) for all $h \in X$, the Hodge structure on $\text{Lie}(G_{\mathbb{R}})$ defined by $\text{Ad} \circ h$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\};$$

- (ii) for all $h \in X$, $\text{ad}(h(i))$ is a Cartan involution of $G_{\mathbb{R}}^{\text{ad}}$;

⁴We acknowledge that this definition is more general than the typical definition of an algebraic group. However, for our purposes, k is always a field of characteristic 0 (and often a number field).

(iii) G^{ad} has no \mathbb{Q} -factor on which the projection of h is trivial.

Once given a Shimura datum (G, X) a compact open subgroup K of $G(\mathbb{A}_{\text{fin}})$, we define a *Shimura variety of level K* relative to (G, X) to be a variety of the form

$$\text{Sh}_K(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\text{fin}})/K,$$

where $G(\mathbb{Q})$ acts on both X and $G(\mathbb{A}_{\text{fin}})$ (induced by the diagonal embedding) on the left and K acts on $G(\mathbb{A}_{\text{fin}})$ on the right. Specifically, for any $q \in G(\mathbb{Q})$, $x \in X$, $a \in G(\mathbb{A}_{\text{fin}})$, and $k \in K$, we have

$$q(x, a) := (qx, qa) \quad \text{and} \quad (x, a)k := (x, ak).$$

We will use the notation $[x, a]_K$ to denote the equivalence class of $(x, a) \in X \times G(\mathbb{A}_{\text{fin}})$ in $\text{Sh}_K(G, X)$.

We outline why $\text{Sh}_K(G, X)$ is indeed a variety as this may not be immediately clear from the definition above. The basic idea is that we can decompose the double coset space $\text{Sh}_K(G, X)$ into a finite disjoint union of right cosets that are known to be varieties. Working towards this goal, for any reductive group G over \mathbb{Q} we define $G(\mathbb{R})_+$ to be the group of elements of $G(\mathbb{R})$ whose image in $G^{\text{ad}}(\mathbb{R})$ lies in its identity component $G^{\text{ad}}(\mathbb{R})^+$ and we further define $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$.

Lemma 4.1.2 ([Mil05], 5.12 and 5.13). Let K be a compact open subgroup of $G(\mathbb{A}_{\text{fin}})$.

- (i) The set $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_{\text{fin}})/K$ is finite.
- (ii) If X^+ is a connected component of X , then

$$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\text{fin}})/K \simeq \coprod_{g \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_{\text{fin}})/K} \Gamma_g \backslash X^+,$$

where $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$. When we endow X with its usual topology and $G(\mathbb{A}_{\text{fin}})$ with its adelic topology, this becomes a homeomorphism.

It can be shown that the subgroups Γ_g are arithmetic and torsion-free. Further, since X is a Hermitian symmetric domain, the quotient $\Gamma_g \backslash X^+$ has a canonical realization as a

Zariski-open subset of a projective algebraic variety by a theorem of Baily and Borel [BB66]. In particular, each $\Gamma_g \backslash X^+$ has a canonical structure of an algebraic variety. In general, an algebraic variety arising as the quotient of a Hermitian symmetric domain by a torsion-free arithmetic subgroup is called a ***locally symmetric variety***. Thus we are justified in calling $\text{Sh}_K(G, X)$ a variety as Lemma 4.1.2 shows that it is a finite disjoint union of locally symmetric varieties.

For an inclusion of sufficiently small compact open subgroups $K' \subseteq K$ of $G(\mathbb{A}_{\text{fin}})$, the natural map $\pi_{K',K} : \text{Sh}_{K'}(G, X) \rightarrow \text{Sh}_K(G, X)$ is regular. Therefore, if we vary (sufficiently small) K , we obtain an inverse system of algebraic varieties $\{\text{Sh}_K(G, X)\}_K$. The system $\{\text{Sh}_K(G, X)\}_K$ has an action of $G(\mathbb{A}_{\text{fin}})$ and is defined by “right multiplication.” To be specific, for each $g \in G(\mathbb{A}_{\text{fin}})$, $K \mapsto g^{-1}Kg$ maps compact open subgroups to compact open subgroups, and $T_g : \text{Sh}_{gKg^{-1}}(G, X) \rightarrow \text{Sh}_K(G, X)$ acts on points as

$$T_g : G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\text{fin}})/gKg^{-1} \longrightarrow G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\text{fin}})/K, \quad [x, a]_{gKg^{-1}} \longmapsto [x, ag]_K.$$

4.2 The General Set-Up

Let F be a totally real field and let E be a CM-extension of F and fix integers r and m that satisfy $m \geq 2$ and $1 \leq r < m$. Let $(\mathbb{V}, \langle -, - \rangle)$ be an m -dimensional incoherent Hermitian space over \mathbb{A}_E that is totally positive at all archimedean places. We define $\mathbb{G} := \text{Res}_{\mathbb{A}_F/\mathbb{A}} \text{U}(\mathbb{V})$, which is a reductive group over \mathbb{A} with $\mathbb{G}_{\text{der}} = \text{Res}_{\mathbb{A}_F/\mathbb{A}} \text{SU}(\mathbb{V})$. For each embedding $\sigma^\circ : E \hookrightarrow \mathbb{C}$ over $\sigma \in \Sigma_\infty$, we will let $(V^{(\sigma)}, \langle -, - \rangle_\sigma)$ denote the ***nearby Hermitian space*** of \mathbb{V} at σ , which is defined to be the unique (up to isometry) Hermitian space over E such that $V_v^{(\sigma)} \simeq \mathbb{V}_v$ for all places $v \neq \sigma$ but $V_\sigma^{(\sigma)}$ is of signature $(m-1, 1)$. If we set $G^{(\sigma)} := \text{Res}_{F/\mathbb{Q}} \text{U}(V^{(\sigma)})$, then $G^{(\sigma)}$ is reductive over \mathbb{Q} and $G^{(\sigma)}(\mathbb{A}_{\text{fin}}) \simeq \mathbb{G}(\mathbb{A}_{\text{fin}})$. For each (sufficiently small) compact open subgroup K of $\mathbb{G}(\mathbb{A}_{\text{fin}})$, there is a Shimura variety X_K of dimension $m-1$ that is defined over the reflex field E . Further, the embedding $\sigma^\circ : E \hookrightarrow \mathbb{C}$ induces the following σ° -adic uniformization

$$X_{K, \sigma^\circ}(\mathbb{C}) \simeq G^{(\sigma)}(\mathbb{Q}) \backslash \mathcal{D}^{(\sigma^\circ)} \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K. \quad (4.2.1)$$

We explain each piece of notation appearing in eq. (4.2.1) below:

- $X_{K,\sigma^\circ}(\mathbb{C})$ denotes the complex analytification of X_K at the place σ° .
- $\mathcal{D}^{(\sigma^\circ)}$ is the Hermitian symmetric domain of all negative \mathbb{C} -lines in $V_\sigma^{(\sigma)}$. To be precise, $V_\sigma^{(\sigma)}$ has the structure of a \mathbb{C} -vector space defined by the action of $F_\sigma \otimes_F E \simeq \mathbb{R} \otimes_F E$, which is identified with \mathbb{C} via σ . If we abuse notation ever so slightly and denote the Hermitian form on $V_\sigma^{(\sigma)}$ by $\langle -, - \rangle_\sigma$, then

$$\mathcal{D}^{(\sigma^\circ)} := \left\{ x \in V_\sigma^{(\sigma)} : q_\sigma(x) < 0 \right\} / \mathbb{C}^\times ,$$

where $q_\sigma(x) := \langle x, x \rangle_\sigma$.

- The group $G^{(\sigma)}(\mathbb{Q})$ acts on $\mathcal{D}^{(\sigma^\circ)}$ via conjugation and on $\mathbb{G}(\mathbb{A}_{\text{fin}})$ by left multiplication via the identification $G^{(\sigma)}(\mathbb{A}_{\text{fin}}) \simeq \mathbb{G}(\mathbb{A}_{\text{fin}})$.
- The subgroup K acts on the second factor, $\mathbb{G}(\mathbb{A}_{\text{fin}})$, via right multiplication.

It is worth pointing out that the data of the incoherent Hermitian space \mathbb{V} over \mathbb{A}_E and the nearby Hermitian $V^{(\sigma)}$ over E are equivalent in the sense that they produce the same Shimura variety X_K . Therefore, rather than using an incoherent Hermitian space over \mathbb{A}_E that is totally positive at each archimedean place to construct X_K , we could have just as easily started with a **standard indefinite** Hermitian space V over E , which is a Hermitian space over E together with a distinguished embedding $\sigma^{(\circ)} : E \hookrightarrow \mathbb{C}$ such that V is of signature $(m-1, 1)$ at the real place lying below $\sigma^{(\circ)}$ and of signature $(m, 0)$ at all other real places. Starting with the incoherent Hermitian space \mathbb{V} over \mathbb{A}_E is preferred because the theory of conjugation on Shimura varieties allows for X_K to be intrinsically defined over E without viewing it as a subfield of \mathbb{C} and gives rise to a system of Shimura varieties $\{X_K\}$ that is canonically defined over E . Starting with a standard indefinite Hermitian space over E gives explicit descriptions of the \mathbb{C} -points of X_K and is particularly useful for explicit computations as well studying geometric invariants of X_K whereas starting with an incoherent Hermitian space over \mathbb{A}_E is better served for studying the arithmetic invariants of X_K .

We also emphasize the importance of the condition that $V^{(\sigma)}$ is standard indefinite or, equivalently, the condition that \mathbb{V} is incoherent. As a primer of what is to come in Section 4.4, this condition provides a straightforward way to construct cycles of codimension 1 on X_K . Namely, let W be the orthogonal complement of a vector $w \in V^{(\sigma)}$ such that $q_\sigma(w) > 0$. Then $(W, \langle -, - \rangle_\sigma|_{W \times W})$ is a Hermitian space over E of signature $(m - 2, 1)$ and therefore yields a Shimura variety Y_{K_w} of dimension $m - 2$ and of level $K_w := K \cap \mathrm{U}(W \otimes_F \mathbb{A}_{F, \mathrm{fin}})$. Moreover the inclusion $W \subseteq V^{(\sigma)}$ induces an embedding $Y_{K_w} \hookrightarrow X_K$, whose image defines an element of $\mathrm{Ch}^1(X_K)$.

For the remainder of the paper, we will assume that K is sufficiently small, i.e. K is contained in some principle congruence subgroup for $N \geq 3$ under the natural embedding $\mathbb{G}(\mathbb{A}_{\mathrm{fin}}) \hookrightarrow \mathrm{GL}(\mathbb{V})$. It is noteworthy that X_K is a Shimura variety of abelian type, rather than a PEL type Shimura variety attached to the group of unitary similitudes. Being of abelian type makes X_K somewhat more complicated than its PEL counterpart in the sense that it admits a moduli description of abelian motives (equipped with additional structure), as opposed to simpler object of abelian varieties as in the case of PEL Shimura varieties.

While we will adopt the geometric description of $\mathcal{D}^{(\sigma^\circ)}$ from above for the remainder of this paper, it is worth noting its group theoretic description as outlined in Definition 4.1.1. In order to be explicit, let d denote the number of embeddings of F into \mathbb{C} whose image is contained in \mathbb{R} . It then follows that $G_{\mathbb{R}}^{(\sigma)} \simeq \mathrm{U}(m - 1, 1)_{\mathbb{R}} \times \mathrm{U}(m, 0)_{\mathbb{R}}^{d-1}$ and so may identify $\mathcal{D}^{(\sigma^\circ)}$ with the $G^{(\sigma)}(\mathbb{R})$ -conjugacy class of the Hodge map $h^{(\sigma)} : \mathbb{S} \rightarrow G_{\mathbb{R}}^{(\sigma)}$ defined by

$$h^{(\sigma)}(z) := \left(\left(\begin{array}{c} \mathbf{1}_{m-1} \\ \bar{z}/z \end{array} \right), \mathbf{1}_m, \dots, \mathbf{1}_m \right). \quad (4.2.2)$$

In the spirit of the more classical language and in accordance with Lemma 4.1.2, fixing a set of coset representatives for $G^{(\sigma)}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\mathrm{fin}})/K$ induces an isomorphism

$$X_{K, \sigma^\circ}(\mathbb{C}) \simeq \bigsqcup_{g \in G^{(\sigma)}(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\mathrm{fin}})/K} \Gamma_g \backslash \mathcal{D}^{(\sigma^\circ)}, \quad (4.2.3)$$

where $\Gamma_g := G^{(\sigma)}(\mathbb{Q}) \cap gKg^{-1}$. Moreover, the selection of a \mathbb{C} -basis for $V_\sigma^{(\sigma)}$ for which the

Hermitian form $\langle -, - \rangle_\sigma$ is represented by the matrix

$$\begin{pmatrix} \mathbf{1}_{m-1} & \\ & -1 \end{pmatrix}$$

induces a diffeomorphism

$$\mathcal{D}^{(\sigma^\circ)} \simeq \{z \in \mathbb{C}^{m-1} : |z| < 1\} \simeq \frac{\mathrm{U}(m-1, 1)}{\mathrm{U}(m-1) \times \mathrm{U}(1)}.$$

In particular, X_K may be written as a disjoint union of arithmetic quotients of the complex open unit ball of dimension $m-1$. Our perspective will remain focused on the adelic double coset description X_K offered in Equation (4.2.1), but we note that quotients of complex open balls by actions of arithmetic subgroups was especially pervasive in the literature prior to widespread adoption of Deligne's axiomatic foundation of Shimura varieties. In particular, the adèle-free work of J.W. Cogdell in [Cog85], who studied a single component $\Gamma_g \backslash \mathcal{D}^{(\sigma^\circ)}$ of the decomposition of $X_{K, \sigma^\circ}(\mathbb{C})$ in Equation (4.2.3) when $F = \mathbb{Q}$ and $m = 3$, is especially relevant to this paper.

