Difference Operators and Pentagram Maps Over Rings

by

Leaha G. Hand

A Dissertation Submitted to the Faculty of the

DEPARTMENT OF MATHEMATICS

In Partial Fulfillment of the Requirements

For the Degree of

DOCTOR OF PHILOSOPHY

In the Graduate College at the

THE UNIVERSITY OF ARIZONA

2024

THE UNIVERSITY OF ARIZONA
GRADUATE COLLEGE

As members of the Dissertation Committee, we certify that we have read the dissertation prepared by: Leaha G. Hand
titled:

and recommend that it be accepted as fulfilling the dissertation requirement for the Degree of Doctor of Philosophy.

Anton Izosimov
Anton Izosimov (Apr 12, 2024 13:14 PDT)
Anton Izosimov — Date: Apr 12, 2024

Doug Pickrell
Doug Pickrell (Apr 12, 2024 16:56 PDT)
Doug Pickrell — Date: Apr 12, 2024

Klaus Lux
Klaus Lux (Apr 12, 2024 13:15 PDT)
Klaus Lux — Date: Apr 12, 2024

Sergey Cherkis
Sergey Cherkis (Apr 12, 2024 15:17 PDT)
Sergey Cherkis — Date: Apr 12, 2024

Final approval and acceptance of this dissertation is contingent upon the candidate's submission of the final copies of the dissertation to the Graduate College.

I hereby certify that I have read this dissertation prepared under my direction and recommend that it be accepted as fulfilling the dissertation requirement.

Anton Izosimov
Anton Izosimov (Apr 12, 2024 13:14 PDT)
Anton Izosimov — Date: Apr 12, 2024
Mathematics

He imagined that sooner or later he would have to think the stars through to the end, and that it would be better to do it now.

-Boris Pasternak in *Doctor Zhivago*

A man is not idle because he is absorbed in thought. There is visible labor and there is invisible labor.

-Victor Hugo in *Les Miserables*

Acknowledgements

First, I would like to thank my advisor, Dr. Anton Izosimov, for the endless support over the last four years. This dissertation would have felt impossible without your patience and willingness to help. I would also like to thank Drs. Klaus Lux, Doug Pickrell, and Sergey Cherkis for their time, consideration, and knowledge. My support at the University of Arizona extends beyond my committee; thank you to Dr. David Glickenstein, Tina Deemer, and Dr. Bob Sims for the guidance while here. I would also like to thank Dr. Mary Smith for showing me that women belong in mathematics, and Dr. Jens Harlander for mentoring me during my time at Boise State.

I would like to thank my friends here at UA. This would have been an impossible feat without my cohort: Eric Roon (friend, co-worker, and landlord), Abigayle Dirdak, Jorge Ledesma, Ben Savoie, Kenny Plante, Tristan Phillips. The countless study sessions, burritos, and coffee breaks with all of you helped to keep me sane. I would especially like to thank Quinton Aboud, for being my partner in crime throughout graduate school, and Rachel Knak, for keeping me laughing and reading. Shout-out to Joel McGuire, Ben Pound, and Olivia Hansen for being our mathless refuge: thank you for the dinners, drinks, and laughs.

A special thanks to my long distance besties: Mackenzie Lawrence and Kortni Clements. You have both been right by my side for over a decade, through some of my biggest academic feats. I do not want to know what life without the two of you would look like.

Next, I would like to thank my crazy family. My parents, Loraine and John, who have always shown me the importance of education and community. Mom, you are the hardest working and most inspiring person I know. You have always known the path that was meant for you and you have not let anyone deter you from it. Dad, you have never let your passions and interests fall to the wayside, no matter how busy life was. Thank you for always teaching us; my lifelong pursuit for knowledge is a direct result.

Thank you to all of my siblings and their partners: Christine Sansotta & Felicia Clarke, Myles & Erin Sansotta, Rorie & Eric Elison, Jared Hand & Lorena Mezini, and Andrew Hand & Chloe Sykes. Christine, thank you for showing me that life is always an adventure. Myles, thank you for showing me what hard-work and dedication look like. Rorie, thank you for showing me that there are many paths in life, and that flexibility might just bring you your true passion. Andrew, thank you for always being there, no matter what. I would like to especially thank Dr. Jared Hand for showing me that a Ph.D. was viable, for always providing guidance and support, and for learning so much math right alongside me.

My aunts, uncles, cousins, nieces, nephews, and non-blood relatives have been my support system from the beginning. Thank you to each and every one of you! To all of my nieces and nephews, thank you for bringing some lightness during life's dark times. To my in-laws, Carolyn & Freddy Guzman and the Wojtkos, thank you for all the truly unforgettable memories (and good beers). To my Schultzies, thank you for always bringing me along and giving me so much love.

Last, but surely not least, I would like to thank my husband, Anthony Guzman, for being by my side through this program. I quite literally could not have done this without you, my rock. Scout and I are endlessly grateful for the love, kindness, and unwavering support you have provided through the highs and lows of our time in Arizona. It is a special thing to get these degrees together. I love and appreciate you more than you will ever know.

Contents

Abstract

The pentagram map, first proposed by R. Schwartz in 1992, is a well studied discrete integrable system on real planar polygons. It has been reframed in the language of difference operators using a known correspondence of certain degree 3 difference operators and polygons in $\mathbb{P}^2$. In this paper, we generalize the notion of a projective space by describing one-dimensional "subspaces" in free modules over stably finite rings. We then define polygons in such spaces and discuss how these projective spaces relate to known objects, such as Grassmannians. With these projective spaces, we generalize the correspondence of difference operators and polygons by proving that there is a one-to-one correspondence between properly bounded left (resp. right) difference operators, whose coefficients are from a stably finite ring R, and polygons in a certain left (resp. right) projective space over R. With this correspondence, we show that some known (and proposed) generalizations of the pentagram map can be understood as the usual pentagram map in a certain projective plane. Finally, we reframe these pentagram maps in the language of difference operators, showing that the pentagram map on pseudo-difference operators is a refactorization, and use this interpretation to find invariants.

Chapter 0

A Brief History

The primary goal of this paper is to relate difference operators to projective polygons and a dynamical system on such polygons, known as the pentagram map. We will begin this paper with a brief discussion of the study of dynamical systems before focusing on our particular dynamical system. The development of the research area "dynamical systems" is often attributed to Henri Poincaré and his publications studying celestial mechanics at the end of the 19th century. The colloquial understanding of the study of dynamical systems is *the study of systems that evolve with time*. Many examples of such systems come from physics, such as Poincaré's study of celestial mechanics. But, dynamical systems are defined in much more generality. Let us define precisely what we mean by a dynamical system. There are numerous ways to do so, but we will follow the definitions outlined by G. Teschl in [Tes12]. A *dynamical system* is a semigroup G (with identity) acting on a set M. In other words, we will have a map $T : G \times M \to M$ which is a semigroup action (i.e. the map T respects semi-group composition and the identity). We often require a structure on M, such as being a metric space, but the definition in general does not require any specific structure. A *continuous dynamical system* is a dynamical system with $G = \mathbb{R}$, $G = \mathbb{R}^+$, or some other continuous Lie group. A *discrete dynamical system* is a dynamical system with $G = \mathbb{Z}$ or $G = \mathbb{Z}^+$, which can be understood as the iteration of maps. For this paper, we are primarily (but not exclusively) interested in discrete dynamical systems.

One is often interested in the orbits of a dynamical system. In particular, one may be interested in whether or not certain types of orbits exist (i.e. periodic, dense, etc). If such orbits exist, are they unique? If not, how many such orbits exist? Are there fixed points under the action? Do the orbits live on specific surfaces? This list is not exhaustive, but illustrates some of the general trends in the study of dynamical systems.

We know that many physical systems are modeled by ordinary or partial differential equations (ODEs and PDEs), which are not dynamical systems, a priori. But, ODEs and PDEs can be understood as dynamical systems. For instance, the flow of a first order, linear, autonomous ODE is a continuous dynamical system. These "translations" of differential equations into dynamical systems usually require solutions. Some methods of solving differential equations bring about "first integrals", or invariants, of the system. With sufficiently many appropriate first integrals, many physical systems can be shown to be *integrable*. What is integrability, and why is it interesting? One may take the approach of N. Hitchin in [N J99] by broadly answering in the following way. A differential (or difference) equation is *integrable* if it involves the three following things:

1. Existence of many conserved quantities;
2. Presence of algebraic geometry;
3. Ability to give solutions.

Many interesting physical systems satisfy these three criteria and therefore the study of integrable systems was born. A particularly famous PDE, and integrable system, is the Korteweg-de Vries (KdV) equation, originally found to model waves in shallow water by D. J. Korteweg and G. de Vries in 1895 [KV95]. In time, it was additionally found to model waves moving through cold plasma [GM60] and ion-acoustic waves [WT66]. A famous integrable system was discovered in 1970 when the study of waves in a nonlinear lattice was initiated by M. Toda [Tod70];

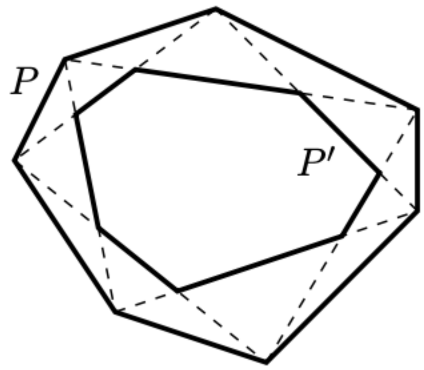

Figure 1: The classical pentagram map sending P to P'

this system was called the *Toda lattice*. The Toda lattice was shown to be a finite dimensional analogue of the KdV equation and its integrability was shown by H. Flaschka in [Fla74].

Now, we bring our focus back to the *pentagram map*, along with some of its known (and potential) generalizations. The pentagram map, first introduced by R. Schwartz in 1992 [Sch92], is a discrete dynamical system on the space of (real) planar polygons, considered up to projective equivalence. The map takes a polygon P to a new polygon P' whose vertices are the intersection points of the lines connecting second nearest neighbors from P. An example of the pentagram map for a closed polygon P is shown in Figure 1. Originally, the map was considered only on closed polygons [Sch92], but it was shown to be defined on a larger class of polygons known as twisted polygons, which are polygons whose vertices have some sort of "quasi-periodic behavior".

Aside from a geometrically satisfying (and comical) construction, what makes this particular dynamical system so interesting? There are a few answers to this question. For instance, it has been shown to be the discretization (in space and time) of the Boussinesq equation, a well known PDE from fluid mechanics [OST10; Sch92]. For now, let us focus on the following answer: it has been shown that the pentagram map on twisted polygons is an integrable system. Originally it was shown to be (Liouville) integrable by finding a Poisson structure and sufficiently many independent first integrals in [OST10]. Then, by finding the Lax representation of the pentagram map, algebraic-geometric integrability was shown in [Sol13].

The pentagram map has been reframed in the language of difference operators [Izo22b], Poisson-Lie groups [FM16] and cluster algebras [GP16; Gek+16]. Furthermore, it has been generalized to higher dimensions [KS13; KS16; Gek+16] as well as to Grassmannians [FB15; Ove20]. Other discrete dynamical systems have also been proposed to be generalizations of the pentagram map. For instance, the "skewer pentagram map" on bi-infinite sequences of lines in $\mathbb{R}^3$ defined in [Tab19].

We were particularly interested in the reframing of the pentagram map in terms of difference operators. In [Izo22b], A. Izosimov showed that the classical pentagram map, and some of its higher-dimensional generalizations, could be interpreted as refactorization type maps on certain classes of pseudo-difference operators. But, this interpretation did not include known generalizations like the Grassmann pentagram map, nor proposed generalizations such as the skewer pentagram map. In an effort to give a ring-theoretic interpretation of such examples, we began an investigation into pentagram type maps in projective spaces over a larger class of (noncommutative) rings.

In Part I, we describe projective spaces over stably finite rings and discuss polygons in such spaces. In Chapter 1, we review some relevant results from ring theory and describe stably finite rings. We provide equivalent definitions of stably finiteness and describe some useful examples. In Chapter 2, we describe "subspaces" in free, finitely generated modules over stably finite rings and use this notion to describe projective spaces. Then, we define polygons in Section 2.2. We recall the notion of an incidence structure and describe some geometry in these projective planes. Finally, in Chapter 3, we discuss what projective spaces over the integers, real square matrices, and the dual numbers will look like, relating them to rational projective spaces, Grassmannians, and skewers, respectively.

In Part II, we introduce and prove a constructive correspondence between difference operators and polygons in our projective spaces. We begin by defining difference operators and proving some results regarding their kernels, in Chapter 4. Then, Chapter 5 is composed of the statement and proof of the correspondence between difference operators and polygons. The correspondence, Theorem 5.1.3, can be paraphrased in the following way:

Theorem A. *Let R be a stably finite ring. Then, there is a one-to-one correspondence between the following sets:*

1. *Polygons in the left (resp. right) projective space of dimension $d-1$ over R, up to projective equivalence.*

2. *Properly bounded left (resp. right) difference operators of degree d with coefficients in R, up to a left/right group action.*

Finally, we discuss a computational example and some other corollaries for certain classes of polygons in Chapter 6.

In Part III, we describe a large class of discrete dynamical systems on polygons in our projective planes, and we use the correspondence above to describe these maps in terms of difference operators. In Chapter 7, we discuss some known pentagram type maps: the "classical" pentagram map in Section 7.1, the Grassmann pentagram map in Section 7.3, and the skewer pentagram map in Section 7.4. In Chapter 8, we describe a discrete dynamical system on polygons in our projective planes, also called the pentagram map. Section 8.2 discusses the relationship between known pentagram maps and our pentagram maps. In particular, throughout Section 8.2, we show the following:

Theorem B. *Let ${}^2P(R)$ denote the left projective plane over a stably finite ring R.*

1. *The "classical" pentagram map on classes of polygons in $\mathbb{P}^2$ is the pentagram map in ${}^2P(\mathbb{R})$.*

2. *The Grassmann pentagram map on classes of polygons in $Gr(m, 3m)$ is the pentagram map in ${}^2P(M_m(\mathbb{R}))$, where $M_m(\mathbb{R})$ denotes square $m \times m$ real matrices.*

3. *The skewer pentagram map on cyclically labeled lines in $\mathbb{R}^3$ is the pentagram map in ${}^2P(\mathbf{D})$, where $\mathbf{D}$ denotes the dual numbers (see Section 3.3).*

In Chapter 8, we use the correspondence of difference operators and polygons to reinterpret the pentagram map in our projective planes as a refactorization of pseudo-difference operators. The refactorization, Theorem 8.5.2, can be paraphrased in the following way:

Theorem C. *Via the correspondence outlined by Theorem A, the following maps coincide:*

1. *The inverse pentagram map on classes of polygons in the projective plane over R;*

2. *The map on of pseudo-difference operators, considered up to a left/right group action, with R coefficients, given by*

$$\mathcal{D}_-^{-1}\mathcal{D}_+ \mapsto \mathcal{D}_+\mathcal{D}_-^{-1}.$$

Then, in Chapter 9, we use this refactorization to highlight invariants of our pentagram maps, in certain cases. Finally, in Chapter 10, we conclude by discussing our results and some possible further research.

Part I

Projective Spaces and Polygons

Chapter 1

Rings and Modules

1.1 A Review of Unital Rings and Their Modules

For the purposes of this paper, R will always be a ring with identity, also called a unital ring, and we will denote the set of two-sided units in R as R^*. Furthermore, we will denote the Cartesian product of n copies of R considered as a right module (resp. left module) as R^n (resp. ${}^nR$). We may consider the dual, $\mathrm{Hom}(R^n, R) =: (R^n)^*$, of the right module R^n. Then, $(R^n)^*$ is a left module. Similarly, $({}^nR)^*$ is a right module. We naturally have a pairing between R^n and ${}^nR$ given by

$$\langle \cdot, \cdot \rangle : {}^nR \times R^n \to R : (\ell, x) \mapsto \sum_{i=1}^{n} \ell_i x_i.$$

This pairing is a nondegenerate, bilinear map and thus induces an isomorphism of (left) modules between ${}^nR$ and $(R^n)^*$ and isomorphism of (right) modules between R^n and $({}^nR)^*$ [Vel81].

Remark 1.1.1 With this pairing, it is natural to consider left module elements as row vectors and right module elements as column vectors. This way, the pairing can be thought of as matrix multiplication of a row vector and a column vector.

The ring $\mathrm{M}_m(R)$ of $m \times m$ matrices over the ring R is an R-bimodule and therefore can be considered as a left or right module, depending on which is appropriate for the given context. We may also consider the space $\mathrm{M}_{n\times m}(R)$ of $n \times m$ matrices over R as either a right or left R-module, since it is also an R-bimodule.

Since we are discussing matrices over unital rings and not fields, some of the usual facts from linear algebra fail (e.g. determinants are not necessarily defined because of potential noncommutativity). Fortunately, we know the following.

Proposition 1.1.2 ***(See [Hun74])*** *For a unital ring R and the free right module R^n, the rings $End_R(R^n)$ and $M_n(R)$ are isomorphic. For the free left module ${}^nR$, the rings $End_R({}^nR)$ and $M_n(R^{op})$ are isomorphic, where R^{op} is the opposite ring.*

Definition 1.1.3 Let R be a ring with identity. An element $a \in R^n$ is said to be *unimodular* if there exists $\ell \in {}^nR$ such that $\langle \ell, a\rangle = 1$. Similarly, an element $\ell \in {}^nR$ is said to be unimodular if there exists $a \in R^n$ such that $\langle \ell, a\rangle = 1$.

We notice that an element $a \in R$ (when R viewed as a right R-module) being unimodular ensures that a has a left inverse. Similarly, if we view R as a left R-module, then a being unimodular means it has a right inverse.

In the following section, we will study a certain class of rings and their free modules. In some of what follows, we rely on the following fact from ring theory. For a proof, see Theorems 3.2 and 3.4 from T. Hungerford's [Hun74].

Theorem 1.1.4. *(See [Hun74]) Let P be a free module over a ring with identity R. Then, P is a projective module. Equivalently, every short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$$

splits; so $B \simeq A \oplus P$.

Since R^n is a projective module by Theorem 1.1.4, we know that any surjective endomorphism $R^n \to R^n$ will have a right inverse, which is also an endomorphism.

1.2 Stably Finite Rings

We now introduce the class of *stably finite rings*, our rings of interest in this paper.

Definition 1.2.1 Let R be a ring with identity. R is called *stably finite* if, for all $n \in \mathbb{N}$, every spanning list of length n in R^n is linearly independent [Coh03].

Remark 1.2.2 Assume R is stably finite. Then, in the right module R^n, the list $\{a_1, ..., a_n\}$ is spanning if and only if it is a basis. Similarly for the left module ${}^nR$.

Such rings are also called *weakly finite* in the literature [Coh03]. Stably finite rings have many nice properties, and can be defined a number of ways. We have the following.

Proposition 1.2.3 ***(See [Lam99; Coh03])*** *The following are equivalent:*

1. *A ring R is stably finite;*
2. *For all $n \in \mathbb{N}$, if $R^n = H \oplus R^n$, then $H = 0$;*
3. *For all $n \in \mathbb{N}$, whenever we have $A, B \in M_n(R)$ with $AB = \mathbb{I}_n$, then $BA = \mathbb{I}_n$;*
4. *Any epimorphism $U : R^n \to R^n$ is necessarily an isomorphism.*

Proof. (1. $\iff$ 2.): Let R be a stably finite ring. For each $n \in \mathbb{N}$, a spanning list $a_1, ..., a_n$ of R^n is necessarily linearly independent (i.e. a basis). For $R^n = H \oplus R^n$, we have $R^n = H + R^n$ and $H \cap R^n = 0$. Let $h \in H$. By hypothesis, we know that $R^n = H \oplus R^n$ so we know $h \in R^n$ (the ambient module), but that means there exists some unique nonzero c_i so that $h = \sum a_i c_i$. But, this ensures $h \in R^n \cap H$, and thus zero.

Let us suppose that $R^n = H \oplus R^n$ implies that $H = 0$. Let e_i denote the standard basis of R^n (i.e. e_i has a 1 in the i-th entry and zeroes elsewhere). Then, for any spanning list $a_1, ..., a_n$ in R^n we have a map where $e_i \mapsto a_i$, denoted by ϕ, inducing the split exact sequence

$$0 \to \ker(\phi) \hookrightarrow R^n \xrightarrow{\phi} R^n \to 0.$$

But, being split means $R^n = \ker(\phi) \oplus R^n$ and by hypothesis $\ker(\phi) = 0$, hence the map is injective. In other words, a_i is a basis.

(2. $\iff$ 4.): Let us now suppose that we know if $R^n = H \oplus R^n$, then $H = 0$. Let us consider a surjective map $U : R^n \to R^n$. We get a split exact sequence,

$$0 \to \ker(U) \hookrightarrow R^n \xrightarrow{U} R^n \to 0.$$

This split exactly sequence gives $R^n = \ker(U) \oplus R^n$. Thus, by hypothesis, $\ker(U) = 0$, giving injectivity.

Suppose that we know every surjective map $R^n \to R^n$ is necessarily an isomorphism. Let us suppose that $R^n \simeq H \oplus R^n$ and let us denote this isomorphism by $\phi : R^n \to H \oplus R^n$. Let p denote the projection from $H \oplus R^n \to R^n$. The map $p \circ \phi$ is surjective, since it is an isomorphism composed with a surjective map. By hypothesis, $p \circ \phi$ is also an isomorphism. Since ϕ is also an isomorphism, p must be an isomorphism. Thus, $0 = \ker(\phi) = H$.

(3. $\iff$ 4.): Suppose we know $AB = \mathbb{I} \implies BA = \mathbb{I}$ for matrices $A, B \in \mathrm{M}_m(R)$. Let us choose a basis for R^n. Suppose that A is the matrix representation of U, a surjective homomorphism, with respect to that basis. Then, since U is a surjective module endomorphism, we know that A has a right inverse. But our hypothesis ensures that this is also the left inverse of A. Therefore, U is injective.

Suppose that we know every surjective map $R^n \to R^n$ is necessarily an isomorphism. From Proposition 1.1.2, we know that endomorphisms from $R^n \to R^n$ are isomorphic to $n \times n$ matrices over R. If an endomorphism is surjective, then its corresponding matrix has a right inverse. But, since we know epimorphisms are isomorphisms, we get that each matrix has a two sided inverse. □

With 3.) from above, we see that another way one can define a stably finite ring is a unitary ring that is Dedekind finite (i.e. for $a, b \in R$, $ab = 1$ implies that $ba = 1$) and whose matrix rings are also Dedekind finite.

We emphasize that stably finite rings need not be commutative (e.g. matrices) so the distinction between left and right objects will be important. Some nice properties of stably finite rings include that their free modules have invariant basis number [Lam99], thus the notion of *rank* makes sense. Additionally, one sided units are necessarily two sided units in stably finite rings. This follows from the Dedekind finiteness condition on R.

Remark 1.2.4 The property of every epimorphism being an isomorphism is known as "Hopfian". Therefore, we know that a ring being stably finite is equivalent to all finitely generated, free modules over the ring being Hopfian. The property of a monomorphism necessarily being an isomorphism is called "co-Hopfian". In general, we know that modules over stably finite rings are not necessarily co-Hopfian. A counterexample to consider is the map $*2 : \mathbb{Z} \to \mathbb{Z} : x \mapsto 2x$. It is injective but clearly not surjective.

Some useful examples of stably finite rings we will discuss in Section 3 are the integers, real square matrices, and dual numbers. But, there are many more examples. For instance, any commutative ring, any Noetherian ring, and any Artinian ring is stably finite [Lam99].

Theorem 1.2.5. *Let R be a ring with identity.*

1. *R is stably finite if and only if its opposite ring R^{op} is stably finite.*

2. *If R is commutative, it is stably finite.*

3. *If R is Noetherian, it is stably finite.*

4. *If R is Artinian, it is stably finite.*

5. *If R is stably finite, then so is $M_m(R)$, for $m \in \mathbb{N}$.*

Proof. 1. We recall that a ring R is stably finite if and only if all of its matrix rings are Dedekind finite. But, a ring is Dedekind finite if and only if its opposite ring is Dedekind finite. One can see this by observing that the two rings are equal as sets, but with opposite multiplication order, and Dedekind finiteness is a symmetry condition on units.

2. If R is commutative, we can use determinants. For $A, B \in \mathrm{M}_m(R)$ satisfy $AB = \mathbb{I}$, A must have a determinant which is a unit. We know that the inverse for a matrix with commutative ring entries will be given by $B = \det(A)^{-1}\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the classical adjoint matrix. Thus, $BA = \mathbb{I}$, as well.

