ON THE EXACTNESS OF THE UNIVERSAL BACKPROJECTION FORMULA
FOR THE SPHERICAL MEANS RADON TRANSFORM

MARK AGRANOVSKY AND LEONID KUNYANSKY

Abstract. The spherical means Radon transform $Mf(x, r)$ is defined by the integral of a function $f$ in $\mathbb{R}^n$ over the sphere $S(x, r)$ of radius $r$ centered at $x$, normalized by the area of the sphere. The problem of reconstructing $f$ from the data $Mf(x, r)$ where $x$ belongs to a hypersurface $\Gamma \subset \mathbb{R}^n$ and $r \in (0, \infty)$ has important applications in modern imaging modalities, such as photo- and thermo-acoustic tomography. When $\Gamma$ coincides with the boundary $\partial \Omega$ of a bounded (convex) domain $\Omega \subset \mathbb{R}^n$, a function supported within $\Omega$ can be uniquely recovered from its spherical means known on $\Gamma$. We are interested in explicit inversion formulas for such a reconstruction.

If $\Gamma = \partial \Omega$, such formulas are only known for the case when $\Gamma$ is an ellipsoid (or one of its partial cases). This gives rise to a question: can explicit inversion formulas be found for other closed hypersurfaces $\Gamma$? In this article we prove, for the so-called “universal backprojection inversion formulas”, that their extension to non-ellipsoidal domains $\Omega$ is impossible, and therefore ellipsoids constitute the largest class of closed convex hypersurfaces for which such formulas hold.

Keywords: Universal backprojection formula, thermoacoustic tomography, explicit inversion formula, spherical means

1. Introduction

1.1. Formulation of the problem and the main result. Given a continuous function $f$ in $\mathbb{R}^n$, spherical means $Mf(x, r)$, $x \in \Gamma$, $r > 0$, are defined by the following formula:

$$Mf(x, r) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x + r\theta) \, dS(\theta),$$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $|S^{n-1}|$ is the $(n-1)$-dimensional area of $S^{n-1}$, $dS$ is the surface Lebesgue measure on $S^{n-1}$.

The spherical mean operator takes functions $f$ in $\mathbb{R}^n$ to functions $Mf(x, r)$ on $\mathbb{R}^n \times [0, \infty]$. The problem of reconstructing $f$ from the spherical means $Mf(x, r)$, with centers $x$ located on a certain hypersurface $\Gamma$ arises in inverse problems and modern modalities of imaging, such as thermo- and photoacoustic tomography (TAT/PAT) [19, 21, 26]. There is an extensive literature on the analytical, computational and applied aspects of this subject; we refer the reader to surveys [17,18] and references therein.

The important case we consider here is when the centers $x$ belong to a closed hypersurface $\Gamma$ which coincides with the boundary $\partial \Omega$ of a bounded domain $\Omega$ that, in turn, contains the support of function $f$. The transform $M$ is invertible in this case [6,7,30], but explicit inversion formulas are only known for ellipsoidal surfaces $\Gamma$ (including spheres, spheroids, etc.) [6–8,11,12,22,25,27–29] .

In this paper we focus on a certain type of inversion formulas, first proposed (in an equivalent form) in [31] for the partial case when $\Gamma$ is a sphere in $\mathbb{R}^3$ and later extended to spheres in arbitrary dimensions and, finally, to arbitrary ellipsoids in $\mathbb{R}^n$ [11,12,15,25].

In the latter case, when $\Omega$ is an ellipsoidal domain $E$ (whose boundary $\partial E$ is an ellipsoid in $\mathbb{R}^n$) or, equivalently, when $\Gamma = \partial E$, the inversion formula for reconstructing function $f$ from the values...
of its spherical means $Mf(x,r)$, $x \in \partial E$, $r \geq 0$, has the following form [15]:

$$f(x_0) = [N_E Mf] (x_0) =$$

$$\frac{(-1)^{n-2} |S^{n-1}|}{2\pi^n} \int_{\partial E} \nu_x \cdot (x_0 - x) \int_0^\infty \frac{[\partial_r D_r^{n-2} p^{n-2}(Mf)] (x, r)}{r^2 - |x_0 - x|^2} dr \, dS(x),$$

(when $n \geq 2$ is even), and

$$f(x_0) = [N_E Mf] (x_0) =$$

$$\frac{(-1)^{n-3} |S^{n-1}|}{4\pi^n} \int_{\partial E} \nu_x \cdot \frac{x_0 - x}{|x_0 - x|} [\partial_r D_r^{n-2} p^{n-2}(Mf)] (x, |x_0 - x|) \, dS(x),$$

(when $n \geq 3$ is odd). Here $f \in C^\infty$ with supp $f \subset E$, $x_0$ is an arbitrary point in $E$, $\nu_x$ denotes the exterior unit normal to $\partial E$, and $D_r = (2r)^{-1}\partial_r$ is the operator of differentiation with respect to $r^2$. By $N_E$, we denote the operator, of backprojection type, in the right hand side, applied to $Mf$. In operator terms, it is the left inverse operator to the restricted spherical means Radon transform $Mf(x,r)_{|\Gamma \times [0,\infty)}$. Versions of the inversion formulas for certain unbounded quadratic surfaces were also obtained [32–34].