We also note that X_K is proper if and only if $V^{(\sigma)}$ is anisotropic by the Borel-Harish-Chandra Theorem. In particular, $F \neq \mathbb{Q}$ implies X_K is proper by the Hasse-Minkowski Theorem, which states that $V^{(\sigma)}$ is isotropic if and only if $V_v^{(\sigma)}$ is isotropic for all $v \in \Sigma$. Indeed, $V_v^{(\sigma)}$ is anisotropic for any $v \in \Sigma_\infty$ with $v \neq \sigma$. As for the case of $F = \mathbb{Q}$, a more careful examination of $(V^{(\sigma)}, q_\sigma)$, the underlying quadratic space over \mathbb{Q} of rank $2m$, is required. When $m \geq 3$, X_K is non-proper because any quadratic form over \mathbb{Q} in at least five variables is necessarily isotropic by Meyer's Theorem (cf. [Ser73, Chapter IV]). When $m = 2$, we appeal to Jacobson's Theorem (c.f. [MH73, Appendix II]), which asserts two Hermitian spaces are isometric over E if and only if the underlying quadratic spaces are isometric over \mathbb{Q} . It follows from the Hasse principle that there is a unique (up to isometry) quaternion algebra B over \mathbb{Q} such that (B, nrd) , where nrd is the reduced norm on B , and (V^σ, q_σ) are isometric as quadratic spaces over \mathbb{Q} . When $\{\infty\} \subsetneq \Sigma(\mathbb{V})$, B is a division algebra and so (B, nrd) is anisotropic. On the other hand, when $\Sigma(\mathbb{V}) = \{\infty\}$, (B, nrd) is isometric to $(M_2(\mathbb{Q}), \det)$, which is isotropic. Thus, for $m = 2$, X_K is non-proper if and only

if $\Sigma(\mathbb{V}) = \{\infty\}$.

For each K , there is a naturally metrized, ample class $\mathcal{L}_K \in \text{Pic}(X_K)$ which we call the **Hodge bundle** on X_K . When $F \neq \mathbb{Q}$ and $\sigma^\circ : E \hookrightarrow \mathbb{C}$ is a fixed embedding, let $\mathcal{L}_{\mathcal{D}^{\sigma^\circ}}$ denote the restriction of the tautological bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V_\sigma^{(\sigma)})$ whose fiber over each point $z \in \mathcal{D}^{\sigma^\circ}$ is the \mathbb{C} -line $\langle z \rangle \subseteq V_\sigma^{(\sigma)}$. The action of $G^{(\sigma)}(\mathbb{R})$ on $\mathcal{D}^{\sigma^\circ}$ lifts to a natural action on $\mathcal{L}_{\mathcal{D}^{\sigma^\circ}}$, which implies that $\mathcal{L}_{\mathcal{D}^{\sigma^\circ}}$ descends to a (holomorphic) line bundle on X_K . This line is precisely the Hodge bundle \mathcal{L}_K and admits the following σ° -adic uniformization:

$$\mathcal{L}_K(\mathbb{C}) \simeq G^{(\sigma)}(\mathbb{Q}) \backslash \mathcal{L}_{\mathcal{D}^{\sigma^\circ}} \times \mathbb{G}(\mathbb{A}_{\text{fin}}) / K \quad (4.2.4)$$

For each pair $K, K' \subseteq \mathbb{G}(\mathbb{A}_{\text{fin}})$ of compact open subgroups such that $K' \subseteq K$, we have an étale covering map

$$\pi_{K',K} : X_{K'} \longrightarrow X_K.$$

The Hodge bundles \mathcal{L}_K are compatible under the pullback of $\pi_{K',K}$ and therefore define an element of $\varinjlim_K \text{Pic}(X_K)_\mathbb{Q}$.

In the case when $F = \mathbb{Q}$ and X_K is non-proper, the definition for the Hodge bundle will need to be modified. When we are working with \widetilde{X}_K , a canonical toroidal compactification X_K , the **compactified Hodge bundle** $\widetilde{\mathcal{L}}_K$ will be a line bundle whose restriction to the open part of \widetilde{X}_K agrees with \mathcal{L}_K . Just as is the case when X_K is proper, we have an étale covering map

$$\widetilde{\pi}_{K',K} : \widetilde{X}_{K'} \rightarrow \widetilde{X}_K$$

for each pair of open compact subgroups $K' \subseteq K \subseteq \mathbb{G}(\mathbb{A}_{\text{fin}})$. Unsurprisingly, the Hodge bundles $\widetilde{\mathcal{L}}_K$ are compatible under the pullback of the covering maps $\widetilde{\pi}_{K',K}$ and define an element of $\varinjlim_K \text{Pic}(\widetilde{X}_K)_\mathbb{Q}$.

4.3 Toroidal Compactifications

Our main focus will be on the case of $F = \mathbb{Q}$; we take the opportunity to fix some notation for this specific case in addition to describing \widetilde{X}_K , a canonical toroidal compactification of X_K for $m = 2$ and $m = 3$. For the remainder of the paper let $\delta \in \mathbb{C}$ be such that $\text{Im } \delta > 0$

and $\delta^2 = -D$ for some square-free positive integer $D \in \mathbb{Z}$. Define $E := \mathbb{Q}(\delta)$ and denote the ring of integers of E by \mathcal{O}_E . Let $\tau : E \rightarrow E$ denote the nontrivial Galois involution on E and define $e^\tau := \tau(e)$ for $e \in E$.

Just as in Section 4.2, we begin with a totally positive definite incoherent Hermitian space $(\mathbb{V}, \langle -, - \rangle)$ over \mathbb{A}_E of rank m for some positive integer $m \geq 2$. Since \mathbb{Q} admits only a single archimedean place, we suppress the σ and σ° in the notation for the objects defined in the previous section. To be precise,

- $\mathbb{G} := \mathrm{U}(\mathbb{V})$;
- V is the unique (up to isometry) Hermitian space over E such that $V \otimes_{\mathbb{Q}} \mathbb{A}_{\mathrm{fin}} \simeq \mathbb{V}_{\mathrm{fin}}$ and $V \otimes_{\mathbb{Q}} \mathbb{R}$ is of signature $(m-1, 1)$;
- $G := \mathrm{U}(V)$;
- \mathcal{D} is the space of all negative \mathbb{C} -lines in $V \otimes_{\mathbb{Q}} \mathbb{R}$;
- $h : \mathbb{S} \rightarrow \mathrm{U}(m-1, 1)_{\mathbb{R}}$ is the Hodge map defined by

$$h(z) := \begin{pmatrix} \mathbf{1}_{m-1} & \\ & \bar{z}/z \end{pmatrix}.$$

We will abuse notation and use $\langle -, - \rangle$ (resp. $\langle -, - \rangle_p$) to denote the hermitian form on V , $V \otimes_{\mathbb{Q}} \mathbb{R}$, and \mathbb{V} (resp. V_p and \mathbb{V}_p). Likewise, we will let $q(x) := \langle x, x \rangle$ (resp. $q_p(x) := \langle x, x \rangle_p$) denote the underlying quadratic form on V , $V \otimes_{\mathbb{Q}} \mathbb{R}$, and \mathbb{V} (resp. V_p and \mathbb{V}_p).

For $m = 2$ and $\Sigma(\mathbb{V}) = \{\infty\}$, constructing the smooth compactification of X_K is just a matter of adding cusps, i.e.

$$\widetilde{X}_K(\mathbb{C}) \simeq G(\mathbb{Q}) \backslash \mathcal{D} \times \mathbb{G}(\mathbb{A}_{\mathrm{fin}}) / K \sqcup \{\mathrm{cusps}\}.$$

For the sake of computation, we fix an explicit description of V in this instance. In this specific case, $(V, \langle -, - \rangle)$ is isometric to $V = E^2$ (viewed as column vectors) equipped with

the Hermitian form $\langle v_1, v_2 \rangle := {}^t \bar{v}_2 H v_1$, where

$$H := \begin{pmatrix} & \delta^{-1} \\ -\delta^{-1} & \end{pmatrix}.$$

Much in the same way cusps are adjoined to modular curves, we compactify X_K by enlarging the space \mathcal{D} by adjoining the set of rational boundary components of \mathcal{D} prior to forming the quotient via the action of $G(\mathbb{Q})$. The set of rational boundary components of \mathcal{D} , which we will denote by \mathcal{D}_0 , is the collection of rational isotropic lines in $\mathbb{P}(V \otimes_{\mathbb{Q}} \mathbb{R})$ which, as a set, is spanned by the set of isotropic vectors in V . The action of $G(\mathbb{Q})$ on \mathcal{D} extends to a continuous action on $\mathcal{D}^{\#} := \mathcal{D} \sqcup \mathcal{D}_0$ when $\mathcal{D}^{\#}$ is equipped with the Satake topology. Moreover, the action of $G(\mathbb{Q})$ on \mathcal{D}_0 is transitive and so $G(\mathbb{Q})/\text{Stab}_{G(\mathbb{Q})}(\ell) \simeq \mathcal{D}_0$ for any $\ell \in \mathcal{D}_0$.

In general, the set of cusps of X_K may be described as

$$\{\text{cusps}\} = G(\mathbb{Q}) \backslash \mathcal{D}_0 \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K,$$

but our concrete description of $(V, \langle -, - \rangle)$ in turn allows for a much more explicit description of the set of cusps. In particular, if we set $\ell_{\infty} \in \mathcal{D}_0$ to be the image of rational isotropic vector $v_{\infty} := {}^t(1 \ 0)$ under the natural projection map $(V \otimes_{\mathbb{Q}} \mathbb{R}) \setminus \{0\} \rightarrow \mathbb{P}(V \otimes_{\mathbb{Q}} \mathbb{R})$, then $\text{Stab}_{G(\mathbb{Q})}(\ell_{\infty}) = P(\mathbb{Q})$, the set of upper triangular matrices in $G(\mathbb{Q})$, and

$$\{\text{cusps}\} = G(\mathbb{Q}) \backslash \mathcal{D}_0 \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K \simeq G(\mathbb{Q}) \backslash (G(\mathbb{Q})/P(\mathbb{Q})) \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K.$$

However the map

$$\begin{aligned} G(\mathbb{Q}) \backslash (G(\mathbb{Q})/P(\mathbb{Q})) \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K &\longrightarrow \ell_{\infty} \times P(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\text{fin}})/K \\ [\gamma P(\mathbb{Q}), g]_K &\longmapsto (\ell_{\infty}, P(\mathbb{Q})\gamma^{-1}gK) \end{aligned}$$

is well-defined and bijective. Hence, when $m = 2$ and $\Sigma(\mathbb{V}) = \{\infty\}$, the set of cusps of \widetilde{X}_K may be represented by points of the form $[\ell_{\infty}, g]_K$, where $g \in P(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\text{fin}})/K$ and so we

may write

$$\widetilde{X}_K(\mathbb{C}) \simeq G(\mathbb{Q}) \backslash \mathcal{D} \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K \sqcup \{[\ell_\infty, g]_K : g \in P(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\text{fin}})/K\}. \quad (4.3.1)$$

The construction of \widetilde{X}_K above is the Bailey-Borel compactification of X_K , which is smooth only when $m = 2$. A smooth compactification of X_K is a necessity for height pairings and so the Bailey-Borel compactification of X_K , which we will denote by $X_K^\#$, is insufficient for our purposes when $m \geq 3$. However, $X_K^\#$ is still relevant because our desired compactification is obtained by blowing-up $X_K^\#$ at the cusps.

For $m \geq 3$, $X_K^\#$ is constructed in essentially the same way as when $m = 2$, namely by adjoining rational boundary components of \mathcal{D} . Similar to before, we let \mathcal{D}_0 denote the set of rational isotropic lines in $\mathbb{P}(V \otimes_{\mathbb{Q}} \mathbb{R})$; that is, if we let V_0 denote the set of rational isotropic vectors in $V \otimes_{\mathbb{Q}} \mathbb{R}$ and let $\text{pr} : (V \otimes_{\mathbb{Q}} \mathbb{R}) \setminus \{0\} \rightarrow \mathbb{P}(V \otimes_{\mathbb{Q}} \mathbb{R})$ denote the natural projection map, then $\mathcal{D}_0 := \text{pr}(V_0)$. If we keep with notation from before and define $\mathcal{D}^\# := \mathcal{D} \sqcup \mathcal{D}_0$, then the Bailey-Borel compactification is

$$X_K^\#(\mathbb{C}) \simeq G(\mathbb{Q}) \backslash \mathcal{D}^\# \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K. \quad (4.3.2)$$

The trick we employed for indexing the cusps in the $m = 2$ case may be applied to index the cusps of $X_K^\#$ for all m . In particular, if $\ell_\infty \in \mathcal{D}_0$ is a line spanned by a rational isotropic vector and $P(\mathbb{Q}) \subseteq G(\mathbb{Q})$ is the parabolic subgroup stabilizing ℓ_∞ , then

$$X_K^\#(\mathbb{C}) \simeq G(\mathbb{Q}) \backslash \mathcal{D} \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K \sqcup \{[\ell_\infty, g]_K : g \in P(\mathbb{Q}) \backslash \mathbb{G}(\mathbb{A}_{\text{fin}})/K\}.$$

When $m = 3$, the smooth toroidal compactification \widetilde{X}_K is then obtained by blowing up each cusp of $X_K^\#$ to an elliptic curve equipped with multiplication by \mathcal{O}_E . Cogdell described this compactification in great detail in [Cog85]. If U is an open neighborhood of a cusp $[\ell_\infty, g]_K$ (with respect to the Satake topology) in the component $\Gamma_g \backslash \mathcal{D}^\#$, then Cogdell showed that $U \setminus \{[\ell_\infty, g]_K\}$ is the total space of a complex analytic disk bundle over an elliptic curve of the form \mathbb{C}/Λ , where $\Lambda \subseteq \mathcal{O}_E$ is a lattice dependent on the choice of basis for V and level K [Cog85, Proposition 2.1]. The compactification \widetilde{X}_K is then obtained by replacing each cusp

on $X_K^\#$ with corresponding elliptic curve. For the remainder of the section, we will let \widetilde{X}_K° denote the open part of \widetilde{X}_K .

As mentioned at the end of Section 4.2, we have an étale covering map

$$\widetilde{\pi}_{K',K} : \widetilde{X}_{K'} \rightarrow \widetilde{X}_K$$

for each pair of open compact subgroups $K' \subseteq K \subseteq \mathbb{G}(\mathbb{A}_{\text{fin}})$. Further, \widetilde{X}_K admits a ***compactified Hodge bundle***, which we denote by $\widetilde{\mathcal{L}}_K$. The key properties of $\widetilde{\mathcal{L}}_K$ are that its restriction to \widetilde{X}_K° agrees with \mathcal{L}_K and that it is compatible under the pullbacks by the maps $\widetilde{\pi}_{K',K}$ and therefore defines an element of $\varinjlim_K \text{Pic}(\widetilde{X}_K)_\mathbb{Q}$.