3. To see that a Noetherian ring is stably finite, we first prove the following: For a Noetherian module M, every epimorphism $U : M \to M$ is an isomorphism. Suppose otherwise. Then, we have $x \in \ker(U)$. For $n \in \mathbb{N}$, find $y \in M$ so that $U^n(y) = x$ (U is surjective). Then, we have $U^{n+1}(y) = U(x) = 0$. But, $y \notin \ker(U^n)$. We can do this for each $n \in \mathbb{N}$, giving a strictly ascending chain of ideals,

$$\ker(U) \subset \ldots \subset \ker(U^n) \subset \ker(U^{n+1}) \subset \ldots$$

contradicting the fact that M is Noetherian. If R is Noetherian, then so is any finitely generated, free R-module. Thus, we know for each $n \in \mathbb{N}$, any epimorphism $U : R^n \to R^n$ is an isomorphism. By Lemma 1.2.3, R is stably finite.

4. Suppose that R is Artinian but is not stably finite. We know that any finitely generated, free R-module is also Artinian. By hypothesis of R not being stably finite, we know that for some $n \in \mathbb{N}$, $R^n \simeq H \oplus R^n$ for $H \neq 0$. R^n being isomorphic to its own submodule; $R^n \simeq H \oplus R^n \gneq R^n$. Repetition gives an infinite descending chain of submodules, contradicting R being Artinian.

5. The square matrix ring over a stably finite ring R, denoted by $\mathrm{M}_m(R)$, is stably finite. This follows from the fact that we can understand matrices over $\mathrm{M}_m(R)$ as matrices over R, of larger size.

□

In [Vel81], F. Veldkamp describes projective planes over *stably rank 2 rings*, also called rings of stable range 1. Many of our definitions are extensions of this work. But, we will see that sometimes the stably rank 2 condition is useful. All stable rank 2 rings are stably finite [Lam91]. We define these rings now, since they are mentioned later on.

Definition 1.2.6 Let R be a ring with identity. We say it has *stable rank* 2 if for all unimodular $x = (a, b) \in R^2$ there exists $c \in R$ such that $a + cb$ is unimodular in R.

N.B. For the remainder of the paper, we will consider stably finite rings unless otherwise specified.

Chapter 2

Projective Spaces and Polygons

2.1 Projective Spaces Over Stably Finite Rings

Now, we explore how to projectivize our modules ${}^nR$ and R^n. Let us first recall our usual intuition with real projective spaces. If we are in the real space $\mathbb{R}^n$, the projective space of dimension $n-1$ is the quotient space obtained by collapsing each one-dimensional subspace to a point. We want to mirror such a construction, but we are in a module and therefore the notion of *subspace* is not usually encountered. We will define it as follows.

Definition 2.1.1 If A, a rank p free submodule, is a direct summand of ${}^nR$, we call A a *p-dimensional subspace* of ${}^nR$. In other words, if there exists a submodule B such that ${}^nR = A \oplus B$, we call A a p-dimensional subspace.

Remark 2.1.2 For R stably finite, we know that any p-dimensional subspace of R^n must satisfy $p < n$ because of Proposition 1.2.3.

The notion of a one-dimensional subspace of a vector space is very intuitive. Unfortunately, some of that intuition is lost when we are in the realm of modules. However, when we focus on the unimodular elements of our modules, some of that intuition remains. In order to see this, we prove a series of statements about unimodular vectors. They are stated and proved in right modules, but can analogously be proven for the left modules. First, we want to see how multiplication by scalars affects unimodularity. We have the following equivalence.

Lemma 2.1.3 *Let R be a stably finite ring, $a \in R^n$, and $c \in R$ be a scalar. Then, the following statements are equivalent,*

1. *c is a unit and a is unimodular.*

2. *ac is unimodular.*

Similarly for $a \in {}^nR$.

Proof. Let us first assume that c is a unit and a is unimodular. Then, we know that there exists some element $\ell \in {}^nR$ so that $\langle \ell, a \rangle = 1$. If we consider ac, we know from bilinearity of our natural pairing that

$$\langle c^{-1}\ell, ac \rangle = c^{-1}\langle \ell, a \rangle c = 1.$$

So, $\langle c^{-1}\ell, ac \rangle = 1$, ensuring ac is unimodular.

Now, suppose that we know ac is unimodular. Then, we know that there exists some $\ell \in {}^nR$ so that $\langle \ell, ac \rangle = 1$. Using bilinearity again, we see that

$$\langle \ell, ac \rangle = \langle \ell, a \rangle c = 1$$

which ensures that c has a left inverse. But, since R is stably finite, this ensures that c is a unit. Therefore, with associativity and invertibility of c, we observe that a is unimodular:

$$1 = c(1)c^{-1} = c\langle \ell, a \rangle cc^{-1} = \langle c\ell, a \rangle.$$

□

Now, we want to compare the spans of unimodular elements and one-dimensional subspaces. We will see that the two coincide, which also allows us to comment on unimodular elements and elements of bases.

Proposition 2.1.4 *Let R be stably finite and let $a \in R^n$. Then, $A = aR$ is a one-dimensional subspace if and only if a is unimodular. Similarly for ${}^nR$.*

Proof. Suppose that we have a one-dimensional subspace A spanned by $a \in R^n$. This choice of a is unique up to multiplication by a unit. Since it is a subspace, we know that we can decompose $R^n = A \oplus B$ for some submodule B. Hence, for each $x \in R^n$ there exists unique $\zeta \in R$ and $b \in B$ such that $x = a\zeta + b$. Let us define

$$\ell : R^n \to R : a\zeta + b \mapsto \zeta.$$

Then, since $\ell(a) = 1$ and $(R^n)^*$ is isomorphic to ${}^nR$ via our pairing in Section 1.1, we know that $\langle \ell, a \rangle = 1$, ensuring a is unimodular.

Now, suppose that we have a unimodular $a \in R^n$ and we denote its span $A := aR$. Since a is unimodular, we know that there exists some $\ell \in (R^n)^*$ so that $\ell(a) = 1$. With this, we know that for any $x \in R^n$, we know that $x = a\ell(x) + (x - a\ell(x))$. Thus, $R^n = A \oplus \ker(\ell)$, as desired. □

Proposition 2.1.5 *Let R be stably finite. If an element $a \in R^n$ is in a basis for R^n, then it is unimodular. Similarly for $a \in {}^nR$.*

Proof. Suppose that $a \in R^n$ is a member of the basis $\{a_1, ..., a_n\}$. Suppose without loss of generality that $a = a_1$. Then, by the definition of a basis, we know that

$$R^n = a_1R \oplus a_2R \oplus \ldots \oplus a_nR.$$

This ensures that a_1R is a one-dimensional subspace. From Proposition 2.1.4, a_1 is unimodular. □

Remark 2.1.6 We notice that by Proposition 2.1.4, any p-dimensional subspace can be found to have unimodular generators. To see this, suppose that we have a p-dimensional subspace A generated by $\{a_1, ..., a_p\}$. Then, we know there is some B so that

$$R^n = a_1R \oplus \ldots \oplus a_pR \oplus B.$$

But, each a_iR is a one-dimensional subspace, and therefore we know that each a_i is unimodular.

Remark 2.1.7 Notice that in this generality, some intuition regarding linear independence and basis fails. For instance, a list of linearly independent vectors of the appropriate length need not be a basis. For example, $(1, 2)$ and $(2, 1)$ are linearly independent over $\mathbb{Z}$ and both are unimodular, but they do not form a basis of $\mathbb{Z}^2$.

We now have some intuition about one-dimensional subspaces back. We notice that we do not necessarily know if the complements to these one-dimensional subspaces, denoted above by B, will be free submodules. There are classes of rings, such as rings of stable rank 2 from Definition 1.2.6, for which B will necessarily be a free submodule. In cases such as this, Proposition 2.1.5 becomes an equivalence statement.

Proposition 2.1.8 *(See [Vel81]) Let R be stable rank 2. An element $a \in R^n$ is a basis element if and only if it is unimodular. Similarly for $a \in {}^nR$.*

We now have the ability to define projective spaces over these modules. Since the right and left module are not necessarily the same, we have to consider right and left projective spaces separately.

Definition 2.1.9 The *right projective space of dimension n over R* is defined as

$$P^n(R) := \left(\{a \in R^{n+1} : a \text{ is unimodular}\}\right) \Big/ \left(a\lambda \sim a\right) \qquad \forall \lambda \in R^*$$

where R^* denotes the units of R. In other words, $P^n(R)$ is the set of one-dimensional subspaces in R^{n+1}.

Definition 2.1.10 The *left projective space of dimension n over R* is defined as

$${}^nP(R) := \left(\{a \in {}^{n+1}R : a \text{ is unimodular}\}\right) \Big/ \left(\lambda a \sim a\right) \qquad \forall \lambda \in R^*$$

where R^* denotes the units of R. In other words, ${}^nP(R)$ is the set of one-dimensional subspaces in ${}^{n+1}R$.

Remark 2.1.11 Since one-dimensional subspaces and spans of unimodular elements are the same in R^{n+1}, this definition of $P^n(R)$ ensures that one-dimensional subspaces of the module projectivize to points. Similarly for the left projective space.

2.2 Polygons in Projective Space

With these projective spaces, we may now define *polygons*.

Definition 2.2.1 A *polygon in* $P^n(R)$ is a bi-infinite sequence of points $\{x_i\}$ satisfying the following "spanning condition": for all $i \in \mathbb{Z}$ there exists a corresponding unimodular $X_i \in R^{n+1}$ where the projectivization of X_iR is x_i and

$$R^{n+1} = X_iR \oplus X_{i+1}R \oplus \ldots \oplus X_{i+n}R.$$

We analogously define a *polygon in* ${}^nP(R)$ as a bi-infinite sequence of points $\{x_i\}$ satisfying the following "spanning condition": for all $i \in \mathbb{Z}$ there exists a corresponding unimodular $X_i \in {}^{n+1}R$ where the projectivization of RX_i is x_i and

$${}^{n+1}R = RX_i \oplus RX_{i+1} \oplus \ldots \oplus RX_{i+n}.$$

In each case, we call the bi-infinite sequence (X_i) a *unimodular lift* of the polygon $\{x_i\}$. For polygons and their lifts, it is our standard that capital letters belong to the module and lower case belong to the corresponding projective space.

Remark 2.2.2 We require a "spanning condition" to generalize the notion of polygons having vertices in general position. We state our "spanning condition" in this way to emphasize that the unimodular lifts decompose the space nicely.

Definition 2.2.3 A *right projective transformation* is a map $M : P^n(R) \to P^n(R)$ induced by a right module automorphism $M : R^{n+1} \to R^{n+1}$. A *left projective transformation* is defined analogously.

In a slight abuse of notation, we will often use the same name for the module automorphism and the projective transformation.

Definition 2.2.4 We say two polygons $\{x_i\}$ and $\{\widehat{x}_i\}$ are in the same class up to projective equivalence (or, projectively equivalent) if there exists a projective transformation that takes each x_i to $\widehat{x}_i$.

For most of the paper, we are interested in polygons considered up to projective equivalence. We notice that our definition of polygons allows either infinite polygons or "closed" polygons (i.e. polygons where there is periodic behavior in the points). We do not require any specific periodic behavior in general for this paper, but we are sometimes interested in specific kinds of polygons, such as "twisted" polygons. Consider the following definition.

Definition 2.2.5 We say a polygon $\{x_i\}$ in ${}^nP(R)$ is a *twisted N-gon* if there exists a fixed projective transformation M on ${}^nP(R)$ so that $x_{i+N} = M(x_i)$ for all $i \in \mathbb{Z}$. If $x_{i+N} = x_i$ for all $i \in \mathbb{Z}$, we call the polygon *closed*.

We call N the *period* of the twisted N-gon and M the *monodromy*. In a slight abuse of notation, we also use M to denote the corresponding homomorphism on ${}^{n+1}R$. We notice that closed polygons occur in the special case when M is the identity. We can analogously define twisted (resp. closed) N-gons in $P^n(R)$.

2.3 A Review of Incidence Structures

Throughout this paper, we will rely on the notion of an *incidence structure*. As we move forward and discuss pentagram type maps in Part III, we will see that this structure is the minimal structure required to describe such maps. Since these structures will be mentioned often throughout the remainder of the paper, we take a moment to formally define them and give a few basic examples.

Definition 2.3.1 An *incidence structure* is a set of "points", denoted by P, a set of "lines", denoted L, and an incidence $I \subset P \times L$. We say $p \in P, \ell \in L$ are "incident" when $(p, \ell) \in I$.

When $(p, \ell) \in I$, we may also say the point "lies on the line". We say that two incidence structures (P, L) and (P', L') are isomorphic if there is a bijective map from $(P, L) \to (P', L')$ for which I maps onto I'.

A somewhat trivial example would be viewing $\mathbb{R}^n$ carrying the "usual" incidence structure: $P = \{\text{points}\}$, $L = \{\text{lines}\}$ and $p \in P$ is incident to $\ell \in L$ if $p \in \ell$. Any time we have a space that has a notion of points and lines (e.g. vector spaces, projective spaces, etc.), the space carries an incidence structure. Therefore incidence geometry is not a departure from some of the usual geometries we encounter. But, we may also view some of these known spaces as carrying alternative incidence structures. An example is the "skewer" incidence structure which lives in the ambient space $\mathbb{R}^3$. We will discuss this particular incidence structure in Section 3.3. For now, we see that our projective planes have minimal geometry, a priori. But, they do carry an incidence structure. One will see how incidence structures become important when discussing pentagram type maps (in Part III). Although the structure is very simple, the construction of these pentagram type maps will only requires a list of objects (our "points") in our projective planes, a notion of a "line" connecting said objects, and a notion of intersecting such lines.

2.4 Some Geometry of ${}^2P(R)$

In [Vel81], F. Veldkamp investigated the geometry of the projective plane over rings with stable rank 2, a class of rings contained in stably finite rings. We use some of his notation in what follows, but we generalize to stably finite rings. We want to formalize some of notions necessary for future sections, such as incidence, but we need to be careful given the generality.

Definition 2.1.1 gives us the notion of a p-dimensional subspace of ${}^3R$. We use this notion to view our projective planes as incidence structures.

Definition 2.4.1 *"Points"* in ${}^2P(R)$ will correspond to projectivizations of 1-dimensional subspaces (as done previously in the paper) and *"lines"* in ${}^2P(R)$ are the projectivizations of 2-dimensional subspaces of the module ${}^3R$. Incidence is given by inclusion. We say that three points are *collinear* if they lie on a common line.

Remark 2.4.2 We can analogously define a line, or other higher dimensional spans, in a higher dimensional projective space. In such cases, incidence would still be given by containment. However, for the purposes of this paper, we are most often interested in projective planes.

In the real projective plane, we have many nice geometric features. For instance, any two distinct points share a unique line and any two distinct lines share a unique intersection point. This is not true in our generality. We now investigate one potential problem: "neighboring" points.

Definition 2.4.3 We say two distinct points $x, y \in {}^2P(R)$ are *neighboring*, denoted $x \approx y$, if for their respective lifts $X, Y \in {}^3R$ the submodule $\text{span}_R(X, Y)$ is not a two-dimensional subspace of ${}^3R$.

In the case of the real projective plane, such a pair does not exist. But, in a stably finite ring, we may have zero divisors causing problems. For example, let $1, a, b \in R$ such that $a \neq b$ but they share a common (left) zero divisor λ. Then, the projectivizations of the vectors $(1, a, 0)$ and $(1, b, 0)$ are distinct but the two vectors are linearly dependent. Therefore, these projective points are "neighboring".

Lemma 2.4.4 *Let R be a stably finite ring. Suppose that we have distinct $x, y \in {}^2P(R)$ so that $x \not\approx y$. Then, x and y lie on a common line in ${}^2P(R)$.*

Proof. The points x and y being non-neighboring means that the $\text{span}_R(X, Y) = RX \oplus RY$ is a two-dimensional subspace, where X, Y denote unimodular lifts of x, y, respectively. Thus, they lie on a common line in ${}^2P(R)$. □

The spanning condition of polygons ensures that nearest and second nearest neighbors are non-neighboring. Therefore, they share a line. Furthermore, the common line for nearest (or second nearest) neighbors is unique.

Lemma 2.4.5 *Let us consider a polygon $\{x_i\}$ in ${}^2P(R)$ and a unimodular lift (X_i) with $X_i \in {}^3R$. Using the spanning condition of polygons, let $B_i, C_i, D_i \in R$ be coefficients so that*

$$X_i = B_i X_{i+1} + C_i X_{i+2} + D_i X_{i+3}.$$

Then, we have the following

1. *For all $i \in \mathbb{Z}$, $X_i - C_i X_{i+2}$ and $X_i - B_i X_{i+1}$ are unimodular.*

2. *The unique line through the pair x_i, x_{i+2} and the unique line through the pair x_{i+1}, x_{i+3} intersect uniquely, and*

$$X_i - C_i X_{i+2} = B_i X_{i+1} + D_i X_{i+3} \in {}^3R$$

   projects to that intersection point. In other words, the intersection point's unimodular lift is $X_i - C_i X_{i+2} = B_i X_{i+1} + D_i X_{i+3}$.

3. *The unique line through the pair x_i, x_{i+1} and the unique line through the pair x_{i+2}, x_{i+3} intersect uniquely, and*

$$X_i - B_i X_{i+1} = C_i X_{i+2} + D_i X_{i+3} \in {}^3R$$

   projects to that intersection point. In other words, the intersection point's unimodular lift is $X_i - B_i X_{i+1} = C_i X_{i+2} + D_i X_{i+3}$.

Proof. 1. From Proposition 2.1.5, we know that it suffices to find a basis of ${}^3R$ for which $X_i + C_i X_{i+2}$ is an element. Consider the set $\{X_i + C_i X_{i+2}, X_{i+1}, X_{i+2}\}$. This is a basis. Similarly, $X_i + B_i X_{i+1}$ is an element of the basis $\{X_i + B_i X_{i+1}, X_{i+1}, X_{i+2}\}$.

2. We know that the unique line going through x_i, x_{i+2} corresponds to $\text{span}_R(X_i, X_{i+2}) \subset {}^3R$. Likewise the line going through x_{i+1}, x_{i+3} and $\text{span}_R(X_{i+1}, X_{i+3}) \subset {}^3R$. We know that these two spans intersect at $X_i - C_iX_{i+2} = B_iX_{i+1} + D_iX_{i+3}$, which is unimodular by the first statement. Therefore, the two lines intersect.

   Suppose these two lines intersect at another point in ${}^2P(R)$, denoted by $\widehat{x}$. We know that a unimodular lift $\widehat{X}$ satisfies

$$\widehat{X} = aX_i + cX_{i+2} = bX_{i+1} + dX_{i+3}$$

   for some $a, b, c, d \in R$. These coefficients are unique, since X_i's are unimodular lifts from a polygon. (i.e. $\{X_{i+1}, X_{i+2}, X_{i+3}\}$ is a basis of ${}^3R$, for all $i \in \mathbb{Z}$).

   Separately, we know, by our hypothesis, that our unimodular lifts satisfy

$$X_i = B_iX_{i+1} + C_iX_{i+2} + D_iX_{i+3}.$$

   We know that $X_i - C_iX_{i+2}$ is unimodular. Additionally, we claim that it is a unit multiple of $\widehat{X} = aX_i + cX_{i+2}$. Suppose that $a(X_i - C_iX_{i+2}) = aX_i + cX_{i+2}$. Then, by Lemma 2.1.3, we get that a is a unit immediately. Suppose instead that they are not equal. Then, we know that $a(X_i - C_iX_{i+2}) - (aX_i + cX_{i+2}) = (-aC_i - c)X_{i+2}$ is nonzero. However, by construction we know that both $a(X_i - C_iX_{i+2})$ and $aX_i + cX_{i+2}$ are in $\text{span}_R(X_{i+1}, X_{i+3})$ and therefore their difference is also in this span. But, this means $(-aC_i - c)X_{i+2} \in \text{span}_R(X_{i+1}, X_{i+3})$, contradicting the fact that $\{x_i\}$ is a polygon. Therefore, we must have $a(X_i - C_iX_{i+2}) = aX_i + cX_{i+2}$ and therefore $X_i - C_iX_{i+2}$ is a unit multiple of $\widehat{X}$, as desired.

3. The proof of this claim is identical with indices $i + 1$ and $i + 2$ exchanged.

□

Chapter 3

Examples of Projective Spaces and Polygons

3.1 Integers vs. Rational Numbers

Let us first consider the cases of $R = \mathbb{Z}$ and $R = \mathbb{Q}$. Since $\mathbb{Z}$ and $\mathbb{Q}$ are commutative, we know by Theorem 1.2.5 that both rings are stably finite. Furthermore, commutativity ensures that the notion of "right" and "left" projective spaces are the same. The first thing we notice is that $P^n(\mathbb{Z}) = P^n(\mathbb{Q})$.

Proposition 3.1.1 *Projective spaces over the integers and the rational numbers are equal:* $P^n(\mathbb{Z}) = P^n(\mathbb{Q})$.

Proof. Given $\mathbb{Z} \subset \mathbb{Q}$, it is clear that any point in $P^n(\mathbb{Z})$ will be a point in $P^n(\mathbb{Q})$. To show the reverse containment, let $[a_0 : a_1 : \ldots : a_n] \in P^n(\mathbb{Q})$. Thus, $a_i \in \mathbb{Q}$ for all $i \in \{0, ..., n\}$. Let λ be the ratio of the least common denominator and greatest common denominator for the list $\{a_0, ..., a_n\}$. Then, $[a_0 : a_1 : \ldots : a_n] = [\lambda a_0 : \lambda a_1 : \ldots : \lambda a_n]$ in $P^n(\mathbb{Q})$ but all of the $\lambda a_i \in \mathbb{Z}$. To show that $[\lambda a_0 : \lambda a_1 : \ldots : \lambda a_n] \in P^n(\mathbb{Z})$, we need to check that $(\lambda a_0, \lambda a_1, ..., \lambda a_n)$ is unimodular in $\mathbb{Z}^{n+1}$. If we know that λa_i and λa_j will be coprime for some $i \neq j$ (by choice of λ), and therefore we can find integers n, m so that $1 = na_i + ma_j$. Therefore, a vector in $\mathbb{Z}^{n+1}$ with n in the i-th entry, m in the j-th entry, and zeroes elsewhere will give us unimodularity of $(\lambda a_0, \lambda a_1, ..., \lambda a_n)$. If no two λa_i and λa_j are coprime, then λa_i are all equal, so we have unimodularity of $(\lambda a_0, \lambda a_1, ..., \lambda a_n)$ by scaling alone. Together, this has shown that $[\lambda a_0 : \lambda a_1 : \ldots : \lambda a_n]$ is in $P^n(\mathbb{Z})$, as desired. Therefore, $P^n(\mathbb{Z}) = P^n(\mathbb{Q})$. □

Remark 3.1.2 This same argument would ensure that the projective space over any principal ideal domain equals the projective space over its field of fractions, since we can construct an appropriate scaling λ. We have to be careful to ensure that scaling in the field of fractions will still result in a unimodular element over the original ring.

Now, let us consider polygons in $P^n(\mathbb{Z})$ and $P^n(\mathbb{Q})$. Although the spaces are equal, we will see that polygons in the spaces look quite different. We know that determinants make sense in the commutative setting, so we can rephrase the "spanning condition" of a polygon $\{x_i\}$ in $P^n(\mathbb{Z})$ as follows. You can perform an analogous computation on a polygon from $P^n(\mathbb{Q})$. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a bi-infinite sequence of points in $P^n(\mathbb{Z})$. Let $X_i \in \mathbb{Z}^{n+1}$ be a unimodular lift for each x_i (viewed as a row). Then, we may consider the matrix

$$\begin{pmatrix} X_i \\ X_{i+1} \\ \vdots \\ X_{i+n} \end{pmatrix}.$$

If this determinant is a unit (i.e. ± 1) for all i, then $\{x_i\}$ form a polygon in $P^n(\mathbb{Z})$.

Now, let us consider the projective plane, $P^2(\mathbb{Z})$. One might ask, is the image in Figure 3.1 part of a polygon in this plane? If we lift these three points into $\mathbb{Z}^3$, we may choose unimodular lifts (considered as rows),

$$(m, n, 1), \qquad (0, 0, 1), \qquad (1, 0, 1).$$

Given our understanding of the polygon spanning condition above, the possibility

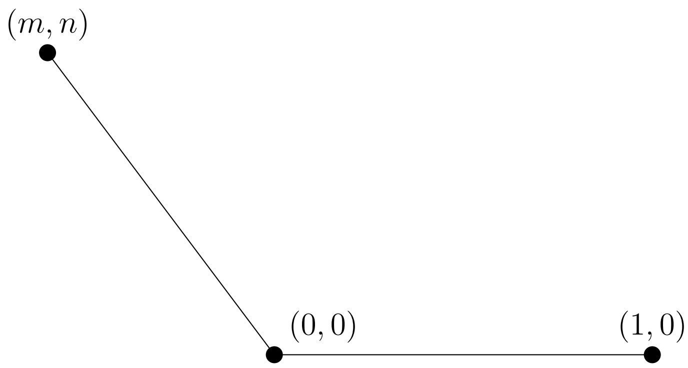

Figure 3.1: Is this part of a polygon in $P^2(\mathbb{Z})$?

of these three points lying on a polygon reduces to the question of whether or not this matrix has unit (± 1 for $\mathbb{Z}$) determinant:

$$\begin{pmatrix} m & n & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

A quick computation shows that the determinant is n, therefore Figure 3.1 may lie on a polygon in $P^2(\mathbb{Z})$ only when $n = \pm 1$. For example, the case of $m = 0, n = 2$ would not possibly result in a part of a polygon, but $m = 0, n = 1$ could.

