Reductions of Some Crystalline Representations in the Unramified Setting

by

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Contents

Abstract

We determine semisimple reductions of irreducible, 2-dimensional crystalline representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f})$. To this end, we provide explicit representatives for the isomorphism classes of the associated weakly admissible filtered φ-modules by concretely describing the strongly divisible lattices which characterize the structure of the aforementioned modules. Using these representatives, we construct Kisin modules canonically associated to Galois stable lattice representations inside our crystalline representations. This allows us to compute the reduction of such crystalline representations with labeled Hodge-Tate weights in the range $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_i \leq p-3$ for $1 \leq i \leq f-1$ and certain parameters having p-adic valuation sufficiently large, unless $f=2$ in which case all reductions in said range are displayed.

Chapter 1

Introduction

1.1 Motivation: Global Context and History

This thesis concerns itself with the study of crystalline representations of absolute Galois groups with base fields being unramified extensions of the p-adic numbers. In particular, we are interested in the behavior of such representations under the operation of reduction modulo the prime p. While the investigation of such things is interesting in and of themselves from a purely representation theory perspective, we would be remiss to ignore the context from which motivation for the study initially arose and the generalizations that drive this very thesis.

We reference the introductory chapters of [BG09] and [Dou10a] in the following discussion.

Global Context

Let $g = \sum_{n\geq 1} a_n q^n$ be a weight $k \geq 2$ cuspidal modular form for $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ for some level $N \geq 1$ and Nebentype character ψ. We will also assume that g is an eigenform at all primes p coprime to N. In other words, g is a simultaneous eigenvector for the Hecke operator action T_p on the space of weight k cusp forms of level N denoted $S_k(N)$ for all $p \nmid N$. If g is normalized so that $a_1 = 1$, then the eigenform encodes the associated eigenvalues via its complex Fourier coefficients in that $T_p(g) = a_p \cdot g$, each of which may be interpreted as algebraic integers. After choosing certain embeddings, the Fourier coefficients a_n may be considered as elements of $\overline{\mathbb{Q}}_p$ so that we may take p-adic valuations of the Hecke eigenvalues $\nu_p(a_p)$ known as the *slope* of the eigenform.

The work of Deligne [Del71] allows us to attach to g, a continuous, two-dimensional p-adic representation

$$\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for any prime p. By fixing an embedding of algebraic closures $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we may choose a place of $\overline{\mathbb{Q}}$ above p for which the decomposition group D_p at this place is isomorphic to the local absolute Galois group $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. By restricting ρ_g to D_p, we give rise to a local representation of $G_{\mathbb{Q}_p}$,

$$\rho_{g,p} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p).$$

If we assume that g has slope $\nu_p(a_p) > 0$ with $a_p^2 \neq 4p^{k-1}$ then $\rho_{g,p}$ is irreducible and crystalline with Hodge-Tate weights $\{0, k-1\}$ by the works of Faltings and Scholl in [Fal89] and [Sch90]. The associated weakly admissible filtered φ-module $\mathbf{D}^*_{\mathrm{cris}}(\rho_{g,p})$ has characteristic polynomial of Frobenius given by $X^2 - a_pX + \psi(p)p^{k-1}$. In [Bre03], Breuil shows that the data of the characteristic polynomial of Frobenius suffices to completely determine the filtered φ-module $\mathbf{D}^*_{\mathrm{cris}}(\rho_{g,p}) = D_{k,a_p}$ up to a twist by an unramified character. In particular, by choosing a suitable basis $\{\eta_1, \eta_2\}$, one is able to describe the Frobenius and filtration structures on D_{k,a_p} by

$$[\varphi]_\eta = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix} \qquad \mathrm{Fil}^j D_{k,a_p} = \begin{cases} D_{k,a_p} & 0 \geq j \\ \mathbb{Q}_p(\eta_1) & 0 < j \leq k-1 \\ 0 & k-1 < j. \end{cases} \tag{1.1}$$

In fact, every irreducible, two-dimensional crystalline $\mathbb{Q}_p$-representation of $G_{\mathbb{Q}_p}$ takes the form $V_{k,a_p} := \mathbf{V}^*_{\mathrm{cris}}(D_{k,a_p})$, up to the aforementioned twist, for some a_p with $\nu_p(a_p) > 0$ and $k \geq 2$ so that we have an isomorphism $\rho_{g,p} \cong V_{k,a_p}$. Hence, the data of the weight k and Hecke eigenvalue of g suffice to completely determine the structure of $\rho_{g,p}$.

The above discussion lends merit to the idea that the local study of p-adic Galois representations may help to provide answers to global questions regarding modular forms. In particular, the residual representations $\overline{\rho}_{g,p} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ have proven to be especially fruitful, having played an important part in the proof of Serre's modularity conjectures by Khare and Wintenberger in [KW09a; KW09b] and are conjectured to provide insight into the distribution of slopes of modular forms, see [Buz05].

History of Reductions

The prospect of answering global questions via studying local phenomenon has driven the investigation of the modulo p reductions $\overline{\rho}_{g,p} \cong \overline{V}_{k,a_p}$ of irreducible crystalline representations of $G_{\mathbb{Q}_p}$. We however are interested in the purely local question posed in [BG09]

Question: What is the isomorphism class of $\overline{V}_{k,a_p}$ as an explicit function of the parameters k and a_p?

The study of this question has proven to be far more complicated than in characteristic 0 setting, recall 1.1. The essential problem is that in order to determine $\overline{V}_{k,a_p}$, we must first describe an integral structure associated to D_{k,a_p}. As a result, only partial results are known which we detail below in a format inspired by [Ber11a].

Let ω_n denote fundamental characters of niveau $n \geq 1$ and $\chi(x)$ denote the unramified character in $G_{\mathbb{Q}_p}$ sending Frobenius to $x \in \overline{\mathbb{F}}_p$. All inductions are from $G_{\mathbb{Q}_{p^2}}$ to $G_{\mathbb{Q}_p}$. We begin with where the study of the shapes of reductions began, with bounded weights.

Theorem 1.1.1: *The representation $\overline{V}_{k,a_p}$ is known in the following situations:*

(a) *If $2 \leq k \leq p+1$ then $\overline{V}_{k,a_p} = \mathrm{Ind}(\omega_2^{k-1})$.*
(b) *For $k = p+2$,*

- *if $0 < \nu_p(a_p) < 1$, then $\overline{V}_{k,a_p} = \mathrm{Ind}(\omega_2^2)$;*
- *if $\nu_p(a_p) \geq 1$, then $\overline{V}_{k,a_p} = \chi(x)\omega_1 \oplus \chi(x)^{-1}\omega_1$ where $x^2 - \overline{(a_p/p)}x + 1 = 0$.*

(c) *For $p+3 \leq k \leq 2p$,*

- *if $0 < \nu_p(a_p) < 1$, then $\overline{V}_{k,a_p} = \mathrm{Ind}(\omega_2^{k-p})$;*
- *if $\nu_p(a_p) = 1$, then $\overline{V}_{k,a_p} = \chi(x)\omega_1^{k-2} \oplus \chi(x)^{-1}\omega_1$ where $x = (k-1)\overline{(a_p/p)}$;*
- *if $\nu_p(a_p) > 1$, then $\overline{V}_{k,a_p} = \mathrm{Ind}(\omega_2^{k-1})$.*

Let us explain the methods used to prove each of these results. The case of $k \leq p$ in (a) were the first reductions computed, attributed to Fontaine and Laffaille in [FL82] using the theory of what is now known as Fontaine-Laffaille modules which relies on an equivalence of categories only valid for $k \leq p$ for technical reasons. For weights $p+2 \leq k \leq 2p$ in (b) and (c), being outside the Fontaine-Laffaille range, the reductions were instead computed by Breuil in [Bre03] using the p-adic local Langlands correspondence for GL_2. We note that the case of $k = 2p+1$ is known by unpublished calculations due to Breuil as well. The reader may have noticed that we neglected to discuss the case of $k = p+1$. This is due to the fact that the reduction at this particular weight can be thought of as a special case of far more general results that we state shortly.

For $k > 2p+1$, one should suspect the number of possible reductions to increase, each depending on the p-adic valuations of a_p in differing ranges. Hence, the computation of reductions for large weights seems to be unfeasible unless however, we put strict bounds on the p-adic valuations of the a_p parameter. To this end, we have the next set of reductions which apply no bounds on their weights with the trade-off being the aforementioned strict bounds on the slopes $\nu_p(a_p)$. In these cases, the possible reductions are restricted heavily.

Theorem 1.1.2: *The representation $\overline{V}_{k,a_p}$ is known in the following situations:*

(d) *For $a_p = 0$,*

- *if $p+1 \nmid k-1$, then $\overline{V}_{k,0} = \mathrm{Ind}(\omega_2^{k-1})$;*
- *if $p+1 \mid k-1$, then $\overline{V}_{k,0} = \chi(\sqrt{-1})\omega_1^{\frac{k-1}{p+1}} \oplus \chi(-\sqrt{-1})\omega_1^{\frac{k-1}{p+1}}$*

(e) *If $\nu_p(a_p) > \lfloor \frac{k-1}{p+1} \rfloor + \lfloor \log_p(k-1) \rfloor$ with $p > 3$ and $p+1 \nmid k-1$, then $\overline{V}_{k,a_p} \cong \overline{V}_{k,0}$*
(f) *For $0 < \nu_p(a_p) < 1$, let $t = [k-1 \pmod{p-1}] + 1$*

- *$\overline{V}_{k,a_p} = \mathrm{Ind}(\omega_2^t)$ if $p-1 \nmid k-3$;*

- $\overline{V}_{k,a_p} \in \{\mathrm{Ind}(\omega_2^t), \chi\omega_1 \oplus \chi^{-1}\omega_1\}$ *for a certain* χ *if* $p-1 \nmid k-3$.

Part (d) is again due to Breuil in [Bre03]. Case (e) is actually the latest in a string of compounding improvements toward the conjecture that $\overline{V}_{k,a_p} \cong \overline{V}_{k,0}$ whenever $\nu_p(a_p) > \lfloor \frac{k-1}{p+1} \rfloor$.[1] The first of these results is due to Berger-Li-Zhu in [BLZ04] where they are able to display the bound $\nu_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$ using the theory of Wach modules. This bound was improved to $\nu_p(a_p) > \lfloor \frac{k-1}{p} \rfloor$ by Bergdall and Levin in [BL20], this time using Kisin modules. The form displayed in (e) is due to Arsovski in [Ars21] who uses the local Langlands correspondence to improve the bounds with slightly more restrictions. The final case of (f) is another local Langlands correspondence result by Buzzard and Gee in [BG09].

While this history serves to illustrate the intricacies of computing reductions and the methods used to do so, this thesis is not concerned with this generality. Rather, we aim to provide similar results in the next natural generalization; the unramified setting.

Generalizations

Hilbert modular forms may be thought of as multi-variate generalizations of modular forms. Hence, the study of Hilbert modular forms motivates the following generalization of the above discussion. Let F be a totally real number field of degree $f > 1$ and let $I = \{\tau_0, \ldots, \tau_{f-1}\}$ denote the set of real embeddings of $F \hookrightarrow \mathbb{C}$. Fix a f-tuple $\mathbf{k} = (k_{\tau_0}, \ldots, k_{\tau_{f-1}})$ and denote the ring of integers of F by $\mathscr{O}$ with nonzero ideal $\mathfrak{n} \subset \mathscr{O}$.

Let g be a Hilbert cuspform of level $\mathfrak{n}$ and weight $\mathbf{k}$ with each $k_{\tau_i} \geq 2$. Let us also assume that g is an eigenform with respect to the Hecke action $T_\mathfrak{p}$ at each prime ideal $\mathfrak{p} \subset \mathcal{O}$ coprime to $\mathfrak{n}$. As before, we may attach to g a continuous, irreducible, two-dimensional Galois representation $\rho_g : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ of $G_F = \mathrm{Gal}(\overline{F}/F)$ by the combined works of Carayol, Taylor, Blasius-Rogawski, and Jarvis [Car86; Tay89; BR89; Jar97]. Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}$ lying above p and denote by $F_\mathfrak{p}$ the completion of F at $\mathfrak{p}$. Upon fixing an embedding $\overline{F} \hookrightarrow \overline{F}_\mathfrak{p}$, we obtain an isomorphism $D_\mathfrak{p} \cong G_{\mathbb{Q}_{p^f}} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f})$ and a local representation of $G_{\mathbb{Q}_{p^f}}$ given by

$$\rho_{g,\mathfrak{p}} : G_{\mathbb{Q}_{p^f}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p).$$

Under the assumption that p is an odd prime unramified in F and coprime to $\mathfrak{n}$, then $\rho_{g,\mathfrak{p}}$ is an irreducible, two-dimensional crystalline representation of $G_{\mathbb{Q}_{p^f}}$ by Breuil [Bre99] and Berger [Ber04b]. The reduction of such representations occur naturally in the context of the generalization of Serre's modularity conjecture of Buzzard, Diamond and Jarvis [BDJ10].

The study of these reductions is far less advanced than in the $G_{\mathbb{Q}_p}$ case detailed in Theorems 1.1.1 and 1.1.2. The initial obstruction is that we lose the nice classification of filtered φ-modules $\mathbf{D}^*_{\mathrm{cris}}(\rho_{g,\mathfrak{p}})$ as we had over $\mathbb{Q}_p$ in 1.1. Indeed, the data of the Hecke

[1] This conjecture is actually a local version of a global conjecture of Gouvêa predicting the distribution of slopes of modular forms, see [Gou01].

eigenvalue of g at $\mathfrak{p}$ and the weights no longer suffice to describe the structure of $\mathbf{D}^*_{\mathrm{cris}}(\rho_{g,\mathfrak{p}})$. In essence, there is not enough data provided by the Hilbert modular form to concretely determine the associated weakly admissible filtered φ-module. In order to compute these reductions, it is therefore first necessary to formulate models for (the isomorphism classes of) weakly admissible filtered φ-modules corresponding to irreducible representations.

We should note that the options available to us for computing such reductions are much more limited in the unramified setting. Most notably is the loss of the p-adic local Langlands correspondence used in (b) and (c) in Theorem 1.1.1 and in (e) and (f) in Theorem 1.1.2. Indeed, a correspondence is conjectured for any p-adic field but the only case known at this time of writing is for GL_2 over $\mathbb{Q}_p$. We do retain Wach and Kisin modules however, the later being our theory of choice.

Some partial results are known however. Notably, the work of Dousmanis in [Dou10b] gives such a description of filtered φ-modules which is used in [Dou10a] and [Dou14] to generalize the results of [BLZ04] and [Ber11b] to compute reductions for certain (infinite) families of crystalline representations whose Frobenius action 'looks like' the Frobenius matrix in 1.1 using Wach modules. The deficit of these methods is that it becomes very difficult to know whether a given representation lies inside one of these families and how many representations these families don't cover.

The purpose of this thesis is to give a novel description of irreducible, rank two weakly admissible filtered φ-modules and by proxy, a description of irreducible, two dimensional crystalline representations of $G_{\mathbb{Q}_{p^f}}$ without bound on the (labeled) Hodge-Tate weights. Moreover, we use the theory of Breuil and Kisin modules to explicitly compute the reductions of such crystalline representations under restrictions on the labeled Hodge-Tate weights. Hence, we aim to provide the first results in a future form of Theorem 1.1.1 for crystalline reductions in the unramified setting.

1.2 Notation and Conventions

We fix the following notation for the remainder of this thesis. Let $p \geq 2$ be a prime and equip $\mathbb{Q}_p$ with a p-adic valuation ν_p normalized so that $\nu_p(p) = 1$.[2] Upon fixing an algebraic closure $\overline{\mathbb{Q}}_p$, we denote the p-adic completion of $\overline{\mathbb{Q}}_p$ by $\mathbb{C}_p$.

Let K be the unique unramified extension of $\mathbb{Q}_p$ with (inertial) degree $f \geq 1$; that is, we have an isomorphism $K \cong \mathbb{Q}_{p^f}$. Denote its ring of integers by $\mathscr{O}_K$ with uniformizer π_K and its residue field $k = \mathscr{O}_K/\pi_K\mathscr{O}_K \cong \mathbb{F}_{p^f}$. Let σ_K denote the absolute Frobenius on K induced from the natural Frobenius on the residue field k. We define the absolute Galois group of K to be $G_K = \mathrm{Gal}(\overline{K}/K)$ and we will let I_K denote the inertia subgroup of G_K.

Remark 1.2.1: For a general finite extension $K/\mathbb{Q}_p$, we define $K_0 = W(k)[1/p]$ where $W(k)$ is the ring of Witt vectors over the (perfect) residue field k. It follows that $K_0/\mathbb{Q}_p$

[2] We exclude $p = 2$ in the second part for technical reasons. See the Descent Algorithm 7.2.4.

is the maximal unramified extension of $\mathbb{Q}_p$ contained in K. Since we are taking K to be unramified, then $K_0 = K$ and $\mathscr{O}_K = W(k)$; however, the field K_0 occurs naturally in much of p-adic Hodge theory so we will sometimes write K_0 when it is relevant to the theory with the understanding that we are actually talking about K. We hope this causes no confusion.

Fix a uniformizer $\pi_K = -p$ of K so that the Eisenstein polynomial of π_K in the formal variable u may be denoted $E \coloneqq E(u) = u + p \in \mathscr{O}_K[u]$. We fix once and for all a p-power compatible sequence $\pi = (\pi_0, \pi_1, \pi_2, \dots)$ in $\overline{K}$ such that $\pi_0 = \pi_K$ and $\pi_n^p = \pi_{n-1}$. We define the field K_∞ to be the compositum $K_\infty = \cup_n K(\pi_n)$ contained in $\overline{K}$ and the absolute Galois group of K_∞ is denoted $G_\infty \coloneqq \mathrm{Gal}(\overline{K}/K_\infty)$.

Let $\Lambda \subset K_0[\![u]\!]$ denote the ring of rigid analytic functions on the open p-adic unit disc in K and set $\mathfrak{S} \coloneqq W(k)[\![u]\!] \subset \Lambda$. The ring $K_0[\![u]\!]$ admits a Frobenius action φ which acts on coefficients by the absolute Frobenius σ_K on K and acts on the formal variable u by $\varphi(u) = u^p$. Observe that the rings $\mathfrak{S} \subset \Lambda \subset K_0[\![u]\!]$ are φ-stable under this definition.

Acting as linear coefficients, let F be a finite extension of $\mathbb{Q}_p$ taken large enough to admit an embedding $\tau_0 : K \hookrightarrow F$. Fix a uniformizer ϖ of F and define its ring of integers by $\mathscr{O}_F$ with residue field $k_F \coloneqq \mathscr{O}_F/\varpi\mathscr{O}_F$. Extending the scalars of our rigid analytic functions, let $\mathfrak{S}_F \coloneqq \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathscr{O}_F$ and $\Lambda_F \coloneqq \Lambda \otimes_{\mathbb{Q}_p} F$. Extending the φ-action on $K_0[\![u]\!]$ by F-linearity, we obtain φ-stable rings $\mathfrak{S}_F \subset \Lambda_F \subset (F \otimes_{\mathbb{Q}_p} K_0)[\![u]\!]$.

1.3 Overview of Thesis

This thesis is broken up into two distinct but logically connected parts. In Part I, we describe ‘simple’ representatives for the isomorphism classes of irreducible, two dimensional crystalline F-representations of G_K without restriction on Hodge-Tate weights. Our methods will not be to attack crystalline representations directly but to rather go after a determination of the structure of certain semi-linear algebraic objects known as weakly admissible filtered φ-modules. These two objects turn out to be naturally connected via equivalence functors from *rational* p-adic Hodge theory.

More precisely, suppose V is a d-dimensional crystalline F-representation of G_K. In Chapter 2 we will discuss how to naturally attach to V, a unique weakly admissible filtered φ-module $D \coloneqq \mathbf{D}^*_{\mathrm{cris}}(V)$ over $F \otimes_{\mathbb{Q}_p} K$ and how this attachment gives us a natural correspondence between the isomorphism classes of such objects. Since K is an unramified extension of $\mathbb{Q}_p$ of degree $f > 1$, the structure of D over the tensor $F \otimes_{\mathbb{Q}_p} K$ is complicated and hence, so are the isomorphism classes. To get a hold of this structure, we utilize the f-embeddings of $K \hookrightarrow F$ to decompose $D = \prod D^{(i)}$ into pieces, each being viewed as filtered φ-modules over F, so that we can write down explicit partial Frobenius matrices and filtration structures on each individual piece. In essence, we choose to trade a mysterious single D for f-many $D^{(i)}$, each of which we aim to understand completely.

The weak admissibility property on our filtered φ-modules can be used to further restrict the structure on D and hence, on each $D^{(i)}$. Indeed, we utilize the results of [Zhu09] in Chapter 3 to prove the existence of a strongly divisible lattice L inside D and detail how the isomorphism classes of strongly divisible lattices determine the structure of weakly admissible filtered φ-modules. This strongly divisible structure will descend along the decomposition $D = \prod D^{(i)}$ so that we may write $L = \prod L^{(i)}$ where each $L^{(i)}$ may be viewed as a strongly divisible lattice inside $D^{(i)}$.

We then choose a basis so that the filtration structure is fixed for our strongly divisible lattice L, and hence it will be fixed on D. In this basis, the structure of L is completely determined by a f-tuple of matrices $(A) = (A^{(0)}, A^{(1)}, \ldots, A^{(f-1)}) \in \mathrm{GL}_d(\mathscr{O}_F)^f$ in that we write $L = L(A) = \prod L(A)^{(i)}$ where the partial Frobenius matrix on $L(A)^{(i)}$ in this chosen basis is determined by $A^{(i)} \in \mathrm{GL}_2(\mathscr{O}_F)$. The result is that the isomorphism classes of strongly divisible lattices are determined by the f-tuple $(A^{(i)}) \in \mathrm{GL}_d(\mathscr{O}_F)^f$. Moreover, we give a concrete method to describe such isomorphism classes in terms of parabolic equivalence classes on $\mathrm{GL}_d(\mathscr{O}_F)^f$ via a bijection $\Theta : [(A)] \leftrightarrow L(A)$.

Part I then comes to a head in Chapter 4 where we use, in the dimension two case, a matrix simplifying algorithm inspired by similar techniques in [Liu21] to find a 'nice' representative of $[(A)]$ in $\mathrm{GL}_2(\mathscr{O}_F)^f$ so that each $A^{(i)}$ takes the form of one of two matrix *Types*:

- Type I: $A^{(i)} = \begin{pmatrix} 0 & a_1^{(i)} \\ 1 & a_2^{(i)} \end{pmatrix}$ where $a_1^{(i)} \in \mathscr{O}_F^*$ and $a_2^{(i)} \in \mathscr{O}_F$.
- Type II: $A^{(i)} = \begin{pmatrix} a_1^{(i)} & 0 \\ a_2^{(i)} & 1 \end{pmatrix}$ where $a_1^{(i)} \in \mathscr{O}_F^*$ and $a_2^{(i)} \in \varpi\mathscr{O}_F$.

Since the equivalence class $[(A)]$ determines an isomorphism class of strongly divisible lattices which in turn dictate the structure of weakly admissible filtered φ-modules, then $D(A) := L(A) \otimes_{\mathscr{O}_F} F$ will be a 'simple' representative of its isomorphism class. We write $V(A)$ to be the crystalline representation whose image under $\mathbf{D}^*_{\text{cris}}$ is $D(A)$. The first of our main results is as follows:

Theorem A (Theorem 4.2.3)**:** *Let $V \in \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/F}(G_K)$ be a two dimensional irreducible crystalline F-representation of G_K with labeled Hodge-Tate weights $\mathbf{k} = (0, k_i)_{i \in \mathbb{Z}/f\mathbb{Z}}$ where $k_i > 0$. Then $V \cong V(A)$ where $D(A) = \mathbf{D}^*_{\mathrm{cris}}(V(A))$ and there exists a basis $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ of $D(A) = \prod D(A)^{(i)}$ such that*

$$[\varphi^{(i)}]_\eta = \begin{cases} \begin{pmatrix} 0 & a_1^{(i)} \\ p^{k_{i-1}} & a_2^{(i)} \end{pmatrix} & \text{Type I} \\ \begin{pmatrix} a_1^{(i)} p^{k_{i-1}} & 0 \\ a_2^{(i)} p^{k_{i-1}} & 1 \end{pmatrix} & \text{Type II} \end{cases} \qquad \mathrm{Fil}^j D(A)^{(i)} = \begin{cases} D(A)^{(i)} & j \leq 0 \\ F(\eta_1^{(i)}) & 0 < j \leq k_i \\ 0 & k_i < j \end{cases}$$

In practice, the utility of this theorem is in reducing the number of unrestricted parameters from $4f$ to $2f$ parameters and we additionally are able to place a p-adic valuation condition on the f-many $a_2^{(i)}$ parameters irreducibility.[3]

With an understanding of the structure of $V(A)$ in place, we then move on to Part II where we are tasked with computing the semi-simple modulo ϖ reductions of irreducible, two-dimensional crystalline representations $V(A)$. By this, we mean the computation of k_F-representations $\overline{V(A)} = (T/\varpi T)^{\mathrm{ss}}$ where $T \subset V(A)$ is a G_K-stable $\mathscr{O}_F$-lattice inside of $V(A)$. The study of such a drastic operation lies in the field of *integral* p-adic Hodge theory which is significantly more complicated than the rational theory discussed in much of the first part. The main obstruction turns out to be the determination of suitable integral semi-linear algebraic structures associated to $D(A) := \mathbf{D}^*_{\mathrm{cris}}(V(A))$. Unfortunately, the integral data of strongly divisible lattices do not suffice to describe the reductions in the generality that we desire.

For these reasons, we are incentivized to introduce the notion of Breuil and Kisin modules into our discussion via Chapter 5. In essence, these modules are defined over power series rings and admit Frobenius and filtration structures. Their main utility is in describing the required semi-linear algebraic data that we need to compute reductions. In particular, a Kisin module $\mathfrak{M}$ over $\mathfrak{S}_F$ can be used to compute $\overline{V(A)}$ via the data provided by the modulo ϖ reduction $\overline{\mathfrak{M}}$ *if* the Kisin module $\mathfrak{M}$ is canonically associated to a Galois-stable $\mathscr{O}_F$-lattice $T \subset V(A)$. The issue that persists is that it is very difficult to determine the structure of $\mathfrak{M}$ from $V(A)$. To get around this, we follow results of [BLL23] in Chapter 6 to detail a way to explicitly construct a Kisin module $\mathcal{M}(\mathcal{A})$ over an enlarged coefficient ring S_F from the weakly admissible filtered φ-module $D(A)$. Here, we mean to say that we construct an f-tuple of matrices $(\mathcal{A})$ from (A) so that $(\mathcal{A})$ determines the Frobenius action on $\mathcal{M}(\mathcal{A})$. The largest labeled Hodge-Tate weights $(0, k_i)$ of $D(A)^{(i)}$ then correspond to the labeled heights k_i of the constructed Kisin module $\mathcal{M}(\mathcal{A})^{(i)}$. Since $\mathcal{M}(\mathcal{A})$ is constructed from $D(A)$, then we may use $\mathcal{M}(\mathcal{A})$ to compute the reduction of $V(A)$.

This leaves us with yet another problem. How are we to use the data of $\mathcal{M}(\mathcal{A})$ over S_F to descend the coefficients to an integral Kisin module $\mathfrak{M}(\mathfrak{A})$ over $\mathfrak{S}_F$ so that we can compute the associated reduction? We give an answer in the form of a *descent algorithm* inspired by [Liu21] under some restrictions on height in Chapter 7.

Theorem B (Theorem 7.2.3)**:** *Let $\mathcal{M}(\mathcal{A})$ be a Kisin module over S_F with labeled heights $k_i \leq 2p-4$ constructed from $D(A) = \mathbf{D}^*_{\mathrm{cris}}(V(A))$. Suppose there exists a sequence of determinant preserving base changes (X_n) over $S_F[1/p]$ such that by setting $X_n^{(i)} *_\varphi \mathcal{A}^{(i)} = \mathcal{A}_n^{(i)}$, there is a finite $m > 0$ such that*

$$\mathcal{A}_m^{(i)} = \mathfrak{A}_0^{(i)} + C^{(i)}$$

with $\mathfrak{A}_0^{(i)} \in \mathrm{Mat}_2(\mathfrak{S}_F)$ and $C^{(i)} \in \mathrm{Mat}_2(I)$ where I is a particular ideal in S_F. Then there

[3] When computing reductions, we may twist by unramified characters in order to fix determinants. Hence, up to such characters, the defining parameters becomes the f-many $a_2^{(i)}$ parameters.

exists a Kisin module $\mathfrak{M}(\mathfrak{A})$ canonically associated to a Galois-stable lattice $T \subset V(A)$ such that $\overline{\mathfrak{A}}^{(i)} \equiv \mathfrak{A}_0^{(i)} \pmod{\varpi}$.

Hence, supposing we are able to find such a base change (X_n), then the reduction of $V(A)$ may be computed from $\overline{\mathfrak{M}(\mathfrak{A})}$ as we understand its structure completely. Our entire strategy can be summarized in the following diagram:

$$\begin{tikzcd}
V(A) \arrow[r, "\mathbf{D}^*_{\mathrm{cris}}"] \arrow[d] & D(A) \arrow[r, "\text{Construct}"] & \mathcal{M}(\mathcal{A}) \arrow[d, "\text{Descend}"] \\
T \arrow[rr, leftrightarrow, "\text{Canonical}"] \arrow[d] & & \mathfrak{M}(\mathfrak{A}) \arrow[d, "\bmod \varpi"] \\
\overline{V(A)} & & \overline{\mathfrak{M}(\mathfrak{A})} \arrow[ll, "\text{Compute}"']
\end{tikzcd}$$

Chapter 8 then forms the technical heart of this thesis as we embark on the daunting task of explicitly constructing the required base change (X_n) under the following technical restriction on the labeled Hodge-Tate weights:

$$p+2 \leq k_0 \leq 2p-4 \qquad\qquad 2 \leq k_i \leq p-3 \text{ for all } 1 \leq i \leq f-1$$

This quest involves exceedingly delicate computations and estimations to ensure that the power series elements of our matrices lie either in $\mathfrak{S}_F$ or the ideal $I \subset S_F$. It quickly becomes evident that the sequence, and hence the shape of the reduction, depends on the p-adic valuations of the $a_2^{(i)}$ parameters arising from the f-tuple $(A^{(i)})$ and the Type combination of $D(A)$. Our methods are not foolproof however and indeed, some cases are out of reach using our strategy for technical reasons.[4] When we are able to use our methods, we explicitly compute reductions in Chapter 9 in the following cases:

- $\nu_p(a_2^{(0)}) \geq 1$;
- $\nu_p(a_2^{(0)}) < 1$ and # Type I is even if we assume $\nu_p(a_2^{(i)}) \geq 1 - \nu_p(a_2^{(f-1)})$ for select $i \in \mathbb{Z}/f\mathbb{Z}$;
- $\nu_p(a_2^{(0)}) < 1$ and # Type I is odd if we assume $\nu_p(a_2^{(i)}) \geq 1$ for select $i \in \mathbb{Z}/f\mathbb{Z}$;
- In all cases for $f = 2$.