We conclude this section by mentioning that the compactification \widetilde{X}_K we constructed for $m = 2, 3$ agrees with the canonical smooth compactification of X_K that M.J. Larsen constructed in [Lar92]. Though it is unnecessary to flesh out the details in this paper, the basic idea behind Larsen's construction is to view V as a \mathbb{Q} -vector space equipped with following symplectic form

$$\langle v, w \rangle^\sim := \text{Tr}_{E/\mathbb{Q}}(\delta \langle v, w \rangle)$$

If we let $\check{G} := \text{GU}(V)$ denote the group of symplectic similitudes relative to the form $\langle -, - \rangle^\sim$ and $\check{h} : \mathbb{S} \rightarrow \check{G}_\mathbb{R} \simeq \text{GU}(m-1, 1)_\mathbb{R}$ denote the Hodge map defined by

$$\check{h}(z) := \begin{pmatrix} z & & \\ & z & \\ & & \bar{z} \end{pmatrix},$$

then the pair (\check{G}, \check{h}) defines a PEL Shimura datum; let $\check{X}_{\check{K}}$ denote the Shimura variety attached to (\check{G}, \check{h}) of level $\check{K} \subseteq \check{G}(\mathbb{A}_{\text{fin}})$. It suffices to compactify the Shimura variety $\check{X}_{\check{K}}$ because it shares the same neutral component of $X_{\check{K} \cap G(\mathbb{A}_{\text{fin}})}$ despite the absence of a map of Shimura data between (G, h) and (\check{G}, \check{h}) . As discussed in Section 3C of [Liu11a], Larsen's construction can be generalized beyond the cases of $m = 2$ and $m = 3$ in a systematic manner; this seems noteworthy to those looking to mimic H. Xue's proof in [Xue19] in order to tackle the arithmetic GGP conjecture for $\text{U}(n) \times \text{U}(n+1)$ for $n \geq 3$ and $F = \mathbb{Q}$.

4.4 Kudla Cycles on Unitary Shimura Varieties

In this section we introduce a rich family of algebraic cycles on X_K and \widetilde{X}_K . Essentially, inclusions of subspaces in the ambient Hermitian space \mathbb{V} induce embeddings of unitary groups, which in turn induce embeddings of Shimura varieties. Moreover, we discuss the intersections of these cycles in Chow group of X_K and \widetilde{X}_K . We make no assumptions on m , the rank of the Hermitian space \mathbb{V} , despite not providing any discussion on the smooth toroidal compactification of X_K for $m > 3$ beyond a surface-level comment at the end of the previous section; merely knowing a compactification \widetilde{X}_K of X_K exists is sufficient for this section.

For the remainder of the section, fix a (sufficiently small) level $K \subseteq \mathbb{G}(\mathbb{A}_{\text{fin}})$ as well as an embedding $\sigma^\circ : E \hookrightarrow \mathbb{C}$. For an E -subspace $W \subseteq V^{(\sigma)}$ such that $\langle w, w \rangle_\sigma$ is totally positive definite for all $w \in W$, we define

- $\widehat{W} := W \otimes_F \mathbb{A}_{F, \text{fin}}$,
- $\mathbb{G}_W := \text{Res}_{\mathbb{A}_F/\mathbb{A}} \text{U}(\widehat{W}^\perp)$ so that $\mathbb{G}_W(\mathbb{A}_{\text{fin}})$ is the subgroup of elements in $\mathbb{G}(\mathbb{A}_{\text{fin}})$ fixing every point in \widehat{W} , and
- $\mathcal{D}_W^{(\sigma^\circ)}$ to be the subset of lines in $\mathcal{D}^{(\sigma^\circ)}$ orthogonal to W .

If $r := \dim_E W$ and $g \in \mathbb{G}(\mathbb{A}_{\text{fin}})$, then we define a ***Kudla cycle***, denoted by $Z(W, g)_K \in \text{Ch}^r(X_K)_\mathbb{Q}$, to be the cycle represented by points $(z, hg) \in \mathcal{D}^{(\sigma^\circ)} \times \mathbb{G}(\mathbb{A}_{\text{fin}})$ where $z \in \mathcal{D}_W^{(\sigma^\circ)}$ and $h \in \mathbb{G}_W(\mathbb{A}_{\text{fin}})$. We note that the cycle $Z(W, g)_K$ depends only on the K -class of the E -subspace $g^{-1}W$ of \mathbb{V}_{fin} . We also note that these cycles $Z(W, g)_K$ have also been referred to as *translated* or *Hecke-translated* cycles in the literature, e.g. [Kud04]. Indeed this is because the inclusion of Shimura data $(\mathbb{G}_W, \mathcal{D}_W^{(\sigma^\circ)}) \hookrightarrow (\mathbb{G}, \mathcal{D}^{(\sigma^\circ)})$ induces an embedding of Shimura varieties and the element $g \in \mathbb{G}(\mathbb{A}_{\text{fin}})$ has the effect of right-translation. To be precise, let $G_W^{(\sigma)} := \text{Res}_{F/\mathbb{Q}} \text{U}(W^\perp)$; note $G_W^{(\sigma)}(\mathbb{A}_{\text{fin}}) \simeq \mathbb{G}_W(\mathbb{A}_{\text{fin}})$. Then there is a Shimura variety Y_K with $\sigma^{(\circ)}$ -adic uniformization

$$Y_{K, \sigma^{(\circ)}}(\mathbb{C}) \simeq G_W^{(\sigma)}(\mathbb{Q}) \backslash \mathcal{D}_W^{(\sigma^\circ)} \times \mathbb{G}_W(\mathbb{A}_{\text{fin}}) / K_W^g,$$

where $K_W^g := \mathbb{G}_W(\mathbb{A}_{\text{fin}}) \cap gKg^{-1}$. Then we have an embedding of Shimura varieties

$$\begin{aligned} G_W^{(\sigma)}(\mathbb{Q}) \setminus \mathcal{D}_W^{(\sigma^\circ)} \times \mathbb{G}_W(\mathbb{A}_{\text{fin}})/K_W^g &\longrightarrow G^{(\sigma)}(\mathbb{Q}) \setminus \mathcal{D}^{(\sigma^\circ)} \times \mathbb{G}(\mathbb{A}_{\text{fin}})/K \\ [z, h]_{K_W^g} &\longmapsto [z, hg]_K, \end{aligned}$$

whose image is precisely $Z(W, g)_K$. When working with the compactification \widetilde{X}_K , we regard $Z(W, g)_K$ as an element of $\text{Ch}^r(\widetilde{X}_K^\circ)_\mathbb{Q}$ and define $\widetilde{Z}(W, g)_K \in \text{Ch}^r(\widetilde{X}_K)_\mathbb{Q}$ to be the Zariski closure of $Z(W, g)_K$ in \widetilde{X}_K .

In general, for a positive integer r and $x := (x_1, \dots, x_r) \in V^{(\sigma)}(E)^r$, we define $V_x^{(\sigma)}$ to be the E -subspace of $V^{(\sigma)}$ generated by the components of x . Then for $g \in \mathbb{G}(\mathbb{A}_{\text{fin}})$ we define $Z(x, g)_K \in \text{Ch}^r(X_K)_\mathbb{Q}$ by

$$Z(x, g) := \begin{cases} Z(V_x^{(\sigma)}, g)_K c_1(\mathcal{L}_K^\vee)^{r-\dim_E V_x^{(\sigma)}} & \langle v, v \rangle_\sigma > 0 \text{ for all } v \in V_x^{(\sigma)} \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

Just as before, when working with \widetilde{X}_K , we regard $Z(x, g)_K$ as an element of $\text{Ch}^r(\widetilde{X}_K^\circ)_\mathbb{Q}$ and define $\widetilde{Z}(x, g)_K \in \text{Ch}^r(\widetilde{X}_K)_\mathbb{Q}$ by

$$\widetilde{Z}(x, g) := \begin{cases} \widetilde{Z}(V_x^{(\sigma)}, g)_K c_1(\widetilde{\mathcal{L}}_K^\vee)^{r-\dim_E V_x^{(\sigma)}} & \langle v, v \rangle_\sigma > 0 \text{ for all } v \in V_x^{(\sigma)} \setminus \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

When $V_x^{(\sigma)}$ is totally positive definite and $Z(x, g)_K$ (resp. $\widetilde{Z}(x, g)_K$) does not vanish, the $c_1(\mathcal{L}_K^\vee)^{r-\dim_E V_x^{(\sigma)}}$ (resp. $c_1(\widetilde{\mathcal{L}}_K^\vee)^{r-\dim_E V_x^{(\sigma)}}$) term appearing in the definition of $Z(x, g)_K$ (resp. $\widetilde{Z}(x, g)_K$) ought to be viewed as a normalization factor to ensure that $Z(x, g)_K$ (resp. $\widetilde{Z}(x, g)_K$) is an element of $\text{Ch}^r(X_K)_\mathbb{Q}$ (resp. $\text{Ch}^r(\widetilde{X}_K)_\mathbb{Q}$), rather than $\text{Ch}^{\dim_E V_x^{(\sigma)}}(X_K)_\mathbb{Q}$ (resp. $\text{Ch}^{\dim_E V_x^{(\sigma)}}(\widetilde{X}_K)_\mathbb{Q}$).

We now consider a broader collection of E -subspaces contained in \mathbb{V}_{fin} , rather than just subspaces of $V^{(\sigma)}$. To this end, we say that an E -subspace $\mathbb{W} \subseteq \mathbb{V}_{\text{fin}}$ is **admissible** if $\langle -, - \rangle|_{\mathbb{W} \times \mathbb{W}}$ takes values in E and $\langle w, w \rangle$ is totally positive for every $w \in \mathbb{W}$. Similarly, we say that $x := (x_1, \dots, x_r) \in \mathbb{V}_{\text{fin}}^r$ is admissible if the E -subspace of \mathbb{V}_{fin} generated by the

components of x is admissible. With the help of the following lemma, which holds regardless of the field F or $\Sigma(\mathbb{V})$, we are able to reduce the broader collection of admissible subspaces of \mathbb{V}_{fin} back to subspaces of $V^{(\sigma)}$.

Lemma 4.4.1 ([Liu11a], 3.1). \mathbb{W} is admissible if and only if for any $\sigma \in \Sigma_\infty$, there exists $g \in \mathbb{G}_{\text{der}}(\mathbb{A}_{\text{fin}})$ such that $g\mathbb{W} \subseteq V^{(\sigma)} \subseteq \mathbb{V}_{\text{fin}}$.

Now for an admissible subspace $\mathbb{W} \subseteq \mathbb{V}_{\text{fin}}$, let $g \in \mathbb{G}_{\text{der}}(\mathbb{A}_{\text{fin}})$ as in Lemma 4.4.1 so that $\mathbb{W} = g^{-1}W$ for some E -subspace W of $V^{(\sigma)}$. Then we have a well-defined Kudla cycle $Z(\mathbb{W})_K := Z(W, g)_K$ or $\tilde{Z}(\mathbb{W}) := \tilde{Z}(W, g)_K$ when working with the compactification \tilde{X}_K . For $x := (x_1, \dots, x_r) \in \mathbb{V}_{\text{fin}}^r$, we let \mathbb{V}_x denote the E -subspace of \mathbb{V}_{fin} generated by the components of x and define

$$Z(x)_K := \begin{cases} Z(\mathbb{V}_x)_K c_1(\mathcal{L}_K^\vee)^{r-\dim_E \mathbb{V}_x} & \text{if } \mathbb{V}_x \text{ is admissible} \\ 0 & \text{otherwise.} \end{cases}$$

In keeping with our conventions, if we are working with the compactification \tilde{X}_K , then we may regard $\tilde{Z}(x)_K$ as an element of $\text{Ch}^r(\tilde{X}_K^\circ)_\mathbb{Q}$ and compactify the cycle as follows:

$$\tilde{Z}(x)_K := \begin{cases} \tilde{Z}(\mathbb{V}_x)_K c_1(\tilde{\mathcal{L}}_K^\vee)^{r-\dim_E \mathbb{V}_x} & \text{if } \mathbb{V}_x \text{ is admissible} \\ 0 & \text{otherwise.} \end{cases}$$

Our primary interest are those special cycles arising from subspaces of the form \mathbb{V}_x for $x \in \mathbb{V}_{\text{fin}}^r$. For admissible \mathbb{V}_x with $g \in \mathbb{G}_{\text{der}}(\mathbb{A}_{\text{fin}})$ as in Lemma 4.4.1 so that $g\mathbb{V}_x \subseteq V^{(\sigma)}$, we write \mathbb{G}_x and $\mathcal{D}_x^{(\sigma^\circ)}$ in place of $\mathbb{G}_{g\mathbb{V}_x}$ and $\mathcal{D}_{g\mathbb{V}_x}^{(\sigma^\circ)}$, respectively.

If we let $Z^\#(-)_K$ denote the Zariski closure of $Z(-)_K$ in $X_K^\#$, the Bailey-Borel compactification of X_K , then Lemma 3.1 of [Cog85] implies that it is possible that $Z^\#(x)_K$ and $Z^\#(y)_K$ may intersect at a cusp of $X_K^\#$ when $m = 3$ and $r = 1$. If this is the case, then Lemma 3.2 of [Cog85] implies that $\tilde{Z}(x)_K$ and $\tilde{Z}(y)_K$ intersect the same boundary component in the blow-up \tilde{X}_K , but do not have a point of intersection in any boundary component. Moreover, Lemma 3.2 of [Cog85] implies that $\tilde{Z}(x)_K$ has no self-crossings on the boundary of \tilde{X}_K . All this to say, intersections of Kudla cycles occur in the open part, \tilde{X}_K° (for $m = 3$ at least).