Thus, among bounded closed convex observation surfaces $\Gamma$, ellipsoids are the only those for which explicit inversion formulas are known by now. The natural question arises, whether nice inversion formulas can be constructed for other surfaces? Specifically, with respect to formulas (1), (2), we would like to answer the following question: are these formulas true for bounded convex smooth hypersurfaces $\Gamma$ other than ellipsoids $\partial E$? The goal of this paper is to show that the answer is negative: ellipsoids constitute the largest class of smooth bounded convex hypersurfaces $\Gamma$ for which these inversion formulas are valid. More precisely, the following theorem holds:

**Theorem 1.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$, $n \geq 2$, and $\Gamma = \partial \Omega \subset C^\infty$. Suppose that the inversion formula (1) or (2), where $\partial E$ is replaced by $\partial \Omega$, holds for all $x_0 \in \Omega$ and for any function $f \in C^\infty(\mathbb{R}^n)$ supported in $\Omega$. Then the hypersurface $\partial \Omega$ is an ellipsoid $\partial E$.

1.2. **“Universal backprojection formulas”**. Inversion formulas (1), (2) and their equivalents have received the name of “universal backprojection formulas” [12,15,32]. Our proof of Theorem 1 relies on the most general form of these formulas given in [12]. Namely, it has been found that if in the right hand side of formulas (1), (2) the ellipsoid $\partial E$ is replaced by the boundary $\Gamma = \partial \Omega$ of an arbitrary bounded convex smooth domain, then an additional “error” term $K_\Omega f$ will appear:

$$f(x_0) = [N_\Omega Mf] (x_0) + [K_\Omega f] (x_0),$$

where $N_\Omega$ is given by (1), (2) (with $\Omega$ in place of $E$) and $K_\Omega$ is the integral operator

$$[K_\Omega f] (x_0) = \int_\Omega k_\Omega(x_0, x_1) f(x_1) \, dx_1,$$

where the kernel $k_\Omega$ associated with $\Omega$ is explicitly given by formulas (8), (9) in Section 2.2.

Of course, the name “universal backprojection formula” for (3) is somewhat misleading, since extending formula (3) to a larger class of the domains $\Omega$, in general, results in the loss of its inversion property. Indeed, equation (3) leads to an integral equation of the second type $(I - K_\Omega)f = N_\Omega (Mf)$ for the unknown function $f$ rather than to an explicit expression for $f$. Therefore, in order to have a true inversion formula $f = N_\Omega (Mf)$ one needs to guarantee that the error term vanishes. Thus, we would like to characterize all domains $\Omega$ for which $K_\Omega = 0$. In the latter case we will say that the universal inversion formula (3) is **exact**. In these terms, Theorem 1 can be translated as follows: non-ellipsoidal domains necessarily produce the non-zero “error” term $K_\Omega$ and hence “universal inversion formula” (3) is **exact if and only if** the boundary of $\Omega$ is an ellipsoid.
We note that other exact inversion formulas have been discovered. Such formulas are known for the case when the surface \( \Gamma \) is a sphere \([6,7]\) or an ellipsoid \([27,29]\). They also exist for certain more complicated surfaces \([27,28]\). All these formulas are not equivalent to the “universal backprojection” formulas considered here, meaning that the corresponding inverse operators coincide only on the image of the spherical mean operator \( \mathcal{M} \) restricted to the surface \( \Gamma \). The results of this paper do not extend to such formulas; they only hold for the formulas in the form \((1), (2)\). A further discussion of our results with relation to other reconstruction techniques can be found in Section 4.

The rest of this paper is arranged as follows. In the next section we define the integral transforms needed to properly present the explicit expression for \( K_\Omega \) and prove two lemmas necessary for the further exposition. Theorem 1 is proven in Section 3. We conclude with the further discussion of our results in Section 4.

2. Preliminaries

2.1. Radon and Hilbert transforms. Below we recall several well known facts about the Radon and Hilbert transforms.

The Radon transform \( \mathcal{R} f \) of a compactly supported smooth function \( f \) is defined \([24]\) as

\[
[\mathcal{R} f](\omega, p) = \int_{\Pi(\omega, p)} f(x) \, dA(x), \quad (\omega, p) \in \mathbb{S}^{n-1} \times \mathbb{R},
\]

where \( \mathbb{S}^{n-1} \) is the unit sphere in \( \mathbb{R}^n \), \( \Pi(\omega, p) \) is the hyperplane defined by the equation \( x \cdot \omega = p \), and \( dA(x) \) is the standard measure on \( \Pi(\omega, p) \). Obviously, \([\mathcal{R} f](\omega, p)\) vanishes for all \((\omega, p)\) such that \( \Pi(\omega, p) \) does not intersect the support \( \Omega \) of \( f \).

The Hilbert transform \( \mathcal{H} F \) of a smooth, sufficiently fast decaying function \( F(t), t \in \mathbb{R} \), is defined by the following formula \([16]\)

\[
[\mathcal{H} F](t) = \frac{1}{\pi} \, p.v. \, \int_{\mathbb{R}} \frac{F(s)}{t-s} \, ds,
\]

where \( p.v. \) stands for the principal value of the integral. The Hilbert transform can be extended to less smooth functions and distributions by continuity. It is self-invertible; more precisely \( \mathcal{H}(\mathcal{H} F) = -F \). The following intertwining relation holds (Section 4.7, \([16]\)):

\[
[\mathcal{H} (s \varphi(s))](t) = t \, [\mathcal{H} \varphi](t) - \frac{1}{\pi} \int_{\mathbb{R}} \varphi(t) \, dt.
\]