If we were to perform the same computation over $\mathbb{Q}$, we would see that Figure 3.1 is on a polygon exactly when $n \neq 0$, which would agree with our usual intuition. For instance, the example above, with $m = 0, n = 2$, could belong to a polygon in $P^2(\mathbb{Q})$, but not in $P^2(\mathbb{Z})$.

3.2 Matrices & Grassmannians

Let us consider the case where $R = \mathrm{M}_m(\mathbb{R})$, square $m \times m$ real matrices. To simplify our notation, we will denote these matrices by M_m, omitting the specification of $\mathbb{R}$. By Theorem 1.2.5, we know that this ring is stably finite. The first thing we will observe is that although M_m is not commutative, the right and left projective spaces are canonically isomorphic. Any right projective space can be viewed as a left projective space over the opposite ring. But, when we consider matrices, we know that the opposite ring is isomorphic to the ring itself by taking the transpose.

We are interested in looking at the projective space ${}^{n-1}P(\mathrm{M}_m)$. For points in this projective space, "lifts" lie in ${}^{n}(M_m)$. In other words, a unimodular "lift" of a point from ${}^{n-1}P(\mathrm{M}_m)$ is a $1 \times n$ matrix over M_m, which can be viewed as a $m \times nm$ real matrix. Unimodularity ensures these lifts have full rank. We aim to relate the row space of such matrix to the original point in ${}^{n-1}P(\mathrm{M}_m)$.

Remark 3.2.1 We know that multiplication on the left is a "row space operation". Therefore, for some collection of $m \times nm$ matrices $X_1, ..., X_k$, the left M_m-span of $\{X_1, ..., X_k\}$ is the $\mathbb{R}$ span of the rows of the matrices $\{X_1, ..., X_k\}$, and vice-versa.

With the goal of thinking of points in our projective space ${}^{n-1}P(\mathrm{M}_m)$ as row spaces of $m \times nm$ matrices, we will use the language of *Grassmannians*. Let us recall what a Grassmannian is and establish some notation.

Definition 3.2.2 The Grassmannian Gr(k,n) is the collection of all k-dimensional subspaces in $\mathbb{R}^n$.

We see that $\mathrm{Gr}(1,3) = \mathbb{P}^2$, the real projective plane. In general, k and n need not be related. But, for our purposes, we are often interested in $\mathrm{Gr}(m, nm)$. For $l \in \mathrm{Gr}(m, nm)$, let us consider $X_l \in \mathrm{M}_{m \times nm}$ so that the rows form a basis for l. We may denote this relationship by $l = \mathrm{Row}(X_l)$ (i.e. $\mathrm{Row}(\cdot)$ references the the row-space(s) of a matrix, or a list of matrices). We call X_l a *lift* of l. This lift is not unique. We consider lifts up to multiplication by $m \times m$ invertible matrices on the left, since this does not change the row-space.

Lemma 3.2.3 *There is a one-to-one correspondence between points in ${}^{n-1}P(M_m)$ and points in Gr(m, nm).*

Proof. For a point $x \in {}^{n-1}P(\mathrm{M}_m)$, we choose unimodular lift $X \in {}^{n}\mathrm{M}_m$, which is a full rank $m \times nm$ matrix. The row space corresponds to an m-dimensional subspace of $\mathbb{R}^{nm}$. Let $l = \text{Row}(X) \in \text{Gr}(m, nm)$. Furthermore, x, our point in the projective space, corresponds to all unit (in M_m) multiples of X, on the left. Such products do not change the row space. Therefore, the point $x \in {}^{n-1}P(\text{M}_m)$ corresponds to l, a unique m-dimensional subspace of $\mathbb{R}^{nm}$, which an element of $\text{Gr}(m, nm)$. Finally, the map $x \mapsto l := \text{Row}(X)$ is surjective because any m-dimensional subspace of $\mathbb{R}^{nm}$ can be found as the row space of some $m \times nm$ matrix. □

Now that we have shown that the points ${}^{n-1}P(\text{M}_m)$ and points in $\text{Gr}(m, nm)$ are in bijection, we want to show that the incidence structures carried by both spaces are isomorphic. We recall that a line in ${}^{n-1}P(\text{M}_m)$ is the (left) M_m-span of two non-neighboring, unimodular elements in ${}^n\text{M}_m$. And, incidence is given by containment. Now, we need to define the incidence structure carried by Grassmannians.

Definition 3.2.4 Let a *"line"* be an element of $\text{Gr}(2m, nm)$, thus a $2m$-dimensional subspace of $\mathbb{R}^{nm}$. Incidence between points from $\text{Gr}(m, nm)$ and lines from $\text{Gr}(2m, nm)$ is given by containment. We say three points in $\text{Gr}(m, nm)$ are *collinear* if they span a $2m$-dimensional subspace.

Now, we can show that the incidence structures carried by ${}^{n-1}P(\text{M}_m)$ and $\text{Gr}(m, nm)$ are isomorphic.

Proposition 3.2.5 *The incidence structures* (P, L)*, with* $P = \{$*points in* ${}^{n-1}P(M_m)\}$ *and* $L = \{$*lines in* ${}^{n-1}P(M_m)\}$*, and* (P', L')*, with* $P' = Gr(m, nm)$ *and* $L' = Gr(2m, nm)$*, are isomorphic.*

Proof. We show that lines in ${}^nP(\text{M}_m)$ (equivalently, 2-dimensional subspaces of ${}^n\text{M}_m$) are in one-to-one correspondence with elements of $\text{Gr}(2m, nm)$. In other words, we build a bijective map $L \to L'$.

Let us begin with a 2-dimensional subspace of ${}^n\text{M}_m$. We know that this subspace is the (left) M_m span of two non-neighboring elements $X, Y \in {}^n\text{M}_m$ and that we have the following decomposition:

$$^n\mathrm{M}_m = \mathrm{M}_m X \oplus \mathrm{M}_m Y \oplus B,$$

for some submodule B. Non-neighboring elements X, Y must have linearly independent row spaces, because of Remark 3.2.1. Thus, if we consider the row space of the matrix $\begin{pmatrix} X \\ Y \end{pmatrix}$, which is a $2m \times nm$ matrix, we get a $2m$-dimensional subspace of $\mathbb{R}^{nm}$, which is an element of $\mathrm{Gr}(2m, nm)$. If we choose different unimodular non-neighboring generators for our subspace, denoted by X', Y', we still have

$$^n\mathrm{M}_m = \mathrm{M}_m X' \oplus \mathrm{M}_m Y' \oplus B,$$

and therefore the M_m-span of X, Y equals the M_m-span of X', Y'. But, by Remark 3.2.1, that means

$$\mathrm{Row}\begin{pmatrix} X \\ Y \end{pmatrix} = \mathrm{Row}\begin{pmatrix} X' \\ Y' \end{pmatrix}.$$

Therefore, we have a map $L \to L'$ that is well-defined and injective. It is surjective because any $2m$-dimensional subspace of $\mathbb{R}^{nm}$ can be found in this way.

By Lemma 3.2.3, we know that there is a one-to-one correspondence between P and P'. Above, we constructed a one-to-one correspondence between L and L'. The map $(P, L) \to (P', L')$ induced by these maps, therefore, is also bijective. We must show that incidence is maintained. Suppose $p \in P$ is incident to $\ell \in L$. Then, for unimodular lift X_p of p, we know that there exists some unimodular Y so that ℓ is the projectivization of $\mathrm{M}_m X_p \oplus \mathrm{M}_m Y$. By Remark 3.2.1, we know that

$$\mathrm{M}_m X_p \oplus \mathrm{M}_m Y = \mathrm{Row}\begin{pmatrix} X_p \\ Y \end{pmatrix}$$

and therefore $\mathrm{Row}(X_p)$ (p's corresponding point in $\mathrm{Gr}(m, nm)$) is contained in the $2m$-dimensional subspace Row $\begin{pmatrix} X_p \\ Y \end{pmatrix}$ (ℓ's corresponding line in $\mathrm{Gr}(2m, nm)$), ensuring incidence is maintained.

□

The last thing we explore is the relationship between polygons in ${}^{n-1}P(\mathrm{M}_m)$ and what a "polygon" would be in $\mathrm{Gr}(m, nm)$. We have a correspondence between points in both of these spaces, so we could use that correspondence to describe polygons in $\mathrm{Gr}(m, nm)$. But, in [FB15], G. Marí Beffa and R. Felipe describe polygons for Grassmannians. We describe their notion of a polygon, but adjust some definitions to be more consistent with our work thus far. More specifically, we transpose most definitions so we may consider left spans. We then show that their definition of a polygon in $\mathrm{Gr}(m, nm)$ corresponds to our notion of a polygon in ${}^{n-1}P(\mathrm{M}_m)$.

Definition 3.2.6 A *polygon in Gr(m,nm)* is a bi-infinite list $l = (l_i)_{i\in\mathbb{Z}}$ such that the following spanning condition is satisfied:

$$\mathbb{R}^{nm} = l_i \oplus l_{i+1} \oplus \ldots \oplus l_{i+n-1}$$

for all $i \in \mathbb{Z}$.

One may also think of spanning condition of a polygon $\{l_i\}$ in the following way: for any lift $X = (X_i)$ of l, the matrix

$$\begin{pmatrix} X_i \\ \vdots \\ X_{i+n-1} \end{pmatrix}$$

has nonzero determinant for all $i \in \mathbb{Z}$.

Remark 3.2.7 In [FB15], a polygon satisfying the stated spanning condition is called "regular" and "polygon" simply refers to a bi-infinite list of elements from the Grassmannian. Furthermore, they show that the "moduli space of twisted polygons" is a $N(m-1)n$-dimensional manifold, and they use matrices to find local coordinates. The coordinates for this moduli space in the special case of $\mathrm{Gr}(m, 3m)$ is discussed in Appendix B.

Lemma 3.2.8 *There is a one-to-one correspondence between the following collections:*

1. *Polygons in* ${}^{n-1}P(M_m)$*;*
2. *Polygons in* $Gr(m, nm)$*.*

Proof. The points of ${}^{n-1}P(\mathrm{M}_m)$ and $\mathrm{Gr}(m, nm)$ are in correspondence by Lemma 3.2.3. In Remark 3.2.1, we observed that the (left) M_m-span of $m \times nm$ matrices $X_1, ..., X_n$ is equal to the row space of the matrix

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

Therefore, the spanning conditions for polygons coincide. □

3.3 Dual Numbers & Lines in $\mathbb{R}^3$

The *dual numbers* are a commutative real algebra defined as

$$\mathbf{D} := \mathbb{R}[\varepsilon] \big/ \langle \varepsilon^2 \rangle.$$

In other words, it is the set of numbers of the form $a + \varepsilon b$ where a and b are real, $\varepsilon^2 = 0$, addition is component-wise, and multiplication corresponds to polynomial multiplication:

$$(a + b\varepsilon)(c + d\varepsilon) = ac + (ad + bc)\varepsilon.$$

This ring is commutative and therefore is stably finite, by Theorem 1.2.5.

One may also discuss *dual vectors* as elements of $\mathbf{D}^n$, for some $n \in \mathbb{N}$. Since $\mathbf{D}$ is a commutative algebra, the distinction between right and left modules is unnecessary. To avoid confusion from overloading terms, we have no discussion of "dual spaces" from linear algebra in this section, and therefore have no "dual" objects.

Remark 3.3.1 Dual numbers have many nice qualities. For instance, they are stably rank 2 which means, by Proposition 2.1.8, an element $a \in \mathbf{D}^n$ is unimodular if and only if it is a basis element. This also ensures that the complement of any p-dimensional subspace of $\mathbf{D}^n$ is a free submodule.

Interestingly, one may notice that the "skewer incidence structure", which lives in $\mathbb{R}^3$, ends up having a close relationship with the dual numbers. Before we discuss this relationship, let us recall the following from geometry and establish some notation.

Definition 3.3.2 Two lines in $\mathbb{R}^3$ are said to be *skew* if they are not coplanar. Any two skew lines share a unique common perpendicular line that is called their *skewer*. For two skew lines ℓ and m, we denote their skewer as $S(\ell, m)$ [Tab16].

Some classical configuration theorems can be rephrased and reproved with skewers as in [Tab19]. However, we are interested in the relationship between dual numbers and skewers as described in [Gug63; Tab16].

Let ℓ be an oriented line in $\mathbb{R}^3$ (the Euclidean space). This line does not necessarily go through the origin. We may assign to this line a corresponding dual vector $\xi_\ell := u + \varepsilon v \in \mathbf{D}^3$ as follows. Let $x, y \in \ell$ be distinct points. Then, we consider the vector $u := \rho(x - y)$ where ρ is a real number chosen to ensure $|u| := u \cdot u = 1$. ρ is not unique, but we choose ρ value so that u is parallel to ℓ with matching orientation. We call this the "direction" vector. Then, define $v = u \times OP$ with OP a vector which originates from the origin in $\mathbb{R}^3$ and whose endpoint is on ℓ. This vector v is independent of choice of P.

Remark 3.3.3 In the case of ℓ going through the origin, our corresponding dual vector will be strictly real since u and OP will always be parallel. We emphasize that the discussion which follows is analogous to the construction of $\mathbb{P}^2$ as unoriented lines through the origin of $\mathbb{R}^3$ after considering points on the two-sphere of $\mathbb{R}^3$ as being oriented lines through the origin.

Since $\mathbf{D}$ is commutative, the left and right modules are the same. Therefore, the canonical pairing between left and right modules from Section 1.1 is the usual dot product of vectors. We observe that $\langle \xi_\ell, \xi_\ell \rangle = \xi_\ell \cdot \xi_\ell = 1$, ensuring that each ξ_ℓ is unimodular and on the unit sphere in $\mathbf{D}^3$. To verify, we observe the following,

$$(u + \varepsilon v) \cdot (u + \varepsilon v) = u \cdot u + 2\varepsilon(u \cdot v) = 1 + \varepsilon(0) = 1.$$

Furthermore, all elements of the dual unit sphere in $\mathbf{D}^3$ correspond to an oriented line in $\mathbb{R}^3$. This relationship was originally proposed by E. Study and was discussed further in [Gug63]. We have the following.

Lemma 3.3.4 *Oriented lines in $\mathbb{R}^3$ (not necessarily through the origin) are in one to one correspondence with points on the dual sphere in $\mathbf{D}^3$ via*

$$\ell \leftrightarrow \xi_\ell.$$

Additionally, unoriented lines in $\mathbb{R}^3$ (not necessarily through the origin) correspond to points in $P^2(\mathbf{D})$ (i.e. lines with opposite orientation correspond to the same point in $P^2(\mathbf{D})$).

Proof. The correspondence between unoriented lines in $\mathbb{R}^3$ and points on the unit sphere of $\mathbf{D}^3$ follows from the construction above. The second correspondence follows from the fact that orientation is determined by the direction vector u and therefore considering unoriented lines is accounted for by projectivization (i.e. scaling by -1). $\square$

Remark 3.3.5 This construction depends on the choice of origin in $\mathbb{R}^3$. This is because v is defined using the origin. However, we observe that changing the origin in $\mathbb{R}^3$ corresponds to a projective transformation on $P^2(\mathbf{D})$. To see this, assume we have two origins O and O'. Then, the change in origin from O to O' in $\mathbb{R}^3$ will result in new $\xi'_\ell = \xi_\ell - \varepsilon(u_\ell \times OO')$. The map $\xi_\ell \mapsto \xi'_\ell$ is linear, since the cross product is linear. And, it is bijective, since both $\ell \mapsto \xi_\ell$ and $\ell \mapsto \xi'_\ell$ are bijective. Together, we have $\xi_\ell \mapsto \xi'_\ell$ is an automorphism of $\mathbf{D}^3$.

From this point on, we only consider the correspondence between unoriented lines in $\mathbb{R}^3$ and points in $P^2(\mathbf{D})$. We want this correspondence to retain some of the underlying geometry from both $\mathbb{R}^3$ and $P^2(\mathbf{D})$. In particular, we have the structure in $\mathbb{R}^3$ determined by the "skewer incidence". Separately, in $P^2(\mathbf{D})$ we have some geometry, as described in Section 2.4. Since both carry incidence structures, we aim to show that they are isomorphic (as incidence structures). To that end, we observe the following.

Proposition 3.3.6 *Consider three distinct lines in $\mathbb{R}^3$, at least two being skew. They share a unique, common skewer if and only if the three corresponding points in $P^2(\boldsymbol{D})$ are collinear.*

Proof. First, we suppose that our three distinct lines ℓ_1, ℓ_2, ℓ_3 all share a unique, common skewer. For this skewer to possibly exist, we know that at least two of the lines must be skew. Without loss of generality, suppose that ℓ_2 and ℓ_3 are skew. Let $s = S(\ell_2, \ell_3)$. From above, we know that each ℓ_i has a corresponding dual vector, denoted by $\xi_i \in \mathbf{D}^3$. Let $[\xi_i]$ denote the point in $P^2(\mathbf{D})$ corresponding to ξ_i. We aim to show that $[\xi_1], [\xi_2], [\xi_3]$ are collinear in $P^2(\mathbf{D})$. First, we notice that $[\xi_2]$ and $[\xi_3]$ are distinct. This is because if they were not distinct, $\xi_2 = \pm\xi_3$ for, from Lemma 3.3.4, which would contradict ℓ_2 and ℓ_3 being skew. Thus, $\text{span}_{\mathbf{D}}(\xi_1, \xi_2, \xi_3)$ has rank greater than or equal to two, over $\mathbf{D}$. In Remark 3.3.1 we noted that $\mathbf{D}$ has stably rank 2 which means, by Proposition 2.1.8, that the complement of a subspace of $\mathbf{D}^3$ is a free submodule. Therefore, it suffices to show that ξ_i are linearly dependent (over $\mathbf{D}$).

Let us consider the direction vectors for our ξ_i, denoted by u_1, u_2, u_3 and let us denote $\Omega = \text{span}_{\mathbb{R}}\{u_1, u_2, u_3\}$. We know that since the lines ℓ_2 and ℓ_3 are skew, $\dim(\Omega) \geq 2$ and since they all share a skewer s, $\dim(\Omega) \leq 2$. Together, we have $\dim(\Omega) = 2$ and s is perpendicular to Ω. This ensures that there exists some real values $\alpha_i \in \mathbb{R}$ such that $\sum \alpha_i u_i = 0$. Let us consider $\sum \alpha_i v_i$. By definition, we have the following,

$$\sum \alpha_i v_i = \sum \alpha_i (u_i \times OP_i) = \sum (\alpha_i u_i) \times OP_i.$$

We may assume without loss of generality that P_i is on s for all i. Let u_s denote the unit vector in the direction of the line s. Since these points are chosen to be collinear, we have scalars $\lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$OP_2 = OP_1 + \lambda_2 u_s,$$

$$OP_3 = OP_1 + \lambda_3 u_s.$$

Therefore, we have

$$\sum (\alpha_i u_i) \times OP_i = \alpha_1 u_1 \times OP_1 + \alpha_2 u_2 \times (OP_1 + \lambda_2 u_s) + \alpha_3 u_3 \times (OP_1 + \lambda_3 u_s)$$

$$= (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) \times OP_1 + \alpha_2 \lambda_2 u_2 \times u_s + \alpha_3 \lambda_3 u_3 \times u_s.$$

We know that $\sum \alpha_i u_i = 0$ and, furthermore, since s is perpendicular to the plane Ω, both $u_2 \times u_s$ and $u_3 \times u_s$ lie in Ω. This ensures that there exists some $\beta_i \in \mathbb{R}$ such that $\sum \alpha_i v_i = -\sum \beta_i u_i$.

Let us define $c_i := \alpha_i + \varepsilon \beta_i$. Then, we have $\sum c_i \xi_i = 0$ since both $\sum \alpha_i u_i = 0$ and $\sum (\beta_i u_i + \alpha_i v_i) = 0$. Therefore, the ξ_i are linearly dependent. Since have already seen that $\text{span}_{\mathbf{D}}(\xi_2, \xi_3)$ has rank greater than or equal to two and we know that the complement of $\text{span}_{\mathbf{D}}(\xi_2, \xi_3)$ is free, we have $\text{span}_{\mathbf{D}}(\xi_2, \xi_3)$ is a 2-dimensional subspace of $\mathbf{D}^3$. Thus, the points $[\xi_1], [\xi_2], [\xi_3]$ are collinear in $P^2(\mathbf{D})$.

Conversely, suppose that we know the points $[\xi_1], [\xi_2], [\xi_3]$ are collinear. Again, we suppose that $[\xi_2] \neq [\xi_3]$. Let us consider $\text{span}_{\mathbf{D}}(\xi_2, \xi_3)$. We claim that this is a 2-dimensional subspace of $\mathbf{D}^3$ and that $\xi_1 \in \text{span}_{\mathbf{D}}(\xi_2, \xi_3)$. To see that $\text{span}_{\mathbf{D}}(\xi_2, \xi_3)$ is a 2-dimensional subspace, we recall that $[\xi_2], [\xi_3]$ are distinct and therefore ξ_2 and ξ_3 are unimodular and linearly independent. Since we are in $\mathbf{D}$, which has stably rank 2, we know that we can extend this list to a basis of $\mathbf{D}^3$, ensuring we have a subspace. Collinearity of the points $[\xi_1], [\xi_2], [\xi_3]$ ensures $\xi_1 \in \text{span}_{\mathbf{D}}(\xi_2, \xi_3)$.

Let $c_i = \alpha_i + \varepsilon\beta_i \in \mathbf{D}$, not all zero, be such that $\sum c_i\xi_i = 0$. Then, we have

$$0 = \sum c_i\xi_i = \sum(\alpha_i + \varepsilon\beta_i)(u_i + \varepsilon v_i) = \sum \alpha_i u_i + \varepsilon \sum(\beta_i u_i + \alpha_i v_i).$$

This forces both

$$\sum \alpha_i u_i = 0 \quad \text{and} \quad \sum(\beta_i u_i + \alpha_i v_i) = 0.$$

We may suppose that $c_1 = 1$, since $\xi_1 \in \text{span}_{\mathbf{D}}(\xi_2, \xi_3)$, which means we can rewrite the equations as

$$u_1 = -(\alpha_2 u_2 + \alpha_3 u_3) \tag{3.1}$$

and

$$v_1 = -(\beta_2 u_2 + \beta_3 u_3 + \alpha_2 v_2 + \alpha_3 v_3) \tag{3.2}$$

(3.1) forces $\Omega := \text{span}_{\mathbb{R}}(u_1, u_2, u_3)$ to have dimension 2.

Case 1: α_2 and α_3 are nonzero: Let P_i be a point on ℓ_i for each i. Then, we know by definition of v_i that $\sum \alpha_i v_i$ can be rewritten as follows,

$$\begin{aligned}
\sum \alpha_i v_i &= \alpha_i(u_i \times OP_i) \\
&= u_1 \times OP_1 + \alpha_2 u_2 \times OP_2 + \alpha_3 u_3 \times OP_3 \\
&= -(\alpha_2 u_2 + \alpha_3 u_3) \times OP_1 + \alpha_2 u_2 \times OP_2 + \alpha_3 u_3 \times OP_3 \\
&= \alpha_2 u_2 \times (-OP_1 + OP_2) + \alpha_3 u_3 \times (-OP_1 + OP_3) \\
&= \alpha_2 u_2 \times P_1P_2 + \alpha_3 u_3 \times P_1P_3 \in \Omega.
\end{aligned} \tag{3.3}$$

Let u_s be a unit vector perpendicular to Ω. Then, we know that both u_1, u_2, u_s and u_1, u_3, u_s will form a basis of $\mathbb{R}^3$. Therefore, there exists $a, b, c, a', b', c' \in \mathbb{R}$ so that

$$P_1P_2 = au_1 + bu_2 + cu_s,$$

$$P_1P_3 = a'u_1 + b'u_3 + c'u_s.$$

Now, if we plug this into (3.3), we have

$$\alpha_2 u_2 \times P_1P_2 + \alpha_3 u_3 \times P_1P_3 = \alpha_2 u_2 \times (au_1 + bu_2 + cu_s) + \alpha_3 u_3 \times (a'u_1 + b'u_3 + c'u_s) \quad .$$

Therefore, we need to have $\alpha_2 a u_2 \times u_1 + \alpha_3 a' u_3 \times u_1 \in \Omega$. We observe the following,

$$\begin{aligned} \alpha_2 a u_2 \times u_1 + \alpha_3 a' u_3 \times u_1 &= \alpha_2 a u_2 \times (-(\alpha_2 u_2 + \alpha_3 u_3)) + \alpha_3 a' u_3 \times (-(\alpha_2 u_2 + \alpha_3 u_3)) \\ &= (-a\alpha_2\alpha_3 + a'\alpha_2\alpha_3) u_2 \times u_3 \end{aligned} \quad .$$

Since $u_2 \times u_3 \notin \Omega$, we know that $-a\alpha_2\alpha_3 + a'\alpha_2\alpha_3 = 0$ which is true as long as $a = a'$.