If $m = 3$, then our discussion above opens the door to previous results on intersections of Kudla cycles of compact Shimura varieties. For example, if $\tilde{Z}(W_1)_K, \tilde{Z}(W_2)_K \in \text{Ch}^1(\tilde{X}_K)$ are Kudla cycles, then their intersection in the Chow group (as well as their scheme-theoretic intersection) is given as a sum of cycles $Z(W)_K$ indexed by admissible classes W in $K \setminus (KW_1 + KW_2)$ [YZZ09, Proposition 2.6]. Results of this type are utilized in Xue's proof of the refined arithmetic GGP-conjecture for $U(2) \times U(3)$ when $F \neq \mathbb{Q}$ (e.g. [Xue19, Propositions 4.4 and 4.5]) and so it is reasonable to assume propositions of this type will be relevant to those seeking to establish a refined result when $F = \mathbb{Q}$. However these results are not necessary for our purposes of proving an unrefined result and will not enter the discussion beyond these comments.

4.5 Generating Series of Special Cycles

Roughly speaking, the terms of the generating series we aim to study will be the special cycles $Z(x)_K$ or $\tilde{Z}(x)_K$ multiplied by values of certain subspace of Schwartz functions. To this end, if $\sigma \in \Sigma_\infty$, then we define $\mathcal{S}(\mathbb{V}_\sigma^r)^{U_\sigma}$ to be the subspace of $\mathcal{S}(\mathbb{V}_\sigma^r)$ consisting of those functions of the form

$$P(Q(x)) \exp(-2\pi \text{Tr } Q(x)),$$

where P is a polynomial function on $\text{Herm}_r(\mathbb{C})$ and $Q(x)$ is the moment matrix as defined in Equation (3.3.1). For each compact open subgroup $K \subseteq \mathbb{G}(\mathbb{A}_{\text{fin}})$, we define

$$\begin{aligned} \mathcal{S}(\mathbb{V}^r)^{U_\infty} &:= \left(\bigotimes_{\sigma \in \Sigma_\infty} \mathcal{S}(\mathbb{V}_\sigma^r)^{U_\sigma} \right) \otimes \mathcal{S}(\mathbb{V}_{\text{fin}}^r) \\ \mathcal{S}(\mathbb{V}^r)^{U_\infty K} &:= \left(\bigotimes_{\sigma \in \Sigma_\infty} \mathcal{S}(\mathbb{V}_\sigma^r)^{U_\sigma} \right) \otimes \mathcal{S}(\mathbb{V}_{\text{fin}}^r)^K. \end{aligned}$$

When X_K is proper, we define a generating series of special cycles for any $\phi \in \mathcal{S}(\mathbb{V}^r)^{U_\infty K}$ by the formula

$$\mathfrak{Z}(h, \phi)_K := \sum_{x \in K \setminus \mathbb{V}_{\text{fin}}^r} (\omega_\chi(h)\phi)(Q(x), x) Z(x)_K. \quad (4.5.1)$$

Here, $h \in H_r(\mathbb{A}_F)$ and $\mathfrak{Z}(h, \phi)_K$ is a formal series with values in $\text{Ch}^r(X_K)_\mathbb{C}$. We write a Schwartz function as $\phi = \phi_\infty \phi_f$ and $\phi(x, Q(x)) = \phi_\infty(y) \phi_f(x)$, where $x \in \mathbb{V}_{\text{fin}}^r$ and $y \in V_\infty^r$

is any element satisfying $Q(x) = Q(y)$; such a y exists when x is admissible and the term vanishes when x is not. The generating series $\mathcal{Z}(h, \phi)_K$ is compatible with the pullback of $\pi_{K', K}$ as K varies and so it defines a formal series with values in $\varinjlim_K \text{Ch}^r(X_K)_{\mathbb{C}}$.

As we alluded to at the beginning of the section, integrating automorphic forms on H_r against $\mathcal{Z}(h, \phi)_K$ produces cycles in $\text{Ch}^r(X_K)$. The integral is over $[H_r]$, which necessitates $\mathcal{Z}(h, \phi)_K$ be modular in some sense. When X_K is compact, Liu [Liu11a, Theorem 3.5 (1)] reduced the problem of modularity to that of convergence; that is, if $\ell : \text{Ch}^r(X_K)_{\mathbb{C}} \rightarrow \mathbb{C}$ is a linear functional and the complex-valued series

$$\ell(\mathcal{Z}(h, \phi)_K) := \sum_{x \in K \setminus \mathbb{V}_{\text{fin}}^r} (\omega_{\chi}(h)\phi)(Q(x), x)\ell(Z(x)_K)$$

is absolutely convergent in the usual sense, then $\ell(\mathcal{Z}(h, \phi)_K)$ is an automorphic form of H_r . Moreover, Liu established modularity of $\mathcal{Z}(h, \phi)_K$ when $r = 1$ by showing it satisfied the desired convergence property [Liu11a, Theorem 3.5 (2)], i.e. $\ell(\mathcal{Z}(h, \phi)_K)$ is convergent for every linear functional $\ell : \text{Ch}^1(X_K)_{\mathbb{C}} \rightarrow \mathbb{C}$.

When working with the compactification \widetilde{X}_K , we define a compactified generating series of special cycles by

$$\widetilde{\mathcal{Z}}(h, \phi)_K := \begin{cases} \sum_{x \in K \setminus \mathbb{V}_{\text{fin}}^r} (\omega_{\chi}(h)\phi)(Q(x), x)\widetilde{Z}(x)_K & m > 2; \\ \sum_{x \in K \setminus \mathbb{V}_{\text{fin}}^r} (\omega_{\chi}(h)\phi)(Q(x), x)\widetilde{Z}(x)_K + W_0\left(\frac{1}{2}, h, \phi\right) c_1\left(\widetilde{\mathcal{L}}_K^{\vee}\right) & m = 2, r = 1. \end{cases} \quad (4.5.2)$$

Here, $W_0(s, h, \phi)$ is a product of local Whittaker functionals defined in [Liu11a, Section 2E]. We do not flesh out this definition because we will work to annihilate these types of terms (i.e. boundary contributions) in the coming section.

What we are more interested in is modularity and the term $W_0\left(\frac{1}{2}, h, \phi\right) c_1\left(\widetilde{\mathcal{L}}_K^{\vee}\right)$ is precisely the correct addition to ensure $\widetilde{\mathcal{Z}}(h, \phi)_K$ is modular for $m = 2$ and $r = 1$ [Liu11b, Corollary 3.3]. Liu makes the insightful observation that the addition of the term $W_0\left(\frac{1}{2}, h, \phi\right) c_1\left(\widetilde{\mathcal{L}}_K^{\vee}\right)$ in this case is analogous to the addition of the term $-\pi/\text{Im } \tau$ to the classical Eisenstein se-

ries $G_2(z)$. Unmodified, $G_2(z)$ is not a modular form! However, $G_2(z) - \pi/\text{Im } z$ becomes modular at the expense of no longer being holomorphic.

For $m > 2$, we may apply [Liu11a, Theorem 3.5] to \widetilde{X}_K° , the open part of \widetilde{X}_K . Specifically, if $\ell : \text{Ch}^r(\widetilde{X}_K^\circ)_{\mathbb{C}} \rightarrow \mathbb{C}$ is a linear functional and

$$\widetilde{\mathcal{Z}}^\circ(h, \phi)_K := \sum_{x \in K \setminus \mathbb{V}_{\text{fin}}^r} (\omega_\chi(h)\phi)(Q(x), x)Z(x)_K \in \text{Ch}^r(\widetilde{X}_K^\circ)$$

denotes the open part of $\widetilde{\mathcal{Z}}(h, \phi)_K$, then convergence of $\ell(\widetilde{\mathcal{Z}}^\circ(h, \phi)_K)$ implies that $\ell(\widetilde{\mathcal{Z}}^\circ(h, \phi)_K)$ is an automorphic form of H_r and $\ell(\widetilde{\mathcal{Z}}^\circ(h, \phi)_K)$ is convergent for all ℓ when $r = 1$. Recently, Xia [Xia22] extended Liu’s work to arbitrary codimension $1 \leq r \leq m - 1$ by proving $\ell(\widetilde{\mathcal{Z}}^\circ(h, \phi)_K)$ converges for all $\ell : \text{Ch}^r(\widetilde{X}_K^\circ) \rightarrow \mathbb{C}$ when $E = \mathbb{Q}(\sqrt{-d})$ for $d \in \{1, 2, 3, 7, 11\}$. However, modularity on the *compactification* is necessary for the arithmetic GGP conjecture since the Beilinson-Bloch height pairing requires a smooth *projective* variety.

Beyond these examples, little is known about the modularity of $\widetilde{\mathcal{Z}}(h, \phi)_K$. Recently, J.H. Bruiner, B. Howard, S. Kudla, M. Rapoport, and T. Yang settled the modularity of the generating series for $m \geq 3$ and $r = 1$ under some rigid assumptions on the level K [BHK⁺20]. However, the work in [BHK⁺20] suggests that the generating series as defined in Equation (4.5.2) is likely *not* modular in $\text{Ch}^r(\widetilde{X}_K)$. In order to achieve the desired modularity property, it is expected that $\widetilde{\mathcal{Z}}(h, \phi)_K$ needs to be modified by the addition of “boundary” terms and it is unclear how to do this at the present time, especially in full generality. The issue of modularity of the compactified generating series is the primary obstacle to a refined result affirming the arithmetic GGP-conjecture for $U(2) \times U(3)$ when $F = \mathbb{Q}$. There would still be challenges after settling the question of modularity, but they could be (presumably) be resolved with “standard” techniques; we address this in more detail in Section 5.2.

5 The Arithmetic GGP Conjecture: Our Main Result

Before proceeding, we fix some notation for the remainder of the section. In keeping with Section 4.3, let

- $E := \mathbb{Q}(\sqrt{\delta})$, where $\delta \in \mathbb{C}$ satisfies $\text{Im } \delta > 0$ and $\delta^2 = -D$ for some square-free positive

integer $D \in \mathbb{Z}$;

- \mathcal{O}_E is the ring of integers in E ;
- $\tau : E \rightarrow E$ is the nontrivial Galois involution on E and $e^\tau := \tau(e)$ for all $e \in E$;

Since $F = \mathbb{Q}$ exclusively in this section, we will only be working with the compactified objects defined in Section 4. We deviate slightly from our previous notation and drop the tildes we used to distinguish the compactified objects from Section 4.3, Section 4.4, and Section 4.5 with the hope that it will make the notation less cumbersome. For example, $X_{K''}$ denotes the toroidal compactification we denoted by $\widetilde{X}_{K''}$ in Section 4.3, $Z(x)_{K''}$ denotes the compactified cycle we wrote as $\widetilde{Z}(x)_{K''}$ in Section 4.4, $\mathcal{Z}(h, \phi)_{K''}$ denotes the compactified generated series we wrote as $\widetilde{\mathcal{Z}}(h, \phi)_{K''}$ in Section 4.5, etc. Likewise, we let $X_{K''}^\circ$, $Z(x)_{K''}^\circ$, $\mathcal{Z}^\circ(h, \phi)_{K''}$ denote the open part of $X_{K''}$, $Z(x)_{K''}$, and $\mathcal{Z}(h, \phi)_{K''}$.

Further, we set some additional notation that is necessary to state our main result and accompanying propositions and lemmas. To that end, For the remainder of the paper, let $(V, \langle -, - \rangle)$ be a Hermitian space over E of rank 3. such that $V_\infty := V \otimes_{\mathbb{Q}} \mathbb{R}$ is of signature $(2, 1)$. Fix $w \in V$ such that $\langle w, w \rangle > 0$ and let W denote the orthogonal complement of w in V ; thus $W_\infty := W \otimes_{\mathbb{Q}} \mathbb{R}$ is of signature $(1, 1)$. Also let \mathbb{V} (resp. \mathbb{W}) denote the totally positive incoherent Hermitian space over \mathbb{A}_E whose nearby Hermitian space at the lone archimedean place of \mathbb{Q} is V (resp. W). We define $G_W := \mathrm{U}(W)$ and $G_V := \mathrm{U}(V)$, which are reductive algebraic groups over \mathbb{Q} . Let $K'' \subseteq G_V(\mathbb{A}_{\mathrm{fin}})$ be a sufficiently small compact open subgroup and denote by $X_{K''}$ the toroidal compactification of the Picard modular surface of level K'' attached to $(V, \langle -, - \rangle)$. Likewise, set $K := G_W(\mathbb{A}_{\mathrm{fin}}) \cap K'' \subseteq G_W(\mathbb{A}_{\mathrm{fin}})$ and let Y_K denote the toroidal compactification of the Shimura curve of level K attached to $(W, \langle -, - \rangle|_{W \times W})$. We henceforth fix the compact open subgroup $K'' \subseteq G_V(\mathbb{A}_{\mathrm{fin}})$ and in so doing fix $K \subseteq G_W(\mathbb{A}_{\mathrm{fin}})$ as well. We also let $G := G_V \times G_W$, $K' := K'' \times K \subseteq G(\mathbb{A}_{\mathrm{fin}})$, and $M_{K'} := X_{K''} \times Y_K$.

