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Why does momentum depend on inertia?

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Abstract: Momentum is characterized in terms of inertial mass for particles moving at less than the speed of light, but entirely in terms of their energy for those lacking inertia. Does this difference suggest a physically distinct origin of momentum in the two cases and, if so, what is actually being conserved in interactions involving both types of particle? In this paper, we consider a recently proposed gravitational origin for rest-mass energy to demonstrate that a single definition of momentum applies to all particles, massless or otherwise. When introduced into this description, inertial mass is merely a surrogate for the particle's 'free' energy, but does not imply an origin of momentum different from that of particles without mass.

Keywords: inertial mass; gravitational mass; cosmology; apparent horizon

1 Introduction

The emergence of a clear distinction between the momenta of particles with and without inertia (m_i) creates some ambiguity about what momentum actually represents. Relativity suggests a straightforward interpretation of a photon's momentum purely in terms of its energy [1], though this was actually already hinted at in the nineteenth century by the properties of electromagnetic fields, while the classical momentum of a particle acted on by a Newtonian force is strictly expressible in terms of its inertial mass [2]. Momentum is completely conserved in any interaction involving both types of particle, however, so why does this formal difference arise and does it signify a real *physical* distinction?

A naive response to this question, invoking rest-mass energy,

$$E_0 = m_i c^2, \tag{1.1}$$

is inadequate because momentum would then be related to E_0 rather than the particle's kinetic (or even total) energy. Nevertheless, a recent explanation for the origin of rest-mass energy [3] bears directly on our understanding of momentum and its apparent dichotomy into systems with and without inertia.

It appears that rest-mass energy is simply the gravitational energy, $m_g c^2$, of a gravitating mass m_g bound to that portion of the Universe within our apparent (or gravitational) horizon, $R_h \equiv c/H_0$, where H_0 is the Hubble parameter [4]. The fact that this energy is also expressible in terms of the inertial mass, m_i, is due to a proportionality between these two masses [5], allowing inertia to act as a 'surrogate' for the gravitational coupling constant.

A possible connection between fundamental particle properties and a coupling with the rest of the Universe has been explored many times in the past, most famously by Berkeley [6] and Mach [7] and those who followed their basic concept, in which inertial frames represent zero acceleration relative to the 'fixed stars' or, more appropriately, with respect to the mean of the matter distribution in the Universe. Even Einstein believed that 'Mach's principle,' as he called it, would be foundational to general relativity, though he later realized that the two are not compatible [8, 9] because all gravitational influence disappears entirely for a particle in free-fall, while it nevertheless still exhibits inertia.

Mach himself never provided a physical mechanism describing how the motion of a local particle is affected by the distant Universe. But many have invoked his principle to develop alternative gravity theories, or alternative geometrical descriptions of motion in the cosmos (see, e.g., ref. [10]). In 1953, Sciama [11] proposed the addition of an acceleration-dependent contribution to Newton's law of gravity, calling this effect an 'inertial induction.' Perhaps more famously, Brans and Dicke [12] introduced Mach's principle into a new version of general relativity, by allowing the gravitational constant, G, to be determined by the contents of the Universe.

Today we know that inertia is not due to a Machian process. But though Mach's principle has never been successfully incorporated into a viable model of gravity, the origin of rest mass energy is entirely independent of the mechanism producing inertia [3]. We are again invoking a coupling between local particles and the rest of the cosmos, though it is now clear that this interaction merely creates a gravitational binding energy without any influence on inertia itself.

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The ultimate source of spacetime curvature in general relativity (and therefore gravity) is energy [1], so this new development with rest-mass energy must impact how momentum relates to m_g and/or m_i. Of course, the fact that momentum is proportional to inertia in the classical domain, and to energy in the case of photons, could in principle mean that there is a fundamental difference between the two classes of particle after all.