Let us define new points

$$P_1' := P_1 - au_1 \in \ell_1$$

$$P_2' := P_2 + bu_2 \in \ell_2$$

$$P_3' := P_3 + b'u_3 \in \ell_3.$$

We know these lie on the appropriate lines by their definition, and we claim that $P_1'P_2'$ and $P_1'P_3'$ are both parallel with u_s. We observe,

$$P_1'P_2' = P_1P_2' - P_1P_1' = P_2P_2' - P_2P_1 - P_1P_1' = bu_2 - (au_1 + bu_2 + cu_s) + au_1 = -cu_s.$$

Similarly,

$$P_1'P_3' = P_1P_3' - P_1P_1' = P_3P_3' - P_3P_1 - P_1P_1' = b'u_3 - (au_1 + b'u_3 + c'u_s) + au_1 = -c'u_s.$$

Therefore, if we let s be the line in $\mathbb{R}^3$ corresponding to $\xi_s = u_s + \varepsilon(u_s \times OP_1')$, it is perpendicular to Ω (thus all u_i) and goes through P_1', P_2', P_3' ensuring that it is a common skewer for ℓ_1, ℓ_2, and ℓ_3.

Case 2: α_2 or α_3 are zero: Suppose without loss of generality that $\alpha_2 = 0$. Let P_i be a point on ℓ_i for each i. Then, (3.3) can be rewritten as follows,

$$\begin{aligned}\sum \alpha_i v_i &= \alpha_i(u_i \times OP_i) \\ &= u_1 \times OP_1 + \alpha_3 u_3 \times OP_3 \\ &= -(\alpha_3 u_3) \times OP_1 + \alpha_3 u_3 \times OP_3 \\ &= \alpha_3 u_3 \times (-OP_1 + OP_3) \\ &= \alpha_3 u_3 \times (P_1P_3) \in \Omega.\end{aligned} \tag{3.4}$$

Now, let $u_s = u_2 \times u_1 = u_2 \times (-\alpha_3)u_3$. Then, it is perpendicular to Ω, so that u_1, u_2, u_s and u_2, u_3, u_s will form a basis of $\mathbb{R}^3$. Therefore, there exists $a, b, c, a', b', c' \in \mathbb{R}$ so that

$$P_2P_1 = au_1 + bu_2 + cu_s$$

$$P_2P_3 = a'u_1 + b'u_2 + c'u_s.$$

Now, if we plug this into (3.4), we have

$$\begin{aligned}\alpha_3 u_3 \times P_1P_3 &= \alpha_3 u_3 \times (P_2P_3 - P_2P_1) \\ &= \alpha_3 u_3 \times (a'u_1 + b'u_2 + c'u_s) - \alpha_3 u_3 \times (au_1 + bu_2 + cu_s) \cdot \\ &= (b'\alpha_3 - b\alpha_3) u_3 \times u_2 + (c'\alpha_3 - c\alpha_3) u_3 \times u_s \in \Omega\end{aligned}$$

But, we know that $u_3 \times u_2$ is not in Ω. Therefore, $b = b'$. Let us define new points

$$P_1' := P_1 - au_1 \in \ell_1$$

$$P_2' := P_2 + bu_2 \in \ell_2$$

$$P_3' := P_3 + a'u_1 \in \ell_3$$

We know these lie on the appropriate lines by their definition, and we claim that $P_1'P_2'$ and $P_2'P_3'$ are both parallel with u_s. We observe,

$$P_1'P_2' = P_1P_2' - P_1P_1' = P_2P_2' - P_2P_1 - P_1P_1' = bu_2 - (au_1 + bu_2 + cu_s) + au_1 = -cu_s.$$

Similarly,

$$P_3'P_2' = P_2P_2' - P_2P_3' = P_2P_2' - P_3P_3' + P_3P_2 = bu_2 + a'u_1 - (a'u_1 + bu_2 + c'u_s) = -c'u_s.$$

Therefore, if we let s be the line in $\mathbb{R}^3$ corresponding to $\xi_s = u_s + \varepsilon(u_s \times OP_2')$, it is perpendicular to Ω (thus all u_i) and goes through P_1', P_2', P_3' ensuring that it is a common skewer for ℓ_1, ℓ_2, and ℓ_3.

□

Now, we want to show that the *skewer incidence structure* will be isomorphic (as incidence structures) to our projective plane $P^2(\mathbf{D})$. Let us consider the following incidence structure.

Definition 3.3.7 The *skewer incidence structure* is defined in the following way: let $P = L = \{\text{lines in } \mathbb{R}^3\}$ and a "point" $p \in P$ is incidental with a "line" $\ell \in L$ if p and ℓ intersect at a right angle in $\mathbb{R}^3$ [Tab16].

It turns out that this incidence structure encodes the incidence of our projective plane $P^2(\mathbf{D})$ perfectly.

Proposition 3.3.8 *There is an isomorphism (of incidence structures) between the skewer incidence structure and* $P^2(\mathbf{D})$.

Proof. Let (P, L) denote the skewer incidence structure: $P = L = \{\text{lines in } \mathbb{R}^3\}$. Let (P', L') denote the canonical incidence structure carried by $P^2(\mathbf{D})$: $P = \{\text{points in } P^2(\mathbf{D})\}$ and $L = \{\text{lines in } P^2(\mathbf{D})\}$. We saw in Lemma 3.3.4 that there is a bijection $\phi : P \to P'$ via $p \mapsto [\xi_p]$, where $[\xi_p]$ denotes the projectivization of ξ_p.

We use this bijection to build a bijection between L and L' in the following way. Let us consider a line ω in L. Then, we choose three points $\ell_1, \ell_2, \ell_3 \in \omega$. Assume, without loss of generality, that the three lines are pairwise skew. Then, consider the corresponding points $[\xi_i]$ in $P^2(\mathbf{D})$. We know that these points are collinear by Proposition 3.3.6. Let us denote their common line by $\omega' \in L'$. We consider the map $\phi : L \to L' : \omega \mapsto \phi(\omega) = \omega'$. This map is well defined. Suppose we choose $p \in \omega$, distinct from ℓ_1. Then, Proposition 3.3.6 still applies and $[\xi_p], [\xi_2], [\xi_3]$ are collinear and therefore $[\xi_p] \in \omega'$. The map is bijective because the map $p \mapsto [\xi_p]$ is bijective.

Now, let us define our map between incidence structures

$$(P, L) \to (P', L') : (p, \omega) \mapsto (\phi[p], \phi[\omega]) = ([\xi_p], \omega').$$

This map is bijective because each component is bijective. Now, we want to check that it is an isomorphism of incidence structures. So, let us consider pair (p, ω) such that $p \in \omega$. In other words, the (geometric) line p intersects ℓ orthogonally. We may then use p, together with two other lines (at least one skew to p), to construct $\phi(\omega) = \omega'$. This ensures that $[\xi_p]$ lies on ω'. Thus, this map is an isomorphism of incidence structures. □

Part II

Difference Operators and the Correspondence

Chapter 4

Difference Operators

4.1 Definition

Let R be a stably finite ring. Let us consider the set of bi-infinite sequences of elements from $\mathfrak{M}_R$, some R-module. We may denote this $(\mathfrak{M}_R)^{\mathbb{Z}}$. Whether $\mathfrak{M}_R$ is a right, left, or bi-module depends on the setting, and we will specify its structure when relevant. In the case when $\mathfrak{M}_R = R$, we will use $R^{\mathbb{Z}}$ and consider bi-infinite sequences as a bi-module.

We define the *left shift operator* as $T : (\mathfrak{M}_R)^{\mathbb{Z}} \to (\mathfrak{M}_R)^{\mathbb{Z}}$ where $(TV)_i := V_{i+1}$. We can iterate this map to produce powers of T. Also, we can create "negative" powers of T by repeatedly shifting sequences to the right. This will be use in Section 8.4. Next, let $A \in R^{\mathbb{Z}}$ be a bi-infinite sequence of elements from R. Then, we define the left and right multiplication operators, respectively, as

$$\lambda_A : (\mathfrak{M}_R)^{\mathbb{Z}} \to (\mathfrak{M}_R)^{\mathbb{Z}} : V \mapsto AV$$

$$\rho_A : (\mathfrak{M}_R)^{\mathbb{Z}} \to (\mathfrak{M}_R)^{\mathbb{Z}} : V \mapsto VA$$

where we multiply the sequences component-wise in the ring. For λ (resp. ρ) to be defined, we need $\mathfrak{M}_R$ to be a left (resp. right) module.

Definition 4.1.1 Consider $(\mathfrak{M}_R)^{\mathbb{Z}}$, where $\mathfrak{M}_R$ is a left module. A left difference operator is a map $\mathcal{D} : (\mathfrak{M}_R)^{\mathbb{Z}} \to (\mathfrak{M}_R)^{\mathbb{Z}}$ defined as

$$\mathcal{D} = \sum_{j=0}^{d} \lambda_{A^{(j)}} T^j$$

where each $A^{(j)} \in R^{\mathbb{Z}}$. We call d the degree of the difference operator.

We denote the i-th element of the bi-infinite sequence $A^{(j)}$ as $A_i^{(j)}$. For $V \in (\mathfrak{M}_R)^{\mathbb{Z}}$, we have $\mathcal{D}V$, a new sequence in $(\mathfrak{M}_R)^{\mathbb{Z}}$, defined as

$$(\mathcal{D}V)_i = A_i^{(0)} V_i + A_i^{(1)} V_{i+1} + ... + A_i^{(d)} V_{i+d}.$$

In a slight abuse of notation, sometimes we write a left difference operator defined above as

$$\mathcal{D} = A^{(0)} + A^{(1)}T + \cdots + A^{(d)}T^d.$$

Definition 4.1.2 Consider $(\mathfrak{M}_R)^{\mathbb{Z}}$ where $\mathfrak{M}_R$ is a right module. A right difference operator is a map $\mathcal{Q} : (\mathfrak{M}_R)^{\mathbb{Z}} \to (\mathfrak{M}_R)^{\mathbb{Z}}$ defined as

$$\mathcal{Q} = \sum_{j=1}^{d} \rho_{A^{(j)}} T^j$$

where each $A^{(j)} \in R^{\mathbb{Z}}$. We call d the degree of the difference operator.

For each $W \in (\mathfrak{M}_R)^{\mathbb{Z}}$ we have a new sequence $\mathcal{Q}W \in (\mathfrak{M}_R)^{\mathbb{Z}}$ defined as

$$(\mathcal{Q}W)_i = W_i A_i^{(0)} + W_{i+1} A_i^{(1)} + ... + W_{i+d} A_i^{(d)}.$$

We emphasize that in both cases, we shift and then multiply by ring elements.

Remark 4.1.3 We note that difference operators are only maps. If we additionally want a left (resp. right) difference operator to be an endomorphism, we need $\mathfrak{M}_R$ to be a right (resp. left) module. In other words, a left difference operator can only be a right module endomorphism, and vice versa. We are most often interested in difference operators acting on bi-infinite sequences of ring elements, but they are defined in much more generality.

If $A^{(0)}$ and $A^{(d)}$ are sequences of units in R, we call $\mathcal{D}$ (resp. $\mathcal{Q}$) a *properly bounded left difference operator* (resp. *properly bounded right difference operator*). We will primarily study properly bounded difference operators in this paper. If all the sequences of coefficient are periodic (with the same period), we call the difference operator *periodic*.

4.2 Kernels and Fundamental Solutions

Let us consider a properly bounded left difference operator $\mathcal{D}$ of degree d with coefficients in R acting on $R^{\mathbb{Z}}$, bi-infinite sequences of elements in R. We are interested in the sequences $V \in R^{\mathbb{Z}}$, such that $\mathcal{D}V = 0$, sometimes called *kernel elements*.

Proposition 4.2.1 *Let $\mathcal{D}$ be a properly bounded, left difference operator of degree d with coefficients in R, acting on $R^{\mathbb{Z}}$, then the kernel of $\mathcal{D}$ is a rank d free right R-module.*

Proof. First, we notice that for $\mathcal{D}$ (a left difference operator) to be an R-module homomorphism, we need to consider $R^{\mathbb{Z}}$ as a right, R-module (see Remark 4.1.3). Then, we observe that our kernel elements, $V \in \ker(\mathcal{D}) \subset R^{\mathbb{Z}}$, have a term wise recursive relationship, guaranteed by the hypothesis that all elements of $A^{(0)}$ and $A^{(d)}$ are units. More specifically, we can write

$$V_i = (-A_i^{(0)})^{-1}(A_i^{(1)}V_{i+1} + ... + A_i^{(d)}V_{i+d})$$

$$V_{i+d} = (-A_i^{(d)})^{-1}(A_i^{(0)}V_i + ... + A_i^{(d-1)}V_{i+d-1}).$$

Thus, if we choose $V_1, ..., V_d \in R$ all other V_i are determined uniquely. This means there is a map from the space of initial conditions to the space of solutions of $\mathcal{D}$, and further, this map is a (right) module homomorphism (per Remark 4.1.3). Lastly, we claim that this map is bijective. We notice that the choice of $V_1, ..., V_d$

together with the recursive relationship determines V uniquely, giving injectivity. And, if we know $V \in \ker(\mathcal{D})$, we can list $V_1, ..., V_d$, ensuring surjectivity.

Since the space of initial conditions has rank d (more specifically, it is R^d), this proves that the kernel of $\mathcal{D}$ also has rank d, as a free right R-module. $\square$

Let us now consider the case where $\mathfrak{M}_R = {}^dR$. We recall that we can think of left module ${}^dR$ elements as "row" vectors, or $1 \times d$ matrices. Let $M = (M_i)$, where each M_i is a $1 \times d$ matrix (or, an element of ${}^dR$), then $\mathcal{D}M$ is a new bi-infinite sequence of $1 \times d$ matrices. We say that a bi-infinite sequence of matrices, M, is a *matrix solution* or *vector solution* to $\mathcal{D}$ when $\mathcal{D}M = 0$. We will see that considering matrix/vector solutions is sometimes preferable, and is ultimately necessary for our constructive correspondence. We are interested in how vector solutions and kernel elements can interact. The idea would be to build vector solutions from kernel elements. Fortunately, this construction is meaningful. We prove the following.

Proposition 4.2.2 *Let $\mathcal{D}$ be a properly bounded left difference operator of degree d. Let $V^{(1)}, ..., V^{(d)} \in \ker(\mathcal{D}) \subset R^{\mathbb{Z}}$. If we consider $\mathcal{D}$ acting on the bi-infinite sequence $M = (M_i)$ of $1 \times d$ matrices defined by*

$$M_i := \begin{pmatrix} V_i^{(1)} & \dots & V_i^{(d)} \end{pmatrix},$$

then we have $\mathcal{D}M = 0$. Furthermore, we have the following:

1. *If $\{M_1, ..., M_d\}$ is a basis when viewed as elements of ${}^dR$ then so is the list $\{M_i, ..., M_{i+d-1}\}$ for all $i \in \mathbb{Z}$. In this case, $\{V^{(1)}, ...V^{(d)}\}$ form a basis of $\ker(\mathcal{D})$.*

2. *If $\{V^{(1)}, ...V^{(d)}\} \subset \ker(\mathcal{D})$ form a basis of the kernel, then for all $i \in \mathbb{Z}$ the list $\{M_i, ..., M_{i+d-1}\}$ forms a basis of ${}^dR$.*

Proof. The fact that $\mathcal{D}M = 0$ follows directly from how we defined a difference operator acting on a matrix together with the fact that each $V^{(i)}$ is in the kernel of $\mathcal{D}$.

1. Let us suppose that $\{M_1, ..., M_d\}$ are a basis when viewed as elements of ${}^dR$. We know, from the recursive relationship dictated by $\mathcal{D}$, that M_0 is given by

$$M_0 = (-A_0^{(0)})^{-1}(A_0^{(1)} M_1 + ... + A_0^{(d)} M_d).$$

But, since $A_0^{(0)}$ and $A_0^{(d)}$ are both units, we know that

$$\text{span}_R\{M_0, ..., M_{d-1}\} = \text{span}_R\{M_1, ..., M_d\} = {}^dR.$$

From Remark 1.2.2, we know that a spanning list in ${}^dR$ of length d is necessarily a basis. Therefore, we have $\{M_0, ..., M_{d-1}\}$ is a basis. Similarly, using

$$M_{d+1} = (-A_1^{(d+1)})^{-1}(A_1^{(1)} M_1 + ... + A_1^{(d)} M_d)$$

we observe that $\{M_2, .., M_{d+1}\}$ is a basis of ${}^dR$. Repeating this process, we have the result for each $i \in \mathbb{Z}$.

The fact that $\{V^{(1)}, ...V^{(d)}\}$ form a basis of $\ker(\mathcal{D})$ follows from the right module isomorphism of $\ker(\mathcal{D})$ with the space of initial conditions. In particular, we know that we have right module isomorphism

$$r : \ker(\mathcal{D}) \to R^d : V \mapsto \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}.$$

Under this map, our kernel elements $\{V^{(1)}, ...V^{(d)}\}$ are sent to,

$$V^{(i)} \mapsto \begin{pmatrix} V_1^{(i)} \\ \vdots \\ V_d^{(i)} \end{pmatrix} \qquad\qquad i \in \{1, ..., n\}.$$

By hypothesis, the list $\{M_1, ..., M_d\}$ forms a basis which ensures that the endomorphism on ${}^dR$, corresponding to

$$A = \begin{pmatrix} M_1 \\ \vdots \\ M_d \end{pmatrix}$$

will be surjective (as a left module endomorphism). Thus, A has a left inverse. Since R is stably finite, the left inverse of A is also the right inverse. Fortunately, the definitions of M_i give us

$$A = \begin{pmatrix} V_1^{(1)} & \dots & V_1^{(d)} \\ \vdots & \ddots & \vdots \\ V_d^{(1)} & \dots & V_d^{(d)} \end{pmatrix}.$$

And, A having a right inverse ensures that the columns are spanning. By Remark 1.2.2, we know that a spanning list in R^d of length d is a basis. Finally, we know that module isomorphisms, such as r, send bases to bases. Since the image of $\{V^{(1)}, ...V^{(d)}\}$ is a basis, we have that $\{V^{(1)}, ...V^{(d)}\}$ is also a basis.

2. Suppose instead $\{V^{(1)}, ...V^{(d)}\}$ forms a basis of $\ker(\mathcal{D})$, then our isomorphism with the space of initial conditions ensures that $\{M_1, .., M_d\}$ is a basis of ${}^dR$. But, by the recursive argument above, this ensures that $\{M_i, ..., M_{i+d-1}\}$ is a basis for any $i \in \mathbb{Z}$.

□

We call a $1 \times d$ matrix solution M to $\mathcal{D}$ for which each d-consecutive elements $M_i, ..., M_{i+d-1}$ form a basis of ${}^dR$ a *fundamental solution*.

Remark 4.2.3 A fundamental solution is not an element of $\ker(\mathcal{D}) \subset R^{\mathbb{Z}}$ but rather is a vector solution built from elements of the kernel.

Chapter 5

The Correspondence of Difference Operators and Polygons

5.1 The Correspondence

We will now establish a constructive correspondence between polygons and difference operators. In the setting of the real numbers, a correspondence of polygons and difference operators has been established (e.g. Proposition 2.2 in [Izo22a]). We paraphrase the result here.

Proposition 5.1.1 *There is a one-to-one correspondence between the following,*

1. *Polygons in the real projective plane, up to projective equivalence.*

2. *Properly bounded difference operators of degree* 3*, with real coefficients, up to multiplication on the left and/or right by bi-infinite sequences of nonzero real numbers.*

We will take a moment to outline the basic case of this correspondence. Let us denote the real projective plane by $\mathbb{P}^2$ and let $\{x_i\}$ be a polygon in $\mathbb{P}^2$. By the spanning condition on polygons, we know that $\{x_i\}$ will have lifts $X := (X_i)$ in $\mathbb{R}^3$ satisfying $\mathbb{R}^3 = \text{span}_{\mathbb{R}}(X_i, X_{i+1}, X_{i+2})$ for all $i \in \mathbb{Z}$. Using this, we can find $a_i, b_i, c_i \in \mathbb{R}$ so that

$$X_i = a_i X_{i+1} + b_i X_{i+2} + c_i X_{i+3}.$$

And, we notice that $c_i \neq 0$. If $c_i = 0$, that creates a linear dependence between $\{X_i, X_{i+1}, X_{i+2}\}$, contradicting the fact that $\{x_i\}$ is a polygon. We can do this for every i, building a properly bounded difference operator of degree 3, given by

$$\mathcal{D} = -1 + aT + bT^2 + cT^3, \qquad a, b \in \mathbb{R}^{\mathbb{Z}}, c \in (\mathbb{R}^*)^{\mathbb{Z}}$$

for which $\mathcal{D}X = 0$.

Remark 5.1.2 This process of "constructing" a properly bounded difference operator from a polygon is related closely to the proof of surjectivity in Section 5.2.3.

The correspondence was generalized to include varying degrees of difference operators, in [Izo22b], by picking projective spaces of a proper dimension. We wanted to continue this investigation and generalize the coefficients of the difference operator beyond real numbers. We initially studied difference operators whose coefficients were square matrices, hoping to relate these difference operators to the Grassmann pentagram maps discussed in [FB15; Ove20]. This correspondence is given in Corollary 6.2.1. However, we found that we could generalize further to difference operators whose coefficients were from stably finite rings. This class of rings would allow for the correspondence to include meaningful examples such as matrices and *skewers* (i.e. lines in $\mathbb{R}^3$ with the skewer incidence relation). The generalized correspondence is stated as follows.

Theorem 5.1.3. *Let R be a stably finite ring. Then, there is a one-to-one correspondence between the following sets:*

1. *Polygons in ${}^{d-1}P(R)$ (resp. $P^{d-1}(R)$), up to projective equivalence.*

2. *Properly bounded left (resp. right) difference operators of degree d with coefficients in R, up to multiplication on the left and right by bi-infinite sequences of units in R.*

Rings within the class of stably finite rings need not be commutative and hence the distinction between left and right objects. We will prove the "left case" explicitly and then show how the "right case" is deduced. Once we have established this correspondence, we will use it to study some known and some new generalizations of the pentagram map in Part III.

5.2 Proof of the Correspondence

In order to prove the correspondence, we will first build a map from the space of all degree d, properly bounded left difference operators with coefficients in a stably finite ring R, to the space of polygons, up to projective equivalence, in the left projective space ${}^{d-1}P(R)$. Then, we will consider difference operators up to the action of multiplying on the left or right by bi-infinite sequences of units and show that our original map factors through this quotient. Finally, we will prove that the map on the quotient is a bijection and deduce the "right" case from the left.

Let us denote the space of all degree d properly bounded difference operators as $\mathfrak{D}$ and let us denote the space of polygons in ${}^{d-1}P(R)$ up to projective equivalence as $\mathfrak{P}$. Then, we first aim to build a map $\varphi : \mathfrak{D} \to \mathfrak{P}$.

5.2.1 Initial Map and the Descent to Quotients

Let $\mathcal{D} \in \mathfrak{D}$, acting on $R^{\mathbb{Z}}$. Then, we know that $\ker(\mathcal{D})$ is a rank d, free right module over R (per Proposition 4.2.1). Thus, we know that $(\ker \mathcal{D})^*$ is a free rank d left module over R. Furthermore, we know that $\alpha_i : R^{\mathbb{Z}} \to R : V \mapsto V_i$ is in $(R^{\mathbb{Z}})^*$. Thus, since elements in $\ker \mathcal{D}$ are also elements in $R^{\mathbb{Z}}$, we can restrict each to $\ker(\mathcal{D})$, we get $\alpha_i|_{\ker(\mathcal{D})} \in (\ker(\mathcal{D}))^*$.

Lemma 5.2.1 *Let $\mathcal{D} \in \mathfrak{D}$. For $\alpha_i|_{\ker(\mathcal{D})} \in (\ker(\mathcal{D}))^*$, the list*

$$\alpha_i|_{\ker(\mathcal{D})}, \ldots, \alpha_{i+d-1}|_{\ker(\mathcal{D})}$$

forms a basis of $(\ker(\mathcal{D}))^*$, *for each* $i \in \mathbb{Z}$.

Proof. We aim to show that there is a recursive relationship on $\alpha := (\alpha_i|_{\ker(\mathcal{D})})_{i\in\mathbb{Z}}$. To this end, we recall that α is a bi-infinite sequence of elements from a left module, and therefore our difference operator acts on α in the usual way.