The inclusion of $W \hookrightarrow V$ induces an embedding of $G_W \hookrightarrow G_V$, which induces a diagonal embedding $G_W \hookrightarrow G$ and further induces an embedding $Y_K \hookrightarrow M_{K'}$ of Shimura varieties; we let y_K denote the element in $\mathrm{Ch}^2(M_{K'})$ defined by the image of Y_K in $M_{K'}$ under this embedding. We will make this more precise in Section 5.1, but there exists a Hecke equivariant projection $\mathrm{Ch}^2(M_{K'}) \rightarrow \mathrm{Ch}^2(M_{K'})_0$ and so we henceforth let $y_{K,0}$ denote the image of y_K in

$\mathrm{Ch}^2(M_{K'})_0.$

	Curve	Surface	Product
Incoherent Hermitian Space over \mathbb{A}_E	W	V	-
Nearby Hermitian Space over E	W of signature $(1, 1)$	V of signature $(2, 1)$	-
Group	$G_W := \mathrm{U}(W)$	$G_V := \mathrm{U}(V)$	$G := G_V \times G_W$
Level	$K \subseteq G_W(\mathbb{A}_{\mathrm{fin}})$	$K'' \subseteq G_V(\mathbb{A}_{\mathrm{fin}})$	$K' := K'' \times K \subseteq G(\mathbb{A}_{\mathrm{fin}})$
Shimura Variety	Y_K	$X_{K''}$	$M_{K'} := X_{K''} \times Y_K$
Representation	π_2 (theta lift of σ_2)	π_3 (theta lift of σ_3)	$\pi := \pi_3 \boxtimes \pi_2$
Automorphic Forms	$\varphi_2, \varphi'_2 \in \pi_2$	$\varphi_3, \varphi'_3 \in \pi_3$	$\varphi = \varphi_3 \otimes \varphi_2$ $\varphi' = \varphi'_3 \otimes \varphi'_2$
Functions	$t_{\varphi_2, \varphi'_2} \in C_c^\infty(G_W(\mathbb{A}_{\mathrm{fin}}))$ $t_{\varphi_2, \varphi'_2} \mapsto \varphi_2 \otimes \overline{\varphi'_2}$	$t_{\varphi_3, \varphi'_3} \in C_c^\infty(G_V(\mathbb{A}_{\mathrm{fin}}))$ $t_{\varphi_3, \varphi'_3} \mapsto \varphi_3 \otimes \overline{\varphi'_3}$	$t_{\varphi, \varphi'} \in C_c^\infty(G(\mathbb{A}_{\mathrm{fin}}))$ $t_{\varphi, \varphi'} \mapsto \varphi \otimes \overline{\varphi'}$

Figure 5.1: A table of all of the relevant data for the remainder of the paper.

Let \mathcal{A} denote the set of irreducible admissible representations of $G(\mathbb{A}_{\mathrm{fin}})$ which appear in $\mathrm{H}^3(M_{K'})$. Then we have a surjective map

$$C_c^\infty(G(\mathbb{A}_{\mathrm{fin}})) \longrightarrow \bigoplus_{\pi \in \mathcal{A}} \pi \otimes \tilde{\pi}.$$

If we fix an inner product on each $\pi \in \mathcal{A}$, then we may identify $\tilde{\pi}$ with $\bar{\pi}$ thereby giving a surjection

$$C_c^\infty(G(\mathbb{A}_{\mathrm{fin}})) \longrightarrow \bigoplus_{\pi \in \mathcal{A}} \pi \otimes \bar{\pi}. \quad (5.0.1)$$

For the remainder of the paper, fix an irreducible tempered representation $\pi \in \mathcal{A}$ and let $\varphi, \varphi' \in \pi$. Further, fix a function $t_{\varphi, \varphi'} \in C_c^\infty(G(\mathbb{A}_{\mathrm{fin}}))$ that maps to $\varphi \otimes \overline{\varphi'}$ under the map

in Equation (5.0.1). Also, write $\pi = \pi_3 \boxtimes \pi_2$ where π_3 and π_2 are irreducible admissible tempered automorphic representations of $G_V(\mathbb{A}_{\text{fin}})$ and $G_W(\mathbb{A}_{\text{fin}})$, respectively.

We also assume $\varphi = \varphi_3 \otimes \varphi_2$ and $\varphi' = \varphi'_3 \otimes \varphi'_2$ for some $\varphi_3, \varphi'_3 \in \pi_3$ and $\varphi_2, \varphi'_2 \in \pi_2$. We have surjective maps analogous to the map appearing in Equation (5.0.1) for $G_W(\mathbb{A}_{\text{fin}})$ and $G_V(\mathbb{A}_{\text{fin}})$; we choose $t_{\varphi_2, \varphi'_2} \in C_c^\infty(G_W(\mathbb{A}_{\text{fin}}))$ and $t_{\varphi_3, \varphi'_3} \in C_c^\infty(G_V(\mathbb{A}_{\text{fin}}))$ in the fiber of $\varphi_2 \otimes \overline{\varphi'_2}$ and $\varphi_3 \otimes \overline{\varphi'_3}$, respectively, under these maps. We also assume that there exist irreducible cuspidal automorphic representations σ_3 and σ_2 of $H_1(\mathbb{A}_{\text{fin}})$ so that π_3 and π_2 are theta lifts of σ_3 and σ_2 , respectively. Figure 5.1 above organizes all of this data.

5.1 Künneth-Chow Decompositions and Pairings

Let X be a smooth projective variety over a number field F of dimension n . Recall from Section 2 that we denote the intersection pairing between $\text{Ch}^i(X)$ and $\text{Ch}_i(X)$ by $\alpha \cdot \beta$ for $\alpha \in \text{Ch}^i(X)$ and $\beta \in \text{Ch}_i(X)$. We let $\text{cl} : \text{Ch}^i(X) \rightarrow \text{H}^{2i}(X)$ be the cycle class map and define $\text{Ch}^i(X)_0 := \ker \text{cl}$, the space of cohomologically trivial cycles. It is conjectured that there exists a splitting

$$\text{Ch}^i(X) \simeq \text{Ch}^i(X)_0 \oplus \text{Im cl} . \quad (5.1.1)$$

Moreover, it is expected that this splitting is a canonical isomorphism of Hecke modules when X is a Shimura variety. The primary objective of this subsection is to introduce Künneth-Chow decompositions, which provide us with a straight-forward way of cohomologically trivializing cycles as well as furnishing the desired Hecke equivariant splitting in Equation (5.1.1) for the Shimura varieties of interest to this paper.

Let Δ denote the diagonal cycle in $X \times X$. By a **Künneth Chow decomposition**, we mean a sum of the form

$$\Delta = \Delta_{X,0} + \Delta_{X,1} + \cdots + \Delta_{X,2n} \in \text{Ch}^n(X \times X)$$

such that the natural map $\Delta_{X,i,*} : \text{H}^*(X) \rightarrow \text{H}^*(X)$ is the projection to the i -th component; we call the map $\Delta_{X,i,*}$ **i -th Künneth-Chow projector**. In the same spirit, we call $\Delta_{X,i}$ the **i -th Künneth-Chow component of X** . We will often write Δ_i and $\Delta_{i,*}$ when the

variety X is clear from context. The existence of a Künneth-Chow decomposition beyond a few isolated and low-dimensional cases remains an open question. Nonetheless many anticipate all smooth projective varieties exhibit a Künneth-Chow decomposition and it is a standard conjectural assumption in the realm of algebraic cycles.

The presence of a Künneth-Chow decomposition allows us to produce cohomologically trivial cycles. Specifically, if $z \in \text{Ch}^r(X)$ is a codimension r cycle on X , then $\Delta_{2r-1,*}z \in \text{Ch}^r(X)_0$ because $\text{cl} \circ \Delta_{2r-1,*}$ factors through $H^{2r-1}(X)$ by the definition of Δ_{2r-1} . Our interest is when X is a product of a curve and a surface and, fortunately, we may give an explicit description in this case. This idea of using Künneth-Chow decompositions to trivialize cohomology classes in certain low dimensional cases is due to Wei Zhang (cf. [RSZ20]).

Keeping in mind our eventual application to the Shimura varieties of interest, we review some general theory and recall the construction of Künneth-Chow decompositions when X is a curve, surface, and a product of a curve and a surface following the exposition in Section 2 of [Xue19]. First, assume X is a smooth projective curve, i.e. $n = 1$. Select an ample class $\xi \in \text{Pic}(X)$ of degree one and let

$$\begin{aligned}\Delta_0 &:= X \times \xi, \\ \Delta_2 &:= \xi \times X, \\ \Delta_1 &:= \Delta - \Delta_0 - \Delta_2.\end{aligned}$$

Then $\Delta = \Delta_0 + \Delta_1 + \Delta_2$ is a Künneth-Chow decomposition for X .

When $n = 2$, that is X is a smooth projective surface, we select an ample class $\xi \in \text{Pic}(X)$ and let $e := (\deg \xi \cdot \xi)^{-1}(\xi \cdot \xi) \in \text{Ch}^2(X)$. Also let $\text{Alb}(X)$ and $\underline{\text{Pic}}^0(X)$ denote the Albanese variety of X and the neutral component of the Picard variety of X , respectively. Let $\alpha : \underline{\text{Pic}}^0(X) \rightarrow \text{Alb}(X)$ be the isogeny defined by $L \mapsto L \cdot \xi$; assume that the degree of α is d and let $\alpha^\vee : \text{Alb}(X) \rightarrow \underline{\text{Pic}}^0(X)$ denote the dual isogeny. If $j : X \rightarrow \underline{\text{Pic}}^0(X)$ is the composition of the natural map

$$X \longrightarrow \text{Alb}(X) \qquad x \longmapsto x - e$$

and the isogeny α^\vee , then we have a morphism

$$j \times \text{id}_X : X \times X \longrightarrow \underline{\text{Pic}}^0(X) \times X,$$

where id_X is the identity morphism on X . Now let \mathcal{P} denote the Poincare bundle on $\underline{\text{Pic}}^0(X) \times X$ and put $\beta = (j \times \text{id}_X)^* \mathcal{P}$. Also let ${}^t z \in \text{Ch}^*(X)$ denote the transpose of $z \in \text{Ch}^*(X)$, which simply reverses the factors of α , i.e. ${}^t(P, Q) := (Q, P)$. If we let $p_i : X \times X \rightarrow X$ denote the projection on the i th factor and define

$$\begin{aligned} \Delta_0 &:= X \times e, \\ \Delta_1 &:= \frac{1}{d} p_1^* \xi \cdot \beta, \\ \Delta_3 &:= {}^t \Delta_1, \\ \Delta_4 &:= e \times X, \\ \Delta_2 &:= \Delta - \Delta_0 - \Delta_1 - \Delta_3 - \Delta_4, \end{aligned}$$

then $\Delta = \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ is a Künneth-Chow Decomposition for X .

Lastly, suppose that $M := X \times Y$, where X is a smooth projective surface and Y is a smooth projective curve. If we choose ample class $\xi \in \text{Pic}(X)$, then we can construct Künneth-Chow decompositions

$$\Delta_Y = \Delta_{Y,0} + \Delta_{Y,1} + \Delta_{Y,2} \quad \text{and} \quad \Delta_X = \Delta_{X,0} + \Delta_{X,1} + \Delta_{X,2} + \Delta_{X,3} + \Delta_{X,4}$$

for Y relative to $\xi|_Y$ and X relative to ξ , respectively, in accordance with the previously stated procedure for curves and surfaces. If we let

$$\Delta_{M,i} := \sum_{k+\ell=i} \Delta_{X,k} \times \Delta_{Y,\ell} \quad i = 0, 1, \dots, 6,$$

then $\Delta_M = \sum_{i=0}^6 \Delta_{M,i}$ is a Künneth-Chow decomposition for M .

Our main interest in a Künneth-Chow decomposition $X = \sum_i \Delta_i$ are the cohomologically

trivial projections

$$\Delta_{2r-1,*} : \mathrm{Ch}^r(X) \longrightarrow \mathrm{Ch}^r(X)_0 \quad z \longmapsto \Delta_{2r-1,*}z$$

because our main theorem involves the Beilinson-Bloch pairing, which requires cohomologically trivial cycles. It is worth noting that Rapoport et al. propose trivializing cycles with so-called *Hecke-Künneth projectors* [RSZ20, Section 6.3] rather than Künneth-Chow projectors. In the general setting, these Hecke-Künneth projectors are preferable because they are known to be Hecke equivariant and exist for unitary Shimura varieties of arbitrary dimension. That being said, we still opt to use Künneth-Chow projectors. The advantage of the low-dimensional setting of our paper is that we can construct Künneth-Chow decompositions for Y_K and $X_{K''}$ using the (compactified) Hodge bundles \mathcal{L}_K and $\mathcal{L}_{K''}$, respectively, which in turn gives a Künneth-Chow decomposition for $M_{K'}$. Moreover, the projections $\Delta_{2r-1,*} : \mathrm{Ch}^r(M_{K'}) \rightarrow \mathrm{Ch}^r(M_{K'})_0$ are known to be Hecke equivariant [Xue19, Section 2].

The following proposition reduces the Beilinson-Bloch pairing to the more well-understood Neron-Tate height. To be precise, let Y be a smooth projective curve and let X be a smooth projective surface equipped with an embedding $Y \hookrightarrow X$. Let $M := X \times Y$ and fix an ample class on X and in order to construct Künneth-Chow decompositions

$$\Delta_Y = \sum_{i=0}^2 \Delta_{Y,i}, \quad \Delta_X = \sum_{i=0}^4 \Delta_{X,i}, \quad \Delta_M = \sum_{i=0}^6 \Delta_{M,i},$$

for Y , X , and M , respectively, as constructed in the manner outlined above. The curve Y defines a cycle $\mathrm{Ch}^2(M)$ via the embedding $Y \hookrightarrow X$ on the first factor and the identity map on Y in the second factor; we let y denote the class of Y in $\mathrm{Ch}^2(M)$.

Proposition 5.1.1 ([Xue19], 2.2). In following with the notation in the preceding paragraph, let L_Y be a degree zero line bundle on Y and let L_X be a line bundle on X . Assume that Y and X have regular models \mathcal{Y} and \mathcal{X} , respectively, over \mathcal{O}_F and that M has a regular model \mathcal{M} together with a dominant map $\pi : \mathcal{M} \rightarrow \mathcal{X} \times \mathcal{Y}$. If $\langle -, - \rangle_{\mathrm{BB}}$ is the Beilinson-Bloch pairing on M and $p_Y : M \rightarrow Y$ and $p_X : M \rightarrow X$ denote the projections to Y and X , respectively,

then

$$\langle p_X^* L_X \cdot p_Y^* L_Y, \Delta_{M,3,*} y \rangle_{\text{BB}}$$

is well-defined and

$$\langle p_X^* L_X \cdot p_Y^* L_Y, \Delta_{M,3,*} y \rangle_{\text{BB}} = \langle \Delta_{Y,1,*}(\Delta_{X,2,*} L_X)|_Y, L_Y \rangle_{\text{NT}}$$

where $\langle -, - \rangle_{\text{NT}}$ is the Neron-Tate height on Y .

Proposition 5.1.1 will be especially useful for us because Y_K , $X_{K''}$, and $M_{K'}$ satisfy the hypotheses in Proposition 5.1.1 by [Lar92] and [Xue19]. Thus, at least in certain instances, we may reduce the Beilinson-Bloch pairing on $M_{K'}$ to the Neron-Tate height on Y_K .