The reader with a willingness to trust will find a summary of reductions for the cases listed above in Section 9.5.

As a bonus, we also include Appendix A which covers a generalization of our methods where we increase the number of labeled Hodge-Tate weights that are allowed to be greater than p. This process requires even stricter conditions on the p-adic valuations of the $a_2^{(i)}$ parameters. The cherry on top is a display of reductions in $f = 2$ for $p+2 \leq k_0, k_1 \leq 2p-4$ if we assume that both parameters have p-adic valuation at least one.

[4]By 'out of reach', we only mean that the computations necessarily are not feasible. When $f = 2$, the *complete* theory breaks up into twelve subcases.

Part I

Models for Irreducible Crystalline Representations

Chapter 2

Preliminaries of p-adic Hodge Theory

We begin with a rapid overview of the relevant topics from *rational* p-adic Hodge theory that we will require. Pioneered by Fontaine in the late 20th century, much of the information contained in this chapter is considered to be classical for the purposes of this thesis. As a result, many proofs are given as references and not stated in full. For a more detailed introduction, the reader can look to any of [BC09; Ber04a; FO]; all of which were invaluable to the author when first learning this material.

Section 2.1 defines our main objects of study, continuous p-adic Galois representations and their semisimple reductions. Section 2.2 gives a rather technical construction of Fontaine's *period rings* with the aim of motivating the presence of various structures that will be omnipresent throughout this thesis. Section 2.3 then defines the categories of crystalline representations and weakly admissible filtered φ-modules before detailing how a foundational theorem of Colmez and Fontaine allows us to naturally associated objects in these two categories via equivalence functors. Finally, Section 2.4 gives us the framework to extend linear coefficients in a concrete manner that further generalizes our results and gives us an explicit way to write down various structures.

2.1 Galois Representations

A *p-adic representation* of G_K is a finite dimensional $\mathbb{Q}_p$-vector space equipped with a linear and continuous action of G_K. Equivalently, one may view such a representation as being a group action $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ and upon choosing a basis for V, it can be interpreted as a map $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$. We denote the category of p-adic representations of G_K by $\mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$ with morphisms being linear, G_K-compatible maps. We make the important observation that this category contains all dual representations V^* where we take V^* to be the typical $\mathbb{Q}_p$-linear dual.

The study of the category $\mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$ is the domain of *rational* p-adic Hodge theory for which this preliminary chapter aims to partially introduce. However, the focus of this thesis is on *integral* p-adic Hodge theory which studies Galois-stable integral lattices inside p-adic representations. The existence of such objects relies on a fundamental result which is now convenient to state.

Proposition 2.1.1: *For any $V \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$, there exists a G_K-stable $\mathbb{Z}_p$-lattice $T \subset V$ such that $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

Proof. For any $\mathbb{Z}_p$-lattice $T_0 \subset V$ we have $V = T_0 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ by definition so that $\mathrm{Aut}_{\mathbb{Z}_p}(T_0)$ is naturally an open subgroup of $\mathrm{Aut}_{\mathbb{Q}_p}(V)$. Let $\rho : G_K \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ be the group action map associated to V so that $H := \rho^{-1}(\mathrm{Aut}_{\mathbb{Z}_p}(T_0))$ is an open subgroup of G_K. Since G_K is compact and H is open, then H has finite index so the quotient G_K/H has a finite number of coset representatives $\{g_i\}$. Define $T := \sum_i \rho(g_i)T_0$ so that T is a $\mathbb{Z}_p$-lattice inside V and is G_K-stable as T_0 is H-stable. Since $G_K = \coprod_i g_i H$ then the result follows as T is guaranteed to be G_K-stable. $\square$

Any lattice $T \subset V$ arising from Proposition 2.1.1 is easily seen to be a $\mathbb{Z}_p$-representation of G_K and we denote the category of such objects by $\mathrm{Rep}_{/\mathbb{Z}_p}(G_K)$. For any $T \in \mathrm{Rep}_{/\mathbb{Z}_p}(G_K)$, restricting scalars $T \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ gives a $\mathbb{F}_p$-representation of G_K and an object in the category $\mathrm{Rep}_{/\mathbb{F}_p}(G_K)$. The act of computing $T \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ from a p-adic representation V is one of the primary focuses of this thesis, called *reduction modulo p* for some $T \subset V$. However, note that Proposition 2.1.1 makes no claim that the lattice T is unique, and hence, this computation may not be well defined. This problem is solved by the process of semi-simplification.

For any $\mathbb{F}_p$-representation $W \in \mathrm{Rep}_{/\mathbb{F}_p}(G_K)$, we denote the *semi-simplification* of W to be the direct sum $W^{\mathrm{ss}} := \oplus W_i/W_{i-1}$ where the W_i makeup the composition series of W as a $\mathbb{F}_p[G_K]$-module.

Proposition 2.1.2: *Let $V \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$ and suppose $T, T' \subset V$ are distinct lattices as described in Proposition 2.1.1. Then there is an isomorphism in $\mathrm{Rep}_{/\mathbb{F}_p}(G_K)$,*

$$(T \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{ss}} \cong (T' \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{ss}}.$$

Proof. Denote the $\mathbb{F}_p$-representations by $\overline{\rho} : G_K \to \mathrm{GL}_n(T \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ and $\overline{\rho'} : G_K \to \mathrm{GL}_n(T' \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ arising as modulo p reductions of $(V, \rho) \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$. Each have finite image since G_K is compact and $\mathbb{F}_p$ is finite. Thus $\overline{\rho}$ factors through a finite quotient $\mathrm{Gal}(L/K)$ for some finite extension $L \subset \overline{\mathbb{Q}_p}$ and $\overline{\rho'}$ factors through $\mathrm{Gal}(L'/K)$ for some finite extension L'. We may therefore view $\overline{\rho} : \mathrm{Gal}(L/K) \to \mathrm{GL}_n(T \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$ so that $\overline{\rho}$ is a $\mathbb{F}_p[\mathrm{Gal}(L/K)]$-module and similarly view $\overline{\rho'}$ as a $\mathbb{F}_p[\mathrm{Gal}(L'/K)]$-module.

The fact that the modulo p reduction of a characteristic polynomial is the same as the characteristic polynomial of the reduction implies that $\mathrm{char}(\overline{\rho}(g)) = \mathrm{char}(\overline{\rho}'(g))$ for all $g \in \mathrm{Gal}(L \cup L'/K)$. An application of the Brauer-Nesbitt Theorem, see [CR62], implies that upon factoring through the finite group $\mathrm{Gal}(L \cup L'/K)$, both $\overline{\rho}$ and $\overline{\rho}'$ will have the same composition series so that $(\overline{\rho})^{\mathrm{ss}} \cong (\overline{\rho'})^{\mathrm{ss}}$. □

Definition 2.1.3: For any $V \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$, we define the *semi-simple reduction modulo* p to be

$$\overline{V} := (T \otimes_{\mathbb{Z}_p} \mathbb{F}_p)^{\mathrm{ss}}$$

for any choice of G_K-stable lattice $T \subset V$.

This definition is independent of the choice of T by Proposition 2.1.2 and hence, it is unique to V. We will commonly abuse language and drop the words 'semi-simple' when referring to reductions of representations.

2.2 Fontaine's Period Rings

Our goal is to define a plethora of rings which will be used to construct so called *period rings* which are used to define the various categories we will encounter, including the category of crystalline representations. Much of the theory was developed by Fontaine in [Fon91; Fon94a; Fon94b] and then formulated into the language we present here by Colmez in [Col99] which we recommend the reader look to for more details.

Define the set

$$\widetilde{\mathbf{E}} := \mathbb{C}_p^\flat = \varprojlim_{x \mapsto x^p} \mathbb{C}_p = \left\{ (x^{(0)}, x^{(1)}, \dots) : (x^{(i+1)})^p = x^{(i)} \right\}$$

with ring operations given by

$$(x+y)^{(i)} = \lim_{j \to \infty} \left(x^{(i+j)} + y^{(i+j)} \right)^{p^j} \qquad\qquad (xy)^{(i)} = x^{(i)} y^{(i)}.$$

The ring $\widetilde{\mathbf{E}}$ is an algebraically closed field of characteristic p which is complete relative to the valuation defined by $\nu_{\mathbf{E}}(x) = \nu_p(x^{(0)})$. The field $\widetilde{\mathbf{E}}$ admits a natural Frobenius action and an action of $G_{\mathbb{Q}_p}$, both of which are given component wise. We define the additional rings $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$ where $W(\cdot)$ denotes the ring of Witt vectors over a perfect field. $\widetilde{\mathbf{B}}$ may then be interpreted as a complete discrete valuation field with ring of integers $\widetilde{\mathbf{A}}$ and residue field $\widetilde{\mathbf{E}}$.

To ensure that the $G_{\mathbb{Q}_p}$-action induced by $\widetilde{\mathbf{E}}$ on both $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ remains continuous, we must specify a weaker topology than the one induced by the valuation $\nu_{\mathbf{E}}$. In particular, we endow $\widetilde{\mathbf{A}}$ with the topology making the following map a homeomorphism

$$\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}}) \to \prod_{n \geq 0} \widetilde{\mathbf{E}} \qquad\qquad x \mapsto (x_i)_{i \geq 0}$$

where the right hand side has the usual product topology. We also give $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$ the topology of the inductive limit. With this identification, the Frobenius and Galois actions on $\widetilde{\mathbf{E}}$ are then continuously extended to actions on both $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$.

Let us fix a sequence of primitive p^n-th roots of unity $(\mu_p, \mu_{p^2}, \dots)$ and define the element $\varepsilon := (1, \mu_p, \mu_{p^2}, \dots) \in \widetilde{\mathbf{E}}$. Let $[\varepsilon]$ denote the unique Teichmüller representative of ε in $\widetilde{\mathbf{A}}$ so that the element $\pi = [\varepsilon] - 1$ lies in $\widetilde{\mathbf{A}}$. Recalling that $\widetilde{\mathbf{E}}$ is endowed with the valuation $\nu_{\mathbf{E}}(x) = \nu_p(x^{(0)})$, let $\widetilde{\mathbf{E}}^+$ denote the ring of integers of $\widetilde{\mathbf{E}}$, i.e. one has that

$$\widetilde{\mathbf{E}}^+ := \mathscr{O}_{\mathbb{C}_p}^\flat = \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbb{C}_p} \cong \varprojlim_{x \mapsto x^p} \mathscr{O}_{\mathbb{C}_p}/(p).$$

There is a natural homomorphism $\theta : \widetilde{\mathbf{E}}^+ \to \mathscr{O}_{\mathbb{C}_p}/(p)$ which sends $(x^{(i)}) \mapsto x^{(0)}$. Setting $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ means that θ lifts to a homomorphism $\theta : \widetilde{\mathbf{A}}^+ \to \mathscr{O}_{\mathbb{C}_p}$ and by setting $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] = \{\sum_{k \gg -\infty} [x_k] p^k : x_k \in \widetilde{\mathbf{E}}^+\}$, one can then lift θ to a map

$$\theta : \widetilde{\mathbf{B}}^+ \to \mathbb{C}_p \qquad\qquad \sum [x_k] p^k \mapsto \sum x_k^{(0)} p^k.$$

This map is surjective with principal kernel generated by $\omega = \pi/\varphi^{-1}(\pi)$.

We define the ring $\mathbf{B}_{\mathrm{dR}}^+$ to be the completion of $\widetilde{\mathbf{B}}^+$ in the $(\ker\theta)$-adic topology, i.e. we write

$$\mathbf{B}_{\mathrm{dR}}^+ := \varprojlim_n \frac{\widetilde{\mathbf{B}}^+}{(\ker\theta)^n} = \left\{ \sum_{n=0}^{\infty} x_n \omega^n : x_n \in \widetilde{\mathbf{B}}^+ \right\}.$$

One can observe that $\theta(-\pi) = 0$ so in the topology of $\mathbf{B}_{\mathrm{dR}}^+$, the element $t = \log([\varepsilon])$ exists. In fact, one can show that the element t is a uniformizer of $\mathbf{B}_{\mathrm{dR}}^+$ with Galois action given by $g(t) = \chi(g)t$ for any $g \in G_{\mathbb{Q}_p}$ and χ denoting the cyclotomic character. One then defines the *de Rham period ring* to be $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$ with a natural filtration structure $\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} = t^i \mathbf{B}_{\mathrm{dR}}^+$ and Galois invariants $B_{\mathrm{dR}}^{G_K} = K$.

Finally, let us define a divided power envelope

$$\mathbf{A}_{\mathrm{cris}} := \left\{ \sum_{n=0}^{\infty} x_n \frac{\omega^n}{n!} : x_n \in \widetilde{\mathbf{A}}^+, x_n \to 0 \right\}.$$

One can show that $t \in \mathbf{A}_{\mathrm{cris}}$ with an embedding $\mathbf{A}_{\mathrm{cris}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}^+$ so we may define the ring $\mathbf{B}_{\mathrm{cris}}^+ = \mathbf{A}_{\mathrm{cris}}[1/p] \subset \mathbf{B}_{\mathrm{dR}}^+$. This allows us to define the *crystalline period ring* to be $\mathbf{B}_{\mathrm{cris}} := \mathbf{B}_{\mathrm{cris}}^+[1/t]$. We have that $\mathbf{B}_{\mathrm{cris}}$ admits a Galois action extending to the one on $\mathbf{B}_{\mathrm{dR}}$ and a Frobenius φ-action lifted from the natural Frobenius on $\widetilde{\mathbf{B}}^+$. Moreover, we have the association $\mathbf{B}_{\mathrm{cris}}^{G_K} = K_0$ and a filtration induced by $\mathbf{B}_{\mathrm{dR}}$ given by the intersection $\mathrm{Fil}^i \mathbf{B}_{\mathrm{cris}} = \mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} \cap \mathbf{B}_{\mathrm{cris}}$.

2.3 Filtered φ-Modules and Crystalline Representations

For any $V \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$, define

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Since $B_{\mathrm{cris}}^{G_K} = K_0$ then $\mathbf{D}_{\mathrm{cris}}(V)$ is a K_0-vector space which admits a Frobenius φ-action and filtration structure induced by those same structures on $\mathbf{B}_{\mathrm{cris}}$ as discussed in Section 2.2. Such objects are called *filtered φ-modules* which we now define.

Definition 2.3.1: A *filtered φ-module* over K is a finite dimensional K_0-vector space D equipped with a φ-semilinear endomorphism $\varphi_D : D \to D$ and a K-valued filtration which is separated and exhaustive, i.e. $\bigcup_i \mathrm{Fil}^i(D \otimes_{K_0} K) = D \otimes_{K_0} K$ and $\bigcap_i \mathrm{Fil}^i(D \otimes_{K_0} K) = 0$.

Note that by Remark 1.2.1, a filtered φ-module over unramified K will be a K-vector space so the filtration and underlying vector space are valued over the same field. For completeness, we will continue to write K_0 with the understanding that under our assumptions, $K = K_0$.

We denote the category consisting of such objects by MF_K^{φ} with morphisms being linear maps of filtered vector spaces which are compatible with the φ-semilinear endomorphisms. In particular, we assume that any morphism $h : D \to D'$ in the category is *strict* in the sense that $h(\mathrm{Fil}^j D) = h(D) \cap \mathrm{Fil}^j D'$.

In general, the Frobenius and filtration structures on an arbitrary filtered φ-module need not be related to one another. The class of filtered φ-modules for which these structures are connected is quite special and in order to quantify how connected the two structures are, we need to encode their data in the form of two polygons.

Let us begin by encoding information about the filtration structures. For any filtered φ-module $D \in \mathrm{MF}_K^{\varphi}$ of finite dimension d, consider the d-tuple $\mathbf{k} := (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ where $k_d \geq \cdots \geq k_1$ possibly repeating and each k_j is such that $\mathrm{Fil}^{k_j} D \supset \mathrm{Fil}^{k_j+1} D$ or in other words, the grading of D at k_i is nonzero. We define the *multiplicity* of each k_j to be the value $r_j := \dim_K(\mathrm{Fil}^{k_j} D / \mathrm{Fil}^{k_j+1} D)$. With this, we define the *Hodge Polygon* of D, denoted $\mathrm{HP}(D)$, to be the convex polygon with left most endpoint $(0, 0)$ and each slope k_j having horizontal length $r_j - r_{j-1}$. Since the d-tuple $\mathbf{k}$ encodes all information about the polygon, we will commonly write that $\mathrm{HP}(D) = \mathbf{k}$.

Switching to encoding information about the Frobenius structures, for any reduced rational number $\alpha = r/s \in \mathbb{Q}$, let D_α denote the K_0-subvector space generated by elements $w \in D$ such that $\varphi_D^s(w) = p^r w$. Such α are known to exist by the Dieudonné–Manin classification, see [Man63, p. II.4.1], and descend uniquely to our situation by [BC09, Lemma 8.1.11]. We then have a decomposition $D = \oplus_a D_a$ which will give a list $\mathbf{a} := (a_1, \ldots, a_d) \in \mathbb{Q}^d$ where each a_j has multiplicity $s_j := \dim_{K_0} D_{a_j}$. With this, we define the *Newton Polygon* of D, denoted $\mathrm{NP}(D)$, to be the convex polygon with left most endpoint $(0, 0)$ and each slope a_j having horizontal length $s_j - s_{j-1}$. Since the d-tuple $\mathbf{a}$ encodes all the information about this polygon, we will often write $\mathrm{NP}(D) = \mathbf{a}$.

Definition 2.3.2: A filtered φ-module $D \in \mathrm{MF}_K^{\varphi}$ is said to be *weakly admissible* if both of the following conditions hold:

- The endpoints of the polygons $\mathrm{HP}(D)$ and $\mathrm{NP}(D)$ agree;
- For every nontrivial subobject $D' \subset D$ in the category MF_K^{φ}, we have that the Hodge polygon $\mathrm{HP}(D')$ lies below the Newton polygon $\mathrm{NP}(D')$.

We denote the proper subcategory of weakly admissible filtered φ-modules over K by $\mathrm{MF}_K^{\varphi,w.a.}$. The representations which are associated to such weakly admissible modules via $\mathbf{D}_{\mathrm{cris}}$ are the prime focus of this thesis and in order to study them, we must take a more functorial approach.

From the above discussion, we may interpret the map $\mathbf{D}_{\mathrm{cris}}$ as a functor

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}_{/\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi} \qquad\qquad V \mapsto (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Definition 2.3.3: We say that a p-adic representation $V \in \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$ is *crystalline* if we have an equality of dimensions

$$\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(\mathbf{D}_{\mathrm{cris}}(V)).$$

We denote the proper subcategory of crystalline representations by $\mathrm{Rep}_{\mathrm{cris}/\mathbb{Q}_p}(G_K)$. Such representations are said to be $\mathbf{B}_{\mathrm{cris}}$*-admissible* and one can show that a representation V being admissible implies that $\mathbf{D}_{\mathrm{cris}}(V)$ is weakly admissible. The converse was conjectured by Fontaine and proven by Colmez and Fontaine. The result is a theorem which will serve as the foundation of much of what we do in this thesis.

Theorem 2.3.4: *There is an equivalence of categories*

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathrm{cris}/\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi,w.a.}$$

with quasi-inverse functor given by $\mathbf{V}_{\mathrm{cris}}(D) = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{cris}} \otimes_{K_0} D)^{\varphi \otimes \varphi_D = 1}$.

Proof. See [CF00, Théorème A]. □

We note that the above equivalence respects the notion of duality; in other words, we have $\mathbf{D}_{\mathrm{cris}}(V^*) = (\mathbf{D}_{\mathrm{cris}}(V))^* =: \mathbf{D}^*_{\mathrm{cris}}(V)$ and similarly for $\mathbf{V}_{\mathrm{cris}}$ so that

$$\mathbf{D}^*_{\mathrm{cris}}(\cdot) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(\cdot, \mathbf{B}_{\mathrm{cris}}) \qquad\qquad \mathbf{V}^*_{\mathrm{cris}}(\cdot) = \mathrm{Hom}_{\varphi,\mathrm{Fil}}(\cdot, \mathbf{B}_{\mathrm{cris}}).$$

Since the functors $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{V}_{\mathrm{cris}}$ are both covariant, it is easy to see that the dual functors will be contravariant. In fact, it is often convenient for the purposes of computing reductions that we use these contravariant functors, see Section 9.1.

For any $V \in \mathrm{Rep}_{\mathrm{cris}/\mathbb{Q}_p}(G_K)$, denote the Hodge polygon of $\mathbf{D}_{\mathrm{cris}}(V)$ by $\mathrm{HP}(\mathbf{D}_{\mathrm{cris}}(V)) = \mathbf{k}$. We call each $k_j \in \mathbf{k}$ a *Hodge-Tate weight* of V and for this thesis, we will always assume that each $k_j \geq 0$. Filtered φ-modules with positive Hodge-Tate weights are said to be *effective.*

By the above definitions, it is easy to see that the Hodge-Tate weight data $\mathbf{k}$ associated to a crystalline representation V is equivalent to the Hodge polygon data of $\mathbf{D}_{\mathrm{cris}}(V)$. For this reason, we may restrict the equivalence of categories of Theorem 2.3.4 to crystalline representations with prescribed Hodge-Tate weight data $\mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/\mathbb{Q}_p}(G_K)$ and weakly admissible filtered φ-modules with the same Hodge polygon data $\mathrm{MF}_K^{\varphi,w.a.,\mathbf{k}}$, denoted

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi,w.a.,\mathbf{k}}.$$

Remark 2.3.5: While this thesis focuses on crystalline representations, certain parts of the theory, see Chapter 5, are more naturally stated in terms of *semistable representations.* At this level, we have an equivalence $\mathbf{D}_{\mathrm{st}} : \mathrm{Rep}_{\mathrm{st}/\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\varphi,N,w.a.}$ with quasi-inverse $\mathbf{V}_{\mathrm{st}}$. Here we are taking $\mathrm{MF}_K^{\varphi,N,w.a.}$ to denote the category of weakly admissible filtered φ-modules admitting a K_0-linear monodromy operator N such that $N \circ \varphi = p\varphi \circ N$. Intuitively, one should think of crystalline representations as being semistable representations with zero monodromy. See [BC09, §9.2, 9.3] for a more detailed discussion.

2.4 Extending Coefficients

To further generalize the results of this thesis, we introduce coefficients in the form of a finite extension F to the various representations and modules we have introduced up to this point. When $K = \mathbb{Q}_p$, this process is relatively trivial as we need only extend the above notions via F-linearity over the tensor $F \otimes_{\mathbb{Q}_p} K$. However, the situation becomes more complicated when $K = \mathbb{Q}_{p^f}$ for $f \geq 2$ in a manner which we now describe.

Recall from Section 1.2 that we have defined F to be a finite extension of $\mathbb{Q}_p$ which admits an embedding $\tau_0 : K \hookrightarrow F$. Denote its ring of integers by $\mathscr{O}_F$ with uniformizer ϖ and residue field $k_F = \mathscr{O}_F/\varpi\mathscr{O}_F$. Define the composition $\tau_i := \tau_0 \circ \sigma_K^{-i} : K \hookrightarrow F$ so that the set $|\tau| := \{\tau_i\}$ for $0 \leq i \leq f-1$ gives a list of f-distinct embeddings of K into F. As we have fixed this sequence of embeddings, we will identify the sets $\mathbb{Z}/f\mathbb{Z} \cong \mathrm{Gal}(K/\mathbb{Q}_p)$ and reserve the index $i \in \mathbb{Z}/f\mathbb{Z}$ to denote the enumeration of this list of embeddings.

Define the set

$$F^{|\tau|} := \prod_{\tau_i : K \hookrightarrow F} F \quad \text{with elements} \quad \sum_{x \in F, y \in K} (x\tau_0(y), x\tau_1(y), \ldots, x\tau_{f-1}(y)).$$

The product $F^{|\tau|}$ is a ring with multiplication induced by component wise multiplication on the principle elements given by $(x\tau_i(y))_i \cdot (x'\tau_i(y'))_i := (xx'\tau_i(yy'))_i$. This gives a natural ring isomorphism

$$\theta : F \otimes_{\mathbb{Q}_p} K \to F^{|\tau|} \qquad\qquad x \otimes y \mapsto (x\tau_i(y))_{i\in\mathbb{Z}/f\mathbb{Z}}.$$

The automorphism $1_F \otimes \sigma_K$ on $F \otimes_{\mathbb{Q}_p} K$ transforms via θ to an automorphism $\varphi : F^{|\tau|} \to F^{|\tau|}$ defined by $\varphi(v_0, v_1, v_2, \dots, v_{f-1}) = (v_1, v_2, \dots, v_{f-1}, v_0)$. We endow $F^{|\tau|}$ with an $F \otimes_{\mathbb{Q}_p} K$-module structure defined by $(a \otimes b)(v_0, \dots, v_{f-1}) = (a\tau_0(b)v_0, \dots, a\tau_{f-1}(b)v_{f-1})$.

Lemma 2.4.1: *The ring isomorphism $\theta : F \otimes_{\mathbb{Q}_p} K \to F^{|\tau|}$ defines a φ-equivariant $F \otimes_{\mathbb{Q}_p} K$-module isomorphism.*

Proof. This map is clearly a module homomorphism. The embeddings τ_i are linearly independent over F and since $F^{|\tau|}$ can be viewed as an f-dimensional F-vector space, then the f-distinct embeddings τ_i will define a maximal linearly independent list. Thus the map θ is an isomorphism of modules. Since the Frobenius action φ on $F^{|\tau|}$ is defined by $1_F \otimes \sigma_K$, then it is clear that $\varphi \circ \theta = \theta \circ (1_F \otimes \sigma_K)$ giving φ-compatibility. □

Define $\mathfrak{e}_i := (0, \dots, 0, 1, 0, \dots, 0)$ to be the idempotent of φ in $F^{|\tau|}$ where the 1 appears in the i-th component of the vector. Set $F^{(i)} := \mathfrak{e}_i F$ and observe that each $F^{(i)}$ is isomorphic to F as F-modules. It follows by Lemma 2.4.1 that we may decompose $F \otimes_{\mathbb{Q}_p} K$ in terms of these idempotents

$$F \otimes_{\mathbb{Q}_p} K \cong \prod_{i\in\mathbb{Z}/f\mathbb{Z}} F^{(i)}$$

where $\varphi := (\varphi^{(0)}, \dots, \varphi^{(f-1)})$ with $\varphi^{(i)} : \varphi^* F^{(i-1)} \to F^{(i)}$, the *i-th partial Frobenius*[5]

Let $D \in \mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p}K}$ be a filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$. Recall that technically speaking, D is a $F \otimes_{\mathbb{Q}_p} K_0$-vector space but by Remark 1.2.1, $K_0 = K$ so we are able to make the following observation. Define $D^{(i)} := \mathfrak{e}_i D$ so that the above discussion implies that $\theta : D \to \prod_i D^{(i)}$ is an isomorphism of $F \otimes_{\mathbb{Q}_p} K$-modules and the Frobenius action on $\prod_i D^{(i)}$ acts via $\varphi_D := (\varphi_D^{(0)}, \dots, \varphi_D^{(f-1)})$ with i-th partial Frobenius $\varphi_D^{(i)} : \varphi^* D^{(i-1)} \to D^{(i)}$.

Proposition 2.4.2: *There is an isomorphism of filtered φ-modules*

$$D \cong \prod_{i\in\mathbb{Z}/f\mathbb{Z}} D^{(i)}$$

so that $\mathrm{Fil}^j D^{(i)} = \mathfrak{e}_i \mathrm{Fil}^j D$.

Proof. The fact that θ is φ_D-compatible follows from the fact that

$$\varphi_D \left(x\tau_{i+1}(y)v\right)_{i\in\mathbb{Z}/f\mathbb{Z}} = \left(x\tau_i(y)\varphi_D^{(i)}(v)\right)_{i\in\mathbb{Z}/f\mathbb{Z}}$$

[5]The fact that partial Frobenius shifts embeddings will play a large part in (complicating) future computations.

for $x \otimes y \in F \otimes_{\mathbb{Q}_p} K$ and $v \in D^{(i+1)}$. Furthermore, we induce a filtration structure on each idempotent piece by taking $\mathrm{Fil}^j D^{(i)} := \mathrm{Fil}^j D \cap D^{(i)} = \mathfrak{e}_i \mathrm{Fil}^j D$. This gives us that θ is compatible with both the Frobenius and filtration structures so we get a morphism of filtered φ-modules. □

With this, we may pass between a filtered φ-module D and its decomposition $\prod_i D^{(i)}$ which we do so freely by the results of Proposition 2.4.2 and we can view each piece $D^{(i)}$ as a filtered φ-module over F via the isomorphism of $F^{(i)}$ and F.[6] A morphism in the category $\mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p} K}$ can then be interpreted as an f-tuple $h := (h^{(0)}, \ldots, h^{(f-1)})$ where each $h^{(i)} : D^{(i)} \to D'^{(i)}$ is a morphism viewed in the category of filtered φ-modules over F respecting the idempotent decomposition.

We define the *Hodge polygon* of $D \in \mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p} K}$ to be $\mathrm{HP}(D) = \mathbf{k} = (\mathbf{k}^{(0)}, \ldots, \mathbf{k}^{(f-1)})$ where each $\mathbf{k}^{(i)}$ is the Hodge polygon associated to the idempotent piece $D^{(i)}$ with filtration as described above so we may write $\mathrm{HP}(D^{(i)}) = \mathbf{k}^{(i)} = (k_{i1}, \ldots, k_{id})$. The Newton polygon is defined similarly but it is not of great importance to this thesis so we will not give the details. The proper subcategory of filtered φ-modules over $F \otimes_{\mathbb{Q}_p} K$ with prescribed Hodge-polygon data will be denoted $\mathrm{MF}^{\varphi,\mathbf{k}}_{F\otimes_{\mathbb{Q}_p} K}$.

There is a natural forgetful functor $\mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p} K} \to \mathrm{MF}^{\varphi}_{K}$ which forgets the F-module structure. We say that a filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$ is *weakly admissible* if it is weakly admissible over K after forgetting the underlying F-module structure. We denote the proper subcategory of such weakly admissible filtered φ-modules by $\mathrm{MF}^{\varphi,w.a.}_{F\otimes_{\mathbb{Q}_p} K}$.