With our difference operator $\mathcal{D}$, we see that for each $i \in \mathbb{Z}$ we have

$$(\mathcal{D}\alpha)_i = A_i^{(0)}\alpha_i|_{\ker(\mathcal{D})} + A_i^{(1)}\alpha_{i+1}|_{\ker(\mathcal{D})} + ... + A_i^{(d)}\alpha_{i+d}|_{\ker(\mathcal{D})}.$$

This new bi-infinite sequence still consists of elements in $(\ker(\mathcal{D}))^*$. With this construction, we claim that in fact $\mathcal{D}\alpha = 0$. Consider $V \in \ker(\mathcal{D})$. Then, we have

$$(\mathcal{D}\alpha)_i(V) = A_i^{(0)}\alpha_i|_{\ker(\mathcal{D})}(V) + A_i^{(1)}\alpha_{i+1}|_{\ker(\mathcal{D})}(V) + ... + A_i^{(d)}\alpha_{i+d}|_{\ker(\mathcal{D})}(V)$$

$$= A_i^{(0)}V_i + A_i^{(1)}V_{i+1} + ... + A_i^{(d)}V_{i+d} = (\mathcal{D}V)_i = 0.$$

Now that we have shown α satisfies $\mathcal{D}\alpha = 0$, we know that there is a recursive relationship on $(\alpha_i|_{\ker(\mathcal{D})})$ analogous to that seen in the proof of Proposition 4.2.1. We aim to use this observation to prove the claim that each d consecutive $\alpha_i|_{\ker(\mathcal{D})}$ form a basis of $(\ker(\mathcal{D}))^*$. From the recursive relationship dictated by $\mathcal{D}$, we know that it suffices to show the list $\{\alpha_1|_{\ker(\mathcal{D})}, ..., \alpha_d|_{\ker(\mathcal{D})}\}$ is a basis of $(\ker(\mathcal{D}))^*$. By an argument analogous to Proposition 4.2.2, it will follow for all $i \in \mathbb{Z}$.

In the proof of Proposition 4.2.1, we saw that $\ker(\mathcal{D})$ is isomorphic to R^d via the recursive relationship dictated by $\mathcal{D}$. In particular, we have a right module isomorphism given by the map

$$r : \ker(\mathcal{D}) \to R^d : V \mapsto \begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}.$$

We know that this induces a left module isomorphism $r^* : (R^d)^* \to (\ker(\mathcal{D}))^*$

under which we have for any $f \in (R^d)^*$ and $V \in \ker(\mathcal{D})$,

$$r^*(f)(V) = f\begin{pmatrix} V_1 \\ \vdots \\ V_d \end{pmatrix}.$$

In the case where $f = \alpha_i$ for any $i \in \{1, ..., d\}$, we have

$$r^*(\alpha_i)(V) = \alpha_i(V) = V_i = \alpha_i|_{\ker(\mathcal{D})}(V).$$

Therefore, $r^*(\alpha_i) = \alpha_i|_{\ker(\mathcal{D})}$. But, $\{\alpha_i\}_{i=1}^d$ is a basis for $(R^d)^*$. And since r^* is an isomorphism, we know a basis is mapped to a basis. Hence, our list $\alpha_1|_{\ker(\mathcal{D})}, ..., \alpha_d|_{\ker(\mathcal{D})}$ is a basis, as desired.

□

Since each $\alpha_i|_{\ker(\mathcal{D})}$ is a basis element, it is unimodular by Proposition 2.1.5. Furthermore, we know that $(\ker(\mathcal{D}))^*$ is a rank d free left module over R. In fact, we know that the isomorphism between $(\ker(\mathcal{D}))*$ and $(R^d)^* = {}^dR$ is given by r^*. Therefore, we know that each $(r^*)^{-1}(\alpha_i|_{\ker(\mathcal{D})}) \in {}^dR$ and we can projectivize to get a sequence, denoted $\{x_i\}$, in ${}^{d-1}P(R)$. To be a polygon, we need to show that the lifts of $x_i, ..., x_{i+d-1}$ span all of ${}^dR$ for each $i \in \mathbb{Z}$. But, this follows from Lemma 5.2.1. We notice that we could choose another map between $(\ker(\mathcal{D}))*$ and ${}^dR$ (i.e. choosing a different basis for $(\ker(\mathcal{D}))^*$), but the resulting polygons will be in the same projective equivalence class.

Together, we have shown that there exists a map $\varphi : \mathfrak{D} \to \mathfrak{P}$ that maps $\mathcal{D}$ to polygon $\{x_i\}$ (considered up to projective equivalence) whose unimodular lifts correspond to some basis representation of $(\alpha_i|_{\ker(\mathcal{D})})$. We aim to show that this map factors through the quotient of $\mathfrak{D}$ dictated by the action of multiplying our properly bounded left difference operators by bi-infinite sequences of units on the left or right. In other words, we need to check that if we take a properly bounded left operator of degree d, $\mathcal{D}$, and multiply it on the left or right by a sequence of units in R, then the image under φ is in the same projective equivalence class as $\mathcal{D}$'s original image. Let us consider $\mathcal{D} \in \mathfrak{D}$ and a bi-infinite sequence $c := (c_i)$ of units in R. Then, we immediately observe that multiplication on the left $c\mathcal{D}$ does not change the kernel and thus does not change the image under φ.

Next, we consider $\mathcal{D}c$. Let $c = (c_i)$ and $c^{-1} = (c_i^{-1})$. We notice first that for each $V \in \ker(\mathcal{D})$, we have $c^{-1}V \in \ker(\mathcal{D}c)$ since $0 = \mathcal{D}V = \mathcal{D}cc^{-1}V$. Similarly, if $V \in \ker(\mathcal{D}c)$, then we know that $0 = \mathcal{D}cV = 0$ which ensures that $cV \in \ker(\mathcal{D})$. Thus, $\ker(\mathcal{D})$ and $\ker(\mathcal{D}c)$ are isomorphic as right modules via multiplication on the left by c^{-1}. With this, we have $c^{-1}\ker(\mathcal{D}) = \ker(\mathcal{D}c)$.

This isomorphism of right modules induces a left module isomorphism of their dual modules $(c^{-1})^* : (\ker(\mathcal{D}c))^* \to (\ker(\mathcal{D}))^*$. Under this left module isomorphism, we have

$$(c^{-1})^*(\alpha_i|_{\ker(\mathcal{D}c)}) = c_i^{-1}\alpha_i|_{\ker(\mathcal{D})}.$$

To see this, let $V \in \ker(\mathcal{D})$. Then, $c_i^{-1}\alpha_i|_{\ker(\mathcal{D})}(V) = c_i^{-1}V_i$ and we also know

$$(c^{-1})^*(\alpha_i|_{\ker(\mathcal{D}c)})(V) = (\alpha_i|_{\ker(\mathcal{D}c)} \circ c^{-1})(V) = \alpha_i|_{\ker(\mathcal{D}c)}(c^{-1}V) = c_i^{-1}V_i.$$

Therefore, the sequence $(\alpha_i|_{\ker(\mathcal{D}c)})$ is mapped to a scalar multiple of the sequence $(\alpha_i|_{\ker(\mathcal{D})})$ under this left module isomorphism. This ensures that the polygons corresponding to $(\alpha_i|_{\ker(\mathcal{D}c)})$ and $(\alpha_i|_{\ker(\mathcal{D})})$ are projectively equivalent.

If we denote the collection of properly bounded, degree d left difference operators up to multiplication on the left and right by sequences of units as the quotient $\mathfrak{D}/_{LR}$ (here, LR indicated "left/right action"), we have shown that φ factors through $\mathfrak{D}/_{LR}$. Equivalently, we have a map $\overline{\varphi} : \mathfrak{D}/_{LR} \to \mathfrak{P}$. To finish the proof of the left hand case of Theorem 5.1.3, we must show $\overline{\varphi}$ is bijective.

5.2.2 Injectivity

To show injectivity, we need the following lemma.

Lemma 5.2.2 *Given a left difference operator $\mathcal{D}$ of degree d, with coefficients* *in R, whose leading term is bi-infinite sequence of units and any left difference operator $\mathcal{G}$, with coefficients in R, there exists left difference operators $\mathcal{Q}$ and $\mathcal{R}$, with coefficients in R, such that* $\deg(\mathcal{R}) < d$ *and*

$$\mathcal{G} = \mathcal{Q}\mathcal{D} + \mathcal{R}.$$

Proof. We consider, without loss of generality, $\mathcal{G} = \lambda_c T^k = cT^k$ where $c = (c_i)$ is a bi-infinite sequence of elements in R. Then, the claim will extend linearly. Let

$$\mathcal{D} = \sum_{j=0}^{d} \lambda_{A^{(j)}} T^j.$$

Let us denote the leading term of $\mathcal{D}$ as $\mathcal{D}_\ell := \lambda_{A^{(d)}} T^d = A^{(d)} T^d$ and the rest of $\mathcal{D}$ by $\mathcal{D}_r$. We will prove the claim by induction on k.

Let $k < d$. Then, we immediately have for $\mathcal{Q} = 0$ and $\mathcal{R} = \mathcal{G}$ that $\mathcal{G} = \mathcal{Q}\mathcal{D} + \mathcal{R}$.

Now, suppose $k \geq d$ and that the claim holds for operators of degree j such that $k > j \geq d$. Then, we compute the following using the fact that $\mathcal{D}$ is properly bounded,

$$cT^k = cT^k \mathcal{D}_\ell^{-1} \mathcal{D}_\ell = cT^k T^{-d} (A^{(d)})^{-1} \mathcal{D}_\ell = c(B^{(d)})^{-1} T^{k-d} \mathcal{D}_\ell$$

where $(B^{(d)})_i^{-1} = (A^{(d)})_{i+k-d}^{-1}$. Let us define $\mathcal{Q} = c(B^{(d)})^{-1} T^{k-d}$. Then, we observe that

$$\mathcal{Q}\mathcal{D} = \mathcal{Q}\mathcal{D}_\ell + \mathcal{Q}\mathcal{D}_r = cT^k + \mathcal{Q}\mathcal{D}_r.$$

Rearranging, we see $cT^k = \mathcal{Q}\mathcal{D} - \mathcal{Q}\mathcal{D}_r$ and further $\deg(\mathcal{Q}\mathcal{D}_r) < k$. By induction hypothesis, we know there exists $\mathcal{Q}'$ and $\mathcal{R}'$ such that $\mathcal{Q}\mathcal{D}_r = \mathcal{Q}'\mathcal{D} + \mathcal{R}'$ with $\deg(\mathcal{R}') < d$. Using all of this, we have

$$cT^k = \mathcal{Q}\mathcal{D} - (\mathcal{Q}'\mathcal{D} + \mathcal{R}') = (\mathcal{Q} - \mathcal{Q}')\mathcal{D} - \mathcal{R}'.$$

Since $\deg(\mathcal{R}') < d$, we have shown the claim. □

Let us consider two properly bounded left difference operators $\mathcal{D}$ and $\widehat{\mathcal{D}}$ of matching degree whose corresponding polygons under $\overline{\varphi} : {}^{\mathfrak{D}}\!/_{LR} \to \mathfrak{P}$ (from Section 5.2.1) are the same, up to projective equivalence. We know immediately by the previous lemma that there exists left difference operators $\mathcal{Q}$ and $\mathcal{R}$ with $\deg(\mathcal{R}) < d$ such that

$$\mathcal{D} = \mathcal{Q}\widehat{\mathcal{D}} + \mathcal{R}.$$

Furthermore, we know immediately that $\deg(\mathcal{Q}) = 0$ since $\mathcal{D}$ and $\widehat{\mathcal{D}}$ have matching degree.

Suppose first that $\mathcal{D}$ and $\widehat{\mathcal{D}}$'s kernels are equal, $\ker(\mathcal{D}) = \ker(\widehat{\mathcal{D}})$. We claim that $\mathcal{R} = 0$, thus $\mathcal{D} = \mathcal{Q}\widehat{\mathcal{D}}$. Suppose $\mathcal{R}$ is not identically zero and let us consider a basis $\{X^{(1)}, ..., X^{(d)}\} \in \ker(\mathcal{D})$. Using this basis of $\ker(\mathcal{D})$, let us build fundamental solution, denoted by M, as was done in Proposition 4.2.2:

$$M_i = \begin{pmatrix} X_i^{(1)} & X_i^{(2)} & \dots & X_i^{(d)} \end{pmatrix}.$$

Then, we know that since the kernels are equal, $0 = \mathcal{D}M = (\mathcal{Q}\widehat{\mathcal{D}} + \mathcal{R})M = \mathcal{R}M$. But, since M is a fundamental solution, we know that $M_i, ...M_{i+d-1}$ are a basis and hence linearly independent for all i. But, since $\mathcal{R}$ has degree strictly less than d, $\mathcal{R}M = 0$ contradicts the linear independence. Hence, $\mathcal{R} = 0$. Thus, we have a degree 0 difference operator $\mathcal{Q}$ so that $\mathcal{D} = \mathcal{Q}\widehat{\mathcal{D}}$. Since both $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are degree d and properly bounded, it forces $\mathcal{Q}$ to be a sequence of units. Hence, $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are equivalent in ${}^{\mathfrak{D}}\!/_{LR}$ and we have injectivity in this case

Now, recall that we assumed $\mathcal{D}$ and $\widehat{\mathcal{D}}$'s respective polygons are in the same projective equivalence class. In other words, we know that there exists a left module isomorphism $H : (\ker(\mathcal{D}))^* \to (\ker(\widehat{\mathcal{D}}))^*$ so that $H(\alpha_i|_{\ker(\mathcal{D})}) = c_i\alpha_i|_{\ker(\widehat{\mathcal{D}})}$ for some $c_i \in R^*$. Let $c := (c_i)$ be the sequence of units from this relation. We aim to show that $\ker(\mathcal{D}) = c(\ker(\widehat{\mathcal{D}})) = \ker(\widehat{\mathcal{D}}c^{-1})$.

Let us consider the map dual to H given by $H^* : \ker(\widehat{\mathcal{D}}) \to \ker(\mathcal{D})$ which is a right module isomorphism. Then, we know that for $V \in \ker(\widehat{\mathcal{D}})$ and $\alpha_i|_{\ker(\mathcal{D})}$,

$$H^*(V)(\alpha_i|_{\ker(\mathcal{D})}) = H(\alpha_i|_{\ker(\mathcal{D})})(V) = c_i\alpha_i|_{\ker(\widehat{\mathcal{D}})}(V) = c_iV.$$

Since this is true for all $i \in \mathbb{Z}$, we have $H^*(V) = cV \in \ker(\mathcal{D})$. Bijectivity of H^* gives $\ker(\mathcal{D}) = H^*(\ker(\widehat{\mathcal{D}})) = c(\ker(\widehat{\mathcal{D}}))$. But, a quick computation shows that $c(\ker(\widehat{\mathcal{D}})) = \ker(\widehat{\mathcal{D}}c^{-1})$. Then, from the argument above, we know that there exists a degree 0 properly bounded left difference operator (i.e. a sequence of units) $\mathcal{Q}$ such that

$$\mathcal{D} = \mathcal{Q}\widehat{\mathcal{D}}c^{-1}$$

ensuring that $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are equivalent in ${\mathfrak{D}}/_{LR}$, giving injectivity.

5.2.3 Surjectivity

Let us consider some polygon $\{x_i\}$ in ${}^{d-1}P(R)$. For each i, let X_i be a unimodular lift of x_i in ${}^dR$. Since we have a polygon, we know that

$$RX_i \oplus RX_{i+1} \oplus \ldots \oplus RX_{i+d-1}$$

is the left module ${}^dR$ for all $i \in \mathbb{Z}$ and therefore we know there exists unique scalars $A_i^{(0)}, ..., A_i^{(d-1)} \in R$ such that

$$X_{i+d} = A_i^{(0)}X_i + ... + A_i^{(d-1)}X_{i+d-1}.$$

We can find such $A_i^{(j)}$ for all pairs of i and j which ensures that if we define $A^{(j)} := (A_i^{(j)})_{i\in\mathbb{Z}}$, we have a left difference operator

$$\mathcal{D} := A^{(0)} + A^{(1)}T ... + A^{(d-1)}T^{i+d-1} - T^d$$

to which $X := (X_i)$ is a fundamental vector solution.

Lemma 5.2.3 *The left difference operator $\mathcal{D}$, defined above, is properly bounded.*

Proof. We already know that $A_i^{(d)} = -1$ for all $i \in \mathbb{Z}$. Thus, we need only show that $A_i^{(0)}$ is a unit for all i. To see this, we first observe that we have two relations from the spanning condition of our polygon. We have one we have already used: there exists $A_i^{(0)}, ..., A_i^{(d-1)} \in R$ (not all zero) such that

$$X_{i+d} = A_i^{(0)} X_i + ... + A_i^{(d-1)} X_{i+d-1}.$$

But, we also know that the list $\{X_{i+1}, ..., X_{i+d}\}$ spans all of the left module ${}^dR$ so there exists some scalars $B_i^{(1)}, ..., B_i^{(d)}$ (not all zero) so that

$$X_i = B_i^{(1)} X_{i+1} + \ldots + B_i^{(d)} X_{i+d}.$$

Substituting the first equation into the second and combining all like terms, we have the coefficient in front of X_i is $(B_i^{(d)} A_i^{(0)} - 1)$. But, this list $X_i, ..., X_{i+d-1}$ is linearly independent which forces

$$B_i^{(d)} A_i^{(0)} - 1 = 0$$

and hence $A_i^{(0)}$ is a unit. Since this is true for all i, we have the sequence $A^{(0)}$ as a sequence of units in R, ensuring that $\mathcal{D}$ is a properly bounded left difference operator. □

Now, we have started with a polygon in ${}^{d-1}P(R)$ and shown that there is a properly bounded difference operator of degree d for which its unimodular lift is a vector solution. We need to check that this procedure is in fact the right inverse to our map from Section 5.2.1.

Let us consider the polygon $\{x_i\}$ in ${}^{d-1}P(R)$ and the difference operator $\mathcal{D}$ from above. We need to check that the polygon from the projectivization of $(\alpha_i|_{\ker(\mathcal{D})})$ is projectively equivalent to the polygon $\{x_i\}$. In other words, we aim to show that there exists a left module isomorphism from $(\ker(\mathcal{D}))^*$ to ${}^dR$ such that $\alpha_i|_{\ker(\mathcal{D})} \mapsto X_i$ for all i. But, this is precisely showing there exists a change of basis map H of $(\ker(\mathcal{D}))^*$ for which $H(\alpha_i|_{\ker(\mathcal{D})}) = X_i$.

Recall that each $X_i \in {}^dR$ and therefore we can think of each as a row vector. We notice that if we write

$$X_i = \begin{pmatrix} X_i^{(1)} & X_i^{(2)} & \dots & X_i^{(d)} \end{pmatrix}$$

for each i, we can consider $X^{(1)}, ..., X^{(d)} \in \ker(\mathcal{D}) \subset R^{\mathbb{Z}}$. And, the set $\{X^{(1)}, ..., X^{(d)}\}$ is a basis of $\ker(\mathcal{D})$ by Proposition 4.2.2 since $(X_i)_{i\in\mathbb{Z}}$ are lifts of a polygon, ensuring the spanning condition is satisfied.

Now, let us consider the basis of $(\ker(\mathcal{D}))^*$ dual to $X^{(j)}$ given by $\beta_1, ..., \beta_d$. Then, for each $\ell \in (\ker(\mathcal{D}))^*$ we know that there exists some $c^j \in R$ such that $\ell = \sum_{j=1}^d c^j\beta_j$ and by definition of dual basis $c^j = \ell(X^{(j)})$. In particular, for $\alpha_i|_{\ker(\mathcal{D})}$ we have c_i^j such that $\alpha_i|_{\ker(\mathcal{D})} = \sum_{j=1}^d c_i^j\beta_j$ and each $c_i^j = \alpha_i|_{\ker(\mathcal{D})}(X^{(j)}) = X_i^j$ by definition of $X^{(j)}$. Thus, the vector representation of $\alpha_i|_{\ker(\mathcal{D})}$ with respect to the β_j basis is equal to X_i, giving us our desired change of basis map and ensuring that the polygon built for $\mathcal{D}$ in Section 5.2.1 is in fact projectively equivalent to $\{x_i\}$. Hence, we have shown that $\overline{\varphi}$ is surjective.

5.2.4 The Right Case

Now that we have shown there is a one-to-one correspondence between degree d, properly bounded left difference operators with coefficients in R (up to multiplication on the left and right by bi-infinite sequences of units) and polygons in ${}^{d-1}P(R)$ (up to projective equivalence), we aim to finish the proof of Theorem 5.1.3. Namely, we aim to show that there is a one-to-one correspondence between degree d, properly bounded right difference operators with coefficients in R (up to left/right action) and polygons in $P^{d-1}(R)$ (up to projective equivalence).

To see this, we recall that in Theorem 1.2.5 we showed a ring R is stably finite if and only if its opposite ring R^{op} is also stably finite. Therefore, we have shown that there is a one-to-one correspondence between degree d properly bounded left difference operators over R^{op} (up to multiplication on left and right by units in R^{op}) and projective equivalence classes of polygons in ${}^{d-1}P(R^{\text{op}})$. We observe the two following lemmas.

Lemma 5.2.4 *The set of degree d properly bounded left difference operators whose coefficients are in R^{op} is equal to the set of degree d properly bounded right difference operators whose coefficients are in R.*

Proof. Since R and R^{op} are equal as sets, we know that an element $A^{(i)} \in R^{\mathbb{Z}}$ is also an element in $(R^{\text{op}})^{\mathbb{Z}}$. To emphasize the origin of the sequence, let us denote $A^{(i)} \in R^{\mathbb{Z}}$ and $A_0^{(i)} \in (R^{\text{op}})^{\mathbb{Z}}$. Since multiplication in R versus R^{op} is in opposite order, the multiplication operators $\lambda_{A_0^{(i)}} = \rho_{A^{(i)}}$. Here, we understand these multiplication operators as acting on bi-infinite sequences of elements from some R-bi-module, such as $R^{\mathbb{Z}}$. Therefore, the coefficients of left properly bounded difference operators over R^{op} are exactly coefficients of a right properly bounded difference operator over R. The result follows. □

Lemma 5.2.5 *(See [Bou89]) Every left module over a ring R is a right module over its opposite ring R^{op}. Similarly, every right module over a ring R is a left module over its opposite ring R^{op}.*

To see where these correspondences come from, let M be a left R-module. Let $m \in M$ and $r \in R$. Then, the left module structure means we have a scalar multiplication defined by $rm \in M$. Now, suppose that we consider the same scalar as being in the opposite ring, $r \in R^{\text{op}}$. Then, $m \cdot r = rm \in M$. Thus, M can be understood as a right R^{op}-module. The other arguments are analogous.

Since $(R^{\text{op}})^{\text{op}} = R$, Lemma 5.2.5 ensures that every left modules over R is a right modules over R^{op} and that every right modules over R is a left modules over R^{op}. Hence, we conclude that there is a canonical correspondence between polygons in ${}^{d-1}P(R^{\text{op}})$ and polygons in $P^{d-1}(R)$.

With these observations and the left hand case proof applied to R^{op}, we have shown that there is in fact a one-to-one correspondence between degree d properly bounded right difference operators with coefficients in R (up to multiplication on the left and right by bi-infinite sequences of units) and polygons in $P^{d-1}(R)$ (up to projective equivalence). Thus, Theorem 5.1.3 has been shown.

Chapter 6

Examples and Applications

6.1 A Computational Example Over $\mathbb{Z}$

Let $R = \mathbb{Z}$. Then, we will consider an example of how one computes the class of polygons corresponding to a particular class of difference operators. Let us consider the properly bounded difference operator $\mathcal{D}$ given by

$$\mathcal{D} = T^3 + 2T^2 + 3T - 1.$$

Then, we can find kernel elements using the recursive relationship, dictated by the properly bounded difference operator, with the initial conditions

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ generated by $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix}$ respectively are

$$X^{(1)} = \begin{pmatrix}\vdots\\11\\3\\1\\0\\0\\1\\-1\\\vdots\end{pmatrix}, X^{(2)} = \begin{pmatrix}\vdots\\7\\2\\0\\1\\0\\-6\\12\\\vdots\end{pmatrix}, X^{(3)} = \begin{pmatrix}\vdots\\3\\1\\0\\0\\1\\-2\\1\\\vdots\end{pmatrix}.$$

If we let $[a : b : c] \in {}^2P(\mathbb{Z})$ denote projectivization of $(a, b, c) \in {}^3\mathbb{Z}$, then the polygon corresponding to $\mathcal{D}$, up to projective equivalence, is given by

$$\{\ldots, [11 : 7 : 3], [3 : 2 : 1], [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : -6 : -2], [-1 : 12 : 1], \ldots\}.$$

6.2 Difference Operators with Matrix Coefficients

We can now relate the discussion from Section 3.2 to difference operators with $m \times m$ matrix coefficients. In Lemma 3.2.8, we saw that there is a one-to-one correspondence between polygons in ${}^{d-1}P(\mathrm{M}_m(\mathbb{R}))$ and polygons in the Grassmannian $\mathrm{Gr}(m, dm)$. Now, we are able to extend the correspondence to include difference operators with matrix coefficients.