Let \( \chi_{[a,b]}(s) \) be the characteristic function of the interval \([a, b]\). The Hilbert transform of the function \( \chi_{[-1,1]}(s)\sqrt{(1-s)(1+s)} \) is well-known (formula 11.343, \([16]\)):

\[
[\mathcal{H} \left( \chi_{[-1,1]}(s)\sqrt{(1-s)(1+s)} \right)](t) = t, \quad t \in [a, b].
\]

By a linear change of variables one obtains the Hilbert transform of \( \chi_{[a,b]}(s)\sqrt{(b-s)(s-a)} \):

\[
[\mathcal{H} \left( \chi_{[a,b]}(s)\sqrt{(b-s)(s-a)} \right)](t) = t - \frac{b+a}{2}, \quad t \in [a, b].
\]

We will also make use of the existence of the so-called finite inverse Hilbert transform \([35]\). Namely, if a continuous function \( F(t) \) is supported on an interval \([a, b]\), then

\[
F(t) = \frac{1}{\pi \sqrt{(b-t)(t-a)}} \left( \int_a^b \frac{[\mathcal{H} F](s)}{s-t} \sqrt{(b-s)(s-a)} \, ds + \int_a^b F(s) \, ds \right), \quad t \in [a, b].
\]
2.2. The error operator. Our analysis of the universal backprojection formula is based on the results of [10,12,25], that give explicit expressions for the error arising when this formula is used under the following assumptions: $f(x)$ is is compactly supported strictly inside of a bounded (strictly) convex open domain $Ω ⊂ \mathbb{R}^n$ with an infinitely smooth boundary $\partial Ω$ and the measuring surface $Γ$ coincides with $\partial Ω$. Then the universal backprojection operator $N_Ω$, when applied to spherical means $Mf$ will produce the error $K_Ω f$, see equation (3), with operator $K_Ω$ in the form (4). The latter kernel, according to [10,12,25], has the following form:

$$k_Ω(x_0, x_1) = c \frac{\partial^n[ℏ(\mathcal{R}χ_Ω)](ω_∗(x_0, x_1), s_∗(x_0, x_1))}{|x_0 - x_1|^{n-1}},$$

if $n$ is even, and

$$k_Ω(x_0, x_1) = c \frac{\partial^n[ℏ(\mathcal{R}χ_Ω)](ω_∗(x_0, x_1), s_∗(x_0, x_1))}{|x_0 - x_1|^{n-1}},$$

if $n$ is odd. Here $\mathcal{R}χ_Ω$ is the Radon transform of the characteristic function $χ_Ω$ of $Ω$, the Hilbert transform $ℏ$ acts with respect to the second variable of the pair $(ω, s)$, and functions $ω_∗(x_0, x_1)$ and $s_∗(x_0, x_1)$ are defined as follows:

$$ω_∗(x_0, x_1) = \frac{x_0 - x_1}{|x_0 - x_1|}, \quad s_∗(x_0, x_1) = \frac{|x_1|^2 - |x_0|^2}{2|x_1 - x_0|}.$$ 

It has been proven that if $Ω$ is an ellipsoidal domain then the error operator $K_Ω$ vanishes [10,12,25], and hence the expression $N_Ω(Mf)$ represents the exact inversion and returns $f$. As it was already mentioned, the main result of the present paper (Theorem 1) is that the converse statement is true: the universal backprojection inversion formula (3) is exact only for ellipsoidal domains.

We proceed with two lemmas which we will need in the proof of Theorem 1.

2.3. Two lemmas.

Lemma 2. Let $[a, b]$ be a segment on the real line and let $F ∈ C(\mathbb{R})$ be supported in the segment $[a, b]$ . If there exists a polynomial $P(t)$ such that $[ℏF](t) = P(t)$ for all $t ∈ [a, b]$ then

$$F(t) = \frac{Q(t)}{\sqrt{(b - t)(t - a)}}, \quad t ∈ [a, b],$$

where $Q(t)$ is a polynomial of the degree $\deg Q ≤ \deg P + 1$.

Proof. Consider the Hilbert transform of a function $χ_{[a,b]}(s)s^k\sqrt{(b - s)(s - a)}$ with integer $k$. If $k = 0$, identity (6) yields

$$ℏ(χ_{[a,b]}(s)s^k\sqrt{(b - s)(a - s)}) (t) = t + c_0, \quad t ∈ [a, b].$$

with some constant $c_0$. For $k > 0$ equation (5) leads to

$$ℏ(χ_{[a,b]}(s)s^k\sqrt{(b - s)(a - s)}) (t) = t \left\{ ℏ(χ_{[a,b]}(s)s^{k-1}\sqrt{(b - s)(a - s)}) (t) + c_k \right\},$$

where $c_k$ is yet another constant. By induction, the above two equations imply that $ℏ(χ_{[a,b]}(s)s^k\sqrt{(b - s)(a - s)})$ is a polynomial of degree $k + 1$. Thus, if $P(s)$ is a polynomial of degree $\deg P$, then $ℏ(χ_{[a,b]}(s)P(s)\sqrt{(b - s)(s - a)}) (t)$ is a polynomial of degree $\deg P + 1$. On the other hand, $(ℏF)(t) = P(t)$ for $t ∈ [a, b]$ and hence formula (7) reads as

$$F(t) = \frac{1}{\sqrt{(b - t)(t - a)}} \left( ℏ(χ_{[a,b]}(s)P(s)\sqrt{(b - s)(s - a)}) (t) + \int_a^b F(s) ds \right)$$

with $t ∈ [a, b]$, thus proving the lemma. □
Given a unit vector \( \omega \in S^{n-1} \), define

\[
    h^+_{\partial \Omega}(\omega) = \sup_{x \in \Omega} x \cdot \omega, \\
    h^-_{\partial \Omega}(\omega) = \inf_{x \in \Omega} x \cdot \omega.
\]

The function \( h_{\partial \Omega}(\omega) = h^+_{\partial \Omega}(\omega) \), \( \omega \in S^{n-1} \), is called the support function of the domain \( \Omega \).