But in this paper we demonstrate that this new perspective on the origin of rest-mass energy instead allows us to apply a single definition of momentum to all particles, massless or otherwise, characterized solely by their 'free' energy in every case. As we shall see, the apparent difference arises only because the total particle energy includes the gravitational binding energy in some cases and not in others. This effect is purely expressible in terms of the inertial mass, which is the sole reason inertia appears in this discussion.

2 The origin of rest mass energy

Let us first briefly review this new interpretation of rest-mass energy. Inertia is created through a variety of mechanisms in the standard model. Most of the visible baryonic 'mass' in the Universe is due to the gluonic back-reaction on accelerated quarks [13]. With the discovery of the Higgs boson [14], we now also know that the latter attribute their own inertia to the Higgs field, which further assigns inertia to several other fundamental particles [15–17]. The origin of rest-mass energy, however, is unrelated to any of these processes, and has largely remained a mystery over the century since Einstein published his seminal paper [18] proposing the now famous Equation (1.1). What is clear, however, is that our current understanding of inertia urges us to think of it as an emergent feature of a particle's response to the applied force rather than some intrinsic property. In the end, the lack of a universal origin for inertia compounds the puzzle of why momentum should be proportional to m_i in some cases, but not in others.

The observed proportionality between m_i and m_g (for a recent review, see ref. [5]) is the basis for the equivalence principle in general relativity, hinting at a possible gravitational origin for rest-mass energy. But could an equivalence principle also exist for other forces? For example, we might argue that the electric charge, q, in an object is proportional to its matter content, so that doubling the quantity of matter (and thus the inertial mass) would also double the Coulomb force, and the ratio q/m_i would always remain the same. Would not this imply an equivalence between inertia and the electric charge, leading to an alternative equivalence principle arising from the indistinguishability of charges accelerated in an electromagnetic field and those viewed in a non-inertial frame uniformly accelerated in the opposite direction?

The answer is no because only gravity has the unique combination of properties allowing the equivalence principle to function in this way. It has a single charge, unlike the others, which have two or more. Gravity is therefore always attractive, while the others vary depending on the net charge. Further, gravitational charge cannot be destroyed. All forms of energy contribute to an effective coupling constant that accumulates—as does inertia—whereas electric charge can be completely cancelled. Gravity is thus the only known force for which its coupling constant is always proportional to m_i. It is also the only force that may extend over a vast volume of space—the cosmos. It is energetically prohibitive to separate electric charges over large distances, so the Universe is neutral on scales larger than, say, the solar system. Thus, an equivalence principle can only be formulated for gravity, and because gravitational energy is proportional to m_g—and therefore to m_i—it is not at all unreasonable to suppose that rest-mass energy is due to a particle's gravitational coupling to that portion of the Universe with which it is causally connected.

Recent work in cosmology has provided some insights into the question of how large this region is [4]. The role played by the apparent (or gravitational) horizon, R_h, in both cosmological theory and the interpretation of observational measurements, suggests that the gravitational coupling of a particle with mass m_g ought to extend over all proper distances $R \leq R_h$ [3]. This critical distance coincides with the better known Hubble radius in a spherically symmetric spacetime.

Throughout this paper, we assume that the cosmic spacetime is described by the Friedmann–Lemaître–Robertson–Walker (FLRW) metric with zero spatial curvature [1, 19],

$$ds^2 = c^2 dt^2 - a(t)^2 \left(dr^2 + r^2 d\Omega^2 \right), \tag{2.1}$$

in which $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$. In these expressions, (r, θ, ϕ) are the comoving coordinates, t is the cosmic time, and $a(t)$ is the universal expansion factor. Proper distance is defined as $R \equiv a(t)r$.