Corollary 6.2.1 *There is a one-to-one correspondence between the following:*

1. *Polygons in ${}^{d-1}P(M_m(\mathbb{R}))$, up to projective equivalence;*
2. *Polygons in $Gr(m, dm)$, up to projective equivalence;*

3. *Properly bounded left difference operators of degree d with $m \times m$ matrix coefficients, up to multiplication on the left and right by bi-infinite sequences of invertible matrices.*

Proof. This follows from Theorem 5.1.3 and Lemma 3.2.8. □

6.3 Twisted and Closed Polygons

In Part III, we will see that we are sometimes interested in *twisted polygons* and *closed polygons* (see Definition 2.2.5). Therefore, we aim to explicitly consider how our correspondence between difference operators and polygons via Theorem 5.1.3 will be affected by the restriction to twisted or closed polygons. In Section 5.2.1, we construct a map from properly bounded, left difference operators of degree d into the space of projective equivalence classes of polygons, and we called it φ. This map uses kernels of difference operators and lifts of polygons to relate the two objects. In the following corollaries, we consider how Theorem 5.1.3 is affected when we have certain conditions on the domain.

Corollary 6.3.1 *Let $\mathfrak{D}_N$ denote the N-periodic, properly bounded, left difference operators of degree d with R coefficients. Furthermore, let $\mathfrak{P}_N$ denote the space of twisted N-gons in ${}^{d-1}P(R)$, up to projective equivalence. Then, the image of $\varphi|_{\mathfrak{D}_N}$ is exactly $\mathfrak{P}_N$.*

Proof. Let us begin with $\mathcal{D}$, an N-periodic, properly bounded, left difference operator of degree d. Since the coefficients are N-periodic, consider the monodromy $M = T^N|_{\ker(\mathcal{D})}$. Since the coefficients are periodic, we know that $\mathcal{D}T^N = T^N\mathcal{D}$ ensuring that the kernel is invariant under T^N. Therefore, if $X = (X_i)$ is in the kernel of $\mathcal{D}$, $X_{i+N} = MX_i$. From the definition of φ in Section 5.2.1, the image of $\mathcal{D}$ will satisfy $x_{i+N} = M(x_i)$.

Suppose that we have a twisted N-gon, $\{x_i\}$, in the left projective space ${}^{d-1}P(R)$ as in Definition 2.2.5. Let M be its monodromy. We then know that

$x_{i+N} = M(x_i)$ for all i and we know by Theorem 5.1.3 that this polygon corresponds to a properly bounded difference operator $\mathcal{D}$ of degree d, given by

$$\mathcal{D} = A^{(0)} + A^{(1)}T + A^{(2)}T^2 + ... + A^{(d)}T^d.$$

Furthermore, if we let $X_i \in {}^{n+1}R$ denote a unimodular lift of x_i for each i, then we know that for $X := (X_i)_{i\in\mathbb{Z}}$, $\mathcal{D}X = 0$. From our proof of surjectivity, we know that our corresponding difference operator's coefficients are chosen via the equations

$$X_i = A_i^{(1)}X_{i+1} + A_i^{(2)}X_{i+2} + ... + A_i^{(d)}X_{i+d}.$$

In an abuse of notation, we know that the lifts must satisfy the monodromy $X_{i+N} = MX_i$. Since this is true for all i, we have the following:

$$\begin{aligned} X_{i+N} &= A_{i+N}^{(1)}X_{i+N+1} + A_{i+N}^{(2)}X_{i+N+2} + ... + A_{i+N}^{(d)}X_{i+N+d} \\ &= A_{i+N}^{(1)}MX_{i+1} + A_{i+N}^{(2)}MX_{i+2} + ... + A_{i+N}^{(d)}MX_{i+d} \end{aligned}.$$

Separately, we have the following:

$$\begin{aligned} MX_i &= M(A_i^{(1)}X_{i+1} + A_i^{(2)}X_{i+2} + ... + A_i^{(d)}X_{i+d}) \\ &= A_i^{(1)}MX_{i+1} + A_i^{(2)}MX_{i+2} + ... + A_i^{(d)}MX_{i+d} \end{aligned}.$$

The last equality follows from the fact that M is a module homomorphism. Since we need $X_{i+N} = MX_i$, we may choose the coefficients $A^{(j)}$ to be N-periodic. □

Now, if we have an N-periodic difference operator and multiply on the left and/or right by bi-infinite sequences of units, what results need not be an N-periodic difference operator, but will still correspond to twisted N-gons, per Theorem 5.1.3. However, we remain interested in what classes of periodic difference operators remain periodic after such multiplication.

Definition 6.3.2 The following subset of $(R^*)^{\mathbb{Z}}$ given by

$$\{a \in (R^*)^{\mathbb{Z}} : \exists z \in R^* \text{ and } N \in \mathbb{N} \text{ so that } a_i = a_{i+N}z, \ \forall i \in \mathbb{Z}\}.$$

are called *quasi-periodic* bi-infinite sequences of units. We call z the monodromy.

Proposition 6.3.3 *Let R be a commutative ring and let $\mathcal{D}$ be an N-periodic, left properly bounded difference operator. Then, $\alpha\mathcal{D}\beta^{-1}$ is also N-periodic if and only if α and β are quasi-periodic bi-infinite sequences with period N and the same monodromy z.*

Proof. If α and β are quasi-periodic with the same monodromy, then $\alpha\mathcal{D}\beta^{-1}$ will be periodic. This is because the ring commutes.

Now, suppose instead that $\alpha\mathcal{D}\beta^{-1}$ is N-periodic. Then, we know that for $\mathcal{D} = \sum_{j=0}^{d} A^{(j)}T^j$ we have

$$\alpha_i A_i^{(0)} \beta_i^{-1} = \alpha_{i+N} A_i^{(0)} \beta_{i+N}^{-1}.$$

But, since $A_i^{(0)}$ is a unit and our ring is commutative, this can be rewritten as

$$\frac{\alpha_i}{\beta_i} = \frac{\alpha_{i+N}}{\beta_{i+N}}.$$

This is true for all $i \in \mathbb{Z}$ if and only if α, β are quasi-periodic with matching monodromy. □

In other words, if R is commutative, then the subgroup of quasi-periodic bi-infinite sequences of units acts naturally on N-periodic difference operators.

Corollary 6.3.4 *Let R be a commutative ring. Let ${}^{\mathfrak{D}_N}\!\big/_{LR}$ denote N-periodic, properly bounded, left difference operators of degree d with R coefficients up to left/right multiplication by quasi-periodic bi-infinite sequences of units. Furthermore, let $\mathfrak{P}_N$ denote the space of projectively equivalent twisted N-gons in ${}^{d-1}P(R)$.*

Then the map

$$\overline{\varphi}|_{\mathfrak{D}_N} : {}^{\mathfrak{D}_N}\!/_{LR} \to \mathfrak{P}_N$$

is bijective.

Proof. This follows from Theorem 5.1.3, Corollary 6.3.1, and Proposition 6.3.3. □

This result will not be true in general. Let us consider an N-periodic difference operator over some generic stably finite ring and suppose we know that $\alpha\mathcal{D}\beta^{-1}$ remains N-periodic. A sufficient condition for this to occur is α and β to be N-periodic, clearly. A more general sufficient condition would be α and β to be quasi-periodic with period N and central monodromy. Unfortunately, there is no constructive necessary condition. Thus, although twisted polygons are a nice restriction geometrically, they do not correspond to a nice quotient of difference operators in our general setting. Therefore, we will shift to focusing on closed polygons.

In Definition 2.2.5, we saw that a closed N-gon is a polygon $\{x_i\}$ such that $x_{i+N} = x_i$ for all $i \in \mathbb{Z}$ (i.e. periodic polygons). These end up corresponding to a specific class of difference operators known as *super-periodic* or *monodromy-free* difference operators. We define them as follows.

Definition 6.3.5 A periodic difference operator is said to be *super-periodic* (or monodromy-free) if its kernel consists entirely of N-periodic bi-infinite sequences.

This class of difference operators is naturally acted upon the left and right by the subgroup of periodic bi-infinite sequences of units.

Proposition 6.3.6 *Let $\mathcal{D}$ denote super-N-periodic, left, properly bounded difference operator. Then, $\alpha\mathcal{D}\beta^{-1}$ will be super-N-periodic if and only if α and β are N-periodic.*

Proof. Suppose first that α and β are N-periodic. Let $X \in \ker(\mathcal{D})$. First, we notice that $\alpha\mathcal{D}X = 0$, therefore $\alpha\mathcal{D}$ will have the same kernel as $\mathcal{D}$. Additionally, we can see that the kernel of $\mathcal{D}\beta^{-1}$ is N-periodic since its elements are βX, and both β and X are N-periodic by hypothesis. Together, $\alpha\mathcal{D}\beta^{-1}$ is super-N-periodic.

Now, suppose instead that $\alpha\mathcal{D}\beta^{-1}$ is super-N-periodic. For any $X \in \ker(\alpha\mathcal{D}\beta^{-1})$, we have $\beta X \in \ker(\mathcal{D})$. Since the hypothesis ensures both $\mathcal{D}$ and $\alpha\mathcal{D}\beta^{-1}$ are super-N-periodic, both X and βX are N-periodic. Thus, we know that

$$X_{i+N} = X_i \text{ and } \beta_{i+N}X_{i+N} = \beta_i X_i \qquad \forall i \in \mathbb{Z}.$$

In other words, we know for all $i \in \mathbb{Z}$, $\beta_{i+N}X_i = \beta_i X_i$. Since this is true for all $X \in \ker(\alpha\mathcal{D}\beta^{-1})$, we can find some nonzero X_i and deduce β is N-periodic. Similarly, since $\mathcal{D}\beta^{-1}$ is N-periodic by above, and $\alpha\mathcal{D}\beta^{-1}$ is N-periodic by hypothesis, we get that α is also periodic. $\square$

Corollary 6.3.7 *Let* ${}^{\mathfrak{D}_N}\!/_{LR}$ *denote the space of super-N-periodic, properly bounded, left difference operators of degree d with coefficients in R, up to left/right multiplication by periodic bi-infinite sequences. And, let $\mathfrak{C}_N$ denote the space of closed N-gons in ${}^{d-1}P(R)$, up to projective equivalence. Then, the map*

$$\overline{\varphi}|_{\mathfrak{D}_N} : {}^{\mathfrak{D}_N}\!/_{LR} \to \mathfrak{C}_N$$

is bijective.

Proof. Let us first show that

$$\varphi|_{\mathfrak{D}_N} : \mathfrak{D}_N \to \mathfrak{C}_N$$

is surjective. Then, the rest of the result will follow from arguments slightly modified from those Theorem 5.1.3 to accommodate the action described in Proposition 6.3.6.

Given the construction of Theorem 5.1.3, the kernel of a super-N-periodic difference operator being periodic ensures that the corresponding polygon is N-periodic, thus closed. Likewise, a closed polygon will have a periodic lift, and therefore will correspond to a super-periodic, properly bounded difference operator. Therefore, this map $\varphi|_{\mathfrak{D}_N}$ has the appropriate codomain and it is surjective.

$\square$

Part III

Pentagram Map &Co

Chapter 7

Known Pentagram Maps

7.1 The Classical Pentagram Map

We recall that the pentagram map, first introduced by R. Schwartz in 1992 [Sch92], is a discrete integrable system on classes of (real) planar polygons, up to projective equivalence. Originally, it is defined as follows: consider a closed polygon $\{x_i\}$ in $\mathbb{P}^2$ (i.e. a bi-infinite sequence of points such that $x_i = x_{i+n}$ for some fixed $n \in \mathbb{Z}$ and for all $i \in \mathbb{Z}$). Then, we define a new polygon by finding the intersection points of lines connecting second nearest neighbors of $\{x_i\}$. An example of the pentagram map is given in Figure 1. We note that the labelling of this new polygon can be done in a number of ways, which is discussed further in [Sch92; OST10]. Labelling schemes are not important for this investigation.

We sometimes consider the *inverse pentagram map* rather than the pentagram map defined above. It is defined as follows: consider a polygon $\{x_i\}$ in $\mathbb{P}^2$. Then, we define a new polygon by finding the intersection points of second nearest lines connecting the neighbors of $\{x_i\}$. One easily checks that this is indeed the inverse of the map defined above, up to re-indexing. In Figure 1, the inverse pentagram map corresponds to sending P' to P.

This pentagram map (and inverse pentagram map) can defined on a larger class of polygons, known as *twisted polygons*.

Definition 7.1.1 A twisted n-gon is a map $\phi : \mathbb{Z} \mapsto \mathbb{P}^2$ such that

$$\phi(n + i) = M \circ \phi(i)$$

for some projective transformation M and all $i \in \mathbb{Z}$ [OST10].

In other words, a twisted n-gon is a bi-infinite sequence of points $\{x_i\}$, setting $x_i := \phi(i)$, so that $x_{i+n} = M(x_i)$ for some fixed projective transformation M, some fixed n, and all i. When M is the identity, we get closed n-gons. The pentagram map is well-defined on twisted n-gons, and may be considered up to projective equivalence since the pentagram map commutes with projective transformations [Sch92].

As discussed in the introduction, there were numerous generalization of the pentagram map beyond twisted polygons in the real projective plane: "higher" pentagram maps [KS13; KS16], Grassmannian pentagram maps [FB15; Ove20], to "skewers" in [Tab19], etc. In this section, our ultimate goal is to understand these pentagram maps in terms of certain difference operators, extending the work of [Izo22b] and using Theorem 5.1.3. In doing so, we aim to explore some of the structure available and find invariants of pentagram maps.

7.2 Higher Pentagram Maps

Although not the focus of the remainder of this paper, we will briefly discuss "higher pentagram maps" and highlight some known results regarding them and difference operators. Introduced concurrently by M. Gekhtman et al. in [Gek+16] and B. Khesin and F. Soloviev in [KS13; KS16], "higher pentagram maps" refer to pentagram type maps in higher dimensional real projective spaces (denoted $\mathbb{P}^n$). Since the pentagram map deals with the intersection of lines, to ensure such intersection is meaningful we either need to restrict the class of polygons, or consider intersections of higher-dimensional subspaces. In pursuit of the first method, we introduce the notion of a "corrugated polygon". Let us consider a bi-infinite list of points in $\mathbb{P}^{k-1}$, denoted by $x = (x_i)_{i\in\mathbb{Z}}$. We take $k \geq 3$. We say that x is a corrugated polygon if the vertices $x_i, x_{i+1}, x_{i+k-1}, x_{i+k}$ span a projective plane for every $i \in \mathbb{Z}$. We note that all polygons in $\mathbb{P}^2$ are corrugated.

With the notion of a corrugated polygon, we can describe a pentagram map in $\mathbb{P}^{k-1}$. The consecutive diagonals connecting the points x_i and x_{i+k-1} of a corrugated polygon intersect, and the intersection points form the vertices of a new polygon. In other words, higher pentagram map on $\mathbb{P}^{k-1}$ sends corrugated polygons x to a new polygon $\widehat{x} = (\widehat{x}_i)_{i\in\mathbb{Z}}$ where $\widehat{x}_i$ is the intersection point of the lines connecting pair x_i, x_{i+k-1} and pair x_{i+1}, x_{i+k}. These higher pentagram maps are integrable. We leave other constructions for another paper.

A. Izosimov investigated the relationship between certain higher pentagram maps and different sets of difference operators in [Izo22b]. This is accomplished by considering difference operators with powers in certain integer sets (e.g. taking higher degree difference operators). Again, these higher pentagram maps are not the focus of this investigation but we mention them for the reader's reference.

7.3 The Grassmann Pentagram Map

At this point, we have mentioned the "Grassmann pentagram map" a few times. In Section 3.2, we reviewed what Grassmannians were and investigated their relationship to our projective spaces (over square matrices). Now, we will define the Grassmann pentagram map on the polygons in $\mathrm{Gr}(m, 3m)$. Just as in Section 3.2, we will denote the (left) matrix span, or equivalently the (real) row-space span, as $\mathrm{Row}(\cdot)$.

Remark 7.3.1 The Grassmann pentagram map can be defined on $\mathrm{Gr}(m, nm)$ and is done in [FB15]. Due to dimension considerations, the definition of these maps depends on if n is even or odd. For simplicity, we restrict to the case of $n = 3$.

Definition 7.3.2 Let $l = \{l_i\}$ be a polygon in $\mathrm{Gr}(m, 3m)$. The lines (i.e. $2m$-

dimensional subspaces of $\mathbb{R}^{3m}$) connecting pairs l_i, l_{i+2} and l_{i+1}, l_{i+3} intersect in at a point, $\widehat{l_i}$. These intersection points give a new polygon in $\mathrm{Gr}(m, 3m)$, denoted $\widehat{l} = \{\widehat{l_i}\}$.

In [FB15], they introduce this definition using the language of lifts. So, the Grassmann pentagram map can equivalently be defined as follows. For polygon $l = (l_i)$ in $\mathrm{Gr}(m, 3m)$ with unimodular lift (X_i), we will consider new polygon, $\widehat{l}$, with vertices whose lifts are given by $\mathrm{Row}(X_i, X_{i+2}) \cap \mathrm{Row}(X_{i+1}, X_{i+3})$. This construction is closely related to that of the classical pentagram map. In Theorem 8.2.2, we show that the Grassmann pentagram map is the usual pentagram map in the projective plane over square matrices.

7.4 The Skewer Pentagram Map

In [Tab19], Tabachnikov defines the "skewer pentagram map" which acts on cyclically labeled tuples of lines in $\mathbb{R}^3$. We aim to show that Tabachnikov's skewer pentagram map is the "classical" pentagram map in a different projective plane: $P^2(\mathbf{D})$. Let us first describe this new discrete dynamical system.

Definition 7.4.1 Let us consider a bi-infinite list of lines $\{\ell_i\}_{i\in\mathbb{Z}}$ in $\mathbb{R}^3$. Then, the skewer pentagram map is the map

$$\{..., \ell_1, \ell_2, ...\} \mapsto \{..., S(S(\ell_1, \ell_3), S(\ell_2, \ell_4)), S(S(\ell_2, \ell_4), S(\ell_3, \ell_5)), ...\}.$$

To ensure that all of these skewers exist, we must require a spanning condition of sorts. In particular, we require that $\{\ell_i, \ell_{i+1}, \ell_{i+2}\}$ are pairwise skew for all $i \in \mathbb{Z}$.

At first glance, it might not be clear that this is a pentagram type map. However, we recall Definition 3.3.7, which describes the skewer incidence structure. This skewer incidence structure relates "incidence" (i.e. intersection) with finding a common skewer. Therefore, this process of finding the common skewer of second nearest neighbors and then finding the common skewer of nearest skewers should seem pentagram-like. Of course this intuition needs to be proved rigorously. This will be shown in Theorem 8.2.4. In particular, we will show that the skewer pentagram map is the usual pentagram map in the projective plane over the dual numbers, as seen in Section 3.3.

Chapter 8

Pentagram Maps in Generalized Projective Planes

8.1 Pentagram Maps in ${}^2P(R)$

In Definition 2.1.1, we were given the definition for a general p-dimensional subspace of a module. We recall that our projective planes carry a "canonical" incidence structure: "points" in ${}^2P(R)$ will correspond to projectivizations of 1-dimensional subspaces, and "lines" in ${}^2P(R)$ are the projectivizations of 2-dimensional subspaces of the module ${}^3R$. With this incidence structure, we may now define the pentagram map on polygons in ${}^2P(R)$.

Definition 8.1.1 The pentagram map in ${}^2P(R)$ is a map on projective equivalence classes of polygons. Let $\{x_i\}$ be a polygon in ${}^2P(R)$. Then, we build a new bi-infinite list, $\{\widehat{x}_i\}$, whose i-th element is the intersection of the lines connecting x_i to x_{i+2} and x_{i+1} to x_{i+3} (i.e. the new "polygon" has vertices given by consecutive intersection points of lines connecting second nearest neighbors of $\{x_i\}$).

Sometimes, we are interested in the "inverse pentagram map". The construction is similar. It is defined as follows.

Definition 8.1.2 The inverse pentagram map in ${}^2P(R)$ is a map on projective equivalence classes of polygons. Let $\{x_i\}$ be a polygon in ${}^2P(R)$. Then, we build a new bi-infinite list, $\{\widehat{x}_i\}$, whose i-th element is the intersection of the lines connecting x_i to x_{i+1} and x_{i+2} to x_{i+3} (i.e. the new "polygon" has vertices given by second nearest intersection points of lines connecting nearest neighbors of $\{x_i\}$).

The inverse pentagram map is the inverse map of the pentagram map, up to shifting indices.

Remark 8.1.3 Notice that if the original polygon is closed, the resulting polygon from both the pentagram map and the inverse pentagram map will also be closed.

One may ask, how do we know that second nearest neighbors (or nearest neighbors) lie on a single line? Or, how do we know that two lines intersect at a single point? Unfortunately, these statements are not always true for arbitrary (distinct) points in ${}^2P(R)$. Fortunately, Lemma 2.4.5 ensured such behavior for points belonging to polygons. We restate the result using our correspondence with difference operators here.

Lemma 8.1.4 *Let us consider a polygon $\{x_i\}$ in ${}^2P(R)$ and a unimodular lift (X_i) with $X_i \in {}^3R$. Let $\mathcal{D} = \sum_{j=0}^{3} \lambda_{A^{(j)}} T^j$ be the polygon's corresponding difference operator, per Theorem 5.1.3.*

1. *For all $i \in \mathbb{Z}$, $A_i^{(0)} X_i + A_i^{(2)} X_{i+2}$ and $A_i^{(0)} X_i + A_i^{(1)} X_{i+1}$ are unimodular.*

2. *The unique line through the pair x_i, x_{i+2} and the unique line through the pair x_{i+1}, x_{i+3} intersect uniquely, and*

$$A_i^{(0)} X_i + A_i^{(2)} X_{i+2} = -(A_i^{(1)} X_{i+1} + A_i^{(3)} X_{i+3}) \in {}^3R$$

projects to that intersection point. In other words, the intersection point can be chosen to have unimodular lift $A_i^{(0)} X_i + A_i^{(2)} X_{i+2} = -(A_i^{(1)} X_{i+1} + A_i^{(3)} X_{i+3})$.

3. *The unique line through the pair* x_i, x_{i+1} *and the unique line through the pair* x_{i+2}, x_{i+3} *intersect uniquely, and*

$$A_i^{(0)} X_i + A_i^{(1)} X_{i+1} = -(A_i^{(2)} X_{i+2} + A_i^{(3)} X_{i+3}) \in {}^3R$$

projects to that intersection point. In other words, the intersection point can be chosen to have unimodular lift $A_i^{(0)} X_i + A_i^{(1)} X_{i+1} = -(A_i^{(2)} X_{i+2} + A_i^{(3)} X_{i+3})$.

We notice that the second statement deals with the definition of the pentagram map whereas the third deals with the definition of the inverse pentagram map. Since we consider both constructions throughout this paper, we consider both statements.

With Lemma 8.1.4, we see that the spanning condition on our polygons ensures the pentagram map (and inverse pentagram map) is defined on polygons in ${}^2P(R)$. More specifically, we know that second nearest neighbors (resp. nearest neighbors) are not neighboring, ensuring that they lie on a unique line. Furthermore, we have also shown that for such lines, the consecutive ones (resp. second nearest) will intersect uniquely. This ensures that every polygon will be in the domain of the pentagram map (resp. the inverse pentagram map). **N.B.** It is not clear whether or not the image of a generic polygon is necessarily a polygon (i.e. if the image will satisfy the spanning condition) for either map. Since we know so little about these generalized projective planes (e.g. they have no topology, a priori), we cannot make any general conclusion. We will restrict our attention to polygons in appropriate domains to avoid such problems.

8.2 Known Pentagram Maps as Pentagram Maps in ${}^2P(R)$

From the definition of the projective plane ${}^2P(\mathbb{R})$ and the definition of the classical pentagram map in Section 7.1, we have the following immediately.

Proposition 8.2.1 *The "classical" pentagram map on classes of polygons in $\mathbb{P}^2$ is the pentagram map in ${}^2P(\mathbb{R})$.*

Proof. By our construction of ${}^2P(\mathbb{R})$, we have ${}^2P(\mathbb{R}) = \mathbb{P}^2$. □

In Section 7, we encountered the Grassmann pentagram map and skewer pentagram map. In Sections 3.2 and 3.3, respectively, we related $\mathrm{Gr}(m, 3m)$ to the projective plane ${}^2P(R)$ where $R = \mathrm{M}_m(\mathbb{R})$ and skewers to the $P^2(R)$ where $R = \mathbf{D}$. We can now conclude that these known pentagram maps are precisely pentagram maps in certain projective planes.

Theorem 8.2.2. *The Grassmann pentagram map on $Gr(m, 3m)$, from Definition 7.3.2, is the pentagram map on the plane ${}^2P(M_m(\mathbb{R}))$.*

Proof. This follows from the isomorphism of incidence structures, Proposition 3.2.5, with $n = 3$. □

Remark 8.2.3 Given the similarities between higher dimensional projective spaces over matrices and Grassmannians (beyond $\mathrm{Gr}(m, 3m)$) that were explored in Section 3.2, it is natural to expect that higher pentagram maps over matrices (i.e. pentagram maps in higher dimensional projective spaces over matrices, analogous to those in Section 7.2) relate to other Grassmann pentagram maps from [FB15]. To prove this, one would need to define these higher pentagram maps rigorously and extend the identification of incidence structures.