The functions \( h^\pm_{\partial \Omega} \) are related by the formula

\[
    h^-_{\partial \Omega}(\omega) = -h^+_{\partial \Omega}(-\omega), \quad \omega \in S^{n-1}.
\]

In the case of ellipsoidal domains, the support function is the square root of a quadratic polynomial. For example, for the domain \( E \) bounded by the ellipsoid

\[
    \partial E = \left\{ \sum_{j=1}^n x_j^2 a_j^2 = 1 \right\}
\]

we have

\[
    h_E(\omega) = \sqrt{\sum_{j=1}^n a_j^2 \omega_j^2}.
\]

A hyperplane \( \{x \cdot \omega = t\} \) meets the domain \( \Omega \) if and only if \( h^-_{\partial \Omega}(\omega) < t < h^+_{\partial \Omega}(\omega) \). The limit cases \( t = h^\pm_{\partial \Omega}(\omega) \) correspond to the tangent hyperplanes to \( \partial \Omega \) at the points \( a^\pm \in \partial \Omega \) where the exterior unit normal vectors are \( \nu_{\partial \Omega}(a^\pm) = \pm \omega \), as illustrated in Figure 1.

The behavior of the Radon transform \( [R\chi_{\Omega}](\omega, t) \) near the tangent planes is given by the following Lemma.

**Lemma 3.** For a dense set of the direction vectors \( \omega \in S^{n-1} \), the following asymptotic relation holds with some nonzero constants \( c^\pm \):

\[
    [R\chi_{\Omega}](\omega, t) = c^\pm (t - h^\pm_{\partial \Omega}(\omega))^\frac{n-1}{2} (1 + o(1)), \quad t \to h^\pm_{\partial \Omega}(\omega) \mp 0.
\]

**Proof.** We will use the notation \( \Gamma = \partial \Omega \). The hypersurface \( \Gamma \) is infinitely differentiable. Let \( \kappa_{\Gamma}(a), a \in \Gamma \) be the Gaussian curvature, i.e. the product of the principal curvatures of the \( C^\infty \) hypersurface \( \Gamma \) at the point \( a \).

Denote by \( \gamma \) the Gauss mapping

\[
    \gamma : \Gamma \ni a \to \nu_{\Gamma}(a) \in S^{n-1},
\]

which maps a point \( a \in \Gamma \) to the exterior unit normal vector \( \gamma(a) = \nu_{\Gamma}(a) \) to \( \Gamma \) at the point \( a \). Since \( \Gamma \) is strictly convex, \( \gamma \) is a one-to-one mapping. It is differentiable and Gaussian curvature \( \kappa_{\Gamma}(a) \) equals to Jacobian determinant \( \kappa(a) = J_{\gamma}(a) \) of \( \gamma \) at the point \( a \). Therefore, the points \( a \)
with \( \kappa_j(a) \neq 0 \) (non-degenerate points) constitute the set \( \text{Reg}_\gamma \) of regular points of the mapping \( \gamma \), while the set of points \( a \) of zero Gaussian curvature coincides with the critical set \( \text{Crit}_\gamma \). By Sard’s theorem (see e.g., [23], Section 2, p.10; Section 3, p.16) \( \gamma(\text{Crit}_\gamma) \) has the Lebesgue measure zero on \( S^{n-1} \), while the set \( \gamma(\text{Reg}_\gamma) \) of regular values is a dense subset of \( S^{n-1} \). This subset consists of the regular directions \( \omega \) which are normal vectors \( \omega = \nu_1(a) \) at non-degenerate points \( a \), i.e. points of nonzero Gaussian curvature.

Let \( \omega \in S^{n-1} \) be a regular direction, \( \omega = \nu_1(a) \), \( a \in \Gamma \). Applying a suitable translation and orthogonal transformation, we can assume that \( a = 0 \), \( \omega = (0, \ldots, 0, 1) \). Then the tangent plane \( T_a(\Gamma) \) is the coordinate plane \( x_n = 0 \) and the domain \( \Omega \) is contained in the half-space \( x_n < 0 \). In this case \( h^+_\Omega(\omega) = 0 \). Moreover, after performing a suitable non-degenerate linear transformation we can make the equation of \( \Gamma \), near \( a = 0 \), to be:

\[
x_n = -\frac{1}{2} \left( c_1 x_1^2 + \cdots + c_{n-1} x_{n-1}^2 \right) + o \left( |x'|^2 \right), \quad (x_1, \ldots, x_{n-1}) = x' \to 0.
\]

The new axes \( x_j \), \( j = 1, \ldots, n - 1 \) are directions of the vectors of principal curvatures and the coefficients \( c_j \) are the values of the principal curvatures at the point \( a = 0 \in \Gamma \). The Gaussian curvature at \( a = 0 \) is \( \kappa_1(0) = c_1 \cdots c_{n-1} \). All the applied transformations preserve regular points, hence \( \kappa_1(0) \neq 0 \). Therefore, none of \( c_j \) is zero and since \( c_j \geq 0 \) due to the convexity of \( \Gamma \), we have \( c_j > 0 \) for all \( j \).