Switching once more to Galois representations, an *F-representation* of G_K is a finite dimensional F-vector space equipped with a linear and continuous action of G_K. We denote the category of F-representations of G_K by $\mathrm{Rep}_{/F}(G_K)$. The results of Section 2.1 carry over to the case of F-representations by F-linearity so we receive the categories of $\mathscr{O}_F$-representations, denoted $\mathrm{Rep}_{/\mathscr{O}_F}(G_K)$ and k_F-representations, denoted $\mathrm{Rep}_{/k_F}(G_K)$, for free. Note that reduction modulo p as in Definition 2.1.3 becomes *reduction modulo* ϖ given by

$$\overline{V} := (T \otimes_{\mathscr{O}_F} k_F)^{\mathrm{ss}}$$

for any G_K-stable $\mathscr{O}_F$-lattice $T \subset V$.

There is a natural forgetful functor $\mathrm{Rep}_{/F}(G_K) \to \mathrm{Rep}_{/\mathbb{Q}_p}(G_K)$ defined by forgetting the F-vector space structure. We say that an F-representation $V \in \mathrm{Rep}_{/F}(G_K)$ is *crystalline* if it is crystalline as a $\mathbb{Q}_p$-representation after forgetting the underlying F structure. We denote the category of crystalline F-representations of G_K by $\mathrm{Rep}_{\mathrm{cris}/F}(G_K)$. This allows us to extend the results of the Colmez-Fontaine Theorem 2.3.4 to the setting of nontrivial coefficients by the work of Breuil and Mézard.

[6] We use 'view' loosely here. Indeed, each $D^{(i)}$ will have its own filtration structure but the partial Frobenius are really maps between $D^{(i-1)}$ and $D^{(i)}$.

Theorem 2.4.3: *There is an equivalence of categories*

$$\mathbf{D}_{\text{cris}} : \text{Rep}_{\text{cris}/F}(G_K) \to \text{MF}^{\varphi,w.a.}_{F\otimes_{\mathbb{Q}_p}K}$$

with quasi-inverse functor denoted by $\mathbf{V}_{\text{cris}}$.

Proof. See [BM02, § 3]. □

By the above discussion, we may denote the Hodge polygon data of the filtered φ-module $\mathbf{D}_{\text{cris}}(V)$ by $\text{HP}(\mathbf{D}_{\text{cris}}(V)) = \mathbf{k} = (\mathbf{k}^{(i)})$ a f-tuple where each $\mathbf{k}^{(i)} = \text{HP}(\mathbf{D}_{\text{cris}}(V)^{(i)})$ for $i \in \mathbb{Z}/f\mathbb{Z}$. We call the d-tuple $\mathbf{k}^{(i)} = (k_{i1}, \dots, k_{id})$ the *labeled Hodge-Tate weights* of V with respect to the embedding τ_i. If we have that $k_{i(j+1)} > k_{ij}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and $1 \leq j < d$ then we say that the labeled Hodge-Tate weights are *regular*. As the labeled Hodge-Tate weight data $\mathbf{k}$ attached to V is equivalent to the Hodge polygon data attached to $\mathbf{D}_{\text{cris}}(V)$ and we get a restriction of the equivalence of Theorem 2.4.3 to prescribed Hodge-Tate weights given by

$$\mathbf{D}_{\text{cris}} : \text{Rep}^{\mathbf{k}}_{\text{cris}/F}(G_K) \to \text{MF}^{\varphi,w.a.,\mathbf{k}}_{F\otimes_{\mathbb{Q}_p}K}.$$

Chapter 3

Strongly Divisible Lattices

We are now ready to begin discussing *integral* p-adic Hodge theory. Restricting to the crystalline case, as is understood in this thesis, recall that for any $V \in \mathrm{Rep}_{\mathrm{cris}/F}(G_K)$ we have a naturally associated weakly admissible filtered φ-module $\mathbf{D}_{\mathrm{cris}}(V)$. If T is a G_K-stable $\mathscr{O}_F$-lattice inside V, then integral p-adic Hodge theory aims to determine a corresponding integral submodule associated in some way to $\mathbf{D}_{\mathrm{cris}}(V)$. The complication that arises is that the functor $\mathbf{D}_{\mathrm{cris}}$ does not naturally descend to the integral case as the crystalline period ring $\mathbf{B}_{\mathrm{cris}}$ is not compatible with the integral structure of T. As a result, the integral theory requires more complicated methods to define the functors and categories associated to such lattice representations. One such theory developed by Fontaine and Laffaille concerns so-called *strongly divisible lattices* inside weakly admissible filtered φ-modules which we cover here. We note that much of this chapter can be viewed as a survey of relevant results from [Zhu09, §3].

Section 3.1 defines the notion of an integral submodule inside of a filtered φ-module that is compatible with the already present Frobenius and filtration structures. When we further enforce weak admissibility, the structure on the integral submodule is further restricted giving rise to something called a strongly divisible lattice. Section 3.2 then gives a concrete classification of the isomorphism classes of such strongly divisible lattices which can be used to determine the structure of the associated weakly admissible filtered φ-modules and, by association, crystalline representations.

3.1 Lattices Inside Filtered φ-Modules

Let $D \in \mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p}K}$ be a filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$. A *$\varphi$-lattice* L inside D is an $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$-module L with Frobenius action φ_L such that

$$L \otimes_{\mathscr{O}_F\otimes_{\mathbb{Z}_p}\mathscr{O}_K} (F \otimes_{\mathbb{Q}_p} K) = D \quad \text{and} \quad \varphi_L(L) \subseteq L.$$

We say that L has an *induced filtration* from D if L admits a filtration structure such that

$$\mathrm{Fil}^j L = \mathrm{Fil}^j D \cap L.$$

From now on, we will always assume that any φ-lattice L inside a filtered φ-module D has an induced filtration.

Recall that by Section 2.4, we may decompose D along the f embeddings $\tau_i : K \hookrightarrow F$ so that $D = \prod_i D^{(i)}$. This relied on the fact that there is an isomorphism $\theta : F \otimes_{\mathbb{Q}_p} K \to \prod_i F^{|\tau|}$ which respects the Frobenius φ_D structure on D. We would like to descend this isomorphism to the integral case which the following lemma from [Zhu09, Lem 3.1] achieves.

Lemma 3.1.1: *The isomorphism $\theta : F \otimes_{\mathbb{Q}_p} K \to F^{|\tau|}$ of Lemma 2.4.1 induces an isomorphism of $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$-modules, $\theta : \mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K \to \prod_{i \in \mathbb{Z}/f\mathbb{Z}} \mathscr{O}_F =: \mathscr{O}_F^{|\tau|}$.*

Proof. The proof of the lemma is completely similar to that of Lemma 2.4.1 once we show that θ is an isomorphism of integral rings. $\mathscr{O}_K$ is generated by a single element over $\mathbb{Z}_p$ so there exists an irreducible polynomial $g \in \mathbb{Z}_p[x]$ of degree f such that $\mathscr{O}_K \cong \mathbb{Z}_p[x]/(g)$. Hence,

$$\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K \cong \mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[x]/(g) \cong \mathscr{O}_F[x]/(g).$$

Since $K \subseteq F$ then $\mathscr{O}_K \subseteq \mathscr{O}_F$ and any roots of g that lie in $\mathscr{O}_K$ will lie in $\mathscr{O}_F$. Thus $\mathscr{O}_F[x]/(g) \cong \prod_{i \in \mathbb{Z}/f\mathbb{Z}} \mathscr{O}_F$. □

Let L be a φ-lattice inside D with induced filtration. The idempotents $\mathfrak{e}_i$ inside $F^{|\tau|}$ can be viewed as lying inside $\mathscr{O}_F^{|\tau|}$. By definition of the induced filtration, we can write $L^{(i)} := \mathfrak{e}_i L = D^{(i)} \cap L$. The map $\theta : L \to \prod_i L^{(i)}$ is an isomorphism of $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$-modules by Lemma 3.1.1 so that we may write $\varphi_L = (\varphi_L^{(0)}, \dots, \varphi_L^{(f-1)})$ with i-th partial Frobenius action $\varphi_L^{(i)} : \varphi^* L^{(i-1)} \to L^{(i)}$.

Proposition 3.1.2: *There is an isomorphism of φ-lattices with induced filtration*

$$L \cong \prod_{i \in \mathbb{Z}/f\mathbb{Z}} L^{(i)}.$$

Proof. θ is compatible with the filtration and Frobenius structures on $D^{(i)}$ by Proposition 2.4.2. The fact that $L^{(i)} = D^{(i)} \cap L$ will imply the same is true on $L^{(i)}$. □

Just as for filtered φ-modules, the results of Proposition 3.1.2 allows us to pass between a φ-module L and its decomposition $\prod_i L^{(i)}$ which we do so freely. Each $L^{(i)}$ may then be viewed as $\mathscr{O}_F$-modules via the isomorphism between $\mathfrak{e}_i \mathscr{O}_F$ and $\mathscr{O}_F$.

An $\mathscr{O}_F$-basis $\{e_1^{(i)}, \dots, e_d^{(i)}\}$ of $L^{(i)}$ is said to be *adapted to the induced filtration* if the set $\{e_1^{(i)}, \dots, e_{\dim \mathrm{Fil}^j L^{(i)}}^{(i)}\}$ forms a basis for $\mathrm{Fil}^j L^{(i)}$ for all j. A filtered φ-module $D = \prod_i D^{(i)}$ is said to have a basis adapted to the filtration if it contains a φ-lattice $L = \prod_i L^{(i)} \subset D$

which has a basis adapted to the induced filtration for all $i \in \mathbb{Z}/f\mathbb{Z}$. The existence of such bases relies on the following proposition using ideas from [Zhu09, Prop 2.1].

Proposition 3.1.3: *Let $D = \prod_i D^{(i)}$ be a d-dimensional filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$ and let $L = \prod_i L^{(i)} \subset D$ be a φ-lattice contained in D. Then there exists a basis of L adapted to the filtration induced from D.*

Proof. Without loss of generality, let us work with a fixed $i \in \mathbb{Z}/f\mathbb{Z}$. We first observe that each $\mathrm{Fil}^{j+1} L^{(i)}$ is a saturated submodule of $\mathrm{Fil}^j L^{(i)}$. Indeed, if we let $r \in \mathscr{O}_F$ and $x \in \mathrm{Fil}^j L^{(i)}$ be such that $rx \in \mathrm{Fil}^{j+1} L^{(i)}$, then $x \in \mathrm{Fil}^{j+1} D^{(i)}$ since $\mathrm{Fil}^{j+1} L^{(i)}$ spans $\mathrm{Fil}^{j+1} D^{(i)}$ over F. Since the filtration of L is induced from D then $x \in \mathrm{Fil}^{j+1} L^{(i)}$.

Take n to be the largest integer such that $\mathrm{Fil}^n L^{(i)}$ is nontrivial and choose a basis $\{e_1^{(i)}, \dots, e_{r_n}^{(i)}\}$ of $\mathrm{Fil}^n L^{(i)}$ where $r_n = \mathrm{Rank}(\mathrm{Fil}^n L^{(i)})$. Since $\mathrm{Fil}^n L^{(i)}$ is saturated in $\mathrm{Fil}^{n-1} L^{(i)}$ then $e_i^{(i)} \in \mathrm{Fil}^{n-1} L^{(i)}$ and there exists a basis of $\mathrm{Fil}^{n-1} L^{(i)}$ such that the quotient can be written

$$\mathrm{Fil}^{n-1} L^{(i)} / \mathrm{Fil}^n L^{(i)} = \langle \bar{e}_{r_n+1}^{(i)}, \dots, \bar{e}_{r_{n-1}}^{(i)} \rangle$$

where $r_{n-1} = \mathrm{Rank}(\mathrm{Fil}^{n-1} L^{(i)})$. By choosing a lift $e_j^{(i)} \in \mathrm{Fil}^{n-1} L^{(i)}$ of $\bar{e}_j^{(i)}$ then $\mathrm{Fil}^{n-1} L^{(i)}$ has basis

$$\{e_1^{(i)}, \dots, e_{r_n}^{(i)}, e_{r_n+1}^{(i)}, \dots, e_{r_{n-1}}^{(i)}\}.$$

Completing an induction up to when $r_i = d$ will give a basis $\{e_1^{(i)}, \dots, e_d^{(i)}\}$ of $L^{(i)}$ that is adapted to the induced filtration. $\square$

As a result of Proposition 3.1.3, we will take a φ-lattice L to always be generated by a basis that is adapted to the induced filtration. This means the filtration structure is completely described by the Hodge polygon data $\mathbf{k}$ and so the Frobenius action determines isomorphism classes of such modules. This idea is expanded upon in Section 3.2.

Recall that the weakly admissible condition on a filtered φ-module D implies some compatibility between the Frobenius and filtration structures on D. As a result, a φ-lattice L inside D should inherit some compatibility between the two induced structures whenever D is weakly admissible. The next definition gives the required condition on such φ-lattices.

Definition 3.1.4: A φ-lattice $L \subset D$ is said to be *strongly divisible* if it satisfies the further properties

$$\varphi_L(\mathrm{Fil}^j L) \subseteq p^j L \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \frac{1}{p^j} \varphi_L(\mathrm{Fil}^j L) = L.$$

We define the category of strongly divisible φ-lattices over $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$ by $\mathrm{SD}^{\varphi}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$. We obtain a natural functor $\mathrm{SD}^{\varphi}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K} \to \mathrm{MF}^{\varphi}_{F \otimes_{\mathbb{Q}_p} K}$ given by scalar extension up to $F \otimes_{\mathbb{Q}_p} K$.

We would like to translate this strongly divisible condition to be in terms of the decomposition $L = \prod_i L^{(i)}$. This is done by the following lemma from [Zhu09, Prop 3.2].

Lemma 3.1.5: *A φ-lattice $L \subset D$ is strongly divisible if and only if when viewing $L = \prod_i L^{(i)}$, we have both*

$$\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) \subseteq p^j L^{(i)} \quad \textit{and} \quad \sum_{j\in\mathbb{Z}} p^{-j}\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = L^{(i)}$$

for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Proof. We know that L is strongly divisible if and only if

$$\varphi_L(\mathrm{Fil}^j L) \subseteq p^j L \quad \text{and} \quad \sum_{i\in\mathbb{Z}} \frac{1}{p^j}\varphi_L(\mathrm{Fil}^j L) = L.$$

Since the idempotent decomposition $L = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} L^{(i)}$ respects both the filtration and Frobenius structures, we may write

$$\prod_{i\in\mathbb{Z}/f\mathbb{Z}} \varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = \varphi_L(\mathrm{Fil}^j L) \subseteq p^j L = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} p^j L^{(i)}$$

allowing us to conclude that $\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) \subseteq p^j L^{(i)}$. Furthermore, we may also write

$$\sum_{j\in\mathbb{Z}} \prod_{i\in\mathbb{Z}/f\mathbb{Z}} \frac{1}{p^j}\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = \sum_{i\in\mathbb{Z}} \frac{1}{p^j}\varphi_L(\mathrm{Fil}^j L) = L = \prod_{i\in\mathbb{Z}/f\mathbb{Z}} L^{(i)}.$$

Orthogonality of the idempotent decomposition will then allow us to deduce that the equality $\sum_{j\in\mathbb{Z}} p^{-j}\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = L^{(i)}$. The only if direction follows by following the above argument in reverse. ∎

In [Laf80], Laffaille showed that the weak admissibility condition on a filtered φ-module D over K is equivalent to the existence of a strongly divisible lattice over $\mathscr{O}_K$ inside D. The next proposition from [Zhu09, Prop 3.3] may be viewed as an extension of this result to F-valued scalars.

Proposition 3.1.6: *Let D be a d-dimensional filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$ with Hodge polygon data $\mathrm{HP}(D) = \mathbf{k}$. The following statements are equivalent:*

(a) D is weakly admissible;
(b) There exists a strongly divisible lattice L over $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$ inside D;
(c) There exists a φ-lattice $L = \prod_i L^{(i)}$ over $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$ inside D such that with respect to a basis adapted to the induced filtration, $\varphi_L^{(i)} = A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ where $A^{(i)} \in \mathrm{GL}_2(\mathscr{O}_F)$ and $\Delta_{\mathbf{k}^{(i)}} = \mathrm{Diag}(p^{k_{i1}}, \ldots, p^{k_{id}})$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Proof. The fact that $(b) \Rightarrow (a)$ is due to [Laf80, §3] and then extending scalars to $\mathscr{O}_F$. To see that $(a) \Rightarrow (b)$, then by [Laf80] again, weak admissibility implies the existence of a strongly divisible lattice L' over $\mathscr{O}_K$ since weak admissibility is a condition on K. By setting $L = L' \otimes_{\mathbb{Z}_p} \mathscr{O}_F := \sum_{j=1}^{n} L' e_i$ where $\mathscr{O}_F = \langle e_1, \ldots, e_n \rangle$ over $\mathbb{Z}_p$, then we have an $\mathscr{O}_K \otimes_{\mathbb{Z}_p} \mathscr{O}_E$ module and since filtrations and Frobenius are F-linear,

$$\sum_{j\in\mathbb{Z}} \frac{1}{p^j}\varphi(\mathrm{Fil}^j L) = \sum_{\ell=0}^{n}\sum_{j\in\mathbb{Z}} \frac{1}{p^j}\varphi(\mathrm{Fil}^j L')e_\ell = \sum_{\ell=0}^{n} L' e_\ell = L.$$

Thus L is strongly divisible over $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$ and we have that $(a) \Leftrightarrow (b)$.

For $(b) \Rightarrow (c)$, Lemma 3.1.5 says that we must have $\sum_{j\in\mathbb{Z}} p^{-j}\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = L^{(i)}$ and $\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) \subseteq p^j L^{(i)}$. By Lemma 3.1.3, we may choose a basis $\{e_1^{(i)}, \ldots, e_d^{(i)}\}$ of $L^{(i)}$ which is adapted to the induced filtration. For $1 \le j \le d$, set $r_{ij} = \dim \mathrm{Fil}^{k_{ij}} L^{(i)}$. Then the fact that $\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) \subseteq p^j L^{(i)}$ implies for each $1 \le j \le d$,

$$\varphi_L^{(i)}(e_1^{(i-1)}, \ldots, e_{r_{ij}}^{(i-1)}) = (e_1^{(i)}, \ldots, e_d^{(i)}) A^{(i)}_{r_{(i-1)j}} \begin{pmatrix} \mathrm{Diag}(p^{k_{(i-1)1}}, \ldots, p^{k_{(i-1)j}}) \\ 0 \end{pmatrix}$$

for some $A^{(i)}_{r_{(i-1)j}} \in \mathrm{Mat}_{r_{(i-1)j}}(\mathscr{O}_F)$. Inducting on j will give us $\mathrm{Mat}(\varphi_L^{(i)}) = A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$. The fact that $\sum_{j\in\mathbb{Z}} p^{-j}\varphi_L^{(i)}(\mathrm{Fil}^j L^{(i-1)}) = L^{(i)}$ will imply that the columns of $A^{(i)}$ will generate $L^{(i)}$ over $\mathscr{O}_F$ so that $A^{(i)} \in \mathrm{GL}_d(\mathscr{O}_F)$. The $(c) \Rightarrow (b)$ direction follows from Lemma 3.1.5. $\square$

As a result, we may refine our scalar extension functor $\mathrm{SD}^{\varphi}_{\mathscr{O}_F\otimes_{\mathbb{Z}_p}\mathscr{O}_K} \to \mathrm{MF}^{\varphi,w.a.}_{F\otimes_{\mathbb{Q}_p}K}$. Restricting to strongly divisible lattices arising out of weakly admissible filtered φ-modules of prescribed Hodge newton polygon data $\mathbf{k}$, we obtain the proper subcategory $\mathrm{SD}^{\varphi,\mathbf{k}}_{\mathscr{O}_F\otimes_{\mathbb{Z}_p}\mathscr{O}_K}$.

In practice, the equivalence of Proposition 3.1.6 give us a concrete way to describe the structure of crystalline representations. Since isomorphism classes of crystalline representations are directly related to isomorphism classes of filtered φ-modules by the Colmez-Fontaine equivalence of Theorem 2.4.3, then in order to determine such isomorphism classes, we need only classify the strongly divisible lattices associated to those filtered φ-modules. The following section gives us the machinery to do just this.

3.2 Classifying Strongly Divisible Lattices

Let D be a weakly admissible filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$ with Hodge polygon data $\mathrm{HP}(D) = \mathbf{k}$. Also let L be a strongly divisible φ-lattice inside D and assume that it has a basis $\{e_1, e_2, \ldots, e_d\}$ which is adapted to the induced filtration which is guaranteed to exist by Proposition 3.1.6. In this case, for each labeled Hodge-Tate weight $k_{ij} \in \mathbf{k}^{(i)} \subset \mathbf{k}$, we define $r_{ij} := \dim_F \mathrm{Fil}^{k_{ij}} D^{(i)}$.

Definition 3.2.1: We define the *parabolic group* of $\mathbf{k}^{(i)}$ over $\mathscr{O}_F$ to be the proper subgroup

$\mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F) \subset \mathrm{GL}_d(\mathscr{O}_F)$ defined by

$$\mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F) := \begin{pmatrix} \mathrm{GL}_{r_{i1}}(\mathscr{O}_F) & \star & \cdots & \star \\ 0 & \mathrm{GL}_{r_{i2}-r_{i1}}(\mathscr{O}_F) & \star & \vdots \\ \vdots & & \ddots & \star \\ 0 & \cdots & 0 & \mathrm{GL}_{d-r_{i(d-1)}}(\mathscr{O}_F) \end{pmatrix}$$

where $\star$ denotes arbitrary elements in $\mathscr{O}_F$.

We can then define the *parabolic group* of $\mathbf{k}$ over $\mathscr{O}_F$ to be the subgroup $\mathscr{P}_{\mathbf{k}}(\mathscr{O}_F) \subset \mathrm{GL}_d(\mathscr{O}_F)^f$ where $\mathscr{P}_{\mathbf{k}} := (\mathscr{P}_{\mathbf{k}^{(0)}}(\mathscr{O}_F), \ldots, \mathscr{P}_{\mathbf{k}^{(f-1)}}(\mathscr{O}_F))$.

We can use our parabolic groups to induce an equivalence relation on matrices. Recall that for any $\mathbf{k}^{(i)} = (k_{i1}, \ldots, k_{id}) \subset \mathbf{k}$, we define $\Delta_{\mathbf{k}^{(i)}} = \mathrm{Diag}(p^{k_{i1}}, \ldots, p^{k_{id}})$. Let $(A) = (A^{(0)}, \ldots A^{(f-1)})$ and $(B) = (B^{(0)}, \ldots, B^{(f-1)})$ be two f-tuples in $\mathrm{GL}_d(\mathscr{O}_F)^f$. We say that (A) and (B) are *parabolic equivalent with respect to* $\mathbf{k}$, written $(A) \sim_{\mathbf{k}} (B)$, if there exists $(C) = (C^{(0)}, \ldots, C^{(f-1)}) \in \mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$ such that for each $0 \leq i \leq f-1$, we have

$$B^{(i)} = C^{(i)} A^{(i)} \Delta_{\mathbf{k}^{(i-1)}} \left(C^{(i-1)}\right)^{-1} \Delta^{-1}_{\mathbf{k}^{(i-1)}}.$$

In this case, we will commonly write $(C) *_{\mathscr{P}} (A) = (B)$. The next theorem inspired from [Zhu09, Thm 3.5] shows that we can use parabolic equivalence classes to define isomorphism classes of strongly divisible lattices.

Theorem 3.2.2: *There is a bijection*

$$\Theta : \mathrm{GL}_d(\mathscr{O}_F)^f / \sim_{\mathbf{k}} \longrightarrow \mathrm{SD}^{\varphi,\mathbf{k}}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$$

defined by sending an f-tuple $(A) = (A^{(0)}, \ldots, A^{(f-1)}) \in \mathrm{GL}_d(\mathscr{O}_F)^f$ to a strongly divisible lattice $L = \prod_i L^{(i)}$ inside $D \in \mathrm{MF}^{\varphi,w.a.,\mathbf{k}}_{F \otimes_{\mathbb{Q}_p} K}$ in a basis adapted to the filtration where $\varphi_L^{(i)} = A^{(i)} \Delta_{\mathbf{k}^{(i-1)}}$.

Proof. We begin with the reverse direction. Let $L, L' \in \mathrm{SD}^{\varphi,\mathbf{k}}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$ be isomorphic strongly divisible lattices inside $D \in \mathrm{MF}^{\varphi,w.a.,\mathbf{k}}_{F \otimes_{\mathbb{Q}_p} K}$ via an isomorphism $h : L \to L'$. Proposition 3.1.6 says that we may write $L = \prod_i L^{(i)}$ and $L' = \prod_i L'^{(i)}$ with Frobenius actions given by $\varphi_L^{(i)} = A^{(i)} \Delta_{\mathbf{k}^{(i-1)}}$ and $\varphi_{L'}^{(i)} = A'^{(i)} \Delta_{\mathbf{k}^{(i-1)}}$ respectively. Restricting our filtration and φ-compatible isomorphism to each embedding piece gives $h^{(i)}(L^{(i)}) \cong L'^{(i)}$, $h^{(i)}(\mathrm{Fil}^j L^{(i)}) \cong \mathrm{Fil}^j L'^{(i)}$ and $h^{(i)} \circ \varphi_L^{(i)} = \varphi_{L'}^{(i)} \circ h^{(i-1)}$. These conditions on $h^{(i)}$ implies that for $1 \leq j \leq d$ if we set $r_{ij} = \dim \mathrm{Fil}^j L^{(i)}$, we have

$$h^{(i)}(e^{(i)}_{r_{i(j-1)}}, \ldots, e^{(i)}_{r_{ij}}) = (e'^{(i)}_{r_{i(j-1)}}, \ldots, e'^{(i)}_{r_{ij}}) C^{(i)}_{r_{ij}}$$

where each $C^{(i)}_{r_{ij}} \in \mathrm{GL}_{r_{ij} - r_{i(j-1)}}(\mathscr{O}_F)$. Hence, we have $h^{(i)}(e_1^{(i)}, \ldots, e_d^{(i)}) = (e_1^{(i)}, \ldots, e_d^{(i)}) C^{(i)}$

where

$$C^{(i)} := \begin{pmatrix} C^{(i)}_{r_{i1}} & \star & \cdots & \star \\ 0 & C^{(i)}_{r_{i2}} & & \vdots \\ \vdots & & \ddots & \star \\ 0 & \cdots & 0 & C^{(i)}_{r_{id}} \end{pmatrix} \in \mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F)$$

which gives $\mathrm{Mat}(h) \in \mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$. We then have the matrix relation from φ-compatibility

$$A^{(i)}\Delta_{\mathbf{k}^{(i-1)}} = C^{(i)}A'^{(i)}\Delta_{\mathbf{k}^{(i-1)}}(C^{(i-1)})^{-1}$$

so we conclude that $(C) *_{\mathscr{P}} (A) = (A')$.

The other direction follows from performing the above argument in reverse. $\square$

As a consequence, we will commonly write $\Theta(A) = L(A) = \prod_i L(A)^{(i)}$ to denote the associated strongly divisible lattice with i-th partial Frobenius action given by $A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$. We define

$$D(A) := L(A) \otimes_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K} F \otimes_{\mathbb{Q}_p} K$$

to be the associated weakly admissible filtered φ-module with Hodge Polygon data $\mathbf{k}$ by Proposition 3.1.6. By the aforementioned proposition, all such weakly admissible filtered φ-modules arise out of this construction. Recalling the equivalence functor $\mathbf{V}_{\text{cris}}$ of Theorem 2.4.3, there exists a crystalline representation with Hodge-Tate weight data $\mathbf{k}$ given by

$$V(A) := \mathbf{V}^*_{\text{cris}}(D(A))$$

which likewise exhausts all such representations. This means we can use parabolic equivalence classes to pick out 'simple' representatives for the isomorphism classes of crystalline representations. More on this in Chapter 4.

We conclude this section with looking that how the bijection of Theorem 3.2.2 behaves with respect to the notion of taking tensor products. This will be important as it will allow us to twist by crystalline characters in order to adjust the labeled Hodge-Tate weights of our representations for more convenient computations. Consider strongly divisible lattices $\Theta(A) = L(A)$ and $\Theta(A') = L(A')$ of $\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K$ dimension d and d' respectively and arising out of weakly admissible filtered φ-modules with Hodge Polygon data $\mathbf{k}$ and $\mathbf{k}'$ respectively.

We define two tensor products

- $\mathbf{k} \otimes \mathbf{k}' := \left((k_{ij} + k'_{ij'})_{j,j'}\right)_{i \in \mathbb{Z}/f\mathbb{Z}}$ where $1 \leq i \leq d$ and $1 \leq i' \leq d'$ reordering if necessary.
- $A \otimes A' := (A^{(i)} \otimes A'^{(i)})_{i \in \mathbb{Z}/f\mathbb{Z}} \in \mathrm{GL}_{dd'}(\mathscr{O}_F)^f$ where $A^{(i)} \otimes A'^{(i)}$ denotes the normal Kronecker tensor product.

The result we desire is given in the following proposition from [Zhu09, Prop 3.7].

Proposition 3.2.3: *Let notation be as above so that* $\Theta(A) = L(A) \in \mathrm{SD}^{\varphi,\mathbf{k}}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$ *and* $\Theta(A') = L(A') \in \mathrm{SD}^{\varphi,\mathbf{k}'}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$. *Then* $\Theta(A \otimes A') = L(A) \otimes L(A') \in \mathrm{SD}^{\varphi,\mathbf{k}\otimes\mathbf{k}'}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$.

Proof. First, note that

$$L(A) \otimes L(A') = \prod_i L(A)^{(i)} \otimes \prod_i L(A')^{(i)} = \prod_i L(A)^{(i)} \otimes L(A')^{(i)}$$

with the last equality following from the orthogonality of the idempotent decomposition. Both $L(A)$ and $L(A')$ are strongly divisible so Proposition 3.1.6 implies the existence of a basis adapted to the filtration so that the i-th partial Frobenius action of $L(A) \otimes L(A')$ is given by

$$(A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}) \otimes (A'^{(i)}\Delta_{\mathbf{k}'^{(i-1)}}) = (A^{(i)} \otimes A'^{(i)})(\Delta_{\mathbf{k}^{(i-1)}} \otimes \Delta_{\mathbf{k}'^{(i-1)}})$$

where the equality follows form the mixed product property of the Kronecker tensor product. After observing the fact that

$$\Delta_{\mathbf{k}^{(i-1)}} \otimes \Delta_{\mathbf{k}'^{(i-1)}} = \mathrm{Diag}(p^{k_{(i-1)1}+k'_{(i-1)1}}, \ldots, p^{k_{(i-1)d}+k'_{(i-1)d'}})$$

then it is clear that $\Theta(A \otimes A') = L(A) \otimes L(A') \in \mathrm{SD}^{\varphi,\mathbf{k}\otimes\mathbf{k}'}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$. □

The results of Proposition 3.2.3 give us a way to simplify the Hodge-Tate weight data $\mathbf{k}$ of a crystalline representation V.