5.2 Hecke Correspondences and Arithmetic Theta Lifts

There is a Hecke action on $X_{K''}$ by $G_V(\mathbb{A}_{\text{fin}})$. To be precise, for $x \in G_V(\mathbb{A}_{\text{fin}})$, there is a natural isomorphism $\gamma_x : X_{xK''x^{-1}} \rightarrow X_{K''}$ given by “right multiplication by x ,” in terms of the complex uniformization induced by the inclusion $\mathbb{Q} \hookrightarrow \mathbb{C}$, γ_x acts on points by

$$\begin{aligned} \gamma_x : G_V(\mathbb{Q}) \backslash \mathcal{D} \times G_V(\mathbb{A}_{\text{fin}})/xK''x^{-1} &\longrightarrow G_V(\mathbb{Q}) \backslash \mathcal{D} \times G_V(\mathbb{A}_{\text{fin}})/K \\ [z, g]_{xK''x^{-1}} &\longmapsto [z, gx]_{K''}. \end{aligned}$$

If we define $K''_x := K'' \cap xK''x^{-1}$, then we define a cycle $T(x)_{K''}$ by image of the map

$$(\pi_{K''_x, K''}, \pi_{K''_x, K''} \circ \gamma_x) : X_{K''_x} \longrightarrow X_{K''} \times X_{K''}$$

If $p_i : X_{K''} \times X_{K''} \rightarrow X_{K''}$ is the projection onto the i th factor, then $T(x)_{K''}$ induces a correspondence on cycles $\alpha \in \text{Ch}^*(X_{K''})$ in the following way

$$\alpha \longmapsto T(x)_{K'',*} \alpha := p_{2,*}(p_1^* \alpha \cdot T(x)_{K''}).$$

In terms of the complex uniformization described above, the pushforward map, $T(x)_{K'',*}$, is given by

$$T(x)_{K'',*}[z, g]_{K''} = \sum_{i=1}^s [z, gx_i]_{K''},$$

where x_1, \dots, x_s is a set of representatives of $K''xK''/K''$. We also note that the transpose of $T(x)_{K''}$ is simply $T(x^{-1})_{K''}$.

Let $C_c^\infty(K'' \backslash G_V(\mathbb{A}_{\text{fin}})/K'')$ denote the subalgebra of $C_c^\infty(G_V(\mathbb{A}_{\text{fin}}))$ consisting of functions that are bi- K'' -invariant. If $\phi \in C_c^\infty(K'' \backslash G_V(\mathbb{A}_{\text{fin}})/K'')$, then we define a cycle $T(\phi)_{K''} \subseteq X_{K''} \times X_{K''}$ by

$$T(\phi)_{K''} := \sum_{x \in K'' \backslash G_V(\mathbb{A}_{\text{fin}})/K''} \phi(x) T(x)_{K''}. \quad (5.2.1)$$

Remark 5.2.1. Up to this point, none of what we have presented in this section requires $X_{K''}$ to be a surface or $F = \mathbb{Q}$. We only restrict our discussion to $X_{K''}$ because that is the setting we will personally use it.

With all of the preliminaries out of the way, we move towards the primary objective of this section, which is to show that the condition of π_3 being a theta lift of σ_3 allows us to reinterpret the projector $T(t_{\varphi_3, \varphi'_3})$ in terms of arithmetic theta lifts. Write

$$K'' = \prod'_{v \in \Sigma} K''_v \quad \text{and} \quad \pi_3 = \bigotimes'_{v \in \Sigma_{\text{fin}}} \pi_{3,v}$$

and fix a large finite set of places $\mathbf{S} \subseteq \Sigma$ such that \mathbf{S} contains

- (i) the archimedean place of \mathbb{Q} ;
- (ii) all finite places $p \in \Sigma_{\text{fin}}$ such that $K''_p \not\cong G_V(\mathbb{Z}_p)$; and
- (iii) all finite places $p \in \Sigma_{\text{fin}}$ where $\pi_{3,v}$ is ramified.

Define

$$K''^{\mathbf{S}} := \prod_{v \notin \mathbf{S}} K''_v$$

and consider the spherical Hecke algebra

$$\mathbb{T}^{\mathbf{S}} := C_c^\infty(K''^{\mathbf{S}} \backslash G_V(\mathbb{A}_{\text{fin}}^{\mathbf{S}})/K''^{\mathbf{S}})$$

of $G_V(\mathbb{Q})$. We may write

$$\mathbb{T}^{\mathbf{S}} = \bigotimes_{v \notin \mathbf{S}} \mathbb{T}_v,$$

where $\mathbb{T}_v := C_c^\infty(K_v'' \backslash G_V(\mathbb{Q}_v)/K_v'')$. Then $\pi_{3,v}^{K_v''}$ is one-dimensional for all $v \notin \mathbf{S}$ and so we may consider the so called **spherical Hecke character**

$$\chi_{\pi_{3,v}} : \mathbb{T}_v \longrightarrow \mathbb{C}, \quad (5.2.2)$$

defined by $\pi_{3,v}(\Phi)\varphi_{3,v} = \chi_{\pi_{3,v}}(\Phi)\varphi_{3,v}$ for all $\Phi \in \mathbb{T}_v$ and $\varphi_{3,v} \in \pi_{3,v}^{K_v''}$ (cf. [Bum97, Section 4.6], [GH11, Section 10.6]). Hence, we may define a global character

$$\chi_{\pi_3}^{\mathbf{S}} : \mathbb{T}^{\mathbf{S}} \longrightarrow \mathbb{C}, \quad (5.2.3)$$

by taking tensor products, i.e. $\chi_{\pi_3}^{\mathbf{S}} := \otimes \chi_{\pi_{3,v}}$. We set $\mathfrak{m}_{\pi_3}^{\mathbf{S}} := \ker \chi_{\pi_3}^{\mathbf{S}}$, which is a maximal ideal on $\mathbb{T}^{\mathbf{S}}$.

Let $B_1, \dots, B_N \subset X_{K''}$ be the set of irreducible boundary components, i.e.

$$X_{K''} \setminus X_{K''}^\circ = \prod_{j=1}^N B_j.$$

The inclusion $X_{K''}^\circ \hookrightarrow X_{K''}$ induces a $\mathbb{T}^{\mathbf{S}}$ -equivariant surjective map

$$\mathrm{Ch}^1(X_{K''}) \longrightarrow \mathrm{Ch}^1(X_{K''}^\circ), \quad (5.2.4)$$

whose kernel is generated by the images of B_1, \dots, B_N in $\mathrm{Ch}^1(X_{K''})$; we let

$$\mathcal{B} := \ker(\mathrm{Ch}^1(X_{K''}) \rightarrow \mathrm{Ch}^1(X_{K''}^\circ)).$$

Then the support of it is finite, and does not contain $\mathfrak{m}_{\pi_3}^{\mathbf{S}}$ as the boundary does not generate cuspidal representations. The payoff of these considerations is that there exists a Hecke operator $\beta \in \mathbb{T}^{\mathbf{S}}$ such that β annihilates \mathcal{B} , but acts as identity on π_3 . Indeed for each boundary component B_j , let ρ_j be a representation appearing in $H^2(X_{K''})$ that contains the subspace spanned by B_j . Then there is a character $\chi_{\rho_j}^{\mathbf{S}} : \mathbb{T}^{\mathbf{S}} \rightarrow \mathbb{C}$ constructed in precisely the

same manner as $\chi_{\pi_3}^{\mathbb{S}}$ for π_3 ; let $\mathfrak{m}_{\rho_j}^{\mathbb{S}} := \ker \chi_{\rho_j}^{\mathbb{S}}$. Moreover, there is a surjective homomorphism of rings

$$\mathbb{T}^{\mathbb{S}} \longrightarrow \frac{\mathbb{T}^{\mathbb{S}}}{\mathfrak{m}_{\pi_3}^{\mathbb{S}}} \oplus \frac{\mathbb{T}^{\mathbb{S}}}{\mathfrak{m}_{\rho_1}^{\mathbb{S}}} \oplus \cdots \oplus \frac{\mathbb{T}^{\mathbb{S}}}{\mathfrak{m}_{\rho_N}^{\mathbb{S}}} \quad \beta \longmapsto \left(\beta \bmod \mathfrak{m}_{\pi_3}^{\mathbb{S}}, \beta \bmod \mathfrak{m}_{\rho_1}^{\mathbb{S}}, \dots, \beta \bmod \mathfrak{m}_{\rho_N}^{\mathbb{S}} \right),$$

and any $\beta \in \mathbb{T}^{\mathbb{S}}$ lifting $(1, 0, \dots, 0) \in \mathbb{T}^{\mathbb{S}}/\mathfrak{m}_{\pi_3}^{\mathbb{S}} \oplus \mathbb{T}^{\mathbb{S}}/\mathfrak{m}_{\rho_1}^{\mathbb{S}} \oplus \cdots \oplus \mathbb{T}^{\mathbb{S}}/\mathfrak{m}_{\rho_N}^{\mathbb{S}}$ under the above map annihilates \mathcal{B} , but acts as the identity on π_3 .

We choose $f_3, f'_3 \in \sigma_3$, $\phi_3, \phi'_3 \in \mathcal{S}(\mathbb{V})$ and an $H_1(\mathbb{A}) \times \mathrm{U}(\mathbb{V})(\mathbb{A})$ equivariant map $p : \overline{\sigma_3} \otimes \omega_\chi \rightarrow \pi_3$ so that $\varphi_3 = p(f_3, \phi_3)$, $\varphi'_3 = p(f'_3, \phi'_3)$. Then we have a generating series $\mathcal{Z}^\circ(h, \phi_3)_{K''}$ valued in $\mathrm{Ch}^1(X_{K''}^\circ)$, which is modular by Theorem 3.5 of [Liu11a]. On the other hand, we also have the compactified generating series $\mathcal{Z}(h, \phi_3)_{K''}$ valued in $\mathrm{Ch}^1(X_{K''})$, which is in general *not* modular. We may also create a new generating series, $\beta_* \mathcal{Z}(h, \phi_3)_{K''}$, by applying the Hecke action of β term-by-term to the Chow cycles:

$$\beta_* \mathcal{Z}(h, \phi_3)_{K''} := \sum_{x \in K'' \setminus \mathbb{V}_{\mathrm{fin}}} (\omega_\chi(h) \phi_3)(Q(x), x) \beta_* Z(x)_{K''}.$$

This generating series *is* modular and is central to our argument.

Lemma 5.2.2. Let $\beta \in \mathbb{T}^{\mathbb{S}}$ and $\beta_* \mathcal{Z}(h, \phi_3)_{K''}$ be as defined above.

(i) (Hecke Exchange) There exists $\beta_* \phi_3 \in \mathcal{S}(\mathbb{V})$ such

$$\beta_* \mathcal{Z}(h, \phi_3)_{K''} = \mathcal{Z}(h, \beta_* \phi_3)_{K''}.$$

(ii) The compactified generating series $\beta_* \mathcal{Z}(h, \phi_3)_{K''} = \mathcal{Z}(h, \beta_* \phi_3)_{K''}$ is modular.

Proof. The first claim follows immediately from [LL21, Lemma 4.4]. In fact, Lemma 4.4 of [LL21] implies that the function $\beta_* \phi_3$ is indeed the result of applying the Hecke action of β to ϕ_3 via the left $\mathbb{T}^{\mathbb{S}}$ -module structure on $\mathcal{S}(\mathbb{V})$ arising from the Weil representation ω_χ . As for our second claim, our choice in $\beta \in \mathbb{T}^{\mathbb{S}}$ implies that we have an isomorphism of Hecke modules:

$$\beta_* \mathrm{Ch}^1(X_{K''}) \simeq \beta_* \mathrm{Ch}^1(X_{K''}^\circ).$$

The claim then follows from the modularity of $\mathcal{Z}^\circ(h, \phi_3)$ in $\mathrm{Ch}^1(X_{K''}^\circ)$ along with the fact that the closure of $\mathcal{Z}^\circ(h, \beta_*\phi_3)_{K''}$ in $\beta_*\mathrm{Ch}^1(X_{K''})$ is $\mathcal{Z}(h, \beta_*\phi_3)_{K''}$. \square

There is one major obstacle to our approach: it has been exceedingly tricky to determine whether $\mathcal{Z}(h, \beta_*\phi_3)_{K''}$ vanishes in $\mathrm{Ch}^1(X_{K''})$. We are unsure at the present time, but we nonetheless expect it to be true. Moving forward, we assume the following conjecture:

Conjecture 5.2.3. There are $\phi_3 \in \mathcal{S}(\mathbb{V})$ and $\beta_3 \notin \mathfrak{m}_{\pi_3}^{\mathfrak{s}}$ such that $\mathcal{Z}(h, \beta_*\phi_3)_{K''} \neq 0$ in $\mathrm{Ch}^1(X_{K''})$.

For the remainder of the paper, we assume Conjecture 5.2.3 holds. We may define the *arithmetic theta lift* by the formula

$$\Theta_{f_3}^{\beta_*\phi_3} = \int_{[H_1]} \overline{f_3(h)} \mathcal{Z}(h, \beta_*\phi_3)_{K''} dh. \quad (5.2.5)$$

The theta lift $\Theta_{f_3}^{\beta_*\phi_3}$ is a well-defined element in $\mathrm{Ch}^1(X_{K''})$ by Lemma 5.2.2 and nontrivial by Conjecture 5.2.3. Moreover, $\Theta_{f_3}^{\beta_*\phi_3}$ is cohomologically trivial and compatible with pull-backs as the level K'' varies [Liu11a, Section 3D].

The assumption that the representation π_3 is tempered has important implications on its contribution to the cohomology of $X_{K''}$. In particular, Matsushima's formula [Mat67] expresses $H^i(X_{K''})$ in terms of automorphic forms on G_V and the $(\mathfrak{g}_V, K_\infty'')$ -cohomology of $\pi_{3,\infty}$, where $K_\infty'' \subseteq G_V(\mathbb{R})$ is a maximal compact subgroup. Moreover, a result of Borel and Wallach [BW80, Theorem III.5.1] implies that the $(\mathfrak{g}_V, K_\infty'')$ -cohomology vanishes outside of the middle degree. All of this to say that the hypothesis that π_3 is tempered implies that it only contributes to the middle cohomology, $H^2(X_{K''})$. We make use of this fact in the proposition that follows.