As shown in ref. [3], the particle's energy, E, along a geodesic in a homogeneous and isotropic Universe (i.e., one satisfying the cosmological principle associated with the FLRW metric), is given by the expression

$$E^2 = \frac{(c\kappa)^2\Phi + (cp)^2}{\left[\Phi + \left(\frac{R}{R_h}\right)\frac{\dot{R}}{c}\right]^2}, \tag{2.2}$$

where

$$\Phi(R) \equiv 1 - \left(\frac{R}{R_h}\right)^2, \tag{2.3}$$

and overdot signifies a derivative with respect to t. In addition, κ^2 is the contraction of the four-vector $p^\mu = (E/c, p^R, p^\theta, p^\phi)$, in terms of the spatial components of the 3-momentum in spherical geometry, whose magnitude is p. Since the goal of this exercise is to *derive* the rest-mass energy associated with m_i—and thus the value of κ—its form is left unspecified at this step.

As shown in ref. [4], a particle's peculiar velocity has no impact on this argument, chiefly because its speed approaches c at $R \to R_h$, regardless of what its velocity is at shorter distances. We therefore place the inertial mass within the Hubble flow and write

$$p^R \equiv m_i \dot{R} \tag{2.4}$$

where, according to Hubble's law [19],

$$\dot{R} = c\frac{R}{R_h}. \tag{2.5}$$

Note that there is no time dilation factor (which reduces to the Lorentz factor γ in Minkowski space) in this expression because the cosmic time t used here to infer the expansion speed $\dot{R}$ is in fact the local proper time at every spacetime point within the medium. Thus, if we ignore the angular motion for now, we can immediately see that $E \to p^R c = m_i c^2$ in the limit $R \to R_h$, due entirely to the transitioning of the 3-momentum to its relativistic limit, $p^R \to m_i c$ (if we set $m_i = m_g$, with an appropriate choice of units for the gravitational constant G). That is, the particle attains escape energy, $E \to E_{esc} \equiv m_i c^2$, as it approaches the gravitational horizon.

Within the Hubble flow, Equation (2.2) may be simplified further to the following expression:

$$E^2 = \Phi(R)(c\kappa)^2 + (cp)^2. \tag{2.6}$$

In principle, the ratio R/R_h could be varying in time, for which E in this equation could not remain constant at any fixed proper radius R, no matter what value κ has.

But the data [19] are telling us that the gravitational horizon appears to be expanding at speed c. In this case, both R and R_h scale linearly with t so, if we set $cp = m_i c^2 (R/R_h)$, the righthand side of Equation (2.6) is entirely independent of time along the worldline of observers for which t is the local proper time. One can also understand this result from the time independence of the g_{tt} and g_{RR} coefficients in the FLRW metric. Energy is conserved along the worldlines of these particular (comoving) observers [1, 20] because a Universe with a linearly expanding R_h has zero active mass ($\rho + 3p = 0$), for which everything within R_h then experiences zero net acceleration. In such a spacetime, particles cannot gain or lose energy from the background as the Universe expands, and E in Equation (2.6) must thus be constant. Under these conditions, the only solution guaranteeing a time-independent energy at all radii is $\kappa = m_i c$.

We emphasize the dependence of this outcome on the background cosmology. In particular, the metric coefficients g_{tt} and g_{RR} are independent of time only for a linearly expanding cosmos. The key question is then how this condition comports with the observations. After two decades of development and testing, the observational and theoretical basis for an FLRW cosmology with a linear expansion is actually quite compelling and fully supported by the data. This has now been demonstrated in over 100 refereed papers and a monograph (ref. [19]). A quick summary of the tests published thus far may be found in Table 2 of ref. [21].

Equation (2.6) tells us that E transitions from purely kinetic at $R = R_h$ to purely static at $R = 0$, where $p^R \to 0$. In other words, the particle's escape energy is 'stored' as a gravitational binding energy at the observer's location.

3 Momentum with and without inertia

Recognizing that the rest-mass energy apparently represents the gravitational energy of a particle bound to the rest of the Universe within our gravitational horizon, R_h, let us now use Equation (2.6) to interpret the origin of momentum p. Rearranging this equation, we see that

$$p^R = \frac{E}{c}\sqrt{1 - \left(\frac{E_{esc}}{E}\right)^2 + \left(\frac{R}{R_h}\right)^2\left(\frac{E_{esc}}{E}\right)^2}. \tag{3.1}$$

This is actually a *universal* equation, in that it applies to all particles, even those for which $m_i \to 0$, i.e., those without inertia, and at all proper radii $R \le R_h$.