Remark 3.2.4: Consider the representation $V(A) \in \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/F}(G_K)$ of dimension d and define a crystalline character $V(I_d) \in \mathrm{Rep}^{\mathbf{k}'}_{\mathrm{cris}/F}(G_K)$ where $\mathbf{k}' = (-k_{id})_i$. Then the above results will imply that $V(A) \otimes V(I_d)$ will have Hodge-Tate weight data given by $(k_{i1} - k_{id}, \ldots, 0)_i$ but leave the matrices (A) unaffected. Hence, up to a twist by a crystalline character, we may always assume that $k_{id} = 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.

Chapter 4

Isomorphism Classes in Dimension Two

The classification of irreducible two-dimensional crystalline representations of $G_{\mathbb{Q}_p}$ was first published by Breuil in [Bre03]. In this case, the isomorphism classes of such representations turn out to only depend on the Hodge-Tate weight k and the trace of Frobenius a_p. This chapter is dedicated to achieving a similar description in the case of $K \cong \mathbb{Q}_{p^f}$ when two parameters are no longer enough to provide a complete classification.

We begin in Section 4.1 by classifying the parabolic equivalence classes of $\mathrm{GL}_2(\mathscr{O}_F)^f$ by choosing representatives made up of matrices of specific *Types*. Using the results of previous chapters, in Section 4.2 we are able to classify the isomorphism classes of strongly divisible lattices in terms of these Types which, in turn, gives us an understanding of the structure for crystalline representations of $G_{\mathbb{Q}_{p^f}}$.

4.1 Parabolic Equivalence Classes

From Section 3.2, recall that we may induce an equivalence relation $\sim_{\mathbf{k}}$ on $\mathrm{GL}_2(\mathscr{O}_F)^f$ by understanding that $(A^{(i)}) \sim_{\mathbf{k}} (B^{(i)})$ whenever there exists $(C^{(i)}) \in \mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$ such that

$$B^{(i)} = C^{(i)} A^{(i)} \Delta_{\mathbf{k}^{(i-1)}} \left(C^{(i-1)}\right)^{-1} \Delta_{\mathbf{k}^{(i-1)}}^{-1}$$

for all $i \in \mathbb{Z}/f\mathbb{Z}$. Under the assumption that dimension $d = 2$ and the labeled Hodge-Tate weights are *regular* in the sense that $k_i > 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$, we may realize that each parabolic group $\mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F)$ is nothing more than the invertible upper triangular matrices over $\mathscr{O}_F$ and $\Delta_{\mathbf{k}^{(i)}} = \mathrm{Diag}(p^{k_i}, 1)$ by Remark 3.2.4. The purpose of this section is to pick equivalence class representatives that are as simple as possible by choosing suitable parabolic matrices $(C^{(i)})$.

Let $[(A^{(i)})] \in \mathrm{GL}_2(\mathscr{O}_F)^f/\sim_{\mathbf{k}}$ define an arbitrary equivalence class where each

$$A^{(i)} = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} \end{pmatrix}.$$

Since each matrix is invertible, then we can safely assume that at least one of $a_{21}^{(i)}$ and $a_{22}^{(i)}$ is invertible. Following the ideas found in [Liu21, § 2], we will explicitly construct parabolic matrices $(C^{(i)}) \in \mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$ by

$$C^{(i)} = \begin{pmatrix} c_{11}^{(i)} & c_{12}^{(i)} \\ 0 & c_{22}^{(i)} \end{pmatrix}$$

such that the result of $(C^{(i)}) *_{\mathscr{P}} (A^{(i)})$ has $\left(\begin{smallmatrix} 0 \\ \beta \end{smallmatrix}\right)$ as one of its columns where $\beta \in \mathscr{O}_F^*$. In order to ease notation, let us begin to write $(xy)^{(i)} = x^{(i)}y^{(i)}$.

Lemma 4.1.1: *Suppose $k_i > 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and $(A^{(i)}) \sim_{\mathbf{k}} (B^{(i)})$. Then $a_{2\ell}^{(i)} \in \mathscr{O}_F^*$ if and only if $b_{2\ell}^{(i)} \in \mathscr{O}_F^*$.*

Proof. Let $(C^{(i)})$ be the parabolic equivalent change of basis turning $(A^{(i)})$ into $(B^{(i)})$ using notation as above. We have

$$b_{21}^{(i)} = \frac{(a_{21}c_{11})^{(i)}}{c_{22}^{(i-1)}} \qquad\qquad b_{22}^{(i)} = \frac{(a_{22}c_{22})^{(i)} + (a_{21}c_{12})^{(i)}p^{k_i}}{c_{22}^{(i-1)}}.$$

The lemma follows from the fact that $c_{11}^{(i)}, c_{22}^{(i)} \in \mathscr{O}_F^*$ for all i. □

Lemma 4.1.2: *For all $i \in \mathbb{Z}/f\mathbb{Z}$, suppose $k_i > 0$ and $a_{2\ell}^{(i)} \in \mathscr{O}_F^*$ for some $\ell \in \{1, 2\}$. Then there exists a parabolic equivalent change of basis $(C^{(i)}) \in \mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$ such that by setting $(C^{(i)}) *_{\mathscr{P}} (A^{(i)}) = (B^{(i)})$, then $b_{1\ell}^{(i)} = 0$.*

Proof. We must treat two cases separately. First suppose that there exists some $r \in \mathbb{Z}/f\mathbb{Z}$ such that $A^{(r)}$ has $a_{21}^{(r)} \in \mathscr{O}_F^*$. For any fixed $i \in \mathbb{Z}/f\mathbb{Z}$, observe that if $a_{2\ell}^{(i)} \in \mathscr{O}_F^*$ then by taking

$$C^{(i)} = \begin{pmatrix} 1 & -\frac{a_{1\ell}^{(i)}}{a_{2\ell}^{(i)}} \\ 0 & \frac{1}{a_{2\ell}^{(i)}} \end{pmatrix} \in \mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F) \quad \text{and} \quad C^{(j)} = I \quad \text{for} \quad j \neq i \tag{4.1}$$

will give $B^{(i)}$ with $b_{1\ell}^{(i)} = 0$ and $b_{2\ell}^{(i)} = 1$. We notice that this change of basis also affects the right column of $B^{(i+1)}$ (but not the invertibility of $b_{22}^{(i+1)}$ by Lemma 4.1.1)and does not affect $B^{(j)}$ for $j \notin \{i, i+1\}$. Since this base change is in $\mathscr{P}_{\mathbf{k}}(\mathscr{O}_F)$ and we are only concerned with matrices up to parabolic equivalence, then we can safely reset $(A^{(i)})$ to be $(B^{(i)})$ and repeat this process of 4.1 for $i+1$ and so on until we return to i taking the indexes modulo f. If we start this process at the aforementioned $A^{(r)}$ then when we perform the final step to base change $A^{(r-1)}$ then the left hand column of $A^{(r)}$ is unaffected. The result is a parabolic equivalent f-tuple $(B^{(i)})$ with one column which looks like $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

The other case is when there is no such r with $a_{21}^{(r)} \in \mathscr{O}_F^*$; Hence, we want to make $a_{12}^{(i)} = 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Here, the above will not work as the final step of the process will make the upper right hand entry of the initial matrix non-zero. Instead, let us take

$$C^{(i)} = \begin{pmatrix} 1 & -\frac{a_{12}^{(i)}}{a_{22}^{(i)}} \\ 0 & \frac{1}{a_{22}^{(i)}} \end{pmatrix} \in \mathscr{P}_{\mathbf{k}^{(i)}}(\mathscr{O}_F) \quad \text{for all} \quad i \in \mathbb{Z}/f\mathbb{Z}.$$

The result will leave $(B^{(i)})$ with $b_{22}^{(i)} \in \mathscr{O}_F^*$ by Lemma 4.1.1 and $\nu_p(b_{12}^{(i)}) \geq k_i$. Since $k_i > 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ then by resetting $(A^{(i)})$ to be $(B^{(i)})$ and repeating the process, we will have that $\nu_p(b_{12}^{(i)}) \to \infty$ and $b_{22}^{(i)} \in \mathscr{O}_F^*$. □

The results of the above lemmas motivates the following definition.

Definition 4.1.3: We define a *Type* matrix to be:

- Type I matrix: $\begin{pmatrix} 0 & m_1 \\ 1 & m_2 \end{pmatrix}$ where $m_1 \in \mathscr{O}_F^*$ and $m_2 \in \mathscr{O}_F$;
- Type II_α matrix: $\begin{pmatrix} m_1 & 0 \\ m_2 & \alpha \end{pmatrix}$ where $m_1, \alpha \in \mathscr{O}_F^*$ and $m_2 \in \varpi\mathscr{O}_F$.

We define the associated Type map to be $\mathcal{T} : \mathrm{GL}_2(\mathscr{O}_F) \to \{\mathrm{I}, \mathrm{II}\}$ where for any $A \in \mathrm{GL}_2(\mathscr{O}_F)$, $\mathcal{T}(A) = \mathrm{I}$ if and only if $a_{21} \in \mathscr{O}_F^*$ and $\mathcal{T}(A) = \mathrm{II}$ otherwise.

In other words, the above definition says that for an f-tuple $(A^{(i)}) \in \mathrm{GL}_2(\mathscr{O}_F)^f$, for each i, $\mathcal{T}(A^{(i)}) = \mathrm{I}$ or II whenever $(A^{(i)}) \sim_{\mathbf{k}} (B^{(i)})$ with $B^{(i)}$ being a Type I or II_α matrix respectively. We will refer to the f-tuple $(\mathcal{T}(A^{(0)}), \ldots, \mathcal{T}(A^{(f-1)}))$ as being the *Type combination* of (A). Since parabolic equivalence does not effect the invertibility of the bottom row by Lemma 4.1.1, then the Type map is well-defined.

Proposition 4.1.4: *The parabolic equivalence classes of* $\mathrm{GL}_2(\mathscr{O}_F)^f / \sim_{\mathbf{k}}$ *for* $k_i > 0$ *may be represented by an* f*-tuple of matrices* $(A^{(i)})$ *satisfying one of two cases:*

(a) For each i*, the matrix* $A^{(i)}$ *is a Type* I *or* II_1 *matrix;*
(b) For all i*, the matrix* $A^{(i)}$ *is a Type* $\mathrm{II}_{\alpha^{(i)}}$ *matrix for some* $\alpha^{(i)} \in \mathscr{O}_F^*$.

Proof. For a general equivalence class $[(B^{(i)})] \in \mathrm{GL}_2(\mathscr{O}_F)^f / \sim_{\mathbf{k}}$, Lemma 4.1.1 ensures that elements of the bottom row of every member of this class has the same invertiblity statuses. The proposition follows from the proof of Lemma 4.1.2. Namely, if for at least one $B^{(i)}$ we have $b_{21}^{(i)} \in \mathscr{O}_F^*$ then we are in the case of (a), otherwise, we are in the case of (b). □

Remark 4.1.5: As a result of our definition for Type, we see that the classification is not unique. In particular, there is ambiguity when both entries on the bottom row are invertible. We make this particular choice as it behaves better with respect to irreducibility, see Proposition 4.2.1.

4.2 Models for Crystalline Representations

Recall by Theorem 3.2.2, we have a bijection

$$\Theta : \mathrm{GL}_2(\mathscr{O}_F)^f / \sim_{\mathbf{k}} \longrightarrow \mathrm{SD}^{\varphi,\mathbf{k}}_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K}$$

given by taking an f-tuple $(A) = (A^{(i)})$ to a strongly divisible lattice $\Theta(A) = L(A)$ with a basis adapted to its induced filtration so that the i-th partial Frobenius action takes the form $\varphi^{(i)} = A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ with $\Delta_{\mathbf{k}^{(i)}} = \mathrm{Diag}(p^{k_i}, 1)$ by Remark 3.2.4 and with the additional assumption that $k_i > 0$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ due to Lemma 4.1.2. In Section 4.1, we observed that we may view each $A^{(i)}$ as being a matrix of one of two Types so that we have effectively chosen simple representatives of the isomorphism classes of strongly divisible lattices.

Recall that we have defined

$$D(A) := L(A) \otimes_{\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K} (F \otimes_{\mathbb{Q}_p} K)$$

to be the weakly admissible filtered φ-module which contains the strongly divisible lattice $L(A)$. All weakly admissible filtered φ-module necessarily arises out of this construction by Proposition 3.1.6. Hence, up to isomorphism, $D(A) = \prod_i D(A)^{(i)}$ admits a basis adapted to the filtration $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ such that

$$[\varphi^{(i)}]_\eta = \begin{cases} \begin{pmatrix} 0 & a_1^{(i)} \\ p^{k_{i-1}} & a_2^{(i)} \end{pmatrix} & \text{Type I} \\ \begin{pmatrix} a_1^{(i)} p^{k_{i-1}} & 0 \\ a_2^{(i)} p^{k_{i-1}} & \alpha^{(i)} \end{pmatrix} & \text{Type II}_{\alpha^{(i)}} \end{cases} \qquad \mathrm{Fil}^j D(A)^{(i)} = \begin{cases} D(A)^{(i)} & j \leq 0 \\ F(\eta_1^{(i)}) & 0 < j \leq k_i \\ 0 & k_i < j. \end{cases}$$

Since we are only concerned with irreducible representations, then we may impose more structure on our partial Frobenius matrices to rid ourselves of some reducible cases and make this description a bit cleaner.

Proposition 4.2.1: *Keeping with notation as above, the weakly admissible filtered φ-module $D(A)$ is reducible in $\mathrm{MF}^{\varphi,w.a.}_{F\otimes_{\mathbb{Q}_p}K}$ in each of the following cases:*

(a) The i-th partial Frobenius $\varphi^{(i)}$ is Type $\mathrm{II}_{\alpha^{(i)}}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$;

(b) The product $\prod_{i\in\mathcal{S}} a_2^{(i)} \in p^w \mathscr{O}_F^*$ where $\mathcal{S} = \{i \in \mathbb{Z}/f\mathbb{Z} : \varphi^{(i)}$ is Type I$\}$ and $w = \sum_{i\in J} k_i$ for some subset $J \subseteq \mathcal{S}$ (if $J = \emptyset$ then $w = 0$).

This proposition relies on the following lemma regarding reducibility of strongly divisible lattices in the rank 2 case.

Lemma 4.2.2: *Continuing to use notations as in Proposition 4.2.1, let* $\varphi^f = \prod_{i=0}^{f-1} A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ *be the (linear) Frobenius on* $L(A)^{(0)}$. *If* φ^f *has an eigenvalue* λ *with* $\lambda \in p^w \mathscr{O}_F^*$ *where* $w = \sum_{i\in J} k_i$ *for some subset* $J \subseteq \mathbb{Z}/f\mathbb{Z}$ *(if* $J = \emptyset$ *then* $w = 0$*), then* $L(A)$ *is reducible in* $\mathrm{SD}^{\varphi}_{\mathscr{O}_F\otimes_{\mathbb{Z}_p}\mathscr{O}_K}$.

Proof. By assumption there exists simultaneously nonzero $x^{(0)}, y^{(0)} \in \mathscr{O}_F$ such that

$$\prod_{i=0}^{f-1} A^{(i)}\Delta_{\mathbf{k}^{(i-1)}} \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} = \varphi^f \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} = \lambda \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} = \prod_{i=0}^{f-1} \lambda^{(i-1)} \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix}$$

where each $\lambda^{(i)} \in \mathscr{O}_F^*$ if $i \notin J$ and $\lambda^{(i)} \in p^{k_i}\mathscr{O}_F^*$ otherwise. Strong divisibility in the sense of Lemma 3.1.5 then provides the existence of simultaneously non-zero $x^{(i)}, y^{(i)} \in \mathscr{O}_F$ such that

$$A^{(i)}\Delta_{\mathbf{k}^{(i-1)}} \begin{pmatrix} x^{(i-1)} \\ y^{(i-1)} \end{pmatrix} = \lambda^{(i-1)} \begin{pmatrix} x^{(i)} \\ y^{(i)} \end{pmatrix}$$

for all $i \in \mathbb{Z}/f\mathbb{Z}$. Hence, each $L(A)^{(i)}$ admits a φ-stable submodule $L'^{(i)} = x^{(i)}\eta_1^{(i)} + y^{(i)}\eta_2^{(i)}$ so we need only show that $L' = \prod_i L'^{(i)} \subset L(A)$ is strongly divisible using Lemma 3.1.5.

For any $i \in \mathbb{Z}/f\mathbb{Z}$, observe that since $L'^{(i)}$ is of rank 1, then $\mathrm{Fil}^j L'^{(i)}$ has only one filtration break from $L'^{(i)}$ to 0. In particular, there is an integer $r \geq 0$ so that

$$\mathrm{Fil}^j L'^{(i)} = \begin{cases} L'^{(i)} & j \leq r \\ 0 & j > r. \end{cases}$$

Hence, by the arguments above we have

$$\varphi^{(i)}(\mathrm{Fil}^j L'^{(i-1)}) = A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}(\mathrm{Fil}^j L'^{(i-1)}) = \begin{cases} \lambda^{(i-1)} L'^{(i)} & j \leq r \\ 0 & j > r. \end{cases}$$

However, since $L'^{(i)} = x^{(i)}\eta_1^{(i)} + y^{(i)}\eta_2^{(i)} \subset L(A)^{(i)}$ then strong divisibility of $L(A)^{(i)}$ forces $r = 0$ whenever $i-1 \notin J$ and $r = k_{i-1}$ when $i-1 \in J$. We may then apply Lemma 3.1.5 to deduce strong divisibility of L'. □

Proof of Proposition 4.2.1. Let $L(A) = \prod_i L(A)^{(i)}$ be the strongly divisible lattice inside $D(A) = \prod_i D(A)^{(i)}$ with a basis adapted to the filtration $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ so that the i-th partial Frobenius is given by $\varphi^{(i)} = A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$. Since reducibility is preserved by scalar extension, then we need only display that we can apply Lemma 4.2.2 to each case.

Consider the product $\varphi^f = \prod_{i=0}^{f-1} A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ and observe that since the left column of each $A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ is divisible by $p^{k_{i-1}}$ then the lower right entry of φ^f must contain the product $\prod_{i,j} a_2^{(i)}\alpha^{(j)}$ for $i \in \mathcal{S}$ and $j \notin \mathcal{S}$ and any other term on the diagonal is necessarily divisible by $p^{w'}$ where $w' = \sum_{i\in J'} k_i$ for some subset $J' \subseteq \mathbb{Z}/f\mathbb{Z}$. Since the trace of φ^f give the sum of the eigenvalues, then one of the eigenvalues takes the form

$v = \prod_{i,j} a_2^{(i)}\alpha^{(j)} + p^{w'}\mathscr{O}_F$. It then becomes clear that we can apply Lemma 4.2.2 to both cases since $\alpha^{(i)} \in \mathscr{O}_F^*$. $\square$

We are now ready to give the first of our main results. Recall that $V(A) := V^*_{\mathrm{cris}}(D(A)) \in \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/F}(G_K)$ is the image of $D(A)$ under the quasi-inverse functor V^*_{cris}.

Theorem 4.2.3: *Let $V \in \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/F}(G_K)$ be an irreducible, two dimensional crystalline representation of G_K with labeled Hodge-Tate weights $\{k_i, 0\}_{i\in\mathbb{Z}/f\mathbb{Z}}$ where $k_i > 0$. Then $V \cong V(A)$ where $D(A) := \mathbf{D}^*_{\mathrm{cris}}(V(A))$ with each $A^{(i)}$ being Type I or II_1. In particular, $D(A) = \prod_i D(A)^{(i)}$ has a basis $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ such that*

$$[\varphi^{(i)}]_\eta = \begin{cases} \begin{pmatrix} 0 & a_1^{(i)} \\ p^{k_{i-1}} & a_2^{(i)} \end{pmatrix} & \text{Type I} \\ \begin{pmatrix} a_1^{(i)}p^{k_{i-1}} & 0 \\ a_2^{(i)}p^{k_{i-1}} & 1 \end{pmatrix} & \text{Type II} \end{cases} \qquad \mathrm{Fil}^j D(A)^{(i)} = \begin{cases} D(A)^{(i)} & j \le 0 \\ F(\eta_1^{(i)}) & 0 < j \le k_i \\ 0 & k_i < j. \end{cases}$$

Proof. Let $D = \mathbf{D}^*_{\mathrm{cris}}(V)$ and observe that Proposition 3.1.6 and Theorem 3.2.2 come together to imply the existence of a f-tuple $(B) = (B^{(i)}) \in \mathrm{GL}_2(\mathscr{O}_F)^f$ such that $L(B)$ is a strongly divisible lattice inside D so that $D = D(B)$. By Proposition 4.1.4 there is a another f-tuple $(A) = (A^{(i)}) \in \mathrm{GL}_2(\mathscr{O}_F)^f$, with each $A^{(i)}$ being Type I or $\mathrm{II}_{\alpha^{(i)}}$, such that $(B) \sim_{\mathbf{k}} (A)$ so that $L(A) \cong L(B)$ by Theorem 3.2.2 once more. Hence $D(B) \cong D(A)$ so that $V \cong V(A)$. Since V is irreducible then $D(A)$ is irreducible so that Lemma 4.2.1 gives each $A^{(i)}$ the desired Type by Proposition 4.1.4 again. $\square$

As a result, we will commonly say that the *Type* of $V(A)$ is the Type of the partial Frobenius matrices of $\mathbf{D}^*_{\mathrm{cris}}(V(A))$. It follows that the f-tuple of matrices (A) completely determines the isomorphism class of representations which $V(A)$ is apart of. Hence, to compute reductions, we need only consider those representations $V(A)$ of one of these $2f-1$ Type combinations. We will now write $\mathrm{II} = \mathrm{II}_1$ due to the exclusion of II_α in the irreducible case for $\alpha \neq 1$.

Remark 4.2.4: It is not difficult to see that our description may be used to recover the $f = 1$ classification of Breuil, see 1.1, as we may use irreducibility to exclude the single Type II case.

Remark 4.2.5: It should be pointed out that our notion of Type is only a property of strongly divisible lattices. In particular, a weakly admissible filtered φ-module may admit distinct strongly divisible lattices of different Types. This most obvious reason for this is due to the fact that parabolic equivalence over F need not preserve invertibility. Indeed, Lemma 4.1.1 does not hold over F.

Part II

Explicit Constructions and Reductions

Chapter 5

Breuil-Kisin Theory

We now describe another integral p-adic Hodge theory introduced by Kisin in [Kis06]. As opposed to finding integral φ-lattices inside filtered φ-modules as we did in Chapter 3, we instead study φ-modules over power series rings in a formal variable u called *Breuil* and *Kisin modules*. In order to return to the filtered φ-module setting, we then need only take modulo u. The motivation for studying these more complicated objects lies in taking semisimple reductions. Indeed, Kisin's theory allows for the (theoretical) computation of reductions without restriction on Hodge-Tate weights; this is as opposed to Fontaine-Laffaille theory which is only able to compute reductions when the Hodge-Tate weights are bounded above by $p-1$. We note that this chapter may be viewed as a survey of relevant results from [BL20, §2] and [BLL23, §3].

Beginning in Section 5.1 we introduce the notion of a Kisin module, the objects we will work with most in the second part of this thesis. In Section 5.2 we learn how to pick out those Kisin modules that are canonically associated to crystalline representations. In Section 5.3, we explain how we may construct an étale φ-module from the data of a Kisin module which can be used to compute semisimple reductions of the crystalline representations that they are associated too. Section 5.4 then introduces the notion of a Breuil module which can be used as an intermediary between filtered φ-modules and Kisin modules. Finally, Section 5.5 gives the main result of this section; An algorithm to explicitly construct a Kisin module over an enlarged coefficient ring from a filtered φ-module by first passing through a Breuil module.

5.1 Kisin Modules

We begin by recalling some notation from Section 1.2. Let $\Lambda \subset K_0[\![u]\!]$ be the ring of rigid analytic functions on the open p-adic unit disc in K and set $\mathfrak{S} = W(k)[\![u]\!] = \mathscr{O}_K[\![u]\!]$. Note the differing notation from [BL20] and [BLL23]. Both rings admit a Frobenius action defined by $\varphi(u) = u^p$ and by extending this action by F-linearity, we get φ-stable rings $\mathfrak{S}_F \subset \Lambda_F \subset (F \otimes_{\mathbb{Q}_p} K_0)[\![u]\!]$. Also let $E \coloneqq E(u) = u + p$ be the minimal polynomial of

$\pi_K = -p$. It is easy to see that all the rings defined previously contain E.

Let $R \subseteq (F \otimes_{\mathbb{Q}_p} K_0)[\![u]\!]$ be a φ-stable subring containing E. We define a *φ-module* over R to be a finite free R-module M admitting an injective φ-semilinear operator $\varphi_M : M \to M$.

Definition 5.1.1: A *Kisin module over $\mathfrak{S}_F$ of (E-)height* $\leq h$ is a φ-module $\mathfrak{M}$ over $\mathfrak{S}_F$ such that the linearization map

$$1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} := \mathfrak{S}_F \otimes_{\varphi, \mathfrak{S}_F} \mathfrak{M} \to \mathfrak{M}$$

has cokernel killed by E^h.

The category of Kisin modules over $\mathfrak{S}_F$ with height $\leq h$ is denoted $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi, \leq h}$ with morphisms being $\mathfrak{S}_F$-linear maps compatible with the Frobenius actions.

Remark 5.1.2: A more concrete notion of finite E-height may be achieved by choosing a basis of $\mathfrak{M}$ so that $\varphi_{\mathfrak{M}}$ in this basis is given by a matrix $\mathfrak{A}$. Then a Kisin module being of height h is equivalent to $E^h\mathfrak{A}^{-1}$ being a matrix over $\mathfrak{S}_F$.

We intend to associate finite height Kisin modules over $\mathfrak{S}_F$ to Galois-stable $\mathscr{O}_F$-lattice representations $T \subset V$ inside our crystalline representations. However, unlike strongly divisible lattices, it is not apparent how a Kisin module as defined arises from a given weakly admissible filtered φ-module. It turns out that we require an intermediate category which is equivalent to filtered φ-modules and captures a notion of 'rational' Kisin modules.

Indeed, by extending scalars we arrive at functor

$$\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi, \leq h} \to \mathrm{Mod}_{\Lambda_F}^{\varphi, \leq h}$$

and a notion of a finite height Kisin module over Λ_F. We will commonly reserve $\mathfrak{M}$ to denote a Kisin module over $\mathfrak{S}_F$ and $\mathcal{M}$ to denote a Kisin module over a larger φ-stable coefficient ring $\mathfrak{S}_F \subset R$ such as but not limited to Λ_F.

Definition 5.1.3: Let R be a φ-stable ring such that $\mathfrak{S}_F \subset R$. When we can write a finite height Kisin module over R by $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S}_F} R$ then we call $\mathfrak{M}$ a *descent* of $\mathcal{M}$ to $\mathfrak{S}_F$.

By [Kis06, Proposition 1.2.8], there is a functor

$$\mathrm{Mod}_{\Lambda_F}^{\varphi, \leq h} \to \mathrm{MF}_{F \otimes_{\mathbb{Q}_p} K}^{\varphi}.$$

As topological spaces, the above functor maps $\mathcal{M} \mapsto \mathcal{M}/u\mathcal{M}$ with the induced φ action and with filtration structure induced by $\mathcal{M}$ being more complicated.[7] However, this functor does not define an equivalence as the category $\mathrm{Mod}_{\Lambda_F}^{\varphi, \leq h}$ is too large. We instead need to cut out a subcategory inside $\mathrm{Mod}_{\Lambda_F}^{\varphi, \leq h}$ to attain the equivalence we require.

[7]The formal definition is not used in this thesis so we will exclude it here, see [Kis06, Rem 1.2.7] for the proper statement

Remark 5.1.4: Kisin's theory is more general than we have hinted at thus far. To capture the notion of *semistable* representations with nonzero monodromy, see Remark 2.3.5, we need only add the existence of an $(F\otimes_{\mathbb{Q}_p}K_0)$-linear monodromy operator $N_{\mathfrak{M}}$ on $\mathfrak{M}/u\mathfrak{M}\otimes_{\mathscr{O}_F} F$ satisfying $N_{\mathfrak{M}}\circ\varphi_{\mathfrak{M}} = p\varphi_{\mathfrak{M}}\circ N_{\mathfrak{M}}$ to Definition 5.1.1. The category of finite height Kisin modules with monodromy is denoted $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,N,\leq h}$. This gives an extension of scalars functor $\mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,N,\leq h}\to\mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ and Kisin then describes a functor $\mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}\to\mathrm{MF}_{F\otimes_{\mathbb{Q}_p}K}^{\varphi,N}$ into the category of filtered (φ,N)-modules.

Define the element

$$\lambda := \prod_{n=0}^{\infty}\varphi^n\left(\frac{E}{p}\right)\in\Lambda_F.$$

The operator $-u\lambda\frac{d}{du}$ is a derivation on Λ_F and observe that we get the relation

$$\left(-u\lambda\frac{d}{du}\right)\circ\varphi = p\left(\frac{E(u)}{E(0)}\right)\varphi\circ\left(-u\lambda\frac{d}{du}\right).$$

Note that since $E(u) = u+p$ in our setting, then $p\left(\frac{E(u)}{E(0)}\right) = E$ and for the rest of this thesis, we will simplify this term likewise with the understanding that there is a more general context for this simplification.

Definition 5.1.5: A finite height (φ,N_∇)-module $\mathcal{M}$ over Λ_F is a finite height Kisin module over Λ_F admitting a differential operator $N_\nabla^{\mathcal{M}}$ over $-u\lambda\frac{d}{du}$ on Λ_F satisfying the relation

$$N_\nabla^{\mathcal{M}}\circ\varphi_{\mathcal{M}} = E\varphi_{\mathcal{M}}\circ N_\nabla^{\mathcal{M}}$$

such that $N_\nabla^{\mathcal{M}}|_{u=0} = N_{\mathcal{M}}$.