Since, after the above transformations, we have \( \omega = (0, \cdots, 0, 1) \) the hyperplane \( x \cdot \omega = t \) now is given by the equation \( x_n = t \), with \( t < 0 \). The main term of \( \text{Vol}_{n-1}(\Omega \cap \{ x_n = t \}) \) near \( t = 0 \), is determined by the main term of expansion (14), i.e., by the volume of the domain bounded by an ellipsoid \( -t = c_1 x_1^2 + \cdots + c_{n-1} x_{n-1}^2 \), which is equal to

\[
c(\pm t)^{\frac{n-1}{2}}; \quad c = \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2}) \sqrt{\kappa_1(0)}}.
\]

Thus, for the specific choice \( a = 0 \) and \( \omega = (0, \ldots, 0, 1) \), we have the following asymptotic formula (see e.g. [9] Ch.1, Section 1, 7):

\[
[R\chi_\Omega](\omega, t) = \text{Vol}_{n-1}\{ x_n = t \} = c(\pm t)^{\frac{n-1}{2}} + o(|t|^{\frac{n-1}{2}}), \quad t \to 0,
\]

Performing the inverse affine transformation, we obtain the asymptotic formula (13) near the point \( h^+\Omega(\omega) \) with some new nonzero constant \( c^+ \). There are two points \( a^\pm \in \Gamma \) with parallel tangent planes and opposite exterior unit normal vectors \( \nu_1(a^\pm) = \pm \omega \), hence, by repeating the argument for the point on \( \Gamma \) with the exterior unit normal vector \(-\omega \), we obtain the similar asymptotic near the point \( -h^+\Omega(\omega) = -h^+\Omega(-\omega) \). Lemma is proved.

\[\square\]

3. PROOF OF THEOREM 1

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) satisfying all the conditions of Theorem 1. The exactness of the universal backprojection formula for any function \( f(x) \) implies that the kernel \( k_\Omega(\omega_0(x_0, x_1), s_0(x_0, x_1)) \) vanishes for all \( x_0, x_1 \in \Omega \). The geometric meaning of the variables \( \omega_0(x_0, x_1) \) and \( s_0(x_0, x_1) \) is that the hyperplane \( x \cdot \omega_0(x_0, x_1) = s_0(x_0, x_1) \) is orthogonal to the segment \( [x_0, x_1] \) and passes through its midpoint. Obviously, every hyperplane intersecting the interior of \( \Omega \) can be obtained by choosing certain \( x_0, x_1 \in \Omega \). Therefore, \( k_\Omega(\omega, t) = 0 \) for all \( (\omega, t) \in S^{n-1} \times (h^+\Omega(\omega), h^+\Omega(\omega)) \). This is also trivially true for \( t \) lying outside of the interval \( (h^+\Omega(\omega) , h^+\Omega(\omega)) \), so \( k_\Omega(\omega, t) = 0 \) for all \( (\omega, t) \in S^{n-1} \times \mathbb{R} \). This in turn, implies that, for a fixed \( \omega \), functions \( [H(R\chi_\Omega)](\omega, t) \) in the even dimensional case and \( [R\chi_\Omega](\omega, t) \) in the odd dimensional case are polynomials in \( t \) of degree not exceeding \( n - 1 \).

Below we consider the cases of even and odd \( n \) separately. We start with odd dimensions.
3.1. The case of odd $n$. The condition $k_{\Omega}(\omega, s) = 0$, $h_{\Omega}^{-}(\omega) < t < h_{\Omega}^{+}(\omega)$ and expression (9) for $k_{\Omega}$ imply that

$$[\mathcal{R}_{\chi_{\Omega}}](\omega, t) = P_{\omega}(t), \ h_{\Omega}^{-}(\omega) < t < h_{\Omega}^{+}(\omega),$$

where

$$P_{\omega}(t) = \sum_{k=0}^{n-1} p_{k}(\omega) t^{k}$$

is a polynomial of degree at most $n - 1$, with coefficients $p_{k}(\omega)$ - continuous functions on the unit sphere $S^{n-1}$.

Domains $\Omega$ with polynomial dependence of $[\mathcal{R}_{\chi_{\Omega}}](\omega, t)$ on $t$ are called polynomially integrable [1]. There is no such domains in even dimensions [1]. Koldobsky, Merkurjev and Yaskin proved in [20] that, in odd dimensions, all polynomially integrable domains with infinitely smooth boundaries are ellipsoidal domains. Therefore, we could refer here to that result. However, in our case we have an additional information about the degree of polynomial $P_{\omega}$ which allows us to use simpler arguments than those in [20]. These arguments are given in [1] (see also [2]). They are based on Lemma 3 and on the range conditions for the Radon transform. We will present them here, to make the presentation self-contained and because these arguments extend to the case of even $n$, where the result of [20] is not directly applicable.