For particles with inertia, we have $E_{esc,i} = m_i c^2$, while photons are always unbounded with $E_{esc,ph} = 0$. The ratio E_{esc}/E in this equation clearly represents the fraction of particle energy **not** associated with the momentum. Quite trivially,

$$p^R = \frac{E}{c} \tag{3.2}$$

for massless particles whose gravitational binding energy is zero. The analogous (equally simple) momentum of a particle with speed less than c is also given by the particle's energy divided by c, but only in terms of the energy in excess of E_{esc}.

For example, consider a particle close to the origin of the observer's coordinates ($R \to 0$), with $E \approx E_{\rm esc} + m_{\rm i}v^2/2 \gtrsim E_{\rm esc}$, in terms of its 'peculiar' velocity v. Then, its momentum, p, may be found from Equation (3.1) with

$$p \approx \frac{1}{c}\sqrt{(E+E_{\rm esc})(E-E_{\rm esc})}, \tag{3.3}$$

which further simplifies to

$$p \approx \frac{1}{c}\sqrt{2m_{\rm i}c^2\left(m_{\rm i}v^2/2\right)} \to m_{\rm i}v. \tag{3.4}$$

The explicit use of inertia in Equation (2.4) has led to the universal Equation (3.1) written entirely in terms of energy. The inertial mass is clearly a 'place holder' in these expressions, acting as a convenient surrogate for the fraction of E not associated with the gravitational binding energy $E_{\rm esc}$, as demonstrated in the derivation of Equation (3.4).

4 Discussion

With these results in hand, we may now consider what has been gained with this updated concept of momentum. The critical point underlying this work (and the preceeding papers) is that rest mass energy appears to be the gravitational binding energy of a particle coupled to the rest of the Universe within our apparent (or 'gravitational') horizon.

The fact that inertia is apparently not an intrinsic particle property therefore suggests that the nature of momentum may be better understood in the context of general relativity, where energy plays a key role in establishing the spacetime curvature. In this context, our demonstration that momentum may be expressed entirely in terms of the free energy in all cases, for massive and massless particles, eliminates any possibility that it may be due to a variety of mechanisms. When used to characterize momentum, inertia is merely a place holder for the difference between the total and free energies, leading to the universal formulation in Equation (3.1).

This conceptualization would eventually impact several other physical disciplines. For example, the formal structure of relativistic quantum field theory takes into account how inertia arises in different circumstances (e.g., via the Higgs field for leptons and a gluonic backreaction within baryons), but it is not yet equipped to probe the distinctions we consider in this paper because it relies directly on the rest mass energy of the particle, rather than its inertia.

But the gravitational coupling in general relativity is the total energy of the particle, not just its rest energy. So a deeper understanding of inertia and momentum may eventually lead to a measurable signature at the highest energy scales, perhaps at the grand unification scale, and certainly toward the Planck regime.

5 Conclusions

Our modern understanding of inertia has supplanted the classical view [2] that it represents an intrinsic, immutable property of matter. As we now know, inertia arises via different means under different circumstances, so there is no single, universal mechanism that defines all of its properties. Inertia thus appears to be more of an emergent particle property, hardly a factor that ought to be determining how much momentum it is carrying. And yet, while momentum is 'more' understandably proportional to energy in the case of massless particles, it has mysteriously been associated with inertia when the particles have mass.

In this paper, we have sought to better understand this dichotomy, with the hope of framing momentum in a unique way that represents all particles, with or without inertia. Indeed, we have found that a single definition of momentum may work under all circumstances, as long as we associate it with the 'free' energy of the particle, instead of its total energy, E. The free energy is of course the same as E for massless particles, but is merely the fraction of E exceeding the gravitational binding energy when $m_{\rm i} \neq 0$.

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