Theorem 8.2.4. *The skewer pentagram map in $\mathbb{R}^3$, from Definition 7.4.1, is the pentagram map in $P^2(\mathbf{D})$.*

Proof. This follows from the isomorphism of incidence structures, Proposition 3.3.8. □

8.3 Pentagram Maps in Terms of Difference Operators

We know from [Izo22b] that the "classical" pentagram map and higher pentagram maps can be expressed as a refactorization of pseudo-difference operators. We aim to extend this result in the more general case of stably finite rings. For this to be possible, we need to consider the inverse pentagram map (from Definition 8.1.2). Once it becomes clear, the reasoning behind this choice is addressed in Remark 8.3.6. From Theorem 5.1.3, we have a correspondence between appropriate classes of difference operators and polygons. Furthermore, we know that the inverse pentagram map is a map on polygons. We are interested in how the inverse pentagram map affects corresponding difference operators. Let us denote the inverse pentagram map on polygons in ${}^2P(R)$ as Ψ.

Since we are only interested in the projective plane, we re-establish some notation. Let us denote the inverse pentagram map as Ψ. We will now let $\mathfrak{P}$ denote the set of polygons in ${}^2P(R)$ (up to projective equivalence) and we will denote the subset of $\mathfrak{P}$ for which the inverse pentagram map is well-defined and whose image is a polygon as $\mathrm{Dom}(\Psi)$ (and $\mathrm{Dom}(\Psi^{-1})$ for the pentagram map).

Remark 8.3.1 Given a general stably finite ring R it is not clear that $\mathrm{Dom}(\Psi)$ or $\mathrm{Dom}(\Psi^{-1})$ are nonempty.

Lemma 8.3.2 *The image of* $\mathrm{Dom}(\Psi)$ *under* Ψ *is* $\mathrm{Dom}(\Psi^{-1})$.

Proof. Suppose $\{\widehat{x}_i\} \in \Psi(\mathrm{Dom}(\Psi))$. Then, it is a polygon and is the inverse pentagram map's image for some $\{x_i\} \in \mathrm{Dom}(\Psi)$. But, that means the pentagram map applied to $\{\widehat{x}_i\}$ is $\{x_i\}$, which is a polygon. So, $\{\widehat{x}_i\} \in \mathrm{Dom}(\Psi^{-1})$.

Now, suppose $\{\widehat{x}_i\} \in \mathrm{Dom}(\Psi^{-1})$. Then, its image under the pentagram map is defined and is a polygon, denoted by $\{x_i\}$. Therefore $\{x_i\}$'s image under the inverse pentagram map is $\{\widehat{x}_i\}$, which is a polygon. □

Again, we will introduce some new notation regarding difference operators given our restriction to projective planes. We denote the set of properly bounded left difference operators of degree 3 as $\text{PBDO}(R)$. These difference operators considered up to multiplication on the right or left by bi-infinite sequences of units is denoted by $\text{PBDO}(R)/_{LR}$. An element, $[\mathcal{D}] \in \text{PBDO}(R)/_{LR}$ will be a class whose representative is of the form,

$$\mathcal{D} = \lambda_a + \lambda_b T + \lambda_c T^2 + \lambda_d T^3 \qquad a, b, c, d \in R^{\mathbb{Z}}.$$

This is a slightly different notation than that in Chapter 4. For $[\mathcal{D}] \in \text{PBDO}(R)/_{LR}$, we define $\mathcal{D}_+$ and $\mathcal{D}_-$ as follows,

$$\mathcal{D}_+ := \lambda_a + \lambda_b T \qquad \mathcal{D}_- := \lambda_c T^2 + \lambda_d T^3. \tag{8.1}$$

We aim to define a relation on $\text{PBDO}(R)/_{LR}$, which can be thought of as pairs $\mathcal{D}_\pm$ modulo our left/right action. In order to do so, we first define a relation $\sim$ on all properly bounded, left difference operators of degree 3 as follows.

Definition 8.3.3 Two properly bounded left difference operators of degree 3, $\mathcal{D}, \widehat{\mathcal{D}} \in \text{PBDO}(R)$, are said to be related, denoted $\mathcal{D} \sim \widehat{\mathcal{D}}$, if

$$\widehat{\mathcal{D}}_+ \mathcal{D}_- = \widehat{\mathcal{D}}_- \mathcal{D}_+. \tag{8.2}$$

This relation is not symmetric, and therefore not an equivalence relation. But, we observe that this relation descends naturally to $\text{PBDO}(R)/_{LR}$. Here, we say two classes $[\mathcal{D}], [\widehat{\mathcal{D}}]$ are related, $[\mathcal{D}] \sim [\widehat{\mathcal{D}}]$, if some pair of representatives satisfy $\mathcal{D} \sim \widehat{\mathcal{D}}$. In other words, if $[\mathcal{D}] \sim [\widehat{\mathcal{D}}]$, it is not necessarily true that $\mathcal{D} \sim \widehat{\mathcal{D}}$ precisely, but $\mathcal{D}$ is related to a multiple of $\widehat{\mathcal{D}}$ by units on the left. We are interested in the subset of $\text{PBDO}(R)/_{LR}$ where, for a fixed $[\mathcal{D}]$, there is a unique class $[\widehat{\mathcal{D}}]$ so that $[\mathcal{D}] \sim [\widehat{\mathcal{D}}]$. We denote such subset $\text{Dom}(\psi)$.

Definition 8.3.4 We define $\psi : \text{Dom}(\psi) \to \text{PBDO}(R)/_{LR}$ as follows. Let $\mathcal{D} \in [\mathcal{D}]$ from $\text{Dom}(\psi) \subset \text{PBDO}(R)/_{LR}$. By definition of $\text{Dom}(\psi)$, there exists a solution $\widehat{\mathcal{D}}$ to the equation (8.2) and all such solutions represent the same class in $\mathrm{PBDO}(R)/_{LR}$. Set $\psi([D]) = [\widehat{\mathcal{D}}]$ to be that class. Thus,

$$\psi : \mathrm{Dom}(\psi) \to \mathrm{PBDO}(R)/_{LR} : [\mathcal{D}] \mapsto [\widehat{\mathcal{D}}]$$

such that $\mathcal{D}$ and $\widehat{\mathcal{D}}$ satisfy (8.2).

We have the inverse pentagram map, denoted by Ψ, on polygons. And, we have the map ψ on classes of difference operators. We are interested in how these two maps relate. Let us first recall some notation: φ denotes the correspondence from Theorem 5.1.3; $\mathrm{Dom}(\Psi)$ denotes the set of polygons in ${}^2P(R)$ for which the inverse pentagram map is defined and whose image is a polygon (up to projective equivalence); $\mathrm{Dom}(\psi)$ denotes the difference operators for which (8.2) has a unique solution (up to class). Now, let us prove a technical lemma.

Lemma 8.3.5 *The classes of difference operators corresponding to* $\mathrm{Dom}(\Psi)$ *via Theorem 5.1.3 are contained in* $\mathrm{Dom}(\psi)$*.*

Proof. Let us suppose that $[\mathcal{D}]$ is the class of difference operators that corresponds to polygon $\{x_i\} \in \mathrm{Dom}(\Psi)$ via Theorem 5.1.3. Then, we aim to show $[\mathcal{D}] \in \mathrm{Dom}(\psi)$. Pick representative $\mathcal{D} \in [\mathcal{D}]$. Let us denote $\Psi(\{x_i\}) =: \{\widehat{x}_i\}$ (the image of $\{x_i\}$ under the inverse pentagram map). So, the unique line through the pair x_i, x_{i+1} intersects the unique line through the pair x_{i+2}, x_{i+3} at $\widehat{x}_i$. By Lemma 2.4.5, we know that this intersection point has unimodular lift $(\mathcal{D}_+X)_i = -(\mathcal{D}_-X)_i$, for all $i \in \mathbb{Z}$. Let us use this unimodular lift to choose $\widehat{\mathcal{D}}$ corresponding to $\{\widehat{x}_i\}$ via Theorem 5.1.3 (i.e. we choose $\widehat{\mathcal{D}}$ such that $\widehat{\mathcal{D}}\mathcal{D}_+X = 0$). Then, we have the following.

$$
\begin{aligned}
0 = \widehat{\mathcal{D}}(\mathcal{D}_+X) &= (\widehat{\mathcal{D}}_+ + \widehat{\mathcal{D}}_-)(\mathcal{D}_+X) \\
&= \widehat{\mathcal{D}}_+(\mathcal{D}_+X) + \widehat{\mathcal{D}}_-(\mathcal{D}_+X) \\
&= \widehat{\mathcal{D}}_+(\mathcal{D}_+X) + \widehat{\mathcal{D}}_-(\mathcal{D}_+X) - \widehat{\mathcal{D}}_+(\mathcal{D}_-X) + \widehat{\mathcal{D}}_+(\mathcal{D}_-X) \qquad (8.3) \\
&= \widehat{\mathcal{D}}_+(\mathcal{D}X) + \widehat{\mathcal{D}}_-(\mathcal{D}_+X) - \widehat{\mathcal{D}}_+(\mathcal{D}_-X) \\
&= (\widehat{\mathcal{D}}_-\mathcal{D}_+ - \widehat{\mathcal{D}}_+\mathcal{D}_-)X
\end{aligned}
$$

Thus, $(\widehat{\mathcal{D}}_-\mathcal{D}_+ - \widehat{\mathcal{D}}_+\mathcal{D}_-)X = 0$. If we expand $\widehat{\mathcal{D}}_-\mathcal{D}_+ - \widehat{\mathcal{D}}_+\mathcal{D}_-$, we see that it is a difference operator with powers T^2, T^3, and T^4. But, when evaluated on X, it is zero. Since X corresponds to a polygon, the fact that any three consecutive elements in X are a basis of ${}^3R$ (i.e. linearly independent) ensures that $\widehat{\mathcal{D}}_-\mathcal{D}_+ - \widehat{\mathcal{D}}_+\mathcal{D}_-$ is identically zero, giving (8.2). Suppose that we have another $\widehat{\mathcal{D}}'$ that satisfies (8.2). Then, we know $\widehat{\mathcal{D}}'_-\mathcal{D}_+ - \widehat{\mathcal{D}}'_+\mathcal{D}_- = 0$, and thus $(\widehat{\mathcal{D}}'_-\mathcal{D}_+ - \widehat{\mathcal{D}}'_+\mathcal{D}_-)X = 0$. By (8.3), this means $\widehat{\mathcal{D}}'\mathcal{D}_+X = 0$, and thus $\widehat{\mathcal{D}}'$ corresponds to $\{\widehat{x}_i\}$. So, $\widehat{\mathcal{D}}$ And $\widehat{\mathcal{D}}'$ must be in the same class in $\mathrm{PBDO}(R)/_{LR}$ by Theorem 5.1.3. Therefore, $\mathcal{D}$ has a unique class of solutions to (8.2), ensuring $[\mathcal{D}] \in \mathrm{Dom}(\psi)$. □

Remark 8.3.6 In the proof above, we see that using the inverse pentagram map rather than the usual pentagram map was necessary. In particular, the definition for $\mathcal{D}_+$ and $\mathcal{D}_-$ was necessary for the spanning condition to come into play when we consider the expanded form of $\widehat{\mathcal{D}}_-\mathcal{D}_+ - \widehat{\mathcal{D}}_+\mathcal{D}_-$. The relationship between this decomposition of $\mathcal{D}_+$ and $\mathcal{D}_-$ and the inverse pentagram map can be enlightened by the third item in Proposition 8.1.4.

Theorem 8.3.7. *With the notation established above, the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{Dom}(\Psi) & \xrightarrow{\quad\Psi\quad} & \mathrm{Dom}(\Psi^{-1}) \\ \Big\downarrow \varphi & & \Big\downarrow \varphi \\ \mathrm{Dom}(\psi) & \xrightarrow{\quad\psi\quad} & \mathrm{PBDO}(R)\big/LR \end{array}$$

Proof. We aim to show that $\varphi \circ \Psi = \psi \circ \varphi$. Let us consider $\{x_i\} \in \mathrm{Dom}(\Psi)$. We know that $\varphi(\Psi(\{x_i\}))$ is the class of difference operators corresponding to inverse pentagram map's image of $\{x_i\}$. Let us denote the image of $\{x_i\}$ under the inverse pentagram map as $\{\widehat{x}_i\}$ and let us denote its corresponding class of difference operators by $[\widehat{\mathcal{D}}]$. Therefore, $\varphi(\Psi(\{x_i\})) = [\widehat{\mathcal{D}}]$.

Separately, we know that there is some class of difference operators $[\mathcal{D}]$ corresponding to our original $\{x_i\}$. By the proof of Lemma 8.3.5, $[\mathcal{D}]$ and the class $[\widehat{\mathcal{D}}]$ from above satisfy $\widehat{\mathcal{D}}_+\mathcal{D}_- = \widehat{\mathcal{D}}_-\mathcal{D}_+$. Thus, $\psi([\mathcal{D}]) = [\widehat{\mathcal{D}}]$, so $\psi(\varphi(\{x_i\}) = [\widehat{\mathcal{D}}]$. Together, we have that $\varphi(\Psi(\{x_i\})) = \psi(\varphi(\{x_i\})$. This is true for any $\{x_i\} \in \mathrm{Dom}(\Psi)$, and therefore the diagram commutes. □

From this theorem, we have a way to express the inverse pentagram map in terms of difference operators. Unsurprisingly, Ψ and ψ both denote the inverse pentagram map, just in the appropriate setting. We have successfully translated our investigation of inverse pentagram maps into the language of difference operators, which was one of the major goals of this paper. With this reinterpretation, we want to study invariants of maps like that determined by (8.2) to hopefully find invariants of our inverse pentagram maps in these generalized projective planes.

Remark 8.3.8 From Remark 8.1.3, we know that closedness of polygons is preserved by the inverse pentagram map. And, from Corollary 6.3.7, we know that closed polygons correspond to super-periodic difference operators (in the sense that there is a super-periodic representative in the corresponding class). Together, we see that the map ψ sends super-periodic difference operators to super-periodic difference operators.

8.4 Pseudo-Difference Operators

We know that the left (and right) difference operators can naturally be added. Furthermore, we can multiply two difference operators. We would like to look at a collection of difference-type operators which form a group under this multiplication operation. To do so, we need the notion of *pseudo-difference operators*. We follow the notations and definitions given in [Izo22b].

Definition 8.4.1 A left *pseudo-difference operator* (over ring R) is a formal Laurent series in terms of the left shift operator T whose coefficients are bi-infinite sequences of elements from R of the form

$$\sum_{i=k}^{\mathbb{Z}} \lambda_{A^{(i)}} T^i$$

for $k \in \mathbb{Z}$ and $A^{(i)} \in R^{\mathbb{Z}}$.

Here, we understand T raised to a negative power as iteration of *the right shift operator* on bi-infinite sequences. We can analogously define a right pseudo-difference operator using our right multiplication ρ operation, as in Section 4.1. Such a pseudo-difference operator can be understood as a formal sum or as an operator acting on various kinds of bi-infinite sequences, as in Section 4.1.

Remark 8.4.2 We choose positive infinity as our upper bound here, but we could instead sum from negative infinity to a finite upper bound.

With these definitions, we see that a properly bounded left (resp. right) difference operator will have an inverse that is a left (resp. right) pseudo-difference operator. In fact, any difference operator whose bi-infinite sequence $A^{(0)}$ is a sequence of units will have an inverse. Furthermore, any pseudo-difference operator whose lowest term $A^{(k)}$ is a bi-infinite sequence of units will have an inverse. Given our noncommutative setting, we do have to be careful about these inverses. Let us see an example of finding an inverse which will generalize nicely.

Let $\mathcal{D} = \lambda_a + \lambda_b T = \lambda_a(1 + \lambda_{a^{-1}b}T)$. We know that $(1 + x)$ has an inverse, as a formal power series in x, $(1 + x)^{-1} = \sum_{k=0}^{\mathbb{Z}}(-1)^k x^k = 1 - x + x^2 - x^3 + ...$ Therefore, we know that $(1 + \lambda_{a^{-1}b}T)$ will also have an inverse, as a formal power series, denoted by $\sum \lambda_{A^{(i)}}T^i$:

$$\mathcal{D}^{-1} = (\sum \lambda_{A^{(i)}}T^i)\lambda_{a^{-1}} = \sum \lambda_{A^{(i)}}\lambda_{\widehat{a}^{-1}}T^i = \sum \lambda_{A^{(i)}\widehat{a}^{-1}}T^i$$

where $(\widehat{a}^{-1})_j = (a^{-1})_{i+j}$. Notice that for $\mathcal{D}^{-1}$, the constant term is given by $\lambda_{a^{-1}}$. In other words, the degree 0 terms are inverses for $\mathcal{D}$ and $\mathcal{D}^{-1}$. This procedure will work for all left difference operators (as long as the degree 0 term is a bi-infinite sequence of units). This is, again, because we know that $(1+x)$ has an inverse, as a formal power series in x. As long as x is a "polynomial" in T (with no constant term), we will be able to plug said "polynomial" into the formal power series and simplify like terms.

If our difference operator is a right difference operator, then the expression for its inverse is analogous. We compute an elementary example here, but we can extend as above. Let $\mathcal{Q} = \rho_a + \rho_b T = \rho_a(1 + \rho_{ba^{-1}}T)$. Then, again we know that there is a formal power series expressing $(1 + \rho_{ba^{-1}}T)^{-1}$, denoted by $\sum \rho_{A^{(i)}}T^i$. Therefore,

$$\mathcal{Q}^{-1} = (\sum \rho_{A^{(i)}}T^i)\rho_{a^{-1}} = \sum \rho_{A^{(i)}}\rho_{\widehat{a}^{-1}}T^i = \sum \rho_{\widehat{a}^{-1}A^{(i)}}T^i$$

where $(\widehat{a}^{-1})_j = (a^{-1})_{i+j}$.

Remark 8.4.3 For λ, composition $\lambda_a\lambda_b = \lambda_{ab}$ whereas for ρ composition $\rho_a\rho_b = \rho_{ba}$. To make up for this odd composition with right pseudo-difference operators, we may alternatively consider right pseudo-difference operators as formal power series of T over the opposite ring. But, we are only really interested in left pseudo-difference operators for this paper.

Now that we have a notion of pseudo-difference operators, and thus the notion of an "invertible" difference operator, let us establish some notation. We will denote the class of all left pseudo-difference operators over a ring R as $\Theta\mathrm{DO}_\lambda(R)$ and all invertible left pseudo-difference operators over a ring R by $\mathrm{I}\Theta\mathrm{DO}_\lambda(R)$. Our properly bounded left difference operators live in $\mathrm{I}\Theta\mathrm{DO}_\lambda(R)$.

We also introduced periodic difference operators (Section 4) and super-periodic difference operators (Definition 6.3.5). The latter, unfortunately, does not make sense in the setting of pseudo-difference operators. But, the former does; A pseudo-difference operator $\mathcal{D}$ is said to be *periodic* if its coefficients are all N-periodic. We call N its *period*.

8.5 Pentagram Maps as Refactorizations

We are interested in using the result from Theorem 8.3.7 to understand the inverse pentagram map (from Definition 8.1.2) as a refactorization of pseudo-difference operators. We define $\mathrm{PBDO}_+(R)$ (resp. $\mathrm{PBDO}_-(R)$) to be the set of $\mathcal{D}_+$ (resp. $\mathcal{D}_-$) as in (8.1).

Lemma 8.5.1 *Let us consider $\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_- \in {}^{\mathrm{PBDO}(R)}\!\big/_{LR}$. Then, the map*

$$\xi : {}^{\mathrm{PBDO}(R)}\!\big/_{LR} \to {}^{\Theta DO_\lambda(R)}\!\big/_{\mathrm{Ad}((R^*)^{\mathbb{Z}})}$$

$$\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_- \mapsto \mathcal{D}_-^{-1}\mathcal{D}_+$$

is a well-defined map, where $(R^*)^{\mathbb{Z}}$ denotes bi-infinite sequences of units.

Proof. This map is well-defined. Consider another element of the class $\mathcal{D} \in {}^{\mathrm{PBDO}(R)}\!\big/_{LR}$ given by $\alpha\mathcal{D}\beta$, where α, β are bi-infinite sequences of units. Then, we get

$$\alpha\mathcal{D}_+\beta + \alpha\mathcal{D}_-\beta \mapsto (\alpha\mathcal{D}_-\beta)^{-1}\alpha\mathcal{D}_+\beta = \beta^{-1}\mathcal{D}_-^{-1}\alpha^{-1}\alpha\mathcal{D}_+\beta = \beta^{-1}\mathcal{D}_-^{-1}\mathcal{D}_+\beta.$$

But, this is in the same class as $\mathcal{D}_-^{-1}\mathcal{D}_+$.

□

With this map, we can reinterpret what (8.2) means. The map ensures that we can now think of a difference operator $\mathcal{D}$ as a (conjugation class of) pseudo-difference operator $\mathcal{D}_-^{-1}\mathcal{D}_+$. Furthermore, using pseudo-difference operators, (8.2) is equivalent to

$$\widehat{\mathcal{D}}_-^{-1}\widehat{\mathcal{D}}_+ = \mathcal{D}_+\mathcal{D}_-^{-1}. \tag{8.4}$$

Theorem 8.5.2. *With the notation established above, the map ψ on difference operators can be understood as a refactorization of pseudo-difference operators:*

$$\psi : \mathcal{D}_-^{-1}\mathcal{D}_+ \mapsto \mathcal{D}_+\mathcal{D}_-^{-1}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Dom}(\Psi) & \xrightarrow{\quad\Psi\quad} & \mathrm{Dom}(\Psi^{-1}) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\
\mathrm{Dom}(\psi) & \xrightarrow{\quad\psi\quad} & \mathrm{PBDO}(R)/LR \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\
\Theta\mathrm{DO}_\lambda(R)/\mathrm{Ad}((R^*)^{\mathbb{Z}}) & \xrightarrow{\quad\psi\quad} & \Theta\mathrm{DO}_\lambda(R)/\mathrm{Ad}((R^*)^{\mathbb{Z}})
\end{array}$$

Proof. We know that the first layer of the diagram commutes by Theorem 8.3.7. Then, using Lemma 8.5.1 and the equation (8.4) we get the second layer. . □

If we define the *Lax operator* for a difference operator to be $\mathcal{L} = \mathcal{D}_-^{-1}\mathcal{D}_+$, then the map ψ on pseudo-difference operators is given by conjugation by $\mathcal{D}_+$,

$$\mathcal{L} \mapsto \mathcal{D}_+\mathcal{L}\mathcal{D}_+^{-1}.$$

Remark 8.5.3 A Lax operator $\mathcal{L}$ for the inverse pentagram map is unique up to conjugation by a bi-infinite sequence of units, by Lemma 8.5.2. This is because when considering a fixed polygon, the choice of corresponding difference operator is unique up to our left/right action and the Lax representation depends on the choice of difference operator.

Chapter 9

Invariants of Our Pentagram Maps

9.1 An Invariant Inner Product

Let us consider periodic difference and pseudo-difference operators. We aim to build an invariant inner product on these spaces that we will use to construct invariants of our inverse pentagram maps from Chapter 8. Let us consider the following.

Definition 9.1.1 For periodic, left pseudo-difference operator $\mathcal{D} = \sum_{j=\ell}^{\mathbb{Z}} \lambda_{A^{(j)}} T^j$ with period N, let us define the trace

$$\mathrm{TR}(\mathcal{D}) = \sum_{k=1}^{N} A_k^{(0)}.$$

In other words, the trace of a periodic, left difference operator is the sum of zero-th coefficients through one period. We notice that this trace operator is R valued (i.e. noncommutative), and therefore unlikely to have nice symmetry properties. Therefore, we consider the *universal trace* of our ring R.

Definition 9.1.2 The universal trace of a ring R is the projection map

$$\mathrm{tr} : R \to {}^{R}\!/_{[R, R]}$$

where $[R, R]$ is the subgroup generated by $[x, y] = xy - yx$ for $x, y \in R$. We call ${}^{R}\!/_{[R, R]}$ the cyclic space of R.

Remark 9.1.3 We notice that the cyclic space is often not a ring itself since $[R, R]$ is not necessarily an ideal. If we restrict to the case where R is a real algebra, then the cyclic space ${}^{R}\!/_{[R, R]}$ is a vector space. In general, it is just an Abelian group. In the literature, this "cyclic space" is sometimes denoted by $R^\natural$ [Ove20; AOS22].

With this universal trace, we may now introduce a new, third trace which is the composition of the two. We will refer to this inner product as the "trace of trace" inner product but denote it by Tr.

Definition 9.1.4 For periodic, left pseudo-difference operator $\mathcal{D} = \sum_{j=\ell}^{\mathbb{Z}} \lambda_{A^{(j)}} T^j$ with period N, let us define the trace

$$\mathrm{Tr}(\mathcal{D}) = \mathrm{tr}(\mathrm{TR}(\mathcal{D})) = \left[\sum_{k=1}^{N} A_k^{(0)}\right] \in {}^{R}\!/_{[R, R]}.$$

In general, this "trace of trace" operation does not interact nicely with our classes of periodic pseudo-difference operators after the left/right action. Fortunately, this "trace of trace" operation on difference operators will allow us to define an (invariant) inner product.