Lemma 3 asserts that for a dense set of $\omega \in S^{n-1}$ polynomial $P_{\omega}(t)$ has zeros at the points $t_{1} = h_{\Omega}^{-}(\omega)$ and $t_{2} = h_{\Omega}^{+}(\omega)$, each zero of multiplicity $\frac{n-1}{2}$. This means $\partial_{t}^{\frac{n-1}{2}} P_{\omega}(t) = 0$ when $t = t_{1}$ or $t = t_{2}$ and by continuity with respect to $\omega$ these properties extend to all $\omega \in S^{n-1}$. On the other hand, we have the upper bound $\deg P_{\omega} \leq n - 1$. Therefore, we conclude that polynomial $P_{\omega}$ can be represented in the form

$$P_{\omega}(t) = A(\omega)(t-t_{1})^{\frac{n-1}{2}}(t_{2} - t)^{\frac{n-1}{2}} = A(\omega)(t-h_{\Omega}^{-}(\omega))^{\frac{n-1}{2}}(h_{\Omega}^{+}(\omega) - t)^{\frac{n-1}{2}}.$$  

Polynomial $P_{\omega}(t) = [\mathcal{R}_{\chi_{\Omega}}](\omega, t)$ belongs to the range of the Radon transform. Hence it satisfies the range conditions for this transform, and, in particular, the moment conditions (e.g. [13,24]). Namely, the $k$-th moment

$$M_{k}(\omega) = \int_{\mathbb{R}} [\mathcal{R}_{\chi_{\Omega}}](\omega, t) t^{k} \, dt = \int_{h_{\Omega}^{-}(\omega)}^{h_{\Omega}^{+}(\omega)} P_{\omega}(t) t^{k} \, dt$$

extends from the unit sphere $|\omega| = 1$ to $\mathbb{R}^{n}$ as a homogeneous polynomial of degree $k$.

Substituting the expression (16) we have

$$M_{k}(\omega) = A(\omega) \int_{h_{\Omega}^{-}(\omega)}^{h_{\Omega}^{+}(\omega)} (t-h_{\Omega}^{-}(\omega))^{\frac{n-1}{2}}(h_{\Omega}^{+}(\omega) - t)^{\frac{n-1}{2}} t^{k} \, dt.$$  

Introduce the functions $B(\omega)$ and $C(\omega)$ as follows:

$$C(\omega) = \frac{h_{\Omega}^{+}(\omega) - h_{\Omega}^{-}(\omega)}{2}, \quad B(\omega) = \frac{h_{\Omega}^{+}(\omega) + h_{\Omega}^{-}(\omega)}{2}.$$  

Let us make a substitution in the integral (18):

$$u = \frac{t - B(\omega)}{C(\omega)}.$$

Then

$$h_{\Omega}^{+}(\omega) = B(\omega) + C(\omega), \quad h_{\Omega}^{-}(\omega) = B(\omega) - C(\omega).$$
Then for the translated domain 

$$h_{\Omega}(\omega) - t = (C(\omega) + B(\omega)) - (C(\omega)u + B(\omega)) = C(\omega)(1 - u),$$

$$t - h_{\Omega}(\omega) = (C(\omega)u + B(\omega)) - (B(\omega) - C(\omega)) = C(\omega)(1 + u).$$

Therefore

(19) \hspace{1cm} M_k(\omega) = 2A(\omega)C^n(\omega) \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} (C(\omega)u + B(\omega))^k \, du.

Take \( k = 0 \):

$$M_0(\omega) = 2A(\omega)C^n(\omega) \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} \, du.$$ 

Since \( M_0(\omega) = \text{const} \), we obtain that the entire factor in front of the integral is constant: \( 2A(\omega)C^n(\omega) = \text{const} = c \), so that

$$M_k(\omega) = c \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} (C(\omega)u + B(\omega))^k \, du.$$ 

Now take \( k = 1 \):

$$M_1(\omega) = c \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} (C(\omega)u + B(\omega)) \, du = c B(\omega) \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} \, du.$$ 

The first moment \( M_1(\omega) \) extends to \( \mathbb{R}^n \) as a linear function, hence so does \( B(\omega) \):

$$B(\omega) = b \cdot \omega + b_0,$$

for some vectors \( b, b_0 \in \mathbb{R}^n \).

However, \( 2B(\omega) = h_{\Omega}^+(\omega) - h_{\Omega}^-(\omega) \) so that \( B(\omega) \) is an odd function of \( \omega \) and, in particular, \( b_0 = 0 \). Moreover, by passing to the translated domain

$$\tilde{\Omega} = \Omega - b,$$

we can make \( B(\omega) = 0 \) for all \( \omega \). Indeed,

$$h_{\Omega}^+(\omega) = \sup_{y \in \tilde{\Omega}} y \cdot \omega = \sup_{x \in \Omega} (x - b) \cdot \omega = h_{\tilde{\Omega}}^+(\omega) - b \cdot \omega,$$

$$h_{\tilde{\Omega}}^-(\omega) = -h_{\tilde{\Omega}}^+(\omega) = h_{\Omega}^-(\omega) - b \cdot \omega.$$ 

Then for the translated domain \( \tilde{\Omega} \) we have

$$\tilde{B}(\omega) = \frac{1}{2}(h_{\tilde{\Omega}}^+(\omega) + h_{\tilde{\Omega}}^-(\omega)) = B(\omega) - b \cdot \omega = 0.$$ 

Thus, applying the translation \( x \rightarrow x - b \) we can assume from the very beginning that \( B(\omega) = 0 \). This implies \( h_{\Omega}^-(\omega) = h_{\Omega}^+(\omega) \) which means that after the translation to the vector \( b \), domain \( \Omega \) becomes centrally symmetric. From now on, we assume that this is the case and \( B = 0 \).