Proposition 5.2.4. There is a nonzero constant C such that

$$\mathrm{cl} \left(\Theta_{f_3}^{\beta_*\phi_3} \times \overline{\Theta_{f'_3}^{\beta_*\phi'_3}} \right) = C \cdot \mathrm{cl}(T(t_{\varphi_3, \varphi'_3})_{K''}) \in H^4(X_{K''} \times X_{K''}). \quad (5.2.6)$$

Proof. Note $\pi_3^{K''} \otimes \overline{\pi_3^{K''}}$ is an irreducible \mathbb{T}^S -module because π_3 is irreducible. It then follows that the image of the Hecke equivariant map

$$\pi_3^{K''} \otimes \overline{\pi_3^{K''}} \longrightarrow H^4(X_{K''} \times X_{K''}), \quad (\varphi_3, \overline{\varphi_3}) \longmapsto \text{cl} \left(T(t_{\varphi_3, \varphi_3'})_{K''} \right)$$

lies in one of the Künneth components of $H^4(X_{K''} \times X_{K''})$ by irreducibility of $\pi_3^{K''} \otimes \pi_3^{K''}$ and Hecke invariance. Since π_3 is tempered, it contributes only to the middle cohomology by our comments above, i.e. $H^2(X_{K''}) \otimes H^2(X_{K''})$. On the other hand, the map

$$\overline{\sigma_3} \otimes \omega_\chi \otimes \sigma_3 \otimes \overline{\omega_\chi} \longrightarrow H^2(X_{K''}) \times H^2(X_{K''}), \quad (f_3, f_3', \phi_3, \phi_3') \mapsto \text{cl} \left(\Theta_{f_3}^{\beta_* \phi_3} \times \overline{\Theta_{f_3'}^{\beta_* \phi_3'}} \right)$$

factors through $\pi_3 \otimes \overline{\pi_3}$ because π_3 is a theta lift of σ_3 . It then follows that both sides of Equation (5.2.6) give Hecke equivariant maps

$$\pi_3^{K''} \otimes \overline{\pi_3^{K''}} \longrightarrow H^2(X_{K''}) \times H^2(X_{K''}).$$

Since the tempered part of $H^2(X_{K''})$ is multiplicity free as a Hecke module (cf. [Mok15]), these two maps must differ by a constant depending only on π_3 that is necessarily nonzero because both sides of Equation (5.2.6) are not zero. \square

We end this subsection by noting some of the challenges regarding the modularity of the compactified generating series that we alluded to at the end of Section 4.5. If $\mathcal{Z}(h, \phi)_{K''}$ is modular, then we can define the arithmetic theta lift, $\Theta_{f_3}^{\phi_3}$, by the same formula in Equation (5.2.5) for arbitrary $f_3 \in \sigma_3$, rather than those of the form $\beta_* f_3$ for some carefully selected Hecke operator β . Though modularity is undoubtedly the most significant obstruction to generalizing results to toroidal compactifications, one would still have the challenge of showing $\Theta_{f_3}^{\phi_3}$ is cohomologically trivial or modify the generating series in some way that maintains modularity and ensures that the arithmetic theta lift is cohomologically trivial (cf. e.g. [Liu11b, Definition 3.1] in the case of non-proper unitary Shimura curves) in addition to showing that theta lift does not vanish. Though this would be no small feat, these problems could presumably be remedied by well-established methods and standard techniques, e.g. doubling zeta integrals, geometric/arithmetical Siegel-Weil formulae, etc.

5.3 L -Functions

First, recall $\pi = \pi_3 \boxtimes \pi_2$, where π , π_2 , and π_3 are as defined at the beginning of Section 5. We let $\text{BC}(\pi_i)$ denote the (unramified) base change of π_i for $i = 2, 3$, which is an irreducible tempered representation of $\text{GL}_i(\mathbb{A}_E)$. Then our main theorem is as follows:

Theorem 5.3.1. Keeping with the notation at the beginning of Section 5, assume that $\text{Hom}_{G_W(\mathbb{A}_{\text{fin}})}(\pi, \mathbb{C}) \neq 0$ and $\mathcal{Z}(h, \beta_* \phi_3)_{K''} \neq 0$. Then the following are equivalent:

(a) The map

$$\begin{aligned} \pi \otimes \bar{\pi} &\longrightarrow \mathbb{C} \\ \varphi \otimes \bar{\varphi}' &\longmapsto \langle T(t_{\varphi, \varphi}')_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle_{\text{BB}} \end{aligned}$$

is not identically zero;

(b) $L' \left(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2) \right) \neq 0$.

Recall that we assumed that π_3 and π_2 are theta lifts of σ_3 and σ_2 , respectively, where σ_3, σ_2 are irreducible cuspidal automorphic representations of $H_1(\mathbb{A}_{\text{fin}})$. This hypothesis has entered the discussion on multiple occasions, but its implication on the L -function $L(s, \text{BC}(\pi_3) \times \text{BC}(\pi_2))$ is especially consequential. Specifically, it is a well-known fact (cf. [Xue19, Lemma 5.5]) that

$$L(s, \text{BC}(\pi_3) \times \text{BC}(\pi_2)) = L \left(s, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1} \right) L \left(s, \sigma_2 \times \chi^2 \right), \quad (5.3.1)$$

where $\chi : E^\times \setminus \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ is the character splitting the quadratic character $\eta_{E/\mathbb{Q}} : \mathbb{Q}^\times \setminus \mathbb{A}^\times \rightarrow \{\pm 1\}$ used to define the Weil representation as described in Section 3.3 (i.e. $\chi_v|_{\mathbb{Q}_v^\times} = \eta_{E_v/\mathbb{Q}_v}^3$). Now, $L \left(\frac{1}{2}, \sigma_2 \times \chi^2 \right) = 0$ because its theta lift, π_2 , is a representation of an incoherent unitary group [Liu11a, Theorem 2.12]. We then have

$$L' \left(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2) \right) = L \left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1} \right) L' \left(\frac{1}{2}, \sigma_2 \times \chi^2 \right), \quad (5.3.2)$$

from the decomposition of $L(s, \text{BC}(\pi_3) \times \text{BC}(\pi_2))$ in Equation (5.3.1) coupled with the

vanishing of $L(s, \sigma_2 \times \chi^2)$ at $s = \frac{1}{2}$. Equation (5.3.2) translates the non-vanishing of $L\left(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2)\right)$ to the non-vanishing of two factors that are essentially known. Specifically the factor $L\left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1}\right)$ appears in the statement of the now settled GGP conjecture for $\text{U}(2) \times \text{U}(2)$ [Har14, Corollary 4.7] and the factor $L'\left(\frac{1}{2}, \sigma_2 \times \chi^2\right)$ appears in the Liu's arithmetic inner product formula on unitary Shimura curves [Liu11b, Theorem 6.10]. We will take the remainder of this subsection to highlight these crucial results.

We first recall Liu's arithmetic inner product formula, which relates the Neron-Tate height of certain cycles on Y_K to $L'\left(\frac{1}{2}, \sigma_2 \times \chi^2\right)$. If $f_2 \in \sigma_2$ and $\phi_2 \in \mathcal{S}(\mathbb{W})$, then we can define an arithmetic theta lift by the same formula in Equation (5.2.5):

$$\Theta_{f_2}^{\phi_2} := \int_{[H_1]} \overline{f_2(h)} \mathcal{Z}(h, \phi_2)_K dh \in \text{Ch}^1(Y_K).$$

As mentioned in Section 4.5, the generating series $\mathcal{Z}(h, \phi_2)_K$ is modular and so $\Theta_{f_2}^{\phi_2}$ is well-defined.⁵ Then, for any totally real field F , Theorem 1.1 of [Liu11b] asserts

$$\left\langle \Theta_{f_2}^{\phi_2}, \overline{\Theta_{f'_2}^{\phi'_2}} \right\rangle_{\text{NT}} = (*) L'\left(\frac{1}{2}, \sigma_2 \times \chi^2\right),$$

where $(*)$ is some explicitly defined nonzero constant. It is worth noting that Liu conjectures a generalization of the equality above relating the Beilinson-Bloch pairing of higher-dimensional arithmetic theta lifts, say valued Ch^r of some unitary Shimura variety of arbitrary dimension, to the central derivative of a (twisted) automorphic L -function attached to $H_r(\mathbb{A}_F)$ [Liu11a, Conjecture 3.7]. While Liu's inner product formula provides us with a wealth of insight, we will make use of the following reformulation due to Xue that is more tailored to our purposes.

Proposition 5.3.2 ([Xue19], 4.8). Assume the notation above, in particular π_2 is a theta lift of σ_2 .

⁵Technically, since we are assuming $F = \mathbb{Q}$, $\Theta_{f_2}^{\phi_2}$ is not cohomologically trivial as it is defined above for arbitrary ϕ_2 (though this definition is just fine for $F \neq \mathbb{Q}$). However Liu modifies $\mathcal{Z}(h, \phi_2)$ by the addition of a multiple of $c_1(\mathcal{L}_K)$ in such a way that maintains modularity while also ensuring $\Theta_{f_2}^{\phi_2}$ [Liu11a, Section 3D]. These details are not necessary for our purposes because the Hecke operator $\beta \in \mathbb{T}^{\mathfrak{s}}$ as defined in Section 5.2 annihilates this "error" term. When $F \neq \mathbb{Q}$, this term vanishes and $\Theta_{f_2}^{\phi_2}$ is cohomologically trivial as defined.

(i) For $\varphi_2, \varphi'_2 \in \pi_2$ and $\phi_2, \phi'_2 \in \mathcal{S}(\mathbb{W})$, the function

$$\left\langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_2), \overline{\Delta_{Y,1,*} \mathcal{Z}(h', \phi'_2)} \right\rangle_{\text{NT}} \quad (h, h') \in H_1(\mathbb{A}_F) \times H_1(\mathbb{A}_F)$$

is an automorphic form in $\sigma_2 \otimes \bar{\sigma}_2$, which is a cusp form. It is independent of the choice of level $K \subseteq G_W(\mathbb{A}_{\text{fin}})$ and the choice of the measure used to define $t_{\varphi_2, \varphi'_2}$.

(ii) We have

$$\int_{[H_1]} \left\langle T(t_{\varphi_2, \varphi'_2})_{K,*} \Delta_{Y_K,1,*} \mathcal{Z}(h, \phi_2)_K, \Delta_{Y_K,1,*} \mathcal{Z}(h, \phi'_2)_K \right\rangle_{\text{NT}} dh = (*) L' \left(\frac{1}{2}, \sigma_2 \times \chi^2 \right),$$

where $(*)$ is an explicitly defined nonzero constant.

As for the factor, $L\left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1}\right)$, let $f_2, f'_2 \in \sigma_2$ and $f_3, f'_3 \in \sigma_3$. For the purposes of presenting the GGP conjecture in the greatest generality, we do *not* assume $F = \mathbb{Q}$ and we temporarily think of σ_2 and σ_3 as arbitrary irreducible cuspidal tempered automorphic representations of $H_1(\mathbb{A}_F)$ that have *no* relation to the representations π_2 and π_3 (though this is obviously the situation we are interested in). Let $\phi_1, \phi'_1 \in \mathcal{S}(\mathbb{A}_E)$ be Schwartz functions and let $\theta(h, \phi_1)$ and $\theta(h, \phi'_1)$ denote the theta functions on $H_1(\mathbb{A}_F)$ attached to ϕ_1 and ϕ'_1 , respectively; that is

$$\theta(h, \phi_1) := \sum_{x \in W_1(F)} (\omega_\chi(h) \phi_1)(x) \quad h \in H_1(\mathbb{A}_F), \quad (5.3.3)$$

where W_r is the standard skew-Hermitian space of dimension $2r$ over E from Section 3.1. Then the GGP conjecture for $U(2) \times U(2)$ is as follows.

Proposition 5.3.3 ([Har14], Corollary 4.7). Keeping with the notation above, the following are equivalent:

- (i) $L\left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1}\right) \neq 0$
- (ii) $\int_{[H_1]} f_1(h) f_2(h) \overline{\theta(h, \phi_1)} dh \cdot \overline{\int_{[H_1]} f'_1(h) f'_2(h) \overline{\theta(h, \phi'_1)} dh} \neq 0$

As we will see shortly, the integrals appearing in Proposition 5.3.3 make an appearance when computing $\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle$.

Remark 5.3.4. If $\mathbf{R} \subseteq \Sigma$ is a finite set of places, then it is sufficient to establish Theorem 5.3.1 for partial L -functions $L^{\mathbf{R}}(s, \mathrm{BC}(\pi_3) \times \mathrm{BC}(\pi_2))$. In general, the L -functions appearing in the “usual” GGP conjectures as well as the L -derivatives appearing in the arithmetic variant are insensitive to the the definition of local L -factors at finitely many (“bad”) places. See [Xue19, Section 5.1] and [Zha14, Remark 1] for more details.

5.4 Proof of Theorem 5.3.1

The proof of Theorem 5.3.1 follows the same reasoning as in [Xue19]. However, before proceeding to the proof we take note of some key facts. In order to make the notation slightly less cumbersome, we drop the level K , K'' , and K' from the subscripts for the remainder of the paper. We also write $\langle -, - \rangle$ for both the Beilinson–Bloch and the Neron–Tate height pairing.