The category of (φ,N_∇)-modules over Λ_F of height $\leq h$ is denoted $\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla,\leq h}$ and it is naturally a proper subcategory $\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla,\leq h}\subset\mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$. An effective criteria for the existence of such an operator N_∇ on a Kisin module $\mathcal{M}\in\mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ is given in [BL20] called the *monodromy condition*, more on this in Section 5.2. Restricting to the category of (φ,N_∇)-modules will give us the desired equivalence.

Theorem 5.1.6: *There is an equivalence of categories*

$$\mathbf{D}_{\mathrm{Kis}}:\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla}\to\mathrm{MF}_{F\otimes_{\mathbb{Q}_p}K}^{\varphi,N}$$

with quasi-inverse functor denoted $\mathbf{M}_{\mathrm{Kis}}$.

Proof. When $F=\mathbb{Q}_p$ this is [Kis06, Theorem 1.2.15] . For general F, we will describe the argument from [BLL23, Remark 2.2] in full as we will use the same idea to extend scalars for other functors.

For the purposes of this proof, let $\mathbf{D} : \mathrm{Mod}_{\Lambda}^{\varphi,N_\nabla} \to \mathrm{MF}_K^{\varphi,N}$ denote the equivalence functor of [Kis06, Theorem 1.2.15] with quasi-inverse $\mathbf{M}$. Let us define $\mathbf{D}_{\mathrm{Kis}}$ and $\mathbf{M}_{\mathrm{Kis}}$ to be the functors making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla} \arrow[d] \arrow[r, shift right, "\mathbf{D}_{\mathrm{Kis}}"'] & \mathrm{MF}_{F\otimes_{\mathbb{Q}_p}K}^{\varphi,N} \arrow[l, shift right, "\mathbf{M}_{\mathrm{Kis}}"'] \arrow[d] \\
\mathrm{Mod}_{\Lambda}^{\varphi,N_\nabla} \arrow[r, shift right, "\mathbf{D}"'] & \mathrm{MF}_K^{\varphi,N} \arrow[l, shift right, "\mathbf{M}"']
\end{tikzcd}$$

where the vertical arrows are the natural forgetful functors forgetting the F-linear structure. We will show that $\mathbf{M}_{\mathrm{Kis}}$ is a left quasi-inverse functor of $\mathbf{D}_{\mathrm{Kis}}$ noting that the other direction is completely similar. Let $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla}$ and observe that commutativity of the diagram implies that there is a natural isomorphism $\alpha : \mathbf{M}_{\mathrm{Kis}}(\mathbf{D}_{\mathrm{Kis}}(\mathcal{M})) \to \mathcal{M}$ in $\mathrm{Mod}_{\Lambda}^{\varphi,N_\nabla}$ upon applying the forgetful functor. Since multiplying by a scalar in F is an endomorphism of $\mathcal{M}$ in $\mathrm{Mod}_{\Lambda}^{\varphi,N_\nabla}$ then the natural isomorphism α lifts to one in $\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla}$ proving that $\mathbf{M}_{\mathrm{Kis}}$ is a left quasi-inverse functor of $\mathbf{D}_{\mathrm{Kis}}$. The reverse direction is completely similar. $\square$

Here we have that the functor $\mathbf{D}_{\mathrm{Kis}}$ defines monodromy on its image by $N = N_\nabla|_{u=0}$. To recover the crystalline case, it follows that we need only consider those (φ, N_∇)-modules where $N_\nabla|_{u=0} = 0$.

5.2 The Monodromy Condition

Recall that in the previous section, we defined the category $\mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla,\leq h}$ to be the proper subcategory of Kisin modules $\mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ admitting a differential operator N_∇ over the derivation $-u\lambda\frac{d}{du}$ on Λ_F satisfying the relation $N_\nabla \circ \varphi = E\varphi \circ N_\nabla$ such that $N_\nabla|_{u=0} = N$. The results of Theorem 5.1.6 show that such Kisin modules are of particular interest to us as they may arise, by weak admissibility, from crystalline (more generally semi-stable) representations. In general, it is impractical to explicitly construct N_∇ which motivates the need for a criteria to deduce whether or not a given Kisin module admits such a differential operator.

Recall that we have the element

$$\lambda = \prod_{n=0}^{\infty} \varphi^n\left(\frac{E}{p}\right) \in \Lambda_F.$$

For a Kisin module $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ over Λ_F, we write $\mathcal{M}[1/\lambda] := \mathcal{M} \otimes_{\Lambda_F} \Lambda_F[1/\lambda]$ for the Kisin module over the enlarged ring $\Lambda_F[1/\lambda]$. The following lemma from [BL20, Lemma 2.2.1] shows that the desired operator always exists after inverting λ.

Lemma 5.2.1: *Let* $(\mathcal{M}, \varphi_{\mathcal{M}}, N_{\mathcal{M}}) \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$. *There exists a unique differential operator*

$$N_{\nabla}^{\mathcal{M}} : \mathcal{M}[1/\lambda] \to \mathcal{M}[1/\lambda]$$

over $-u\lambda\frac{d}{du}$ *such that* $N_{\nabla}^{\mathcal{M}}|_{u=0} = N_{\mathcal{M}}$ *and* $N_{\nabla}^{\mathcal{M}} \circ \varphi_{\mathcal{M}} = E(u)\varphi_{\mathcal{M}} \circ N_{\nabla}^{\mathcal{M}}$.

Proof. The existence of such an operator was known to Kisin in [Kis06, Lemma 1.3.10]. As for uniqueness, suppose that $N_{\nabla,1}^{\mathcal{M}}$ and $N_{\nabla,2}^{\mathcal{M}}$ are two such operators and set $H = N_{\nabla,1}^{\mathcal{M}} - N_{\nabla,2}^{\mathcal{M}}$. Since both differential operators are over $-u\lambda\frac{d}{du}$ and satisfy $N_{\nabla,j}^{\mathcal{M}} \circ \varphi_{\mathcal{M}} = E\varphi_{\mathcal{M}} \circ N_{\nabla,j}^{\mathcal{M}}$, then we see that H is an $\Lambda_F[1/\lambda]$-linear endomorphism on $\mathcal{M}[1/\lambda]$ such that $H(\mathcal{M}[1/\lambda]) \subseteq u\mathcal{M}[1/\lambda]$ and $H \circ \varphi_{\mathcal{M}} = E(u)\varphi_{\mathcal{M}} \circ H$.

Since $\mathcal{M}$ has finite height and E is invertible in $\mathcal{M}[1/\lambda]$ then $\mathcal{M}[1/\lambda] \subseteq \varphi_{\mathcal{M}}(\mathcal{M}[1/\lambda])$. Hence, $H(\mathcal{M}[1/\lambda]) \subseteq \varphi_{\mathcal{M}}(H(\mathcal{M}[1/\lambda])) = \varphi_{\mathcal{M}}(u\mathcal{M}[1/\lambda])$ and repeating this process will give us that $H(\mathcal{M}[1/\lambda]) \subseteq u^{p^{\ell}}\mathcal{M}[1/\lambda]$. As $\ell \to \infty$, then it becomes clear that $H = 0$ giving uniqueness. $\square$

Definition 5.2.2: Consider a Kisin module $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ and let $N_{\nabla}^{\mathcal{M}}$ be the differential operator obtained in Lemma 5.2.1. We say that $\mathcal{M}$ satisfied the *monodromy condition* if $N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}$.

For a Kisin module $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,N,\leq h}$, we will say that $\mathfrak{M}$ satisfies the monodromy condition if its extension of scalars $\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F$ satisfies the monodromy condition.

We will soon see that it suffices to check the monodromy condition on a slightly larger coefficient ring. Indeed, for $r > 0$, let $\Lambda_{F,r}$ denote the Λ_F-algebra of rigid analytic functions converging on f-many open p-adic disc of radius p^{-r}. For $n \geq 0$, we write $\mathfrak{S}_{F,n}$to denote the completion of $\mathfrak{S}_F$ at the ideal generated by $\varphi^n(E)$. For each embedding $\tau : K \hookrightarrow F$, the roots of $\tau(\varphi^n(E))$ lie on the p-adic circle of radius p^{1/p^n} so there is a natural inclusion

$$\iota_n : \Lambda_F \hookrightarrow \mathfrak{S}_{F,n}$$

which factors through $\Lambda_{F,r}$ whenever $r \leq p^{-n}$. For any $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$, we write the extension of scalars $\mathcal{M}_n := \mathcal{M} \otimes_{\Lambda_F,\iota_n} \mathfrak{S}_{F,n}$. The following lemma from [BL20, Prop 2.2.4] shows that it suffices to check the monodromy condition on $\mathfrak{S}_{F,0}$.

Lemma 5.2.3: *Let* $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ *and let* $N_{\nabla}^{\mathcal{M}}$ *be as in Lemma 5.2.1. Then* $\mathcal{M}$ *satisfies the monodromy condition if and only if* $\iota_0 N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}_0$.

Proof. The forward direction is clear as the monodromy condition is checked on $\Lambda_F \subset \mathfrak{S}_{F,0}$ so let us suppose that $\iota_0 N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}_0$. Our plan will be to induct on n so that $\iota_n N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}_n$ and as $n \to \infty$, we will achieve the monodromy condition.

Suppose for some $m \geq 0$ we have $\iota_n N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}_n$. By definition, φ induces a natural map $\varphi : \mathfrak{S}_{F,n} \to \mathfrak{S}_{F,n+1}$ and $\varphi_{\mathcal{M}}$ induces a φ-semilinear operator $\varphi_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ so that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\varphi_{\mathcal{M}}"] \arrow[d, "\iota_n"'] & \mathcal{M} \arrow[d, "\iota_{n+1}"] \\ \mathcal{M}_n \arrow[r, "\varphi_n"] & \mathcal{M}_{n+1} \end{tikzcd}$$

The relation $N_{\nabla}^{\mathcal{M}} \circ \varphi_{\mathcal{M}} = E\varphi_{\mathcal{M}} \circ N_{\nabla}^{\mathcal{M}}$ together with the above diagram will imply that

$$\iota_{n+1} N_{\nabla}^{\mathcal{M}} \varphi_{\mathcal{M}}(\mathcal{M}) = \iota_{n+1}(E)\varphi_n(\iota_n N_{\nabla}^{\mathcal{M}}(\mathcal{M})) \subset \mathcal{M}_{n+1}.$$

At the same time, we know that $\mathcal{M}$ has height $\leq h$ so that $E^h\mathcal{M}$ is contained in the Λ_F-span of $\varphi_{\mathcal{M}}(\mathcal{M})$. Thus we get that $\iota_{n+1}N_{\nabla}^{\mathcal{M}}(E^h\mathcal{M}) \subset \mathcal{M}_{n+1}$. Since $N_{\nabla}^{\mathcal{M}}$ is a differential operator, we may apply the Leibniz rule to get

$$\iota_{n+1} N_{\nabla}^{\mathcal{M}} \varphi_{\mathcal{M}}(\mathcal{M}) = \iota_{n+1}(E^h)\iota_{n+1} N_{\nabla}^{\mathcal{M}}(\mathcal{M}) + \iota_{n+1} N_{\nabla}^{\mathcal{M}}(E^h)\iota_{n+1}(\mathcal{M}) \subset \mathcal{M}_{n+1}.$$

Since $\iota_{n+1}(E^h)$ is a invertable in $\mathfrak{S}_{F,n+1}$, we get that $\iota_{n+1}N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subset \mathcal{M}_{n+1}$ as desired. □

The two previous lemmas can now be used to give an alternative criterion for the monodromy condition from [BL20, Corollary 2.2.5].

Proposition 5.2.4: *Let $\mathcal{M} \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N,\leq h}$ and set $\mathcal{M}_r = \mathcal{M} \otimes_{\Lambda_F} \Lambda_{F,r}$ for some $r \leq 1$. The following are equivalent:*

(a) $\mathcal{M}$ satisfies the monodromy condition;
(b) There exists a differential operator $N_{\nabla}^{\mathcal{M}_r} : \mathcal{M}_r \to \mathcal{M}_r$ such that $N_{\nabla}^{\mathcal{M}_r}|_{u=0} = N_{\mathcal{M}}$ and $N_{\nabla}^{\mathcal{M}_r} = E\varphi_{\mathcal{M}_r} N_{\nabla}^{\mathcal{M}_r}$.

Proof. The $(a) \Rightarrow (b)$ direction is once again clear. To see $(a) \Leftarrow (b)$, observe that Lemma 5.2.1 describes the existence of an operator $N_{\nabla}^{\mathcal{M}}$ and uniqueness will imply that $N_{\nabla}^{\mathcal{M}} = N_{\nabla}^{\mathcal{M}_r}$ with the right hand operator being the one arising from $\mathcal{M}_r[1/\lambda]$. Hence we have $N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subseteq \mathcal{M}_r \cap \mathcal{M}[1/\lambda]$. On the other hand, since $r \leq 1$ then the natural inclusion $\iota_0 : \Lambda_F \hookrightarrow \mathfrak{S}_{F,0}$ will factor through $\Lambda_{F,r}$ allowing us to write $\iota_0 N_{\nabla}^{\mathcal{M}}(\mathcal{M}) \subseteq \mathcal{M}_r \cap \mathcal{M}[1/\lambda] \subset \mathcal{M}_0$. Applying Lemma 5.2.3 then allows us to conclude (a). □

5.3 Reductions via Kisin Modules

As mentioned in the beginning of this chapter, Kisin modules over $\mathfrak{S}_F$ may be used to compute reductions of crystalline (or more generally semi-stable) representations. This is done via a canonical association Kisin modules satisfying the monodromy condition to Galois stable $\mathscr{O}_F$ lattices inside such representations which we detail in the following Theorem summarizing results of Kisin in [Kis06]. Let $V \in \mathrm{Rep}_{\mathrm{cris}/F}(G_K)$ be a crystalline representation and recall that $G_\infty = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K_\infty)$ so that $G_\infty \subset G_K$ is a subgroup.

Theorem 5.3.1: *If $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\leq h}$ satisfies the monodromy condition, then $\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F)$ is a weakly admissible filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$. Moreover, if $\mathbf{V}^*_{\mathrm{cris}}(\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F)) = V \in \mathrm{Rep}_{\mathrm{cris}/F}(G_K)$ then $\mathfrak{M} = \mathfrak{M}(T)$ is canonically attached to some G_∞-stable lattice $\mathscr{O}_F$-lattice $T \subset V$.*

Proof. This is a rewording of [Liu07, Theorem 5.4.1] which itself is a summary of results from [Kis06]. □

Recall that for any $s > 0$, we let $\Lambda_{F,s}$ denote the Λ_F-algebra of rigid analytic functions converging over $|u| < p^{-s}$. The following proposition from [BLL23, Prop. 2.1] allows us to make the association of Theorem 5.3.1 using a slightly larger coefficient ring.

Proposition 5.3.2: *For $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\leq h}$ and $V \in \mathrm{Rep}_{\mathrm{cris}/F}(G_K)$, suppose there exists s such that $1/p < s < 1$ and*

$$\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_{F,s} \cong \mathbf{M}_{\mathrm{Kis}}(\mathbf{D}^*_{\mathrm{cris}}(V)) \otimes_{\Lambda_F} \Lambda_{F,s}$$

in $\mathrm{Mod}_{\Lambda_{F,s}}^{\varphi,\leq h}$. Then $\mathfrak{M} = \mathfrak{M}(T)$ is canonically associated to some G_∞-stable $\mathscr{O}_F$-lattice $T \subset V$.

Proof. By Theorem 5.1.6 we know that $\mathbf{M}_{\mathrm{Kis}}(\mathbf{D}^*_{\mathrm{cris}}(V)) \otimes_{\Lambda_F} \Lambda_{F,s} \in \mathrm{Mod}_{\Lambda_{F,s}}^{\varphi,N_\nabla}$ and since $s < 1$ then Proposition 5.2.4 allows us to conclude that $\mathfrak{M}$ satisfies the monodromy condition and hence, $\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F \in \mathrm{Mod}_{\Lambda_F}^{\varphi,N_\nabla}$. Theorem 5.3.1 implies the existence of a crystalline representation $V' \in \mathrm{Rep}^{\mathbf{k}}_{\mathrm{cris}/F}(G_K)$ such that $\mathfrak{M} = \mathfrak{M}(T)$ for some G_∞-stable $\mathscr{O}_F$-lattice $T \subseteq V'$. To conclude the proposition, we need only show that $V \cong V'$.

Since $1/p < s < 1$ then [Kis06, Lemma 1.2.6] implies that $\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F)$ is completely determined by $\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_{F,s}$ and hence,

$$\mathbf{D}^*_{\mathrm{cris}}(V') \cong \mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F) \cong \mathbf{D}_{\mathrm{Kis}}(\mathbf{M}_{\mathrm{Kis}}(\mathbf{D}^*_{\mathrm{cris}}(V))) \cong \mathbf{D}^*_{\mathrm{cris}}(V)$$

with the final isomorphism holding due to Theorem 5.1.6. Since $\mathbf{D}^*_{\mathrm{cris}}$ is a fully faithful functor, we conclude that $V \cong V'$ as desired. □

The reader may have noticed that both Theorem 5.3.1 and Proposition 5.3.2 only allow us to associate Kisin modules to a G_∞-stable $\mathscr{O}_F$-lattices $T \subset V$ as opposed to our usual G_K-stable lattices. This is due to the fact that Kisin modules are naturally associated to *étale φ-modules* in a manner in which we now describe.

Let $\mathcal{O}_{\varepsilon,F}$ denote the p-adic completion of $\mathfrak{S}_F[1/u]$ with Frobenius action extended from $\mathfrak{S}_F[1/u]$ by continuity. Then $\mathcal{O}_{\varepsilon,F}$ is a semi-local ring with $\mathcal{O}_{\varepsilon,F} \otimes_{\mathscr{O}_F} k_F = k_F((u))$.

Definition 5.3.3: An *étale φ-module* over $\mathcal{O}_{\varepsilon,F}$ is a finite dimensional $\mathcal{O}_{\varepsilon,F}$-module M such that:

(a) M admits a φ-semilinear Frobenius endomorphism $\varphi_M : M \to M$;

(b) The linearization $\varphi_M^* = \mathcal{O}_{\varepsilon,F} \otimes_{\varphi_M, \mathcal{O}_{\varepsilon,F}} M \to M : c \otimes m \mapsto c\varphi_M(m)$ is an isomorphism.

We denote the category consisting of such objects by $\mathrm{Mod}_{\mathcal{O}_{\varepsilon,F}}^{\varphi,\text{ét}}$ with morphisms being linear maps compatible with the Frobenius endomorphisms. By replacing $\mathcal{O}_{\varepsilon,F}$ with $k_F((u))$, we receive the category of étale φ-modules over $k_F((u))$ denoted $\mathrm{Mod}_{k_F((u))}^{\varphi,\text{ét}}$. The following theorem of Fontaine shows how the absolute Galois group G_∞ appears in the theory.

Theorem 5.3.4: *There exists covariant equivalence functors*

$$\mathbf{V}_{\mathscr{O}_F} : \mathrm{Mod}_{\mathcal{O}_{\varepsilon,F}}^{\varphi,\text{ét}} \to \mathrm{Rep}_{/\mathscr{O}_F}(G_\infty) \qquad \mathbf{V}_{k_F} : \mathrm{Mod}_{k_F((u))}^{\varphi,\text{ét}} \to \mathrm{Rep}_{/k_F}(G_\infty)$$

satisfying the following compatibility relation for $M \in \mathrm{Mod}_{\mathcal{O}_{\varepsilon,F}}^{\varphi,\text{ét}}$,

$$\mathbf{V}_{\mathscr{O}_F}(M) \otimes_{\mathscr{O}_F} k_F \cong \mathbf{V}_{k_F}(M \otimes_{\mathscr{O}_F} k_F).$$

Proof. This is a summary of results from [Fon91, § A1] □

Returning to the Kisin module settings, given $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\leq h}$ then $\mathfrak{M} \otimes_{\mathfrak{S}_F} \mathcal{O}_{\varepsilon,F}$ defines an étale φ-module over $\mathcal{O}_{\varepsilon,F}$ as upon inverting u, the polynomial $E(u)$ becomes invertible so the finite height condition on $\mathfrak{M}$ implies the étale condition on $\mathfrak{M} \otimes_{\mathfrak{S}_F} \mathcal{O}_{\varepsilon,F}$, see Remark 5.1.2. In particular, by the results of Theorem 5.3.4, we have

$$\mathbf{V}_{\mathscr{O}_F}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \mathcal{O}_{\varepsilon,F}) \in \mathrm{Rep}_{/\mathscr{O}_F}(G_\infty) \qquad \mathbf{V}_{k_F}(\mathfrak{M} \otimes_{\mathscr{O}_F} k_F[1/u]) \in \mathrm{Rep}_{/k_F}(G_\infty).$$

Proposition 5.3.5: *Let* $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_F}^{\varphi,\leq h}$ *and* $V \in \mathrm{Rep}_{\mathrm{cris}/F}(G_K)$ *be such that* $\mathfrak{M} = \mathfrak{M}(T)$ *is the canonically associated Kisin module to a* G_∞*-stable* $\mathscr{O}_F$*-lattice* $T \subset V$ *from Theorem 5.3.1. Then* $(\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathscr{O}_F} k_F[1/u]))^{\mathrm{ss}} \cong \overline{V}|_{G_\infty}$.

Proof. By assumption we know that $\overline{\mathbf{V}_{\mathrm{cris}}^*(\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F))} = \overline{V}$ is a semi-simple reduction. On the other hand, since the restriction res : $\mathrm{Rep}_{\mathrm{cris}/F}(G_K) \to \mathrm{Rep}_{\mathrm{cris}/F}(G_\infty)$ is a fully faithful functor by [Kis06, Cor. 2.1.14] then semi-simplicity if preserved so to prove the proposition, we need only show that $\overline{\mathbf{V}_{\mathrm{cris}}^*(\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F))}|_{G_\infty} \cong (\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathscr{O}_F} k_F[1/u]))^{\mathrm{ss}}$.

We may write

$$\begin{aligned}
\overline{\mathbf{V}_{\mathrm{cris}}^*(\mathbf{D}_{\mathrm{Kis}}(\mathfrak{M} \otimes_{\mathfrak{S}_F} \Lambda_F))}|_{G_\infty} &= ((\mathbf{V}_{\mathscr{O}_F}^*(\mathfrak{M} \otimes_{\mathfrak{S}_F} \mathcal{O}_{\varepsilon,F}) \otimes_{\mathscr{O}_F} k_F)^{\mathrm{ss}} \\
&\cong (\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathfrak{S}_F} (\mathcal{O}_{\varepsilon,F} \otimes_{\mathscr{O}_F} k_F))^{\mathrm{ss}} \\
&= (\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathfrak{S}_F} k_F((u)))^{\mathrm{ss}}
\end{aligned}$$

with the isomorphism in the second line following from Theorem 5.3.4. Since $\mathfrak{M}$ is a finite height Kisin module then we can write $(\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathfrak{S}_F} k_F((u)))^{\mathrm{ss}} = (\mathbf{V}_{k_F}^*(\mathfrak{M} \otimes_{\mathscr{O}_F} k_F[1/u]))^{\mathrm{ss}}$ and the proposition follows. □

With this, we now have a method to compute reductions of crystalline representations V so long as we can identify the Kisin module associated to some Galois stable $\mathscr{O}_F$-lattice $T \subset V$. Unfortunately, finding such a Kisin module is generally very difficult. This motivates the need for an auxiliary category that allows us to translate the data of strongly divisible lattices in weakly admissible filtered φ-modules into Kisin modules. The trade off will be that this can only be done over more complicated coefficients.

5.4 Breuil Modules

The theory of Breuil modules was developed by its namesake in [Bre97]. Roughly, a Breuil module is an analogue of a filtered (φ, N)-module over a more complicated *Breuil ring* S_{Bre} defined to be the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. In actuality, we may replace this complicated ring by a much more concrete subring S_F which we define below using the ideas found in [BLL23, §2.3].

Recall that $K_0[\![u]\!]$ comes equipped with a Frobenius φ and define an operator $N = -u\frac{d}{du}$ on $K_0[\![u]\!]$. Letting $S := W(k)[\![u, \frac{E^p}{p}]\!] = \mathfrak{S}[\![\frac{E^p}{p}]\!]$, we may notice that S is closed under both φ and N. We also endow S with a filtration structure given by $\mathrm{Fil}^j S = E^j S$. Extending scalars to F, define $S_F := S \otimes_{\mathbb{Z}_p} F$ and extend the the structures of φ, N, and $\mathrm{Fil}^\bullet$ by F-linearity.

Definition 5.4.1: A *Breuil module* $\mathcal{D}$ over S_F is a φ-module over S_F such that

- The linearization of $\varphi_{\mathcal{D}}$ is an isomorphism;
- $\mathcal{D}$ is equipped with a decreasing filtration $\mathrm{Fil}^\bullet\mathcal{D}$ with $\mathrm{Fil}^0\mathcal{D} = \mathcal{D}$ and $\mathrm{Fil}^j S_F \cdot \mathrm{Fil}^\ell \mathcal{D} \subseteq \mathrm{Fil}^{j+\ell}\mathcal{D}$;
- $\mathcal{D}$ is equipped with a derivation $N_{\mathcal{D}}$ over N such that both $N_{\mathcal{D}}\varphi_{\mathcal{D}} = p\varphi_{\mathcal{D}}N_{\mathcal{D}}$ and $N_{\mathcal{D}}(\mathrm{Fil}^j\mathcal{D}) \subseteq \mathrm{Fil}^{j-1}\mathcal{D}$.

We denote the category of Breuil modules over S_F by $\mathrm{MF}_{S_F}^{\varphi,N}$ with morphisms being S_F-linear maps compatible with the φ, N, and $\mathrm{Fil}^\bullet$ structures.

Consider the functor
$$\mathbf{D}_{\mathrm{Bre}} : \mathrm{MF}_{F\otimes_{\mathbb{Q}_p}K}^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$$
defined by $\mathbf{D}_{\mathrm{Bre}}(D) := S_F \otimes_{K_0\otimes_{\mathbb{Q}_p}F} D = \mathcal{D}$ with $\varphi_{\mathcal{D}} = \varphi \otimes \varphi_D$, $N_{\mathcal{D}} = N \otimes 1 + 1 \otimes N_D$ and
$$\mathrm{Fil}^j\mathcal{D} = \{x \in \mathcal{D} : N_{\mathcal{D}}(x) \in \mathrm{Fil}^{j-1}\mathcal{D} \text{ and } (\mathrm{eval}_\pi \otimes 1)(x) \in \mathrm{Fil}^j D\}$$
where $\mathrm{eval}_\pi : S_F \to F \otimes_{\mathbb{Q}_p} K$ is the scalar extension of the evaluation at π_K map $\mathrm{eval}_\pi : W(k)[u] \to \mathscr{O}_K$.

Theorem 5.4.2: *The functor* $\mathbf{D}_{\mathrm{Bre}} : \mathrm{MF}_{F\otimes_{\mathbb{Q}_p}K}^{\varphi,N} \to \mathrm{MF}_{S_F}^{\varphi,N}$ *defines an equivalence of categories.*

Proof. In [Bre97, §6], Breuil proves that $\mathbf{D}_{\mathrm{Bre}}$ is an equivalence of categories provided that $F = \mathbb{Q}_p$ and the ring S is replaced by the Breuil ring S_{Bre}. The arguments found in the proof of Theorem 5.1.6 allow us to extend scalars to general F. In order to see why we may replace S_{Bre} with S, it turns out that one needs only replace [Bre97, Prop 6.2.1.1] with [BLL23, Lemma 2.4]. For more details, please see [BLL23, Theorem 2.3] □

Since we are primarily concerned with crystalline representations, we would like to work with Breuil modules without monodromy. However, since the filtration structure is connected to monodromy for a Breuil module, we need a new definition.

Definition 5.4.3: A *Breuil module without monodromy* over S_F is a φ-module $\mathcal{D}$ over S_F such that

- the linearization of $\varphi_{\mathcal{D}}$ is an isomorphism;
- There exists $h \geq 0$ such that $\mathcal{D}$ contains a finite free S_F-submodule $\mathrm{Fil}^h\mathcal{D} \subseteq \mathcal{D}$ such that $\mathrm{Fil}^h S_F \cdot \mathcal{D} \subseteq \mathrm{Fil}^h\mathcal{D}$.

We denote the category of Breuil modules without monodromy by $\mathrm{MF}^{\varphi,h}_{S_F}$. By [BLL23, Lemma 2.5] we then have a natural forgetful functor $\mathrm{MF}^{\varphi,N}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$ which forgets the monodromy structure.

5.5 Comparing Breuil and Kisin Modules

The results of Theorems 5.1.6 and 5.4.2 give rise to a threefold equivalence of categories

$$\mathrm{Mod}^{\varphi,N\nabla}_{\Lambda_F} \xleftarrow{\mathbf{M}_{\mathrm{Kis}}} \mathrm{MF}^{\varphi,N}_{F\otimes_{\mathbb{Q}_p}K} \xrightarrow{\mathbf{D}_{\mathrm{Bre}}} \mathrm{MF}^{\varphi,N}_{S_F}.$$

We have seen that the functor $\mathbf{D}_{\mathrm{Bre}}$ gives us a natural way to construct a Breuil module from a filtered φ-module but this avenue is unsuitable to compute reductions. On the other hand, the functor $\mathbf{D}_{\mathrm{Kis}}$ is difficult to get a hold on structurally, but once done, we have a natural way to compute reductions. Our goal is then to bridge the gap between these two categories by first using $\mathbf{D}_{\mathrm{Bre}}$ to construct a Breuil module without monodromy before noticing that this can really be interpreted as a finite height Kisin module over some suitably large coefficient ring.

Define the functor

$$\mathbf{D}_{\mathrm{BK}} : \mathrm{Mod}^{\varphi,\leq h}_{S_F} \to \mathrm{MF}^{\varphi,h}_{S_F}$$

by setting $\mathbf{D}_{\mathrm{BK}}(\mathcal{M}) \coloneqq S_F \otimes_{\varphi,S_F} \mathcal{M}$ such that $\varphi_{\mathbf{D}_{\mathrm{BK}}(\mathcal{M})} = \varphi \otimes \varphi_{\mathcal{M}}$ and

$$\mathrm{Fil}^h\mathbf{D}_{\mathrm{BK}}(\mathcal{M}) = \{x \in \mathbf{D}_{\mathrm{BK}}(\mathcal{M}) : (1 \otimes \varphi_{\mathcal{M}})(x) \in \mathrm{Fil}^h S_F \cdot \mathcal{M}\}.$$

By [BLL23, Lemma 2.5], $\mathrm{Fil}^h\mathbf{D}_{\mathrm{BK}}(\mathcal{M})$ is finite free over S_F. Let us define the element $\gamma = \varphi(E)/p \in S_F^*$. The following theorem appears as [BLL23, Proposition 2.6].