Then

$$h_{\Omega}^+(\omega) = C(\omega).$$

At last, for \( k = 2 \) formula (19) turns into

$$M_2(\omega) = c \cdot C^2(\omega) \int_{-1}^{1} (1 - u^2)^{\frac{n-1}{2}} u^2 \, du.$$ 

Thus, \( C^2(\omega) \) differs from \( M_2(\omega) \) by a nonzero factor, and since \( M_2(\omega) \) is the restriction to \( S^{n-1} \) of a quadratic homogeneous polynomial, \( C^2(\omega) \) has the same property.

After applying an orthogonal transformation we can reduce the quadratic form \( C^2 = (h_{\Omega}^+)^2 = h_{\Omega}^2 \) to the diagonal form:

$$h_{\Omega}^2(\omega) = \sum_{j=1}^{n} a_j \omega_j^2.$$
Since the left hand side is strictly positive on \( S^{n-1} \) (indeed, \( h_\Omega(\omega) = 0 \) is impossible since it would mean \( 0 \in \partial \Omega \) which is not the case because \( \Omega \) is centrally symmetric and 0 is its interior point), we have \( a_j > 0 \) for all \( j = 1, \ldots, n \). We write \( a_j = \alpha_j^2 \). Then we have

\[
h_\Omega(\omega) = |A_\omega|,
\]

where the matrix \( A \) is the non-degenerate diagonal matrix \( A = \text{diag}(\alpha_1, \ldots, \alpha_n), j = 1, \ldots, n \).

The hyperplane \( x \cdot \omega = h_\Omega(\omega) \) is tangent to \( \partial \Omega \) and the convex domain \( \Omega \) coincides with the intersection of the open half-spaces \( x \cdot \omega < h_\Omega(\omega) = |A_\omega|^2, \omega \in S^{n-1} \), i.e.,

\[
\Omega = \{ x \in \mathbb{R}^n : x \cdot \omega < |A_\omega|, \forall \omega \in S^{n-1} \}.
\]

Taking \( \omega = \frac{A^{-1} \eta}{|A^{-1} \eta|} \), where \( \eta \in S^{n-1} \) is arbitrary, we obtain

\[
\Omega = \{ x \in \mathbb{R}^n : (A^{-1} x) \cdot \eta = x \cdot (A^{-1} \eta) < 1, \eta \in S^{n-1} \}.
\]

The inequality \( (A^{-1} x) \cdot \eta < 1 \) for all \( \eta \), \( |\eta| = 1 \), is equivalent to \( |A^{-1} x| < 1 \) and hence

\[
\Omega = \{ x \in \mathbb{R}^n : |A^{-1} x| < 1 \}
\]

is a domain whose boundary is an ellipsoid \( \sum_{j=1}^n \frac{x_j^2}{\alpha_j^2} = 1 \).

3.2. The case of even \( n \). If \( k_\Omega = 0 \) and \( n \) is even, then by (8)

\[
[H(\mathcal{R}_\omega \chi_\Omega)](\omega, t) = P_\omega(t), \quad h_\Omega^- (\omega) < t < h_\Omega^+ (\omega),
\]

where \( P_\omega \) is a polynomial of degree at most \( n - 1 \). By Lemma 2, with \( a = h_\Omega^- (\omega) \) and \( b = h_\Omega^+ (\omega) \)

\[
[H(\mathcal{R}_\omega \chi_\Omega)](\omega, t) = \frac{Q_\omega(t)}{\sqrt{(t - h_\Omega^-(\omega))(h_\Omega^+(\omega) - t)}}
\]

where \( Q_\omega(t) \) is a polynomial of degree at most \( n \).

By Lemma 3, we have \( [H(\mathcal{R}_\omega \chi_\Omega)](\omega, t) = c(t - h_\Omega^- (\omega))^{n-1} (1 + o(1)) \), as \( t \to h_\Omega^- (\omega) + 0 \), and a similar asymptotic is true for \( t \to h_\Omega^+ (\omega) - 0 \). This implies that \( t_1 = h_\Omega^- (\omega) \) and \( t_2 = h_\Omega^+ (\omega) \) are zeros of the polynomial \( Q_\omega(t) \), each of multiplicity \( \frac{n-1}{2} + \frac{1}{2} = \frac{n}{2} \).

Since, on the other hand, \( \deg Q_\omega \leq n \), the polynomial \( Q_\omega \) has the representation

\[
Q_\omega(t) = A(\omega)(t - t_1)^{\frac{n}{2}} (t_2 - t)^{\frac{n}{2}} = A(\omega)(t - h_\Omega^-(\omega))^{\frac{n}{2}} (h_\Omega^+(\omega) - t)^{\frac{n}{2}}
\]

and, correspondingly,

\[
[H(\mathcal{R}_\omega \chi_\Omega)](\omega, t) = A(\omega)(t - h_\Omega^- (\omega))^{\frac{n-1}{2}} (h_\Omega^+(\omega) - t)^{\frac{n-1}{2}}.
\]

Thus, as in the case of odd \( n \), the Radon transform of function \( \chi_\Omega \) admits representation (16). We now repeat the argument of Section 3.1, using the moment conditions for the range of the Radon transform. Namely, the first three moments \( M_k(\omega), k = 0, 1, 2 \), (see (17)) extend from the unit sphere \( |\omega| = 1 \) as homogeneous polynomials of degrees 0, 1, 2, correspondingly. This implies (see Section 3.1) that the square of the support function \( h_\Omega = h_\Omega^+ \) extends from \( |\omega| = 1 \) as a quadratic homogeneous polynomial and thus we conclude that \( \partial \Omega \) is an ellipsoid. The proof of Theorem 1 is complete.
Below we discuss the connections between our results and other known inversion formulas.