Definition 9.1.5 Let $\mathcal{D}$ and $\widehat{\mathcal{D}}$ be two periodic, left pseudo-difference operators with period N. Then, we consider the inner product $\langle \cdot, \cdot \rangle$ defined as

$$\langle \mathcal{D}, \widehat{\mathcal{D}} \rangle = \mathrm{Tr}(\mathcal{D}\widehat{\mathcal{D}}).$$

This inner product is *invariant* in that it satisfies $\langle \mathcal{D}\mathcal{Q}, \widehat{\mathcal{D}} \rangle = \langle \mathcal{D}, \mathcal{Q}\widehat{\mathcal{D}} \rangle$ for N-periodic left difference operators $\mathcal{D}, \mathcal{Q}, \widehat{\mathcal{D}}$. Furthermore, this "trace of trace" inner product is symmetric.

Lemma 9.1.6 *Let $\mathcal{D}$ and $\widehat{\mathcal{D}}$ be two periodic, left pseudo-difference operators with period N. Then,*

$$\langle \mathcal{D}, \widehat{\mathcal{D}} \rangle = \langle \widehat{\mathcal{D}}, \mathcal{D} \rangle.$$

Proof. Suppose we have pseudo-difference operators

$$\mathcal{D} = \sum_{j=-k}^{\mathbb{Z}} \lambda_{A^{(j)}} T^j, \qquad \widehat{\mathcal{D}} = \sum_{j=-\widehat{k}}^{\mathbb{Z}} \lambda_{\widehat{A}^{(j)}} T^j.$$

Let $-\kappa = \min\{-k, -\widehat{k}\}$. Then, we have traces equal to

$$\operatorname{Tr}(\mathcal{D}\widehat{\mathcal{D}}) = \sum_{j=-\kappa}^{\kappa} \sum_{i=1}^{N} A_i^{(j)} \widehat{A}_{i+j}^{(-j)}$$

and

$$\operatorname{Tr}(\widehat{\mathcal{D}}\mathcal{D}) = \sum_{j=-\kappa}^{\kappa} \sum_{i=1}^{N} \widehat{A}_i^{(-j)} A_{i-j}^{(j)}.$$

Although these sums are not equal in R, a quick re-indexing shows that they equal in the cyclic space $R \big/ [R, R]$ since $xy = yx \bmod [R, R]$ for all $x, y \in R$. □

From this, we can also see that if we have periodic pseudo-difference operators $\mathcal{D}$ and $\widehat{\mathcal{D}}$, then we have

$$\operatorname{Tr}(\mathcal{D}^{-1}\widehat{\mathcal{D}}\mathcal{D}) = \operatorname{Tr}(\widehat{\mathcal{D}}).$$

In other words, our inner product is invariant under conjugation by a pseudo-difference operator.

9.2 Invariants of the Inverse Pentagram Map

In Section 8.5, we reinterpret the inverse pentagram map as a refactorization of pseudo-difference operators. Using Lax operators, we saw that the inverse pentagram map on pseudo-difference operators is given by $\mathcal{L} \mapsto \mathcal{D}_+ \mathcal{L} \mathcal{D}_+^{-1}$ for

$\mathcal{D}_+ \in \mathrm{PBDO}_+(R)$. We know that

$$\mathrm{Tr}(\mathcal{L}) = \mathrm{Tr}(\mathcal{D}_+ \mathcal{L} \mathcal{D}_+^{-1}),$$

i.e. our "trace of trace" operator applied to $\mathcal{L}$ is invariant under the inverse pentagram map.

In [Izo22b], A. Izosimov shows that the central functions for the space of real (invertible) N-periodic difference operators are given by

$$f_{ij}(\mathcal{D}) = \mathrm{Tr}(T^{Ni}\mathcal{D}^j), \qquad i, j \in \mathbb{Z}_{\geq 0}$$

and that invariant functions of the refactorization of real difference operators (corresponding to the pentagram map) are given by

$$f_{ij}(\mathcal{L}) = \mathrm{Tr}(T^{Ni}\mathcal{L}^j), \qquad i, j \in \mathbb{Z}_{\geq 0}. \tag{9.1}$$

This will be true in some more generality. In the case where we have a commutative ring R, we know that periodic difference operators are naturally acted upon by quasi-periodic bi-infinite sequences, as seen in Section 6.3. This action translates to the Lax operator being conjugated by quasi-periodic sequences. If we conjugate $\mathcal{L}$ by quasi-periodic bi-infinite sequence α with monodromy z, the functions f_{ij} is transformed by

$$f_{ij}(\alpha\mathcal{L}\alpha^{-1}) = \mathrm{Tr}(T^{Ni}\alpha\mathcal{L}^j\alpha^{-1}) = \mathrm{Tr}(\alpha z^i T^{Ni}\mathcal{L}^j\alpha^{-1}) = z^i\mathrm{Tr}(T^{ni}\mathcal{L}^j)$$

and therefore these f_{ij} functions are not invariant under conjugation. But, polynomials in z over these invariant functions are invariant under the action of conjugation by quasi-periodic sequences.

Now, we will restrict to the case of closed polygons. We say in Remarks 8.1.3 and 8.3.8 that closed polygons are invariant under the inverse pentagram map and that they correspond to super-periodic difference operators, ensuring that the refactorization preserves super-periodicity. Furthermore, in Section 6.3 we saw that super-periodic difference operators are naturally acted upon by periodic bi-infinite sequences. In this setting, we will see that our refactorization has invariant functions given by (9.1). Together, we have the following.

Theorem 9.2.1. *Let $\mathcal{D}$ be a super-N-periodic, left, properly bounded difference operator and consider its Lax operator $\mathcal{L} = \mathcal{D}_-^{-1}\mathcal{D}_+$. If the functions f_{ij} are defined by (9.1), then*

1. *The functions f_{ij} are independent of the choice of $\mathcal{D}$;*
2. *The functions f_{ij} are invariant under the pentagram map.*

Proof. Suppose we have another super-periodic representative of $[\mathcal{D}]$ given by $\alpha\mathcal{D}\beta^{-1}$. Since $\mathcal{D}$ is super-periodic, we know that α and β are N-periodic (i.e. they commute with N-powers of T). Furthermore, the Lax operator is changed via conjugation by β. We check that $f_{ij}(\mathcal{L}) = f_{ij}(\beta\mathcal{L}\beta^{-1})$:

$$f_{ij}(\beta\mathcal{L}\beta^{-1}) = \mathrm{Tr}(T^{Ni}(\beta\mathcal{L}\beta^{-1})^j) = \mathrm{Tr}(\beta T^{Ni}(\mathcal{L})^j\beta^{-1}) = \mathrm{Tr}(T^{Ni}(\mathcal{L})^j) = f_{ij}(\mathcal{L}).$$

Now, we recall that the inverse pentagram map, in terms of the Lax operator, is the map $\mathcal{L} \mapsto \mathcal{D}_+\mathcal{L}\mathcal{D}_+^{-1}$. $\mathcal{D}_+$ will be periodic, since it comes from a super-periodic difference operator. Therefore,

$$f_{ij}(\mathcal{D}_+\mathcal{L}\mathcal{D}_+^{-1}) = \mathrm{Tr}(T^{Ni}(\mathcal{D}_+\mathcal{L}\mathcal{D}_+^{-1})^j) = \mathrm{Tr}(\mathcal{D}_+T^{Ni}\mathcal{L}^j\mathcal{D}_+^{-1}) = \mathrm{Tr}(T^{Ni}\mathcal{L}^j) = f_{ij}(\mathcal{L}).$$

Hence, the functions f_{ij} are independent of choice of $\mathcal{D}$ and invariant under the inverse pentagram map. □

Chapter 10

Conclusion and Further Work

In this paper, we were able to accomplish many of our major goals. We described projective spaces over stably finite rings, polygons in such spaces, and related such objects to difference operators through a constructive correspondence (Theorem 5.1.3). These constructions were quite delicate, since we did not rely on much structure in our rings. Therefore, intuition from linear algebra and commutative ring theory was lost.

Once our correspondence was established, we were able to describe a new class of discrete dynamical systems on polygons in our generalized projective planes. Using our correspondence, we were able to reinterpret these dynamical systems as refactorizations of difference operators. This introduced some invariants of our system (in some cases). If one's ultimate goal is to prove integrability of these dynamical systems, the next step might be to find a Poisson structure on the space of difference operators.

Throughout this project, there were some questions and routes of investigation that were not attempted, or were inconclusive. For the reader's convenience, we list them here.

Higher pentagram maps: One may continue the investigation from Chapter 8.2 into the relationship between higher pentagram maps, like those from [KS13; KS16], and projective spaces over the Grassmannians. In Remark 8.2.3, we state what we believe the relationship will be, which relates to A. Izosimov's work in [Izo22b] where he uses higher degree difference operators to understand higher pentagram maps.

Coordinates: One may investigate coordinate expressions for these generalized pentagram maps. Over a general ring, it is unclear that anything useful will result. However, in the case of R being an algebra for which there is a useful classification for elements (such as conjugation classes of square matrices), one might be able to say something conclusive about coordinate expressions of these maps. In Appendix B, we review some coordinate expressions for the classical and Grassmannian pentagram maps.

Domains and density: In the case where R is a real algebra or some other topological space, one might be able to address some topological questions. For instance, for the classical pentagram map, we know that the domain is dense in the set of twisted polygons. One might be able to conclude something similar in our more general setting.

Poisson structure for pseudo-difference operators: The most obvious continuation of this paper might be attempting to address the unanswered question: is there a natural Poisson structure on the space of periodic difference operators and periodic pseudo-difference operators?

Appendix A

Notation List

$\mathbb{R}$	Real numbers
$\mathbb{Z}$	Integers
$\mathbb{N}$	Natural numbers
$\mathbb{Q}$	Rational numbers
R	Unital ring (i.e. ring with identity), usually stably finite
R^{op}	Opposite ring of R
${}^nR$	Left free module over R of rank n
R^n	Right free module over R of rank n
$\mathrm{M}_m(R)$	Space of $m \times m$ matrices with entries from R
$\mathrm{M}_{n\times m}(R)$	Space of $n \times m$ matrices with entries from R
$\mathfrak{M}_R$	R-module, side specified when relevant
$(\mathfrak{M}_R)^{\mathbb{Z}}$	Bi-infinite sequences elements from $\mathfrak{M}_R$
${}^nP(R)$	Left projective space over R of dimension n
$P^n(R)$	Right projective space over R of dimension n
$\{x_i\}_{i\in\mathbb{Z}}$	Polygon in projective space
$(X_i)_{i\in\mathbb{Z}}$	Unimodular lift of polygon (lives in relevant module)

λ_A	Component-wise multiplication by bi-infinite sequence A on the left
ρ_A	Component-wise multiplication by bi-infinite sequence A on the right
T	Left shift operator
$\mathcal{D}, \widehat{\mathcal{D}}, \mathcal{Q}, \mathcal{R}$	Difference operators, side specified when relevant
$\ker(\mathcal{D})$	Kernel of operator $\mathcal{D}$
$\mathfrak{D}$	The space of all degree d properly bounded, left difference operators
$\mathfrak{D}/_{LR}$	$\mathfrak{D}$ up to left/right multiplication of bi-infinite sequences of units
$\mathfrak{P}$	Polygons in ${}^{d-1}P(R)$, up to projective equivalence
$\mathbb{P}^n$	Real projective space of dimension n
$\mathrm{Gr}(k, n)$	Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$
$\mathbf{D}$	Dual numbers
$x \approx y$	Neighboring points in a projective plane
Ψ	Inverse pentagram map for polygons in ${}^2P(R)$
$\mathrm{Dom}(\Psi)$	Polygons in ${}^2P(R)$ whose image under Ψ is also a polygon, up to projective equivalence
$\mathrm{PBDO}(R)$	Properly bounded, left degree 3 difference operators
$\mathrm{PBDO}(R)/_{LR}$	$\mathrm{PBDO}(R)$, up to left/right action
$\mathrm{PBDO}_+(R)$	Collection of $\mathcal{D}_+ := \lambda_a + \lambda_b T$
$\mathrm{PBDO}_-(R)$	Collection of $\mathcal{D}_- := \lambda_c T^2 + \lambda_d T^3$
$\mathcal{D} \sim \widehat{\mathcal{D}}$	Degree 3 difference operators satisfying $\widehat{\mathcal{D}}_+ \mathcal{D}_- = \widehat{\mathcal{D}}_- \mathcal{D}_+$
$\mathrm{Dom}(\psi)$	Difference operators $\mathcal{D}$ for which there is a unique class of difference operators so that $\mathcal{D} \sim \widehat{\mathcal{D}}$, domain of ψ
ψ	Inverse pentagram map on difference operators
$\Omega DO_\lambda(R)$	Left pseudo-difference operators over R
$I\Omega DO_\lambda(R)$	Left invertible pseudo-difference operators over R

Appendix B

Coordinate Expressions

The process of finding a difference operator corresponding to a given polygon is closely related to the process of finding the coordinates for the moduli space of twisted polygons. At least, this has been shown in the case of real twisted polygons in $\mathbb{P}^2$ [OST10] and twisted polygons in $\mathrm{Gr}(m, km)$ [FB15; Ove20]. One might find it enlightening to see how these processes compare. Therefore, we include a brief discussion of how to find coordinates for these moduli spaces, and how they yield coordinate expressions for the classical pentagram map.

B.1 Coordinate Expression of the Classical Pentagram Map

In [OST10], Ovsienko, Schwartz, and Tabachnikov find coordinates for the space of twisted n-gons in $\mathbb{RP}^2$, the real projective plane. They use these coordinates to find invariants of the pentagram map which ultimately leads to them showing integrability of the system. The construction of coordinates is adapted in [Ove20] to the case of the Grassmann pentagram map. We will see that these coordinate definitions seem closely related to our difference operators and therefore one might find them to be adaptable in more generality. We begin with discussing coordinates which are generically defined on the twisted n-gons in the real projective plane via [OST10].

Remark B.1.1 Two coordinate systems are developed in this paper. We are referring to the second system defined in Section 4 of [OST10].

Let us consider two n-periodic, bi-infinite sequences (a_i) and (b_i) where $a_i, b_i \in \mathbb{R}$ for all $i \in \mathbb{Z}$. Assume that n is not divisible by 3. Then, we consider the following equations,

$$V_{i+3} = a_i V_{i+2} + b_i V_{i+1} + V_i \tag{B.1}$$

A solution, $V = (V_i)$, is also a bi-infinite sequence of real numbers. The space of solutions to such a system is 3-dimensional, since V is uniquely determined by V_0, V_1, V_2. As a result, we will often consider solutions as vectors in $\mathbb{R}^3$. The periodicity of the system ensures that there is a matrix $M \in \mathrm{SL}_3(\mathbb{R})$ so that $V_{i+n} = MV_i$. We call this matrix the monodromy.

Proposition B.1.2 *(See [OST10]) If n is not divisible by 3, then the space of twisted n-gons (up to projective equivalence) in $\mathbb{P}^2$ is isomorphic to the space of the equations (B.1).*

The proof of this proposition is a bit technical and can be found in [OST10] (proof of Proposition 4.1). Essentially, through lifting our twisted n-gons, we can always rescale the lifts in such a way to get equations like those in (B.1).

Now that we have a coordinate system for our space of twisted n-gons, we can discuss what the pentagram map looks like in this (a, b) coordinate system. This proposition allows us to observe that the pentagram map is a *rational map*.

Proposition B.1.3 *(See [OST10]) Assume that $n = 3m + 1$ or $n = 3m + 2$. In both cases,*

$$T^*(a_i) = a_{i+2} \prod_{k=1}^{m} \frac{1 + a_{i+3k+2} b_{i+3k+1}}{1 + a_{i-3k+2} b_{i-3k+1}} \qquad T^*(b_i) = b_{i-1} \prod_{k=1}^{m} \frac{1 + a_{i-3k-2} b_{i-3k-1}}{1 + a_{i+3k-2} b_{i+3k-1}}$$

where T^* is the standard pull-back of the coordinate functions by the pentagram map T.

We might notice that in the context of our paper, this construction of coordinates is quite natural. In fact, this is the construction we used to find difference operators corresponding to polygons in Chapter 5. However, there is more nuance when we consider that our systems have coefficients from a stably finite ring. These coefficients being from such a general class of rings leads to problems when attempting to generalize this coordinate system. In some cases, such as real matrices, we can use known properties of the ring to generalize this coordinate system. We discuss this process in Section B.2.

B.2 Coordinates for the Moduli Space of Twisted Polygons in Gr$(m, 3m)$

In [Ove20], N. Ovenhouse describes a few different coordinates on the moduli space of twisted polygons in Gr$(m, 3m)$. A more general set of coordinates for the moduli space of twisted polygons in Gr(m, km) is given in [FB15], but we will not include them here. The goal of this discussion is to solidify the relationship between the moduli space of twisted polygons in Gr$(m, 3m)$ and difference operators of degree 3 with matrix coefficients. This was the original motivation for our work discussed in Sections 3.2 and 6.2.

Let $X = (X_i)$ be a twisted lift of the twisted, regular N-gon l in Gr$(m, 3m)$. Then, we know by the spanning condition that any three consecutive elements X_i, X_{i+1}, X_{i+2} will satisfy

$$\begin{pmatrix} X_i \\ X_{i+1} \\ X_{i+2} \end{pmatrix}$$

has nonzero determinant. But, we recall that each X_i is a $m \times 3m$ matrix whose row space corresponds to l_i in the polygon (therefore in Gr$(m, 3m)$). In other words, this nonzero determinant ensures that the row space of the matrix is all of $\mathbb{R}^{3m}$. So, we can find $m \times m$ matrices A_i, B_i, C_i (which are invertible) so that

$$X_{i+3} = A_i X_i + B_i X_{i+1} + C_i X_{i+2}.$$

Remark B.2.1 We notice that if we instead chose our lifts to be $3m \times m$ matrices, we would be dealing with column spaces and therefore these A_i, B_i, C_i would be multiplied on the right. This is how right difference operators can appear.

Since we are dealing with twisted N-gons, these coefficents can be chosen to be periodic (with period N). Furthermore, we have the following.

Proposition B.2.2 *(See [Ove20]) The lift (X_i) of (l_i) can be chosen so that $C_i = \mathbb{I}_m$ for all $i \in \mathbb{Z}$.*

Unfortunately, A_i, B_i do not form coordinates for the moduli space of twisted, regular N-gons. This is because when we rescale in the proposition above to get $C_i = \mathbb{I}$, the A_i and B_i will not end up being periodic. But, they are "almost" periodic. To accommodate this, we introduce the cyclic product $Z = C_{N-1} C_N C_1 \dots C_{N-2}$. Then, shifting the index of A_i and B_i by the period N will result in conjugation by Z. Since matrices are able to be classified up to conjugation using normal forms, we have a coordinate system given by A_i, B_i, Z. This ensures that the moduli space of twisted N-gons in $\mathrm{Gr}(m, 3m)$ has dimension $2N + 1$.

The process of finding these A_i's and B_i's might look similar to our construction of the correspondence of polygons and difference operators in Section 5.2.3. This is partially what inspired the idea that the pentagram map could be defined in a larger class of projective planes. Furthermore, integrability of the Grassmann pentagram map encouraged the idea that investigation in such pentagram maps might be fruitful.

Bibliography

[AOS22] S. Arthamonov, N. Ovenhouse, and M. Shapiro. *Noncommutative Networks on a Cylinder*. 2022. DOI: https://doi.org/10.48550/arXiv.2008.02889. arXiv: 2008.02889.

[Bou89] N. Bourbaki. *Algebra 1: Chapters 1-3*. Springer, 1989.

[Coh03] P.M. Cohn. *Basic Algebra: Groups, Rings, Fields*. Springer, 2003, pp. 105–110.

[FB15] R. Felipe and G. Marí Beffa. "The pentagram map on Grassmannians". In: *Annales de l'Institut Fourier* (2015). DOI: 10.5802/AIF.3248.

[Fla74] H. Flaschka. "The Toda lattice. II. Existence of integrals". In: *Phys. Rev. B* 9 (4 Feb. 1974), pp. 1924–1925. DOI: 10.1103/PhysRevB.9.1924.

[FM16] V.V. Fock and A. Marshakov. ""Loop Groups, Clusters, Dimers and Integrable Systems"". In: *Geometry and Quantization of Moduli Spaces*. Springer International Publishing, 2016. DOI: 10.1007/978-3-319-33578-0_1.

[Gek+16] M. Gekhtman, M. Shapiro, S. Tabachnikov, and A. Vainshtein. "Integrable cluster dynamics of directed networks and pentagram maps"". In: *Advances in Mathematics* 300 (2016), pp. 390–450. DOI: https://doi.org/10.1016/j.aim.2016.03.023.

[GM60] C. S. Gardner and G. K. Morikawa. "Similarity In the Asymptotic Behavior of Collision-Free Hydromagnetic Waves and Water Waves". In: (May 1960). eprint: https://www.osti.gov/biblio/4172979.

[GP16] M. Glick and P. Pylyavskyy. "Y-meshes and generalized pentagram maps". In: *Proceedings of the London Mathematical Society* 112.4 (Mar. 2016), pp. 753–797. DOI: https://doi.org/10.1112\%2Fplms\%2Fpdw007.

[Gug63] H. Guggenheimer. *Differential Geometry*. Dover, 1963, pp. 162–165.

[Hun74] T. A. Hungerford. *Algebra*. Vol. GTM 73. Springer, 1974, pp. 190–192, 328–335.

[Izo22a] A. Izosimov. ""The pentagram map, Poncelet polygons, and commuting difference operators"". In: *Compositio Mathematica* 158 (5 2022), pp. 1084–1124. DOI: https://doi.org/10.1112/S0010437X22007345.

[Izo22b] A. Izosimov. "Pentagram maps and refactorization in Poisson-Lie groups". In: *Advances in Mathematics* 404 (2022). DOI: https://doi.org/10.1016/j.aim.2022.108476.

[KS13] B. Khesin and F. Soloviev. "Integrability of higher pentagram maps". In: *Mathematische Annalen* 357 (2013), pp. 1005–1047. DOI: https://doi.org/10.1007/s00208-013-0922-5.

[KS16] B. Khesin and F. Soloviev. "The geometry of dented pentagram maps". In: *Journal of the European Mathematical Society* 18 (2016), pp. 147–179. DOI: 10.4171/JEMS/586.

[KV95] D. J. Korteweg and G. de Vries. "XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 39.240 (1895), pp. 422–443. DOI: https://doi.org/10.1080/14786449508620739.

[Lam91] T.Y. Lam. *A First Course in Noncommutative Rings*. Vol. GTM 131. Springer, 1991, pp. 314–317.

[Lam99] T.Y. Lam. *Lectures on Modules and Rings*. Vol. GTM 189. Springer, 1999, pp. 5–9.

[N J99] R.S. Ward N. J. Hitchin G. B. Segal. *Integrable Systems: Twistors, Loop Groups, and Riemann Surfaces*. Vol. GTM 4. Oxford University Press, 1999.

[OST10] V. Ovsienko, R. Schwartz, and S. Tabachnikov. "The Pentagram Map: A Discrete Integrable System". In: *Communications in Mathematical Physics* 299 (2 2010), pp. 409–446. DOI: 10.1007/s00220-010-1075-y.

[Ove20] N. Ovenhouse. "Non-Commutative Integrability of the Grassmann Pentagram Map". In: *Advances in Mathematics* 373 (2020). DOI: https://doi.org/10.1016/j.aim.2020.107309.

[Sch92] R. Schwartz. "*The Pentagram Map*". In: *Experimental Mathematics* (1992), pp. 71–81.

[Sol13] F. Soloviev. "*Integrability of the pentagram map*". In: *Duke Mathematical Journal* 162.15 (Dec. 2013). ISSN: 0012-7094. DOI: 10.1215/00127094-2382228.

[Tab16] S. Tabachnikov. "*Skewers*". In: *Arnold Mathematical Journal* 2 (2016), pp. 171–193. DOI: https://doi.org/10.1007/s40598-016-0037-7.

[Tab19] S. Tabachnikov. "Projective Configuration Theorems: Old Wine Into New Wineskins". In: *Geometry in History*. Springer International Publishing, 2019, pp. 401–434. DOI: 10.1007/978-3-030-13609-3_9.

[Tes12] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Vol. GSM 140. American Mathematical Society, 2012, pp. 187–192.

[Tod70] M. Toda. "Waves in Nonlinear Lattice". In: *Progress of Theoretical Physics Supplement* 45 (1970), pp. 174–200. DOI: https://doi.org/10.1143/PTPS.45.174.

[Vel81] F.D. Veldkamp. "*Projective planes over rings of stable rank 2*". In: *Geometriae Dedicata* 11 (1981), pp. 285–308. DOI: https://doi.org/10.1007/BF00149352.

[WT66] H. Washimi and T. Taniuti. "Propagation of Ion-Acoustic Solitary Waves of Small Amplitude". In: *Phys. Rev. Lett.* 17 (19 Nov. 1966), pp. 996–998. DOI: 10.1103/PhysRevLett.17.996.