Theorem 5.5.1: *The functor* $\mathbf{D}_{\mathrm{BK}} : \mathrm{Mod}_{S_F}^{\varphi,\leq h} \to \mathrm{MF}_{S_F}^{\varphi,h}$ *defines an equivalence of categories.*

Proof. We will first show that $\mathbf{D}_{\mathrm{BK}}$ is fully faithful before displaying essential subjectivity. Let $\mathcal{M}$ and $\mathcal{M}'$ be finite height Kisin modules over S_F in the category $\mathrm{Mod}_{S_F}^{\varphi,\leq h}$. Set $\mathcal{D} := \mathbf{D}_{\mathrm{BK}}(\mathcal{M})$ and $\mathcal{D}' := \mathbf{D}_{\mathrm{BK}}(\mathcal{M}')$ to be Breuil modules without monodromy over S_F.

Let us choose a basis $\{e_1, \ldots, e_d\}$ of $\mathcal{M}$ so that $\varphi_{\mathcal{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A$ for some $A \in \mathrm{Mat}_d(S_F)$. Since $\mathcal{M}$ has finite height $\leq h$, then we know that there exists a matrix $B \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^h I_d$. By the definition of $\mathbf{D}_{\mathrm{BK}}$, we know that $\{\tilde{e}_j := 1 \otimes e_j\}_{j=1}^d$ forms a basis of $\mathcal{D}$ and since $E^h\mathcal{D} \subseteq \mathrm{Fil}^h\mathcal{D}$, then there exists a basis $\{\eta_j\}_{j=1}^d$ of $\mathrm{Fil}^h\mathcal{D}$ such that $(\eta_1, \ldots, \eta_d) = (\tilde{e}_1, \ldots, \tilde{e}_d)B$. In the same manner, we will get A', B' and $\{\tilde{e}_j'\}_{j=1}^{d'}$ from a chosen basis $\{e_j\}_{j=1}^{d'}$ of $\mathcal{M}'$ as well.

Let $g : \mathcal{D} \to \mathcal{D}'$ be a morphism of Breuil modules in the category $\mathrm{MF}_{S_F}^{\varphi,h}$. That is, we may write $g(\tilde{e}_1 \ldots, \tilde{e}_d) = (\tilde{e}_1', \ldots, \tilde{e}_{d'}')X$ for some $X \in \mathrm{Mat}_d(S_F)$. By definition, g is φ-equivariant implies that $X\varphi(A) = \varphi(A')\varphi(X)$. In addition, g is compatible with the filtration structures meaning $g(\mathrm{Fil}^h\mathcal{D}) \subseteq \mathrm{Fil}^h\mathcal{D}'$ implying that there exists some $Y \in \mathrm{Mat}_d(S_F)$ such that $XB = B'Y$. Rewriting $X = B'Y(B)^{-1}$, plugging into $X\varphi(A) = \varphi(A')\varphi(X)$ and using the fact that $AB = E^h I_d$ and $A'B' = E^h I_{d'}$ will imply that

$$\varphi(E^h) \cdot \varphi(Y) = \varphi(E^h) \cdot X.$$

Since $\varphi(E)$ is invertible in S_F then $X = \varphi(Y)$ and we may realize that $YA = A'\varphi(Y)$. If we define $G : \mathcal{M} \to \mathcal{M}'$ by $G(e_1, \ldots, e_d) = (e_1', \ldots, e_{d'}')Y$ then G is φ-equivariant and $g = \mathbf{D}_{\mathrm{BK}}(G)$ by the above discussion. Hence $\mathbf{D}_{\mathrm{BK}}$ is full and since Y completely determines X, we see that $\mathbf{D}_{\mathrm{BK}}$ is faithful.

As mentioned prior, we conclude with proving essential subjectivity. Suppose we are given a Breuil module $\mathcal{D} \in \mathrm{MF}_{S_F}^{\varphi,h}$ and choose bases $\{e_j\}_{j=1}^d$ of $\mathcal{D}$ and $\{\eta_j\}_{j=1}^d$ of $\mathrm{Fil}^h\mathcal{D}$. Under these bases we may write $(\eta_1, \ldots, \eta_d) = (e_1, \ldots, e_d)B$ and $\varphi_{\mathcal{D}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X$ where $B \in \mathrm{Mat}_d(S_F)$ and $X \in \mathrm{GL}_d(S_F)$ due to the fact that the linearization of $\varphi_{\mathcal{D}}$ is necessarily invertible. Since $E^h\mathcal{D} \subseteq \mathrm{Fil}^h\mathcal{D}$ then there exists a matrix $A \in \mathrm{Mat}_d(S_F)$ such that $AB = BA = E^h I_d$. As a result, since $\varphi(E) = p\gamma \in S_F^*$ then $\varphi(B) \in \mathrm{GL}_d(S_F)$ so that

$$\varphi_{\mathcal{D}}(\eta_1, \ldots, \eta_d) = (e_1, \ldots, e_d)X\varphi(B)$$

where $X\varphi(B) \in \mathrm{GL}_d(S_F)$.

We introduce another basis $\{\mu_j\}_{j=1}^d$ defined by $(\mu_1, \ldots, \mu_d) = (e_1, \ldots, e_d)X\varphi(B)\varphi(E)^{-h}$ so that $\varphi_{\mathcal{D}}(\eta_1, \ldots, \eta_d) = (\mu_1, \ldots, \mu_d)\varphi(E)^h$. We necessarily then have that

$$(\eta_1, \ldots, \eta_d) = (\mu_1, \ldots, \mu_d)(X\varphi(B)\varphi(E)^{-h})B$$

and the finite height condition will imply the existence of $A' \in \mathrm{Mat}_d(S_F)$ such that the product $A'(X\varphi(B)\varphi(E)^{-h})B = E^h I_d$. Let us construct a finite free S_F-module $\mathcal{M} = \oplus_{j=1}^d S_F\rho_j$ such that $\rho_j := 1 \otimes \mu_j$ and set $\varphi_{\mathcal{M}}(\rho_1, \ldots, \rho_d) = (\rho_1, \ldots, \rho_d)A'$. The above discussion shows that $\mathcal{M}$ is a finite height Kisin module in $\mathrm{Mod}_{S_F}^{\varphi,\leq h}$ and that $\mathbf{D}_{\mathrm{BK}}(\mathcal{M}) = \mathcal{D}$ as desired. $\square$

To this point, we have seen that the functors $\mathbf{D}_{\mathrm{Kis}}$ and $\mathbf{D}_{\mathrm{Bre}}$ give us a way go to from filtered φ-modules to either Kisin or Breuil modules respectively. Additionally, the functor $\mathbf{D}_{\mathrm{BK}}$ allows us to switch between Kisin and Breuil modules over S_F. The following proposition from [BLL23, Theorem 2.7] does the job of connecting all of these functors together.

Proposition 5.5.2: *Let $D \in \mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p}K}$ be a filtered φ-module. Set $\mathcal{D} := \mathbf{D}_{\mathrm{Bre}}(D) \in \mathrm{MF}^{\varphi,h}_{S_F}$ to be the image of D under the forgetful functor forgetting monodromy and set $\mathcal{M} := \mathbf{M}_{\mathrm{Kis}}(D) \otimes_{\Lambda_F} S_F \in \mathrm{Mod}^{\varphi,\leq h}_{S_F}$. Then there is a natural isomorphism $\mathbf{D}_{\mathrm{BK}}(\mathcal{M}) \cong \mathcal{D}$.*

Proof. This is another technical exercise in using known results from [Liu07, §3.2] after replacing S_{Bre} with S and $\mathbb{Q}_p$ with F. For details, see [BLL23, Thm 2.7]. $\square$

We summarize the interaction of all these functors in the following diagram:

$$\begin{tikzcd}
 & \mathrm{MF}^{\varphi}_{F\otimes_{\mathbb{Q}_p}K} \arrow[dl, "\mathbf{M}_{\mathrm{Kis}}"'] \arrow[dr, "\mathbf{D}_{\mathrm{Bre}}"] & \\
\mathrm{Mod}^{\varphi,\leq h}_{\Lambda_F} \arrow[d, "\mathrm{Extend}"'] & & \mathrm{MF}^{\varphi,N}_{S_F} \arrow[d, "\mathrm{Forget}"] \\
\mathrm{Mod}^{\varphi,\leq h}_{S_F} \arrow[rr, "\mathbf{D}_{\mathrm{BK}}"] & & \mathrm{MF}^{\varphi,h}_{S_F}
\end{tikzcd}$$

Remark 5.5.3: The second half of the proof of Theorem 5.5.1 together with Proposition 5.5.2 allows us to describe an algorithm to explicitly compute a finite height Kisin module over S_F from a filtered φ-module by passing first through the category of Breuil modules. In particular, we follow the steps:

1) Choose S_F-bases $\{e_1, \ldots, e_d\}$ of $\mathcal{D}$ and $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathrm{Fil}^h\mathcal{D}$;
2) Write $\varphi_{\mathcal{D}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X$ and $(\alpha_1, \ldots, \alpha_d) = (e_1, \ldots, e_d)B$ for $X, B \in \mathrm{Mat}_d(S_F)$;
3) Then, $\mathcal{M}$ admits an S_F-basis $\{\eta_1, \ldots, \eta_d\}$ in which $\varphi_{\mathcal{M}}(\eta_1, \ldots, \eta_d) = (\eta_1, \ldots, \eta_d)A$ where
$$A = E^h B^{-1} X\varphi(B)\varphi(E)^{-h}.$$

Chapter 6

Determining Explicit Kisin Modules

The results of Chapter 5 gives us a way to explicitly generate a Kisin module over S_F from the data of a weakly admissible filtered φ-module. In Chapter 4, we provided models for irreducible, two-dimensional crystalline representations of G_K in terms of the Type of Frobenius matrix on strongly divisible lattices inside the associated rank-two weakly admissible filtered φ-modules. In this chapter, we combine these results to compute the rank-two Kisin modules associated to a suitable representative of isomorphism classes of irreducible crystalline representations.

In Section 6.1, we begin by using the algorithm displayed at the end of the last chapter to construct the Frobenius matrix over S_F on our Kisin modules. In Section 6.2 we simplify this Frobenius matrices into a form that will be necessary for later computations. Finally, Section 6.3 provides an explicit description of the Frobenius matrices of Kisin modules when we restrict to the $f = 2$ case.

6.1 Initial Frobenius Matrices

We use the algorithm described in Remark 5.5.3 to explicitly describe the Frobenius matrix on a Kisin module $\mathcal{M}$ over S_F from the strongly divisible lattices described in Chapter 4. We briefly recall the algorithm adjusted to our situation below.

Let $(A) = (A^{(i)}) \in \mathrm{GL}_2(\mathscr{O}_F)^f$. For a rank two, weakly admissible filtered φ-module $D(A) \in \mathrm{MF}^{\varphi,w.a.,\mathbf{k}}_{F\otimes_{\mathbb{Q}_p}K}$ with each $\mathbf{k}^{(i)} = (k_i, 0)$ by Remark 3.2.4, let $\mathcal{D}(A) \coloneqq \mathbf{D}_{\mathrm{Bre}}(D(A))$ denote the corresponding Breuil module over S_F. Also let $\mathcal{M}(A) \coloneqq \mathbf{M}_{\mathrm{Kis}}(D(A)) \otimes_{\mathscr{O}_F} S_F$ denote the corresponding finite height Kisin module over S_F. By the results of Proposition 5.5.2 and the compatibility of the decomposition $D(A) = \prod_i D(A)^{(i)}$ with respect to the Breuil and Kisin equivalence functors, we have that $\mathbf{D}_{\mathrm{BK}}(\mathcal{M}(A)) \cong \mathcal{D}(A)$ and we may describe the Frobenius matrix on $\mathcal{M}(A)$ in terms of the idempotent decomposition by the following steps:

1) Choose S_F-bases $\{e_1^{(i)}, e_2^{(i)}\}$ of $\mathcal{D}(A)^{(i)}$ and $\{\alpha_1^{(i)}, \alpha_2^{(i)}\}$ of $\mathrm{Fil}^{k_i}\mathcal{D}(A)^{(i)}$;
2) Write $\varphi^{(i)}_{\mathcal{D}(A)}(e_1^{(i-1)}, e_2^{(i-1)}) = (e_1^{(i)}, e_2^{(i)})X^{(i)}$ and $(\alpha_1^{(i)}, \alpha_2^{(i)}) = (e_1^{(i)}, e_2^{(i)})B^{(i)}$ for matrices $X^{(i)}, B^{(i)} \in \mathrm{Mat}_2(S_F)$;
3) Then the Kisin module $\mathcal{M}(A)$ admits an S_F-basis $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ with i-th partial Frobenius given by $\varphi^{(i)}_{\mathcal{M}(A)}(\eta_1^{(i-1)}, \eta_2^{(i-1)}) = (\eta_1^{(i)}, \eta_2^{(i)})\mathcal{A}^{(i)}$ where

$$\mathcal{A}^{(i)} = E^{k_i}(B^{(i)})^{-1}X^{(i)}\varphi(B^{(i-1)})\varphi(E)^{-k_{i-1}}.$$

Remark 6.1.1: To be completely accurate, recall that our Kisin modules are defined over rings previously defined in Section 1.2 by $\mathfrak{S}_F = (\mathscr{O}_F \otimes_{\mathbb{Z}_p} \mathscr{O}_K)[\![u]\!]$, $\Lambda_F \subset (F \otimes_{\mathbb{Q}_p} K)[\![u]\!]$ and $S_F = \mathfrak{S}[\![\frac{E^p}{p}]\!] \otimes_{\mathbb{Z}_p} F$. When passing to the embedding decomposition, we may use the ring isomorphisms of Lemmas 2.4.1 and 3.1.1 to get Kisin modules which decompose into pieces with coefficients in $\mathscr{O}_F[\![u]\!]$ and inside of $F[\![u]\!]$. We will abuse notation and continue to write $\mathfrak{S}_F$ and Λ_F with the understanding that we are no longer defined over the aforementioned tensor when working with decomposed pieces. We also make the identification $S_F = \mathfrak{S}_F[\![\frac{E^p}{p}]\!]$ at this level as well. We hope this causes no confusion.

Proposition 6.1.2: *The finite height Kisin module $\mathcal{M}(A)$ admits an S_F-basis $\{\eta_1^{(i)}, \eta_2^{(i)}\}$ such that $\varphi^{(i)}_{\mathcal{M}(A)}(\eta_1^{(i-1)}, \eta_2^{(i-1)}) = (\eta_1^{(i)}, \eta_2^{(i)})\mathcal{A}^{(i)}$ where*

$$\mathcal{A}^{(i)} = \begin{pmatrix} E^{k_i} & 0 \\ 0 & 1 \end{pmatrix} A^{(i)}\Delta_{\mathbf{k}^{(i-1)}} \begin{pmatrix} \varphi(E)^{-k_{i-1}} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. To begin, we recall that by definition, the weakly admissible filtered φ-module $D(A)$ contains a strongly divisible lattice $L(A)$ with a basis adapted to the induced filtration, see Proposition3.1.6. In particular, $D(A)$ admits an F-basis $\{e_1^{(i)}, e_2^{(i)}\}$ such that the i-th partial Frobenius is $\varphi^{(i)}_{D(A)}(e_1^{(i-1)}, e_2^{(i-1)}) = (e_1^{(i)}, e_2^{(i)})A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}$ with $\Delta_{\mathbf{k}^{(i)}} = \mathrm{Diag}(p^{k_i}, 1)$.

For each $e_j^{(i)} \in D(A)$, set $\hat{e}_j^{(i)} := 1_S \otimes e_j^{(i)} \in \mathcal{D}(A)$. Then $\mathcal{D}(A)$ is a free S_F-module with basis $\{\hat{e}_1^{(i)}, \hat{e}_2^{(i)}\}$ such that

$$\varphi^{(i)}_{\mathcal{D}(A)}(\hat{e}_1^{(i-1)}, \hat{e}_2^{(i-1)}) = (\hat{e}_1^{(i)}, \hat{e}_2^{(i)})A^{(i)}\Delta_{\mathbf{k}^{(i-1)}}.$$

Since $D(A)$ has trivial monodromy then it is easy to compute $\mathrm{Fil}^{k_i}(\mathcal{D}(A)^{(i)}) = S_F\hat{e}_1^{(i)} \oplus S_F E^{k_i}\hat{e}_2^{(i)}$. Hence, we may choose the basis $\{\hat{e}_1^{(i)}, E^{k_i}\hat{e}_2^{(i)}\}$ of $\mathrm{Fil}^{k_i}(\mathcal{D}(A)^{(i)})$ so that the matrix B of step (2) above is given by $(B) = (B^{(i)})$ where $B^{(i)} = \mathrm{Diag}(1, E^{k_i})$. A brief matrix computation then gives us that the matrix $(\mathcal{A}) = (\mathcal{A}^{(i)})$ of step (3) is as described. □

We can now explicitly describe the Frobenius matrices $\mathcal{A}^{(i)}$ on a Kisin module $\mathcal{M}(A) = \prod_i \mathcal{M}(\mathcal{A})^{(i)}$ in terms of the matrix Types of $(A^{(i)})$. Recall $\gamma = \varphi(E)/p \in S_F^*$ so that $\varphi(E) = p\gamma$.

- Suppose $A^{(i)}$ is of Type I. We may then write $A^{(i)} = \begin{pmatrix} 0 & a_1^{(i)} \\ 1 & a_2^{(i)} \end{pmatrix}$ so that

$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ \gamma^{-k_{i-1}} & a_2^{(i)} \end{pmatrix};$$

- Suppose $A^{(i)}$ is of Type II. We may then write $A^{(i)} = \begin{pmatrix} a_1^{(i)} & 0 \\ a_2^{(i)} & 1 \end{pmatrix}$ so that

$$\mathcal{A}^{(i)} = \begin{pmatrix} \gamma^{-k_{i-1}} E^{k_i} a_1^{(i)} & 0 \\ \gamma^{-k_{i-1}} a_2^{(i)} & 1 \end{pmatrix}.$$

Since the partial Frobenius matrices $(\mathcal{A}^{(i)})$ on $\mathcal{M}(A)$ are completely determined by the f-tuple $(A^{(i)})$, we will commonly write $\mathcal{M}(A) = \mathcal{M}(\mathcal{A})$ with the understanding that $(\mathcal{A})$ comes from (A) which in turn comes from some strongly divisible lattice $\Theta(A)$. We will call each k_i a *labeled height* corresponding to $\mathcal{M}(\mathcal{A})^{(i)}$.

6.2 Change of Basis

From the previous section, we explicitly defined finite height, rank-two Kisin modules over S_F arising from irreducible, weakly admissible filtered φ-modules. The purpose of this section is to change base so that the determinants of partial Frobenius are in a particular form. This special basis will be a critical piece of the puzzle when we later compute reductions.

For a base change f-tuple $(X^{(i)})$, we will write $(X^{(i)}) *_\varphi (\mathcal{A}^{(i)}) = (\mathcal{B}^{(i)})$ to denote

$$\mathcal{B}^{(i)} = X^{(i)} \cdot \mathcal{A}^{(i)} \cdot \varphi(X^{(i-1)})^{-1}$$

for all $i \in \mathbb{Z}/f\mathbb{Z}$. As we are now dealing with φ-twisted base changes, we would like to first introduce some simplifying notation. For $f(\varphi) = \sum c_j \varphi^j \in \mathbb{Z}[\varphi]$ we will write $g^{f(\varphi)} = \prod_j \varphi^j(g)^{c_i}$. Recall $\gamma = \varphi(E)/p \in S_F^*$.

Lemma 6.2.1: *For $f(\varphi) \in \mathbb{Z}[\varphi]$, consider the equation $x^{\varphi^b - 1} = \gamma^{-f(\varphi)}$ over S_F for some integer $b > 0$. Then there exists a solution $x = \lambda_b^{f(\varphi)}$ where*

$$\lambda_b = \prod_{n \geq 0} \varphi^{bn}(\gamma) \in S_F^*.$$

Proof. It suffices to show that $\varphi^b(\lambda_b^{f(\varphi)}) = \gamma^{-f(\varphi)} \lambda_b^{f(\varphi)}$. The right hand side of the equation gives

$$\gamma^{-f(\varphi)} \lambda_b^{f(\varphi)} = \left(\frac{p^{f(\varphi)}}{\varphi(E)^{f(\varphi)}} \right) \left(\prod_{n \geq 0} \varphi^{bn} \left(\frac{\varphi(E)^{f(\varphi)}}{p^{f(\varphi)}} \right) \right) = \prod_{n \geq 1} \varphi^{bn} \left(\frac{\varphi(E)^{f(\varphi)}}{p^{f(\varphi)}} \right) = \varphi^b(\lambda_b^{f(\varphi)})$$

The fact that $\lambda_b \in S_F^*$ follows form the fact that $\gamma \in S_F^*$. $\square$

Recall that we have decided to write $(xy)^{(i)} = x^{(i)}y^{(i)}$.

Proposition 6.2.2: *There exists an f-tuple $(X^{(i)}) \in \mathrm{GL}_2(S_F)^f$ such that $(X^{(i)}) *_\varphi (\mathcal{A}^{(i)}) = (\mathcal{B}^{(i)})$ with $\det(\mathcal{B}^{(i)}) \in E^{k_i}\mathscr{O}_F^*$ for all $i \in \mathbb{Z}/f\mathbb{Z}$.*

Proof. Consider the matrices $X^{(i)} = \mathrm{Diag}(x_1^{(i)}, x_2^{(i)})$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Computing $(X^{(i)}) *_\varphi (\mathcal{A}^{(i)})$ in each embedding gives two cases depending on the Type of $\mathcal{A}^{(i)}$:

$$\text{Type I}: \begin{pmatrix} 0 & \frac{E^{k_i}(a_1x_1)^{(i)}}{\varphi(x_2^{(i-1)})} \\ \frac{x_2^{(i)}}{\gamma^{k_i-1}\varphi(x_1^{(i-1)})} & \frac{(a_2x_2)^{(i)}}{\varphi(x_2^{(i-1)})} \end{pmatrix} \qquad \text{Type II}: \begin{pmatrix} \frac{E^{k_i}(a_1x_1)^{(i)}}{\gamma^{k_i-1}\varphi(x_1^{(i-1)})} & 0 \\ \frac{(a_2x_2)^{(i)}}{\gamma^{k_i-1}\varphi(x_1^{(i-1)})} & \frac{x_2^{(i)}}{\varphi(x_2^{(i-1)})} \end{pmatrix}.$$

In the case of Type I, we set

$$x_1^{(i)} = \varphi(x_2^{(i-1)}) \qquad x_2^{(i)} = \gamma^{k_{i-1}}\varphi(x_1^{(i-1)}),$$

and in the case of Type II, we set

$$x_1^{(i)} = \gamma^{k_{i-1}}\varphi(x_1^{(i-1)}) \qquad x_2^{(i)} = \varphi(x_2^{(i-1)}).$$

Hence, depending on the Type of each $\mathcal{A}^{(i)}$, we get a recursive system of $2f$ equations. When there is an even number of Type I matrices, the system admits two disjoint f-cycles of which $x_1^{(i)}$ is apart of one and $x_2^{(i)}$ is apart of the other. On the other hand, when the number of Type I matrices is odd, then the system consists of a single $2f$-cycle.[8]

Solving for $x_1^{(i)}$ and $x_2^{(i)}$ will give

$$(x_1^{(i)})^{\varphi^b-1} = \gamma^{-g^{(i)}(\varphi)} \qquad (x_2^{(i)})^{\varphi^b-1} = \gamma^{-h^{(i)}(\varphi)}$$

for some $g^{(i)}(\varphi), h^{(i)}(\varphi) \in \mathbb{Z}[\varphi]$. When there is an even number of Type I matrices, the two disjoint f-cycles forces $b = f$ and when there is an odd number of Type I, the single $2f$-cycle means $b = 2f$.

By Lemma 6.2.1, there exists solutions $x_1^{(i)} = \lambda_b^{g^{(i)}(\varphi)}$ and $x_2^{(i)} = \lambda_b^{h^{(i)}(\varphi)}$ in S_F^*. The result is a matrix $\mathcal{B}^{(i)}$ given by

- Type I:
$$\mathcal{B}^{(i)} = \begin{pmatrix} 0 & E^{k_i}a_1^{(i)} \\ 1 & a_2^{(i)}\lambda_b^{h^{(i)}(\varphi)-g^{(i)}(\varphi)} \end{pmatrix};$$
- Type II:
$$\mathcal{B}^{(i)} = \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ a_2^{(i)}\lambda_b^{h^{(i)}(\varphi)-g^{(i)}(\varphi)} & 1 \end{pmatrix}.$$

[8]This is the first sign that there is a dichotomy of behavior depending on the even or oddness of the number of Type I matrices in the Type combination. Indeed, we will see that the shape of the reductions depend on this as well.

Hence, we see $\det(\mathcal{B}^{(i)}) = E^{k_i} a_1^{(i)} \in E^{k_i}\mathscr{O}_F^*$ for each $i \in \mathbb{Z}/f\mathbb{Z}$ as desired. □

While the proof of Proposition 6.2.2 is constructive, it is difficult to compute explicit examples for general f as the polynomials $g^{(i)}(\varphi), h^{(i)}(\varphi)$ that are generated by solving the system depend on the Type combination of the matrices $\mathcal{A}^{(i)}$ as well as the weights.[9] When f is small however, it is possible to compute all cases explicitly as we have done for $f = 2$ in the following section.

6.3 An Example when $f = 2$

When $f = 2$, we have three Type combinations: (I, I), (I, II) and (II, I) upon restricting to irreducible representations. Following the proof of Proposition 6.2.2, we will explicitly compute the base change matrices $(X^{(0)}, X^{(1)})$ so that upon φ-twisted conjugation, we have Frobenius matrices with the correct determinant.

- Type (I, I): Following the proof of Proposition 6.2.2, we get a system of four questions

$$x_1^{(0)} = \varphi(x_2^{(1)}) \qquad x_2^{(0)} = \gamma^{k_1}\varphi(x_1^{(1)})$$
$$x_1^{(1)} = \varphi(x_2^{(0)}) \qquad x_2^{(1)} = \gamma^{k_0}\varphi(x_1^{(0)}).$$

Since in this case there is an even number of Type I matrices, then this system will contain two distinct 2-cycles. Indeed, traversing these two cycles gives

$$(x_1^{(0)})^{\varphi^2-1} = \gamma^{-k_0\varphi} \qquad (x_2^{(0)})^{\varphi^2-1} = \gamma^{-k_1}$$
$$(x_1^{(1)})^{\varphi^2-1} = \gamma^{-k_1\varphi} \qquad (x_2^{(1)})^{\varphi^2-1} = \gamma^{-k_0}.$$

By Lemma 6.2.1 we have solutions over S_F given by

$$x_1^{(0)} = \lambda_2^{k_0\varphi} \qquad x_2^{(0)} = \lambda_2^{k_1}$$
$$x_1^{(1)} = \lambda_2^{k_1\varphi} \qquad x_2^{(1)} = \lambda_2^{k_0}.$$

Hence, by setting $(X^{(0)}, X^{(1)}) *_\varphi (\mathcal{A}^{(0)}, \mathcal{A}^{(1)}) = (\mathcal{B}^{(0)}, \mathcal{B}^{(1)})$ where $\mathcal{A}^{(i)}$ are as in Section 6.1 of Type (I, I), we get that

$$\mathcal{B}^{(0)} = \begin{pmatrix} 0 & E^{k_0} a_1^{(0)} \\ 1 & a_2^{(0)}\lambda_2^{k_1-k_0\varphi} \end{pmatrix} \qquad \mathcal{B}^{(1)} = \begin{pmatrix} 0 & E^{k_1} a_1^{(1)} \\ 1 & a_2^{(1)}\lambda_2^{k_0-k_1\varphi} \end{pmatrix}.$$

- Type (I, II): Following the same process as above, we begin with a system of equations

$$x_1^{(0)} = \varphi(x_2^{(1)}) \qquad x_2^{(1)} = \gamma^{k_1}\varphi(x_1^{(1)})$$

[9] We will later see that these polynomials do not affect the shape of the reduction. In fact, the only effect they may have is given by an unramified character.

$$x_1^{(1)} = \gamma^{k_0}\varphi(x_1^{(0)}) \qquad\qquad x_2^{(1)} = \varphi(x_2^{(0)})$$

Unlike the previous case, we now have an odd number of Type I matrices meaning that our system consists of a single 4-cycle. Indeed we see that

$$\begin{aligned} (x_1^{(0)})^{\varphi^4-1} &= \gamma^{-\varphi^2(k_1+k_0\varphi)} & \qquad (x_2^{(0)})^{\varphi^4-1} &= \gamma^{-(k_1+k_0\varphi)} \\ (x_1^{(1)})^{\varphi^4-1} &= \gamma^{-(k_1\varphi^3+k_0)} & \qquad (x_2^{(1)})^{\varphi^4-1} &= \gamma^{-\varphi(k_1+k_0\varphi)}. \end{aligned}$$

Just as before, Lemma 6.2.1 gives us solutions given by

$$\begin{aligned} x_1^{(0)} &= \lambda_4^{\varphi^2(k_1+k_0\varphi)} & \qquad x_2^{(0)} &= \lambda_4^{k_1+k_0\varphi} \\ x_1^{(1)} &= \lambda_4^{k_1\varphi^3+k_0} & \qquad x_2^{(1)} &= \lambda_4^{\varphi(k_1+k_0\varphi)}. \end{aligned}$$

As a result, by setting $(X^{(0)}, X^{(1)}) *_\varphi (\mathcal{A}^{(0)}, \mathcal{A}^{(1)}) = (\mathcal{B}^{(0)}, \mathcal{B}^{(1)})$, we get that

$$\mathcal{B}^{(0)} = \begin{pmatrix} 0 & E^{k_0}a_1^{(0)} \\ 1 & a_2^{(0)}\lambda_4^{(1-\varphi^2)(k_1+k_0\varphi)} \end{pmatrix} \qquad \mathcal{B}^{(1)} = \begin{pmatrix} E^{k_1}a_1^{(1)} & 0 \\ a_2^{(1)}\lambda_4^{(1-\varphi^2)(k_1\varphi-k_0)} & 1 \end{pmatrix}.$$

- Type $(\mathrm{II}, \mathrm{I})$: This case is very similar to the case of $(\mathrm{I}, \mathrm{II})$ so we will exclude many of the details. Solving the associated system as before gives solutions

$$\begin{aligned} x_1^{(0)} &= \lambda_4^{k_1+k_0\varphi^3} & \qquad x_2^{(0)} &= \lambda_4^{\varphi(k_1\varphi+k_0)} \\ x_1^{(1)} &= \lambda_4^{\varphi^2(k_1\varphi+k_0)} & \qquad x_2^{(1)} &= \lambda_4^{k_1\varphi+k_0}. \end{aligned}$$

As a result, we get base changed matrices over S_F

$$\mathcal{B}^{(0)} = \begin{pmatrix} E^{k_0}a_1^{(0)} & 0 \\ a_2^{(0)}\lambda_4^{(1-\varphi^2)(k_0\varphi-k_1)} & 1 \end{pmatrix} \qquad \mathcal{B}^{(1)} = \begin{pmatrix} 0 & E^{k_1}a_1^{(1)} \\ 1 & a_2^{(1)}\lambda_4^{(1-\varphi^2)(k_1\varphi+k_0)} \end{pmatrix}.$$

Chapter 7

A Descent Algorithm for Kisin Modules

Chapter 6 has given us an explicit description of finite height, rank-two Kisin modules over S_F arising from irreducible, weakly admissible filtered φ-modules. In order to use these Kisin modules to compute reductions of the associated crystalline representations, we will need to *descend*, in the sense of Definition 5.1.3, the coefficients of our Kisin modules from S_F to $\mathfrak{S}_F$ where reduction modulo ϖ is possible as detailed in Section 5.3. The aim of this chapter is to display an algorithm to do just this under some assumptions on the labeled heights.