- As it was mentioned in Introduction, the problem of reconstructing a function from its spherical means centered on a hypersurface $\Gamma$ arises in TAT/PAT. The forward problem of TAT/PAT is modeled by the Cauchy problem for the wave equation (see, e.g., [18])

\begin{align}
    u_{tt} &= c^2(x) \Delta u, \quad t \geq 0, \quad x \in \mathbb{R}^n, \\
    u(x, 0) &= f(x), \quad u_t(x, 0) = 0,
\end{align}

where $u(x, t)$ is the pressure in the propagating acoustic wave, $f(x)$ is the initial pressure and $c(x)$ is the speed of sound. Depending on the type of transducers, one measures either pressure $u$ or its normal derivative $\frac{\partial u}{\partial \nu}$ on a measurement surface $\Gamma \subset \mathbb{R}^n$. The TAT/PAT inverse problem consists of reconstructing the initial value $f(x) = u(x, 0)$ from the Dirichlet data $u|_{\Gamma \times [0, \infty)}$. An alternative version is to find $f(x)$ from the Neumann data $\frac{\partial}{\partial \nu}u|_{\Gamma \times [0, \infty)}$.

For a constant speed of sound $c(x)$, solution $u(x, t)$ of (20), (21) can be expressed through the spherical means $[\mathcal{M}f](x, t)$ of the initial data $f(x)$ by the Kirchhoff-Poisson formula (e.g., Section 2.4, Thm 2 and 3, [5]). This formula allows one to reduce the problem of finding the initial value $f(x) = u(x, 0)$ from the data $u(x, t)|_{\Gamma \times [0, \infty)}$ to the problem of recovering the function $f(x)$ from its spherical means $[\mathcal{M}f](x, t)|_{\Gamma \times [0, \infty)}$ with the centers on $\Gamma$. The solution of the latter problem is given by formulas (1) and (2), and Theorem 1 establishes that, if $\Gamma = \partial \Omega$, these formulas are only valid in the case of ellipsoidal surfaces $\partial \Omega$.

An inversion formula recovering the initial data $f(x) = u(x, 0)$ from the Dirichlet data $u|_{\Gamma \times [0, \infty)}$ was proposed in [31–34] for several different acquisition surfaces $\Gamma$. Taking into account the relation between $u|_{\Gamma \times [0, \infty)}$ and $\mathcal{M}f|_{\Gamma \times [0, \infty)}$, one can show that the above formula is equivalent to formulas (1) and (2). Therefore, in the case when $\Gamma$ is a boundary $\partial \Omega$ of a convex domain with a smooth boundary, formula given in [31–34] holds if and only if $\partial \Omega$ is an ellipsoid.

- In [3, 4], inversion formulas for ellipsoids were obtained for the problem of reconstructing $f(x)$ from the Neumann data $\frac{\partial}{\partial \nu}u|_{\Gamma \times [0, \infty)}$. In particular, it has been shown that, as in the case of Dirichlet data, if these formulas are applied to an arbitrary convex domain $\Omega$ with a smooth boundary, then an error terms $K_{\Omega}$ appears, still given by equations (4), (8), (9). Since we have proven in Theorem 1 that $K_{\Omega} = 0$ is equivalent to $\Gamma = \partial \Omega$ being an ellipsoid, we conclude that the Neumann data version of Theorem 1 is also true, i.e., inversion formula [3, 4] is exact for ellipsoids only.

- Inversion formulas (1), (2) have been also shown [14, 15] to hold in the case of some unbounded quadratic surfaces, e.g., for parabolic and elliptic cylinders. Since an expression for the error operator is not known for the of unbounded surfaces $\Gamma$, our analysis cannot be extended to these cases.

- Other exact inversion formulas (formulated either in terms of the spherical means $\mathcal{M}f|_{\Gamma \times [0, \infty)}$ or in terms of the Dirichlet data $u|_{\Gamma \times [0, \infty)}$) have been discovered. They hold when the surface is a sphere [6, 7] or, more generally, an ellipsoid [27, 29], and also for certain more complicated surfaces [27, 28]. These formulas are not equivalent to the universal backprojection formulas considered here, meaning that the corresponding inverse operators coincide only on the image of the spherical mean operator $\mathcal{M}$ restricted to the surface $\Gamma$. In our opinion, it might be interesting to understand what is the largest class of hypersurfaces $\Gamma$ for which these formulas hold.
Acknowledgments

The question answered in this paper was posed by Professor Haltmeier in a conversation with the first author at the 9th Conference “Inverse Problems, Modeling and Simulation”, IPMS 2018 held on Malta. The preparation of the present paper started in 2022, during the 10th occurrence of this conference. The authors thank Professor Haltmeier for the interesting question and the organizers of IPMS-2022 for creating a stimulating environment and excellent conditions for collaboration. The second author acknowledges support by the NSF, through the award NSF/DMS 1814592. Finally, the authors are thankful to anonymous referees for helpful suggestions that noticeably improved clarity of this paper.

References