First recall from the beginning of the section that we wrote $\varphi, \varphi' \in \pi$ as follows

$$\varphi = \varphi_3 \otimes \varphi_2 \quad \text{and} \quad \varphi' = \varphi'_3 \otimes \varphi'_2,$$

where $\varphi_2, \varphi'_2 \in \pi_2$ and $\varphi_3, \varphi'_3 \in \pi_3$. The decomposition of φ and φ' implies

$$T(t_{\varphi, \varphi'}) = T(t_{\varphi_3, \varphi'_3}) \times T(t_{\varphi_2, \varphi'_2})$$

and so it follows that

$$\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \left\langle \left(\Theta_{f_3}^{\beta_* \phi_3} \times \overline{\Theta_{f'_3}^{\beta_* \phi'_3}} \times T(t_{\varphi_2, \varphi'_2}) \right)_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \right\rangle,$$

by Proposition 5.2.4. Further, we have

$$\left(\Theta_{f_3}^{\beta_* \phi_3} \times \overline{\Theta_{f'_3}^{\beta_* \phi'_3}} \times T(t_{\varphi_2, \varphi'_2}) \right)_* \Delta_{M,3,*} y = \overline{\Theta_{f'_3}^{\beta_* \phi'_3}} \times T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \Theta_{f_3}^{\beta_* \phi_3} \Big|_Y,$$

by [Xue19, Lemma 5.7] and so

$$\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \left\langle \overline{\Theta_{f'_3}^{\beta_* \phi'_3}} \times T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \Theta_{f_3}^{\beta_* \phi_3} \Big|_Y, \Delta_{M,3,*} y \right\rangle.$$

On the other hand, Proposition 5.1.1 implies

$$\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \left\langle T(t_{\varphi_2, \varphi'_2})_* \left(\Delta_{Y,1,*} \Theta_{f_3}^{\beta_* \phi_3} \Big|_Y \right), \overline{\Delta_{Y,1,*} \left(\Delta_{X,2,*} \Theta_{f'_3}^{\beta_* \phi'_3} \right) \Big|_Y} \right\rangle.$$

Since

$$\Delta_{X,2,*} \Theta_{f'_3}^{\beta_* \phi'_3} = \Theta_{f'_3}^{\beta_* \phi'_3}$$

by Lemma 5.6 of [Xue19], we finally arrive at

$$\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle = \left\langle T(t_{\varphi_2, \varphi'_2})_* \left(\Delta_{Y,1,*} \Theta_{f_3}^{\beta_* \phi_3} \Big|_Y \right), \overline{\Delta_{Y,1,*} \Theta_{f'_3}^{\beta_* \phi'_3} \Big|_Y} \right\rangle, \quad (5.4.1)$$

thereby reducing the Beilinson-Bloch pairing to the Neron-Tate height (at least for the cycles of interest).

Now write

$$\beta_* \phi_3 = \sum_i \phi_{2,i} \otimes \phi_{1,i} \quad \text{and} \quad \beta_* \phi'_3 = \sum_i \phi'_{2,i} \otimes \phi'_{1,i},$$

where $\phi_{2,i}, \phi'_{2,i} \in \mathcal{S}(\mathbb{W})$ and $\phi_{1,i}, \phi'_{1,i} \in \mathcal{S}(\mathbb{A}_E)$; it then follows that

$$\mathcal{Z}(h, \beta_* \phi_3) = \sum_i \mathcal{Z}(h, \phi_{2,i} \otimes \phi_{1,i}).$$

If we let $j : Y \rightarrow X$ denote the embedding of Y into X induced by the inclusion $W \rightarrow V$ and $j^\circ : Y^\circ \hookrightarrow X^\circ$ denote the restriction of j to the open-parts of Y and X , then Proposition 4.5 of [Xue19] implies

$$j^{\circ,*} \left(\sum_i \mathcal{Z}^\circ(h, \phi_{2,i} \otimes \phi_{1,i}) \right) = \sum_i \mathcal{Z}^\circ(h, \phi_{2,i}) \theta(h, \phi_{1,i}),$$

where $\theta(h, \phi_{1,i})$ is the theta-function from Equation (5.3.3). The difference, regarded as an element in $\text{Ch}^1(Y)$,

$$j^* \mathcal{Z}(h, \beta_* \phi_3) - \sum_i \mathcal{Z}^\circ(h, \phi_{2,i}) \theta(h, \phi_{1,i}),$$

is supported on the cusps of Y and therefore must be a multiple of the Hodge bundle on Y because the Hodge bundle is supported on the cusps and any two cusps are rationally equivalent by the Manin-Drinfeld Theorem. If we let $e(h)$ denote this multiple of the hodge

bundle, then it follows

$$j^* \mathcal{Z}(h, \beta_* \phi_3) = \sum_i \mathcal{Z}^\circ(h, \phi_{2,i}) \theta(h, \phi_{1,i}) + e(h) c_1(\mathcal{L}_Y^\vee).$$

Note that since $\mathcal{Z}(h, \beta_* \phi_3)$ and each $\mathcal{Z}^\circ(h, \phi_{2,i})$ are all automorphic on H_1 , $e(h)$ must be an automorphic form on H_1 as well. It then follows that

$$\Delta_{Y,1,*} \Theta_{f_3}^{\beta_* \phi_3} \Big|_Y = \sum_i \int_{[H_1]} \overline{f_3(h)} \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \theta(h, \phi_{1,i}) dh - \left(\int_{[H_1]} \overline{f_3(h)} e(h) dh \right) \Delta_{Y,1,*} c_1(\mathcal{L}^\vee).$$

However, note $\Delta_{Y,1,*} c_1(\mathcal{L}^\vee) = 0$ by the definition of $\Delta_{Y,1,*}$. We may apply precisely the same reasoning to $\Delta_{Y,1,*} \Theta_{f'_3}^{\beta_* \phi'_3}$ and combine with Equation (5.4.1) in order to deduce that the value of $\langle T(t_{\varphi,\varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle$ is equal to the expression below:

$$\sum_{i,j} \iint_{[H_1]^2} \overline{f_3(h)} f'_3(h') \left\langle T(t_{\varphi_2,\varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \overline{\Delta_{Y,1,*} \mathcal{Z}(h', \phi'_{2,j})} \right\rangle \theta(h, \phi_{1,i}) \overline{\theta(h', \phi'_{1,j})} dh dh'. \quad (5.4.2)$$

With these considerations in mind, we proceed to the proof of Theorem 5.3.1, which follows along the same lines as in [Xue19].

Proof of Theorem 5.3.1. First assume that $\langle T(t_{\varphi,\varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle_{\text{BB}} \neq 0$. Thus the expression in Equation (5.4.2) is nonzero and so there exists some i, j such that

$$\iint_{[H_1]^2} \overline{f_3(h)} f'_3(h') \left\langle T(t_{\varphi_2,\varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \overline{\Delta_{Y,1,*} \mathcal{Z}(h', \phi'_{2,j})} \right\rangle \theta(h, \phi_{1,i}) \overline{\theta(h', \phi'_{1,j})} dh dh' \neq 0.$$

This immediately implies

$$L\left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1}\right) \neq 0$$

by Proposition 5.3.3 and

$$\left\langle T(t_{\varphi_2,\varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \overline{\Delta_{Y,1,*} \mathcal{Z}(h', \phi'_{2,j})} \right\rangle$$

is not identically zero. The non-degeneracy of the Neron-Tate height coupled with (ii) implies that

$$\int_{[H_1]} \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle dh \neq 0$$

By the Projection Formula, we have

$$\begin{aligned} & \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle \\ &= \langle T(t_{\varphi_2, \varphi'_2})^* T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle \\ &= (*) \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle, \end{aligned}$$

where $(*)$ is a positive constant. It then follows

$$\begin{aligned} 0 &\neq \int_{[H_1]} \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle dh \\ &\iff \\ 0 &\neq \int_{[H_1]} \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle dh. \end{aligned}$$

However,

$$\int_{[H_1]} \langle T(t_{\varphi_2, \varphi'_2})_* \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}), \Delta_{Y,1,*} \mathcal{Z}(h, \phi_{2,i}) \rangle dh \neq 0 \iff L' \left(\frac{1}{2}, \sigma_2 \times \chi^2 \right) \neq 0$$

by Proposition 5.3.2. Since

$$L \left(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1} \right) \neq 0 \quad \text{and} \quad L' \left(\frac{1}{2}, \sigma_2 \times \chi^2 \right) \neq 0,$$

we have

$$L' \left(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2) \right) \neq 0$$

by our remarks in Section 5.3 and, specifically, Equation (5.3.2).

Conversely, assume $L' \left(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2) \right) \neq 0$; as we have established, the statement

$L'(\frac{1}{2}, \text{BC}(\pi_3) \times \text{BC}(\pi_2)) \neq 0$ is equivalent to

$$L(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \times \chi^{-1}) \neq 0 \quad \text{and} \quad L'(\frac{1}{2}, \sigma_2 \times \chi^2) \neq 0.$$

We fix a large finite set of places $\mathbf{R} \subseteq \Sigma$ such that $\mathbf{S} \subseteq \mathbf{R}$ and let $\beta = \text{id}^{\mathbf{R}} \otimes \beta_{\mathbf{R}} \in \mathbb{T}$ where $\text{id}^{\mathbf{R}} = \bigotimes_{v \notin \mathbf{R}} \text{id}_v \in \mathbb{T}^{\mathbf{R}}$ is the identity element in $\mathbb{T}^{\mathbf{R}}$, the spherical Hecke algebra away from \mathbf{R} , and $\beta_{\mathbf{R}} = \bigotimes_{v \in \mathbf{R}} \beta_v$ is an element in $\mathbb{T}_{\mathbf{R}}$, the spherical Hecke algebra at \mathbf{R} .

Due to the calculation in [Xue19, Section 5.4] (specifically the proof of Theorem 5.3), we see that the right-hand side of Equation (5.4.2) is (up to a nonzero factor) equal to

$$L^{\mathbf{R}}(\frac{1}{2}, \text{BC}(\sigma_3) \times \text{BC}(\sigma_2) \otimes \chi^{-1}) \times (L^{\mathbf{R}})'(\frac{1}{2}, \sigma_1 \times \chi^2) \times I_{\mathbf{R}},$$

where

$$I_{\mathbf{R}} := \int_{G_W(\mathbb{A}_{\mathbf{R}})} \int_{H_1(\mathbb{A}_{\mathbf{R}})} \overline{\langle \sigma_{3,\mathbf{R}}(h) f_{3,\mathbf{R}}, f'_{3,\mathbf{R}} \rangle} \langle \omega_{\chi,\mathbf{R}}(h, g) \beta_{\mathbf{R},*} \phi_{3,\mathbf{R}}, \beta_{\mathbf{R},*} \phi'_{3,\mathbf{R}} \rangle \overline{\langle \pi_{2,\mathbf{R}}(g) \varphi_{2,\mathbf{R}}, \varphi'_{2,\mathbf{R}} \rangle} dh dg.$$

Recall from Section 5.2 that we choose $f_3, f'_3 \in \sigma_3$, $\phi_3, \phi'_3 \in \mathcal{S}(\mathbb{V})$ and an $H_1(\mathbb{A}) \times \text{U}(\mathbb{V})(\mathbb{A})$ -equivariant map $p: \overline{\sigma_3} \otimes \omega_{\chi} \rightarrow \pi_3$ so that $\varphi_3 = p(f_3, \phi_3)$, $\varphi'_3 = p(f'_3, \phi'_3)$. Following [Xue19, Section 4.6], we may normalize the map p place by place so that

$$\langle \varphi_{3,v}, \varphi'_{3,v} \rangle = \int_{H_1(\mathbb{Q}_v)} \overline{\langle \sigma_{3,v}(h) f_{3,v}, f'_{3,v} \rangle} \langle \omega_{\chi_v}(h) \phi_{3,v}, \phi'_{3,v} \rangle dh.$$

The integral expression for $\langle \varphi_{3,v}, \varphi'_{3,v} \rangle$ coupled with $H_1(\mathbb{A}) \times \text{U}(\mathbb{V})(\mathbb{A})$ -equivariance gives

$$\begin{aligned} I_{\mathbf{R}} &:= \int_{G_W(\mathbb{A}_{\mathbf{R}})} \int_{H_1(\mathbb{A}_{\mathbf{R}})} \overline{\langle \sigma_{3,\mathbf{R}}(h) f_{3,\mathbf{R}}, f'_{3,\mathbf{R}} \rangle} \langle \omega_{\chi,\mathbf{R}}(h, g) \beta_{\mathbf{R},*} \phi_{3,\mathbf{R}}, \beta_{\mathbf{R},*} \phi'_{3,\mathbf{R}} \rangle \overline{\langle \pi_{2,\mathbf{R}}(g) \varphi_{2,\mathbf{R}}, \varphi'_{2,\mathbf{R}} \rangle} dh dg \\ &= \int_{G_W(\mathbb{A}_{\mathbf{R}})} \overline{\langle \pi_{2,\mathbf{R}}(g) \varphi_{2,\mathbf{R}}, \varphi'_{2,\mathbf{R}} \rangle} \int_{H_1(\mathbb{A}_{\mathbf{R}})} \overline{\langle \sigma_{3,\mathbf{R}}(h) f_{3,\mathbf{R}}, f'_{3,\mathbf{R}} \rangle} \langle \omega_{\chi,\mathbf{R}}(h, g) \beta_{\mathbf{R},*} \phi_{3,\mathbf{R}}, \beta_{\mathbf{R},*} \phi'_{3,\mathbf{R}} \rangle dh dg \\ &= (*) \int_{G_W(\mathbb{A}_{\mathbf{R}})} \overline{\langle \pi_{2,\mathbf{R}}(g) \varphi_{2,\mathbf{R}}, \varphi'_{2,\mathbf{R}} \rangle} \langle \pi_{3,\mathbf{R}}(g) \beta_{\mathbf{R},*} \varphi_{3,\mathbf{R}}, \beta_{\mathbf{R},*} \varphi'_{3,\mathbf{R}} \rangle dg, \end{aligned}$$

where $(*)$ is a nonzero constant. However β acts by a nonzero constant on π_3 by our choice

in β , which implies

$$I_{\mathbf{R}} = (**) \int_{G_W(\mathbb{A}_{\mathbf{R}})} \overline{\langle \pi_{2,\mathbf{R}}(g)\varphi_{2,\mathbf{R}}, \varphi'_{2,\mathbf{R}} \rangle} \langle \pi_{3,\mathbf{R}}(g)\varphi_{3,\mathbf{R}}, \varphi'_{3,\mathbf{R}} \rangle dg,$$

where $(**)$ is a nonzero constant. Since $\text{Hom}_{G_W(\mathbb{A}_{\text{fin}})}(\pi, \mathbb{C}) \neq 0$, we can select data such that $I_{\mathbf{R}}$ is nonzero. Hence, the expression in Equation (5.4.2) is nonzero, which implies that $\langle T(t_{\varphi, \varphi'})_* \Delta_{M,3,*} y, \Delta_{M,3,*} y \rangle \neq 0$. \square

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