We begin in Section 7.1 with an expository overview of our methods in the hopes of building up intuition. Section 7.2 then displays the aforementioned descent algorithm from S_F to $\mathfrak{S}_F$ and also details the assumptions that are need to be made to carry out this algorithm.

7.1 Overview of our Methods

The practice of explicitly computing descents of Kisin modules is an exceedingly delicate process that may seem unintuitive to the inexperienced reader. To alleviate this, we first would like to provide some exposition on our methods in hopes of providing said intuition.

The methods displayed in this thesis are heavily inspired by Liu's reduction algorithm in [Liu21] which are in turn, inspired by the algorithms of Bergdall and Levin in [BL20] and Bergdall, Levin, and Liu in [BLL23]. The common trope between all of these is a two step process that begins with 'preparation' computations followed by a theoretic 'descent' stage. In simple terms, one is typically forced to begin the process with a Kisin module over enlarged coefficients that are not able to be directly reduced, such as $S_F = \mathfrak{S}_F[\![\frac{E^p}{p}]\!]$. A preparation stage is used to find a basis where we can split our matrix into an integral $\mathfrak{S}_F$ piece and a more complicated piece that is sufficiently small in some adic sense inside

S_F. The descent stage then justifies why we can descend our coefficients so that the more complicated piece is disregarded and we are left only with the integral piece. Our methods follow this recipe closely.

Recall that by Section 6.2, we have achieved an explicit description of Kisin modules $\mathcal{M}(\mathcal{A})$ of finite height over S_F by describing their i'th partial Frobenius actions by the matrix $\mathcal{A}^{(i)}$ in terms of Types given by:

- Type I:
$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & a_2^{(i)} \lambda_b^{h^{(i)}(\varphi) - g^{(i)}(\varphi)} \end{pmatrix};$$
- Type II:
$$\mathcal{A}^{(i)} = \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ a_2^{(i)} \lambda_b^{h^{(i)}(\varphi) - g^{(i)}(\varphi)} & 1 \end{pmatrix}.$$

A quick glance at our coefficients make it obvious that reduction modulo ϖ is not possible for such Kisin modules. In Chapter 8, we will (under some restrictions) explicitly construct a change of basis X over $\mathrm{GL}_2(S_F[1/p])$ so that $X *_\varphi \mathcal{A} = \mathfrak{A}_0 + C$ with $\mathfrak{A}_0 \in \mathrm{GL}_2(\mathfrak{S}_F)$ and $C \in \mathrm{Mat}_2(I)$ where $I \subset S_F$ is a particularly chosen ideal which is sufficiently large in both a ϖ and E-adic sense. This will constitute our 'preparation' stage and involves fairly delicate computations to ensure that elements are either integral or are adically small enough to lie in I.

In the following section, we will prove some results making up the 'descent' stage of our algorithm. The idea will be that if we assume that the basis achieved from the preparation stage above exists, then we can prove the existence of another change of basis Y over $R[1/p] = \mathfrak{S}_F[\![\varpi E/p]\!][1/p]$ so that $Y *_\varphi \mathcal{A} = \mathfrak{A}$ where $\mathfrak{A} \in \mathrm{GL}_2(\mathfrak{S}_F)$ and moreover, we will have $\mathfrak{A} \equiv \mathfrak{A}_0 \pmod{\varpi}$. Hence, we will have descended the coefficients of $\mathcal{A}$ to $\mathfrak{S}_F$ and have a complete understanding of the reduction modulo ϖ. We note that this stage of the algorithm is not explicit and indeed, we have little knowledge of what $\mathfrak{A}$ actually looks like; however, we know exactly what its reduction of Frobenius is and this is enough for our purposes.

7.2 The Descent Algorithm

As told in Section 7.1, the descent stage of our reduction algorithm will depend on the existence of a certain base change which allows us to split each partial Frobenius $\mathcal{A}^{(i)}$ into a reducible part and a sufficiently adically small part. Hence, before stating the descent algorithm, we will need to establish a set of assumptions we will call the *descent assumptions* under which descent may be carried out.

Let us begin with two very important sets that will give us our notion of elements that are 'sufficiently adically small'. Recall that $\mathrm{Fil}^j S_F = E^j S_F$.

Definition 7.2.1: Define the sets $I := \varpi p S_F + \varpi \mathrm{Fil}^{2p} S_F + E\mathrm{Fil}^{2p} S_F \subset S_F$ and $J := (\varpi, E, E^2/p)\mathrm{Fil}^{2p} S_F \subset S_F[1/p]$.

It is not difficult to see that $I \subset S_F$ is an ideal, however the same cannot be said of J. We are now ready to state the descent assumptions that will be verified to hold under certain restrictions in Chapter 8.

Assumption 7.2.2 (Descent Assumptions)**:** Let $\mathcal{M}(\mathcal{A}) = \prod_i \mathcal{M}(\mathcal{A})^{(i)}$ be a rank-two Kisin module over S_F with labeled heights k_i. Suppose there exists a sequence of base changes (X_n) so that by setting $(X_n^{(i)}) *_\varphi (\mathcal{A}^{(i)}) = (\mathcal{A}_n^{(i)})$, the following holds:

(a) We have $2p - 4 \geq k_i$ for all $i \in \mathbb{Z}/f\mathbb{Z}$;
(b) Either $X_n^{(i)} \in \mathrm{GL}_2(S_F[1/p])$ with $\det(X_n^{(i)}) = 1$ or $X_n^{(i)} = \mathrm{Diag}(x, y) \in \mathrm{GL}_2(F)$;
(c) There exists a finite $m > 0$ such that

$$\mathcal{A}_m^{(i)} = \mathfrak{A}_0^{(i)} + C^{(i)}$$

where $\mathfrak{A}_0^{(i)} \in \mathrm{Mat}_2(\mathfrak{S}_F)$ and $C^{(i)} \in \mathrm{Mat}_2(I + J)$.

In the hopes of sustaining this fleeting sense of intuition, let us state the descent theorem and temporarily forgo the proof until we have built up the needed results. Recall that $S_F = \mathfrak{S}_F[\![E^p/p]\!]$ and define an auxiliary ring $R = \mathfrak{S}_F[\![\varpi E/p]\!]$. For any ring $\mathcal{S}$ such that $\mathfrak{S}_F \subset \mathcal{S} \subset F[\![u]\!]$ and $h \geq 0$, define the subset of *height-h* matrices:

$$\mathrm{Mat}_d^h(\mathcal{S}) := \{A \in \mathrm{Mat}_d(\mathcal{S}) : \text{there exists } \ B \in \mathrm{Mat}_d(\mathcal{S}) \ \text{ with } \ AB = BA = E^h I_d\}.$$

We are now able state our main result of this chapter, the theoretical foundation which permits descent. The following theorem may be viewed as a version of [Liu21, Prop. 3.3] in the dimension two and $f > 1$ case.

Theorem 7.2.3: *Suppose the Descent Assumptions 7.2.2. Then there exists a base change $(Y^{(i)}) \in \mathrm{GL}_2(R[1/p])^f$ such that $(Y^{(i)}) *_\varphi (\mathcal{A}^{(i)}) = (\mathfrak{A}^{(i)}) \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$ with $\mathfrak{A}^{(i)} \equiv \mathfrak{A}_0^{(i)}$* (mod ϖ).

Two things may stand out to the reader about this theorem. First, it is unclear how R appears in the base change and second, how are we able to ensure that $\mathfrak{A}^{(i)}$ is in $\mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$. Let us walk through the results that answer these two questions before displaying the proof.

Of critical importance to our reduction process will be the following algorithm. We note that it is stated in far more generality than what is allowed in the Descent Assumptions 7.2.2. Indeed, we make no restriction on dimension or height. This may be viewed as a generalization of [Liu21, Prop 3.2] to the $f > 1$ case. Note that it is at this point where we begin to exclude $p = 2$ from the proceedings.

Proposition 7.2.4: *Let $p > 2$ and consider an f-tuple of matrices $(A^{(i)})$ with each $A^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(S_F)$. Suppose we may write $A^{(i)} = A_1^{(i)} + C_1^{(i)}$ with $A_1^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ and $C_1^{(i)} \in \mathrm{Mat}_d(\varpi \mathrm{Fil}^{h_1^{(i)}} S_F)$ where $h_1^{(i)} \geq \frac{\max_i(h^{(i)})}{1-2p^{-1}}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Then there exists a sequence of base change matrices $(Y_n^{(i)}) \in \mathrm{Mat}_d(R)$ such that:*

- *As $n \to \infty$, $Y_n^{(i)}$ converges to $Y^{(i)}$ in $\mathrm{GL}_d(R)$;*
- *$Y^{(i)} *_\varphi A^{(i)} = \mathfrak{A}^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$;*
- *$\mathfrak{A}^{(i)} \equiv A_1^{(i)} \pmod{\varpi}$.*

Proof. Without loss of generality, let us fix an $i \in \mathbb{Z}/f\mathbb{Z}$ and construct the sequence of matrices $Y_n^{(i)}$ by successive approximation. By assumption we know that $C_1^{(i)} \in \mathrm{Mat}_d(\varpi \mathrm{Fil}^{h_1^{(i)}} S_F)$ so we may write $C_1^{(i)} = C_1'^{(i)} E^{h^{(i)}}$ since $h_1^{(i)} > h^{(i)}$. As $A_1^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ then there exists $B_1^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ such that $C_1^{(i)} = C_1'^{(i)} E^{h^{(i)}} = C_1'^{(i)} B_1^{(i)} A_1^{(i)}$. Setting $Y_0^{(i)} = I_d$, we may write

$$(Y_0^{(i)}) *_\varphi (A^{(i)}) = A_1^{(i)} + C_1^{(i)} = A_1^{(i)} + C_1'^{(i)} B_1^{(i)} A_1^{(i)} = \left(I_d + C_1'^{(i)} B_1^{(i)}\right) A_1^{(i)}$$

We will set $Y_1^{(i)} := (I_d + C_1'^{(i)} B_1^{(i)})^{-1} Y_0^{(i)}$ and observe that

$$\begin{aligned}
(Y_1^{(i)}) *_\varphi (A^{(i)}) &= (I_d + C_1'^{(i)} B_1^{(i)})^{-1} Y_0^{(i)} \cdot A^{(i)} \cdot \varphi(Y_0^{(i-1)})^{-1} (I_d + \varphi(C_1'^{(i-1)} B_1^{(i-1)})) \\
&= (I_d + C_1'^{(i)} B_1^{(i)})^{-1} \left(I_d + C_1'^{(i)} B_1^{(i)}\right) A_1^{(i)} (I_d + \varphi(C_1'^{(i-1)} B_1^{(i-1)})) \\
&= A_1^{(i)} (I + \varphi(C_1'^{(i-1)} B_1^{(i-1)})).
\end{aligned}$$

We now embark on the task of writing $A_1^{(i)}(I + \varphi(C_1'^{(i-1)} B_1^{(i-1)})) = A_2^{(i)} + C_2^{(i)}$ so we can repeat the above process.

Observe that any element of S_F may be written as $\sum_{j \geq 0} a_j \frac{E^j}{p^{\lfloor j/p \rfloor}}$ where $a_j \in \mathscr{O}_F[u]$. Since $C_1^{(i)} = C_1'^{(i)} E^{h^{(i)}}$ then any element of $C_1'^{(i)}$ will take the form $x = \sum_{j \geq h_1^{(i)}} a_j \frac{E^{j-h^{(i)}}}{p^{\lfloor j/p \rfloor}}$ where each $a_j \in \varpi \mathfrak{S}_F$. We have $\varphi(E) = p\gamma$ with $\gamma \in S_F^*$ so that $\varphi(x) \in \varpi p^{\ell_1^{(i)}} S_F$ where $\ell_1^{(i)} = h_1^{(i)} - h^{(i)} - \lfloor \frac{h_1^{(i)}}{p} \rfloor$. Hence, we can write $\varphi(x) = \sum_{j \geq 0} b_j p^{\ell_1^{(i)}} \frac{E^j}{p^{\lfloor j/p \rfloor}} = y + z$ where $y \in \varpi \mathfrak{S}_F \cap p^{\ell_1^{(i)}} S_F$ and $z \in \varpi \mathrm{Fil}^{p(\ell_1^{(i)}+1)} S_F$. We set $h_2^{(i)} = p(\ell_1^{(i)} + 1)$. In fact, we have $h_2^{(i)} > h_1^{(i)}$ which can be seen by noticing that proving the previous inequality is equivalent to proving the following inequality after expanding $\ell_1^{(i)}$,

$$h_1^{(i)}(1 - p^{-1}) - \left\lfloor \frac{h_1^{(i)}}{p} \right\rfloor + 1 > h^{(i)}.$$

Hence, we may write $h_1^{(i)}(1 - p^{-1}) - \lfloor \frac{h_1^{(i)}}{p} \rfloor + 1 > h_1^{(i)}(1 - 2p^{-1}) \geq \max_i(h^{(i)}) \geq h^{(i)}$ which gives the desired conclusion since $p > 2$.

Since $B_1^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ then the above discussion implies that $\varphi(C_1'^{(i)} B_1^{(i)}) = D_2^{(i)} +$

$D_2'^{(i)}$ with $D_2^{(i)} \in \mathrm{Mat}_d(\varpi\mathfrak{S}_F \cap p^{\ell_1^{(i)}} S_F)$ and $D_2'^{(i)} \in \mathrm{Mat}_d(\varpi \mathrm{Fil}^{h_2^{(i)}} S_F)$. Let us set

$$A_2^{(i)} = A_1^{(i)}(I_d + D_2^{(i-1)}) \qquad\qquad C_2^{(i)} = A_1^{(i)} D_2'^{(i-1)}.$$

Since $D_2^{(i)} \in \mathrm{Mat}_d(\varpi\mathfrak{S}_F \cap p^{\ell_1^{(i)}} S_F)$ the geometric series $\sum_{j\geq 0}(-D_2^{(i)})^j = (I_d + D_2^{(i)})^{-1}$ converges in $\mathrm{Mat}_d(\mathfrak{S}_F)$. Hence, if $B_1^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ is the matrix such that $A_1^{(i)} B_1^{(i)} = E^{h^{(i)}} I_d$ then by setting $B_2^{(i)} = (I + D_2^{(i-1)})^{-1} B_1^{(i)}$ we get that $A_2^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$. We also notice that by construction it is clear that $C_2^{(i)} \in \mathrm{Mat}_d(\varpi \mathrm{Fil}^{h_2^{(i-1)}} S_F)$. Since $h_2^{(i-1)} > h_1^{(i-1)} > \max_i(h^{(i)}) \geq h^{(i)}$ then we can repeat this process $(f-1)$-times to get a base change matrix $Y_f^{(i)} = \prod_{j=0}^{f}(I_d + C_j'^{(i)} B_j^{(i)})^{-1}$ so that

$$Y_f^{(i)} *_\varphi A^{(i)} = A_{f+1}^{(i)} + C_{f+1}^{(i)}$$

where $A_{f+1}^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ and $C_{f+1}^{(i)} \in \mathrm{Mat}_d(\varpi \mathrm{Fil}^{h_{f+1}^{(i)}} S_F)$ where $h_{f+1}^{(i)} > h_1^{(i)}$. Hence, we have returned to the original assumptions with the only difference being our $C_{f+1}^{(i)}$ matrix is E-adically smaller than $C_1^{(i)}$.

Let us set $Y^{(i)} = \lim_{n\to\infty} Y_n^{(i)} = \prod_{j=0}^{\infty}(I_d + C_j'^{(i)} B_j^{(i)})^{-1}$. Recall that we have written $Y_n^{(i)} = (I_d + C_n'^{(i)} B_n^{(i)})^{-1} Y_{n-1}^{(i)}$ and that we may denote any element of the matrix $C_n'^{(i)} B_n^{(i)}$ by $x = \sum_{j\geq h_n^{(i)}} a_j \frac{E^{j-h^{(i)}}}{p^{\lfloor j/p\rfloor}}$ with $a_j \in \varpi\mathfrak{S}_F$. Since $h_n^{(i)} > \max(h^{(i)}) \geq h^{(i)}$ then $j - h^{(i)} \geq \lfloor j/p \rfloor$ for any $j \geq h_n^{(i)}$. Hence $x \in R$ and $(I_d + C_n'^{(i)} B_n^{(i)})^{-1} \in \mathrm{GL}_d(R)$ so as $n \to \infty$, $Y^{(i)} = \prod_{j=0}^{\infty}(I_d + C_j'^{(i)} B_j^{(i)})^{-1} \in \mathrm{GL}_d(R)$ converges as desired proving (a) since $C_j'^{(i)} B_j^{(i)} \to 0$ in the E-adically closed ring R.

Finally, observe that by construction we have that $A_{n+1}^{(i)} \equiv A_n^{(i)} \pmod{\varpi}$ and $A_{n+1}^{(i)} - A_n^{(i)} \in \mathrm{Mat}_d(p^{\ell_n^{(i+1)}} S_F)$ so as $n \to \infty$, we have that $A_n^{(i)}$ converges to some $\mathfrak{A}^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(\mathfrak{S}_F)$ with $\mathfrak{A}^{(i)} \equiv A_1^{(i)} \pmod{\varpi}$. Moreover, as $n \to \infty$, the above analysis shows that $h_n^{(i)} \to \infty$ so that $C_n^{(i)} \to 0$ in the E-adically closed ring S_F. This proves both (b) and (c). □

Remark 7.2.5: In the postponed proof of Theorem 7.2.3, we will set $h_1^{(i)} = 2p$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ so that the previous proposition makes clear the necessity of the upper bound $2p-4$ for the labeled heights k_i. It is conceivable that a more general theory, say with $h_1^{(i)} = 3p$, that the upper bound could be increased to $3p-6$. This would require a new version of the ideal I however and as we will see later, the theory is already complicated enough for $2p$.

To prove Theorem 7.2.3, we will need to justify the use of Proposition 7.2.4 on $\mathcal{A}_m^{(i)} = \mathfrak{A}_0^{(i)} + C^{(i)}$. However, two problems remain in the way. First, we need to fit $\mathfrak{A}_0^{(i)}$ (or some version of it) into $\mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$ and second, $C^{(i)}$ takes coefficients in $I + J$ which do not lie in $\varpi\mathrm{Fil}^{2p} S_F$. The following two lemmas will give us a way to fix these two problems.

Lemma 7.2.6: *For any element $x \in I$, we may write $x \in \varpi\mathfrak{S}_F + J$. Moreover, by assuming the Descent Assumptions 7.2.2 and decomposing $C^{(i)} \in \mathrm{Mat}_2(I+J)$ in this way, then*

$\mathcal{A}_m^{(i)} = \mathfrak{A}'^{(i)} + C'^{(i)}$ where $\mathfrak{A}'^{(i)} \in \mathrm{Mat}_2(\mathfrak{S}_F)$ and $C'^{(i)} \in \mathrm{Mat}_2(J)$ with $\mathfrak{A}'^{(i)} \equiv \mathfrak{A}_0 \pmod{\varpi}$ and $\det(\mathfrak{A}'^{(i)}) \in E^{k_i}\mathfrak{S}_F^*$.

Proof. Since both $\varpi \mathrm{Fil}^{2p} S_F, E\mathrm{Fil}^{2p} S_F \in J$ then we need only show $\varpi p S_F \in \varpi \mathfrak{S}_F + J$. This is easily seen once we notice that any element of $\varpi p S_F$ takes the form

$$\varpi p \sum_{j=0}^{\infty} \alpha_j \frac{E^{jp}}{p^j} = \varpi(p\alpha_0 + E^p\alpha_1 + p\mathrm{Fil}^{2p} S_F) \in \varpi\mathfrak{S}_F + \varpi\mathrm{Fil}^{2p} S_F \subset \varpi\mathfrak{S}_F + J.$$

By the first part of this lemma, we need only show that $\det(\mathfrak{A}'^{(i)}) \in E^{k_i}\mathfrak{S}_F^*$. By the definitions of $(X_n^{(i)})$, we know that $\det(\mathcal{A}_m^{(i)}) = \det(\mathcal{A}^{(i)}) = E^{k_i}a_1^{(i)}$. It is easy to see that $\det(\mathfrak{A}'^{(i)}) = E^{k_i}a_1^{(i)} \pmod{\mathrm{Fil}^{2p} S_F}$ and hence, $\det(\mathfrak{A}'^{(i)}) - E^{k_i}a_1^{(i)} \in \mathrm{Fil}^{2p} S_F \cap \mathfrak{S}_F = E^{2p}\mathfrak{S}_F$. Since each $k_i < 2p$ then $\det(\mathfrak{A}'^{(i)}) \in E^{k_i}a_1^{(i)} + E^{2p}\mathfrak{S}_F \in E^{k_i}\mathfrak{S}_F^*$. $\square$

Lemma 7.2.7: *We have $\mathcal{A}^{(i)} \in \mathrm{Mat}_2^{k_i}(S_F)$ and $\mathcal{A}_m^{(i)} \in \mathrm{Mat}_2^{k_i}(S_F[1/p])$.*

Proof. Any Type I Frobenius matrix will take the form

$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i}a_1^{(i)} \\ 1 & a_2^{(i)}\lambda_b^{g(\varphi)} \end{pmatrix}.$$

A brief matrix computation will verify that the matrix

$$B^{(i)} = \begin{pmatrix} -a_2^{(i)}\lambda_b^{g(\varphi)} & E^{k_i}a_1^{(i)} \\ 1 & 0 \end{pmatrix}$$

is enough to show that $\mathcal{A}^{(i)} \in \mathrm{Mat}_2^{h^{(i)}}(S_F)$. Similarly, in the Type II case, the matrices

$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i}a_1^{(i)} \\ 1 & a_2^{(i)}\lambda_b^{g(\varphi)} \end{pmatrix} \qquad B^{(i)} = \begin{pmatrix} 1 & 0 \\ -a_2^{(i)}\lambda_b^{g(\varphi)} & E^{k_i}a_1^{(i)} \end{pmatrix}$$

will imply that $\mathcal{A}^{(i)} \in \mathrm{Mat}_2^{k_i}(S_F)$.

For the second part of the lemma, observe that

$$\mathcal{A}_m^{(i)} = X_m^{(i)}X_{m-1}^{(i)}\cdots X_1^{(i)}\mathcal{A}\varphi(X_1^{(i)}\cdots X_{m-1}^{(i)}X_m^{(i)})^{-1}.$$

Hence, for any of the Types detailed above, by setting

$$B_m^{(i)} = \varphi(X_m^{(i)}\cdots X_1^{(i)})B^{(i)}(X_1^{(i)}\cdots X_{m-1}^{(i)}X_m^{(i)})^{-1}$$

then the fact that $\mathcal{A}_m^{(i)} \in \mathrm{Mat}_2^{k_i}(S_F[1/p])$ follows from the first part of the lemma and the fact that $\mathcal{A}^{(i)}B^{(i)} = B^{(i)}\mathcal{A}^{(i)} = E^{k_i}I_2$ in $\mathrm{Mat}_2(S_F)$. $\square$

Proof of Theorem 7.2.3. Our plan is to justify the use of Proposition 7.2.4 on $\mathcal{A}_m^{(i)}$ with $h_1^{(i)} = 2p$. At which point, the existence of $(Y^{(i)})$ with the desired properties will be implied.

We begin by invoking Lemma 7.2.6 in order to write $\mathcal{A}_m^{(i)} = \mathfrak{A}'^{(i)} + C'^{(i)}$ where $\mathfrak{A}'^{(i)} \in \mathrm{Mat}_2(\mathfrak{S}_F)$ and $C'^{(i)} \in \mathrm{Mat}_2(J)$ with $\mathfrak{A}'^{(i)} \equiv \mathfrak{A}_0 \pmod{\varpi}$ and $\det(\mathfrak{A}'^{(i)}) = E^{k_i}\alpha$ for some $\alpha \in \mathfrak{S}_F^*$. To see that $\mathfrak{A}'^{(i)} \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$, we need only define $B'^{(i)} = \alpha^{-1}\det(\mathfrak{A}'^{(i)})(\mathfrak{A}'^{(i)})^{-1} \in \mathrm{Mat}_2(\mathfrak{S}_F)$. A brief matrix computation will show that $\mathfrak{A}'^{(i)}B'^{(i)} = B'^{(i)}\mathfrak{A}'^{(i)} = E^{k_i}I_2$ so that $\mathfrak{A}'^{(i)} \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$.

The only hold up to the application of Proposition 7.2.4 lies in the fact that $C'^{(i)} \in \mathrm{Mat}_2(J)$ and J is not in $\varpi\mathrm{Fil}^{2p}S_F$. To do this, we follow similar ideas to the proof of the aforementioned proposition. We may decompose $C'^{(i)} = \tilde{C}^{(i)}E^{k_i}I_2$ as $k_i < 2p$. Hence by the above analysis, we may write $C'^{(i)} = \tilde{C}^{(i)}E^{k_i}I_2 = \tilde{C}^{(i)}B'^{(i)}\mathfrak{A}'^{(i)}$. Let $Z^{(i)} = (I_2 + \tilde{C}^{(i)}B'^{(i)})^{-1} \in \mathrm{GL}_2(S_F)$ and observe that

$$\begin{aligned}
(Z^{(i)}) *_\varphi (A_m^{(i)}) &= (I_d + \tilde{C}^{(i)}B'^{(i)})^{-1} \cdot A_m^{(i)} \cdot (I_d + \varphi(\tilde{C}^{(i-1)}B'^{(i-1)})) \\
&= (I_d + \tilde{C}^{(i)}B'^{(i)})^{-1}(I_d + \tilde{C}^{(i)}B'^{(i)})\mathfrak{A}'^{(i)}(I_d + \varphi(\tilde{C}^{(i-1)}B'^{(i-1)})) \\
&= \mathfrak{A}'^{(i)}(I_d + \varphi(\tilde{C}^{(i-1)}B'^{(i-1)})).
\end{aligned}$$

Since $C'^{(i)} \in \mathrm{Mat}_2(J)$ then any element of $\tilde{C}^{(i)}$ will take the form $x = \sum_{j\geq 2p} a_j \frac{E^{j-k_i}}{p^{\lfloor j/p \rfloor}}$ with $a_j \in (\varpi, E, E^2/p)\mathfrak{S}_F$. Since $\varphi(E) = p\gamma \in \varpi S_F$ and $k_i \leq 2p-4$ then $j - k_i \geq 4$ so that $\varphi(E)^{j-k_i} \in p^4 S_F$. Hence, $\varphi(\tilde{C}^{(i)}) \in \mathrm{Mat}_2(\varpi p^2 S_F)$ so we decompose $\varphi(\tilde{C}^{(i)}) = D_1^{(i)} + D_2^{(i)}$ where $D_1^{(i)} \in \mathrm{Mat}_2(\varpi\mathfrak{S}_F)$ and $D_2^{(i)} \in \mathrm{Mat}_2(\varpi\mathrm{Fil}^{2p}S_F)$. By setting

$$A_1^{(i)} = \mathfrak{A}'^{(i)}(I_2 + D_1^{(i-1)}\varphi(B'^{(i-1)})) \qquad\qquad C_1^{(i)} = \mathfrak{A}'^{(i)}D_2^{(i-1)}\varphi(B'^{(i-1)}),$$

it is easy to see that $A_1^{(i)} \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$ with $A_1^{(i)} \equiv \mathfrak{A}'^{(i)} \pmod{\varpi}$ and $C_1^{(i)} \in \mathrm{Mat}_2(\varpi\mathrm{Fil}^{2p}S_F)$. Since $k_i \leq 2p-4$ then setting $h_1^{(i)} = 2p$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ allows us to use Proposition 7.2.4 on $\mathcal{A}_m^{(i)} = A_1^{(i)} + C_1^{(i)}$ with $\mathcal{A}_m^{(i)} \in \mathrm{Mat}_2^{k_i}(S_F[1/p])$ by Lemma 7.2.7. We conclude the existence of a matrix $Y \in \mathrm{GL}_2(R[1/p])$ such that $Y^{(i)} *_\varphi \mathcal{A}_m^{(i)} = \mathfrak{A}^{(i)} \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F)$ with $\mathfrak{A}^{(i)} \equiv A_1^{(i)} \equiv \mathfrak{A}'^{(i)} \equiv \mathfrak{A}_0^{(i)} \pmod{\varpi}$. $\square$

Chapter 8

Preparatory Computations for Descent

Recall that in Chapter 6, we were able to achieve an explicit description of Kisin modules $\mathcal{M}(\mathcal{A})$ of finite height over $S_F = \mathfrak{S}_F[\![E^p/p]\!]$ by describing their i-th partial Frobenius actions by the matrix $\mathcal{A}^{(i)}$ in terms of Types given by:

- Type I:
$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} \end{pmatrix};$$
- Type II:
$$\mathcal{A}^{(i)} = \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} & 1 \end{pmatrix}.$$

for some polynomials $g^{(i)}(\varphi) \in \mathbb{Z}[\varphi]$. Let us identify two disjoint subsets

$$\mathcal{S} \coloneqq \{i \in \mathbb{Z}/f\mathbb{Z} : \mathcal{A}^{(i)} \text{ is Type I }\} \qquad \mathcal{T} \coloneqq \{i \in \mathbb{Z}/f\mathbb{Z} : \mathcal{A}^{(i)} \text{ is Type II}\}.$$

Hence, we have decomposed the set $\mathbb{Z}/f\mathbb{Z} = \mathcal{S} \sqcup \mathcal{T}$. We then have that $b = f$ if and only if the order $|\mathcal{S}|$ is even and $b = 2f$ otherwise by Proposition 6.2.2. Additionally by Proposition 4.2.1, we see that $a_2^{(i)} \in \varpi\mathscr{O}_F$ for all $i \in \mathcal{T}$ and $\prod_{i\in\mathcal{S}} a_2^{(i)} \in \varpi\mathscr{O}_F$. We will continue to simplify notation by defining $(xy)^{(i)} = x^{(i)}y^{(i)}$.

As an implementation of the descent algorithm found in Section 7.2, we would like to display the computations necessary to show that the Descent Assumptions 7.2.2 hold for $\mathcal{M}(\mathcal{A})$ with labeled heights in the range

$$p + 2 \le k_0 \le 2p - 4 \qquad\qquad 2 \le k_i \le p - 3 \text{ for all } 1 \le i \le f - 1$$

and the p-adic valuations $\nu_p(a_2^{(i)})$ are sufficiently large for select $i \in \mathbb{Z}/f\mathbb{Z}$. Moreover, we will provide a complete description in the case of $f = 2$. Our strategy will be to perform a finite number of successive row and scaling operations $(X_n) *_\varphi (\mathcal{A}) = (\mathcal{A}_n)$ until each entry of $\mathcal{A}_n^{(i)}$ is in $\mathfrak{S}_F + I$ at which point the Descent Assumptions 7.2.2 will be satisfied.

Remark 8.0.1: There is nothing special other than an ease of indexing for why we choose the (0)-labeled Hodge-Tate weight to be large. Indeed, by choosing another set of f-distinct embeddings of $\tau_i : K \hookrightarrow F$, one can move the index which has weight greater than p to any index.

Section 8.1 deals with the case of $\nu_p(a_2^{(0)}) \geq 1$ which forms the starting point of all the cases we display in this chapter. Section 8.2 then allows $\nu_p(a_2^{(0)}) < 1$ which in turn forces us to make assumptions on the p-adic valuations of other $a_2^{(i)}$ parameters depending on the even or oddness of the set $\mathcal{S}$. Finally, Section 8.3 gives a complete description in all cases where the methods of the previous section had failed in general $f > 2$.

8.1 The Large Valuation Case

We begin with the case of $\nu_p(a_2^{(0)}) \geq 1$ which we collectively call the *large valuation* cases. Please note that we will not make this assumption quite yet as this first step is indeed valid for any valuation and will form the first step in all proceeding arguments.

Looking at $\mathcal{A}^{(i)}$ regardless of Type, it is apparent that our only obstacle is the term $\lambda_b^{g^{(i)}(\varphi)} \in S_F^*$. Hence, our first move will be a row operation over $S_F[1/p]$ designed to strip off as many terms from each $\lambda_b^{g^{(i)}(\varphi)}$ as we can. Since $\lambda_b \in S_F^*$, let us write $\lambda_b^{g^{(i)}(\varphi)} = \sum_{j=0}^{\infty} \alpha_j^{(i)} (E^p/p)^j$ where each $\alpha_j^{(i)} \in \mathscr{O}_F[u]$ of maximal degree $p-1$. For each $i \in \mathbb{Z}/f\mathbb{Z}$, define elements in $S_F[1/p]$ given by

$$x^{(i)} = \begin{cases} -\left(\dfrac{a_2}{a_1}\right)^{(i)} \displaystyle\sum_{j=2}^{\infty} \alpha_j^{(i)} \frac{E^{jp-k_i}}{p^j} & \text{if } i = 0 \\ -\left(\dfrac{a_2}{a_1}\right)^{(i)} \displaystyle\sum_{j=1}^{\infty} \alpha_j^{(i)} \frac{E^{jp-k_i}}{p^j} & \text{for } 1 \leq i \leq f-1. \end{cases}$$

Notice that the only difference between the two is the fact that we have to start at $j = 2$ when $i = 0$ since $k_0 > p$.

Set $X_1^{(i)} = \left(\begin{smallmatrix} 1 & 0 \\ x^{(i)} & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(S_F[1/p])$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and observe that by setting $(X_1) *_\varphi (\mathcal{A}) = (\mathcal{A}_1)$ we will obtain

$$\mathcal{A}_1^{(0)} = \begin{cases} \begin{pmatrix} E^{k_0} a_1^{(0)} \varphi(x^{(f-1)}) & E^{k_0} a_1^{(0)} \\ 1 + ((a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p}) \varphi(x^{(f-1)}) & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} E^{k_0} a_1^{(0)} & 0 \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \varphi(x^{(f-1)}) & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_1^{(i)} = \begin{cases} \begin{pmatrix} E^{k_i} a_1^{(i)} \varphi(x^{(i-1)}) & E^{k_i} a_1^{(i)} \\ 1 + (a_2\alpha_0)^{(i)} \varphi(x^{(i-1)}) & (a_2\alpha_0)^{(i)} \end{pmatrix} & \text{for } i \neq 0 \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \varphi(x^{(i-1)}) & 1 \end{pmatrix} & \text{for } i \neq 0 \text{ and } i \in \mathcal{T}. \end{cases}$$

We now require a lemma that makes sure none of the $\varphi(x^{(i)})$-terms introduced as a result of φ-conjugation are non-integral or non-I factors.

Lemma 8.1.1: *If $i \neq 0$ and $k_i \leq p - h$ for $h > 0$, then $\varphi(x^{(i)}) \in a_2^{(i)} p^{h-1} S_F$. Likewise, if $k_0 \leq 2p - 4$ then $\varphi(x^{(0)}) \in a_2^{(0)} p^2 S_F$.*

Proof. For the first part of the lemma, since each $\alpha_j \in \mathscr{O}_F[u]$, then we need only prove that $\varphi(E^{pj-k_i}/p^j) \in p^{h-1} S_F$ for each $j \geq 1$. Let us write

$$\varphi\left(\frac{E^{pj-k_i}}{p^j}\right) = \frac{\varphi(E)^{pj-k_i}}{p^j} = p^{pj-k_i-j}\gamma^{pj-k_i}.$$

Since $\gamma \in S_F^*$ and $pj - k_i - j \geq h - 1$ for all $j \geq 1$ then the first result follows. The second part follows the same arguments for $j \geq 2$. $\square$

Using the fact that $a_2^{(i)} p \in \varpi \mathscr{O}_F$ for all $i \in \mathbb{Z}/f\mathbb{Z}$, we can see that by Lemma 8.1.1, when $\nu_p(a_2^{(i)}) > 0$ we can write $\varphi(x^{(i)}) \in p^2 \varpi S_F \subset pI$ and whenever $\nu_p(a_2^{(i)}) = 0$, we have $\varphi(x^{(i)}) \in p\varpi S_F \subset I$. To reduce notation, let us define a nonzero valuation indicator function in the following way

$$v(i) = \begin{cases} 1 & \text{if } \nu_p(a_2^{(i)}) > 0 \\ 0 & \text{if } \nu_p(a_2^{(i)}) = 0. \end{cases}$$

We then have that $\varphi(x^{(i)}) \in \varpi^{v(i)} p^2 S_F \subset p^{v(i)} I$. In an effort to reduce notation, let us now begin to gather elements of the ideal I into terms denoted $\varepsilon_{jk}^{(i)}$ and $\delta_{jk}^{(i)}$. Since Theorem 7.2.3 allows us to disregard such elements when computing reductions, then there is not harm in making this simplifying notation. By the above discussion, we write

$$\mathcal{A}_1^{(0)} = \begin{cases} \begin{pmatrix} \varepsilon_{11}^{(0)} & E^{k_0} a_1^{(0)} \\ 1 + \varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} E^{k_0} a_1^{(0)} & 0 \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_1^{(i)} = \begin{cases} \begin{pmatrix} \varepsilon_{11}^{(i)} & E^{k_i} a_1^{(i)} \\ 1 + \varepsilon_{21}^{(i)} & (a_2\alpha_0)^{(i)} \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{T} \end{cases}$$

- $\varepsilon_{11}^{(i)} = E^{k_i} a_1^{(i)} \varphi(x^{(i-1)}) \in E^{k_i} p^{v(i-1)} I$;
- $\varepsilon_{21}^{(i)} = (a_2\alpha_0)^{(i)} \varphi(x^{(i-1)}) \in \varpi^{v(i)+v(i-1)} p^2 S_F \subset I$ for $1 \leq i \leq f-1$;
- $\delta_{21}^{(i)} = \varphi(x^{(i-1)}) \in \varpi^{v(i-1)} p^2 S_F \subset p^{v(i-1)} I$;
- $\varepsilon_{21}^{(0)} = ((a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p}) \varphi(x^{(f-1)}) \in \varpi^{v(0)+v(f-1)} p^2 S_F \subset I$.

With this, we observe that the only non-integral and non-I term is $(a_2\alpha_1)^{(0)} \frac{E^p}{p}$ in $\mathcal{A}_1^{(0)}$. If we assume that $\nu_p(a_2^{(0)}) \geq 1$ then this term becomes integral and we get our first case of satisfied descent.

Proposition 8.1.2: *Suppose $\mathcal{M}(\mathcal{A})$ is a rank-two Kisin module over S_F with labeled heights $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_i \leq p-3$ for $i \neq 0$. If $\nu_p(a_2^{(0)}) \geq 1$, then the Frobenius matrix $(\mathcal{A})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0^{(0)} = \begin{cases} \begin{pmatrix} 0 & E^{k_0} a_1^{(0)} \\ 1 & (a_2\alpha_0)^{(0)} + \left(\frac{a_2^{(0)}}{p}\right) \alpha_1^{(0)} E^p \end{pmatrix} & \textit{if } 0 \in \mathcal{S} \\ \begin{pmatrix} E^{k_0} a_1^{(0)} & 0 \\ (a_2\alpha_0)^{(0)} + \left(\frac{a_2^{(0)}}{p}\right) \alpha_1^{(0)} E^p & 1 \end{pmatrix} & \textit{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathfrak{A}_0^{(i)} = \begin{cases} \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & (a_2\alpha_0)^{(i)} \end{pmatrix} & \textit{if } i \neq 0 \textit{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} & 1 \end{pmatrix} & \textit{if } i \neq 0 \textit{ and } i \in \mathcal{T}. \end{cases}$$

Remark 8.1.3: The above methods may actually be applied even if we allow more than just the (0)-labeled Hodge-Tate weight to be greater than p. See Appendix A for this generalization. Unfortunately, the upcoming small valuation cases do not generalize in the same way.

8.2 The Small Valuation Case

We now assume that $0 \leq \nu_p(a_2^{(0)}) < 1$ and will continue from $\mathcal{A}_1^{(i)}$ as given in the previous section. Recall that at this point we have

$$\mathcal{A}_1^{(0)} = \begin{cases} \begin{pmatrix} \varepsilon_{11}^{(0)} & E^{k_0} a_1^{(0)} \\ 1 + \varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} E^{k_0} a_1^{(0)} & 0 \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_1^{(i)} = \begin{cases} \begin{pmatrix} \varepsilon_{11}^{(i)} & E^{k_i} a_1^{(i)} \\ 1+\varepsilon_{21}^{(i)} & (a_2\alpha_0)^{(i)} \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{T} \end{cases}$$

where $\varepsilon_{11}^{(i)} \in E^{k_i} p^{v(i-1)} I$, $\varepsilon_{21}^{(i)} \in \varpi^{(v(i)+v(i-1))} p^2 S_F \subset I$ and $\delta_{21}^{(i)} \in p^{v(i-1)} I$. Our strategy will be to kill the E^{k_0} term followed by a scaling operation and concluding (if necessary) with a row operation to kill the E^p/p term which, as we may recall, is the only non-integral and non-I term in the matrices above. The real danger in this process is the aforementioned scaling operation as we will have to be very careful to not introduce non-integral factors which will require careful estimates of certain terms.

Notice that since $0 \leq \nu_p(a_2^{(0)}) < 1$, we can write $\nu_p(p/a_2^{(0)}) > 0$. Consider the row operation over $\mathrm{GL}_2(\mathfrak{S}_F)$ designed to kill the E^{k_0}-term in $\mathcal{A}_1^{(0)}$ given by

$$X_2^{(0)} = \begin{pmatrix} 1 & -\frac{a_1^{(0)} p}{(a_2\alpha_1)^{(0)}} E^{k_0-p} \\ 0 & 1 \end{pmatrix} \qquad X_2^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i \neq 0.$$

We see that this base change is isolated as to only affect $\mathcal{A}_1^{(0)}$ and $\mathcal{A}_1^{(1)}$ (since φ-twisted conjugation means row operations in (i) will induce column operations in $(i+1)$) so the result of $(X_2) *_\varphi (\mathcal{A}_1)$ will be given by $\mathcal{A}_2^{(i)} = \mathcal{A}_1^{(i)}$ for $2 \leq i \leq f-1$ and

$$\mathcal{A}_2^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ 1+\varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_2^{(1)} = \begin{cases} \begin{pmatrix} \varepsilon_{11}^{(1)} & E^{k_1} a_1^{(1)} + \varepsilon_{12}^{(1)} \\ 1+\varepsilon_{21}^{(1)} & (a_2\alpha_0)^{(1)} + \varepsilon_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \\ \begin{pmatrix} E^{k_1} a_1^{(1)} & \delta_{12}^{(1)} \\ (a_2\alpha_0)^{(1)} + \delta_{21}^{(1)} & 1+\delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \end{cases}$$

- Both $\varepsilon_{j2}^{(1)}$ and $\delta_{j2}^{(1)}$ are the result of adding a $(p/a_2^{(0)})\varphi(E)^{k_0-p}$-multiple of the first column of $\mathcal{A}_2^{(1)}$ to the second column. Since $\varphi(E) \in pS_F$, $p/a_2^{(0)} \in \varpi\mathscr{O}_F$ and $k_0 - p \geq 2$ then $(p/a_2^{(0)})\varphi(E)^{k_0-p} \in \varpi p^2 S_F \subset pI$. Hence, $\varepsilon_{j2}^{(1)}, \delta_{j2}^{(1)} \in pI$;
- $\tilde{\varepsilon}_{11}^{(0)} = \varepsilon_{11}^{(0)} - \varepsilon_{21}^{(0)} \left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} \in I$;
- $\delta_{11}^{(0)} = -\delta_{21}^{(0)} \left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} \in pI$ since $\delta_{21}^{(0)} \in p^2 S_F$ and $p/a_2^{(0)} \in \varpi\mathscr{O}_F$.

As mentioned above, our next step will be to scale the anti-diagonals of all f-matrices by an appropriate element, either p or $p/a_2^{(0)}$ depending on the situation. The difficulty here is that any Type I matrix will be made non-integral if we do this as they stand now. Our answer to this problem will be to perform row and column permutation operations depending on the Type combinations present in order to 'line up' the anti-diagonals so they have sufficient powers of p as to not become non-integral after scaling. We note that we are not able to cover all cases with our methods however.[10]

Definition 8.2.1: Recall that $\mathcal{S} \subset \mathbb{Z}/f\mathbb{Z}$ denotes the set of those i for which $\mathcal{A}^{(i)}$ is a Type I Frobenius matrix. We define a subset $\mathcal{W} \subset \mathbb{Z}/f\mathbb{Z}$ with the following steps:

1. Label the elements of $\mathcal{S} \setminus \{0\}$ in $\mathbb{Z}/f\mathbb{Z}$ by r_j so that $r_1 < r_2 < \cdots < r_d$ for some d in the range $1 \leq d \leq f-2$. Note that this set may be empty, in which case $\mathcal{W}$ is empty.
2. Include $i \in \mathcal{W}$ for all $r_1 \leq i < r_2$.
3. Now include $i \in \mathcal{W}$ for all $r_3 \leq i < r_4$.
4. Continue this pattern in that we include $i \in \mathcal{W}$ for all $r_{1+n} \leq i < r_{2+n}$ for $1 < n < d-1$ and n even.
5. If d is odd, include $i \in \mathcal{W}$ for all $r_d \leq i \leq f-1$.

The effect of these steps is to include, starting at $i = 1$, all $i \in \mathcal{W}$ which correspond to type combinations of the form $(\mathrm{I}, \mathrm{II}, , \mathrm{II}, \ldots, \mathrm{II}, \mathrm{I})$ excluding the index corresponding to the right most Type I. We will lovingly refer to these combinations as *Type* II *sandwiches*. When d is odd then there is a left over Type I leaving us with an open-faced Type II sandwich meaning that the right most Type I does not exist. As a result we are forced to include all i up to $i = f-1$ and perform a column permutation on the $i = 0$ matrix.

Example 8.2.2: Suppose $f = 9$ and $\mathcal{M}(\mathcal{A})$ has Type combination given by

$$(\mathrm{I}, \mathrm{I}, \mathrm{II}, \mathrm{II}, \mathrm{I}, \mathrm{II}, \mathrm{I}, \mathrm{I}, \mathrm{I})$$

Then $d = 5$ with $r_1 = 1$, $r_2 = 4$, $r_3 = 6$, $r_4 = 7$ and $r_5 = 8$. Hence, we get the set $\mathcal{W} = \{1, 2, 3, 6, 8\} \subset \mathbb{Z}/9\mathbb{Z}$ with $i = 8$ being included since d is odd.

Consider the base change over $\mathrm{GL}_2(F)$, intended to line up the anti-diagonals of each matrix given by $X_3^{(i)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $i \in \mathcal{W}$ and $X_3^{(i)} = I_2$ for $i \notin \mathcal{W}$. With heartfelt apologies

[10] As we will see later in Section 8.3, there are ways around this problem when $f = 2$ but those methods require matrix by matrix considerations that are impractical for the general $f > 2$ case.

to the reader, the result of $(X_3) *_\varphi (\mathcal{A}_2)$ will be given by

$$\mathcal{A}_3^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} & 1 + \varepsilon_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \text{ and } f-1 \in \mathcal{W} \\ \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ 1 + \varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \text{ and } f-1 \notin \mathcal{W} \\ \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} \\ 1 & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \delta_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{T} \text{ and } f-1 \in \mathcal{W} \\ \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} + \delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \text{ and } f-1 \notin \mathcal{W} \end{cases}$$

$$\mathcal{A}_3^{(1)} = \begin{cases} \begin{pmatrix} 1 + \varepsilon_{21}^{(1)} & (a_2\alpha_0)^{(1)} + \varepsilon_{22}^{(1)} \\ \varepsilon_{11}^{(1)} & E^{k_1} a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \text{ so that } 1 \in \mathcal{W} \\ \begin{pmatrix} E^{k_1} a_1^{(1)} & \delta_{12}^{(1)} \\ (a_2\alpha_0)^{(1)} + \delta_{21}^{(1)} & 1 + \delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \text{ so that } 1 \notin \mathcal{W} \end{cases}$$

$$\mathcal{A}_3^{(i)} = \begin{cases} \begin{pmatrix} 1 + \varepsilon_{21}^{(i)} & (a_2\alpha_0)^{(i)} \\ \varepsilon_{11}^{(i)} & E^{k_i} a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ with } i \in \mathcal{S} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & \varepsilon_{11}^{(i)} \\ (a_2\alpha_0)^{(i)} & 1 + \varepsilon_{21}^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ with } i \in \mathcal{S} \text{ and } i \notin \mathcal{W} \\ \begin{pmatrix} 1 & (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} \\ 0 & E^{k_i} a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ with } i \in \mathcal{T} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } 2 \le i \text{ with } i \in \mathcal{T} \text{ and } i \notin \mathcal{W}. \end{cases}$$

As hinted in step 4 of Definition 8.2.1, there is a dichotomy with respect to the even or oddness of the order of the set $\mathcal{S}$. Indeed, these two distinct cases must be handled separately.

8.2.3 Subcase: $|\mathcal{S}|$ Even

When $|\mathcal{S}|$ is even, then either d in Definition 8.2.1 is even or d is odd with $\mathcal{A}^{(0)}$ being Type I. As a result, we may write

$$\mathcal{A}_3^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} & 1+\varepsilon_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} + \delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T}. \end{cases}$$

We may notice that regardless of Type, scaling $\mathcal{A}_3^{(0)}$ by $p/a_2^{(0)} \in \varpi\mathscr{O}_F$ along the diagonal will do the job of making all the terms either integral or lie the ideal I. Consider the base change over $\mathrm{GL}_2(F)$ intended to do just this given by $X_4^{(i)} = \mathrm{Diag}(1, p/a_2^{(0)})$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The result of $(X_4) *_\varphi (\mathcal{A}_3)$ will be given by

$$\mathcal{A}_4^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1}{\alpha_1}\right)^{(0)} E^{k_0-p} + \frac{a_2^{(0)}}{p}\tilde{\varepsilon}_{11}^{(0)} \\ p\alpha_0^{(0)} + \alpha_1^{(0)} E^p & 1+\varepsilon_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} & -\left(\frac{a_1}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ p\alpha_0^{(0)} + \alpha_1^{(0)} E^p + \frac{p}{a_2^{(0)}}\delta_{21}^{(0)} & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_4^{(1)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(1)} & \frac{a_2^{(0)}}{p}(a_2\alpha_0)^{(1)} + \frac{a_2^{(0)}}{p}\varepsilon_{22}^{(1)} \\ \frac{p}{a_2^{(0)}}\varepsilon_{11}^{(1)} & E^{k_1}a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \\ \begin{pmatrix} E^{k_1}a_1^{(1)} & \frac{a_2^{(0)}}{p}\delta_{12}^{(1)} \\ \frac{p}{a_2^{(0)}}(a_2\alpha_0)^{(1)} + \frac{p}{a_2^{(0)}}\delta_{21}^{(1)} & 1+\delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_4^{(i)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(i)} & \frac{a_2^{(0)}}{p}(a_2\alpha_0)^{(i)} \\ \frac{p}{a_2^{(0)}}\varepsilon_{11}^{(i)} & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & \frac{a_2^{(0)}}{p}\varepsilon_{11}^{(i)} \\ \frac{p}{a_2^{(0)}}(a_2\alpha_0)^{(i)} & 1+\varepsilon_{21}^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \notin \mathcal{W} \\ \begin{pmatrix} 1 & \frac{a_2^{(0)}}{p}(a_2\alpha_0)^{(i)} + \frac{a_2^{(0)}}{p}\delta_{21}^{(i)} \\ 0 & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ \frac{p}{a_2^{(0)}}(a_2\alpha_0)^{(i)} + \frac{p}{a_2^{(0)}}\delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \notin \mathcal{W}. \end{cases}$$

Since $p/a_2^{(0)}$ is integral then we need only be concerned with the top right entry of each matrix. To this end, let us now make the following assumption:

$$\nu_p(a_2^{(i)}) \geq 1 - \nu_p(a_2^{(0)}) \quad \text{for all} \quad i \in \mathcal{W}.$$

Notice that under this assumption, we now have $v(i) = 1$ for all $i \in \mathcal{W}$ where $v(i)$ denotes the non-zero valuation indicator function for $a_2^{(i)}$. Make the observation that if $i \notin \mathcal{W}$ but $i \in \mathcal{S}$ then we are guaranteed to have $i-1 \in \mathcal{W}$. We analyze the potentially problematic top right entries individually:

- $\frac{a_2^{(0)}}{p}\varepsilon_{11}^{(i)} \in \frac{a_2^{(0)}}{p}E^{k_i}p^{v(i-1)}I \subset I$ since $i-1 \in \mathcal{W}$;
- $\frac{a_2^{(0)}}{p}\delta_{21}^{(i)} \in \frac{a_2^{(0)}}{p}p^{v(i-1)}I \subset I$ since $i-1 \in \mathcal{W}$;
- $\frac{a_2^{(0)}}{p}\varepsilon_{22}^{(f-2)}, \frac{a_2^{(0)}}{p}\delta_{12}^{(f-2)} \in \frac{a_2^{(0)}}{p}pI \subset I$;
- $\frac{a_2^{(0)}}{p}\tilde{\varepsilon}_{11}^{(0)} = \frac{a_2^{(0)}}{p}(\varepsilon_{11}^{(0)} - \varepsilon_{21}^{(0)}\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)}E^{k_0-p})$ so that the only possible problem would occur in $\frac{a_2^{(0)}}{p}\varepsilon_{11}^{(0)} \in \frac{a_2^{(0)}}{p}E^{k_0}p^{v(f-1)}I$. However, we know that $f-1 \in \mathcal{W}$ since d is odd. Hence, $\frac{a_2^{(0)}}{p}\tilde{\varepsilon}_{11}^{(0)} \in I$.

Reapplying the $\mathcal{W}$-derived row swap $X_5^{(i)} = X_3^{(i)}$ has the effect of putting ourselves back in the original orientation prior to the lineup of anti-diagonals for scaling. As a result, we have our next case of satisfied descent assumptions.

Proposition 8.2.4: *Suppose $\mathcal{M}(\mathcal{A})$ is a rank-two Kisin module over S_F with labeled heights $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_i \leq p-3$ for $i \neq 0$. If $|\mathcal{S}|$ is even and $\nu_p(a_2^{(i)}) \geq 1 - \nu_p(a_2^{(0)})$ for all $i \in \mathcal{W}$, then the Frobenius matrix $(\mathcal{A})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0^{(0)} = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)}E^{k_0-p} & -\left(\frac{a_1}{\alpha_1}\right)^{(0)}E^{k_0-p} \\ p\alpha_0^{(0)} + \alpha_1^{(0)}E^p & 1 \end{pmatrix}$$

$$\mathfrak{A}_0^{(i)} = \begin{cases} \begin{pmatrix} 1 & \frac{a_2^{(0)}}{p}(a_2\alpha_0)^{(i)} \\ 0 & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 1 \leq i \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ \frac{p}{a_2^{(0)}}(a_2\alpha_0)^{(i)} & 1 \end{pmatrix} & \text{if } 1 \leq i \text{ and } i \notin \mathcal{W}. \end{cases}$$

8.2.5 Subcase: $|\mathcal{S}|$ Odd

Let us return ourselves to $\mathcal{A}_3^{(i)}$ as stated prior to the even case 8.2.3. When $|\mathcal{S}|$ is odd, then either d in Definition 8.2.1 is odd or d is even with $\mathcal{A}^{(0)}$ being Type I. Hence, we have that

$$\mathcal{A}_3^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ 1+\varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} \\ 1 & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} + \delta_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_3^{(1)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(1)} & (a_2\alpha_0)^{(1)} + \varepsilon_{22}^{(1)} \\ \varepsilon_{11}^{(1)} & E^{k_1}a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \\ \begin{pmatrix} E^{k_1}a_1^{(1)} & \delta_{12}^{(1)} \\ (a_2\alpha_0)^{(1)} + \delta_{21}^{(1)} & 1+\delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_3^{(i)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(i)} & (a_2\alpha_0)^{(i)} \\ \varepsilon_{11}^{(i)} & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & \varepsilon_{11}^{(i)} \\ (a_2\alpha_0)^{(i)} & 1+\varepsilon_{21}^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \notin \mathcal{W} \\ \begin{pmatrix} 1 & (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} \\ 0 & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \notin \mathcal{W}. \end{cases}$$

Our strategy will be to scale by the anti-diagonals of each matrix by p followed by a row operation that uses the E^{k_0-p}-term in $\mathcal{A}_3^{(0)}$ to kill the non-integral E^p/p-term. Indeed, consider the base change over $\mathrm{GL}_2(F)$ given by $X_4^{(i)} = \mathrm{Diag}(1,p)$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. The result of $(X_4) *_\varphi (\mathcal{A}_3)$ will be given by

$$\mathcal{A}_4^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ p+p\varepsilon_{21}^{(0)} & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)^{(0)} E^{k_0-p} + \frac{1}{p}\delta_{11}^{(0)} \\ p & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} + \delta_{21}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_4^{(1)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(1)} & \frac{1}{p}(a_2\alpha_0)^{(1)} + \frac{1}{p}\varepsilon_{22}^{(1)} \\ p\varepsilon_{11}^{(1)} & E^{k_1}a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \\ \begin{pmatrix} E^{k_1}a_1^{(1)} & \frac{1}{p}\delta_{12}^{(1)} \\ p(a_2\alpha_0)^{(1)} + p\delta_{21}^{(1)} & 1+\delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_4^{(i)} = \begin{cases} \begin{pmatrix} 1+\varepsilon_{21}^{(i)} & \frac{1}{p}(a_2\alpha_0)^{(i)} \\ p\varepsilon_{11}^{(i)} & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & \frac{1}{p}\varepsilon_{11}^{(i)} \\ p(a_2\alpha_0)^{(i)} & 1+\varepsilon_{21}^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{S} \text{ and } i \notin \mathcal{W} \\ \begin{pmatrix} 1 & \frac{1}{p}(a_2\alpha_0)^{(i)} + \frac{1}{p}\delta_{21}^{(i)} \\ 0 & E^{k_i}a_1^{(i)} \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ p(a_2\alpha_0)^{(i)} + p\delta_{21}^{(i)} & 1 \end{pmatrix} & \text{if } 2 \le i \text{ and } i \in \mathcal{T} \text{ and } i \notin \mathcal{W}. \end{cases}$$

As with the analogous scaling operation we did in the even case 8.2.3, the only threats to non-integrality lie in the upper right hand entries of each matrix. As a result, we will now make the following assumption:

$$\nu_p(a_2^{(i)}) \ge 1 \quad \text{for all} \quad i \in \mathcal{W}.$$

Under these assumptions, we need only ensure that the I-terms remain in I after division by p. The analysis of these terms is largely the same as in the even case 8.2.3 save for the $i=0$ matrix which is actually easier to see. Indeed, it remains true that $\frac{1}{p}\varepsilon_{11}^{(i)}, \frac{1}{p}\delta_{21}^{(i)}, \frac{1}{p}\varepsilon_{22}^{(1)}, \frac{1}{p}\delta_{12}^{(1)} \in I$ for $2 \le i$. Since $\delta_{11}^{(0)} \in pI$ then it is easy to see that $\frac{1}{p}\delta_{11}^{(0)} \in I$.

As mentioned prior, we want to kill the E^p/p-term in $\mathcal{A}_4^{(0)}$ which by the above analysis, is the only non-integral or non-I term remaining. Consider the base change over $\mathrm{GL}_2(S_F)$ given by

$$X_5^{(0)} = \begin{pmatrix} 1 & 0 \\ (\frac{a_2\alpha_1^2}{a_1\alpha_0 p})^{(0)}E^{2p-k_0} & 1 \end{pmatrix} \qquad\qquad X_5^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i \neq 0.$$

The result of $(X_5) *_\varphi (\mathcal{A}_4)$ will leave $\mathcal{A}_5^{(i)} = \mathcal{A}_4^{(i)}$ for $2 \le i \le f-1$ and

$$\mathcal{A}_5^{(0)} = \begin{cases} \begin{pmatrix} -\left(\frac{a_1p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)^{(0)} E^{k_0-p} \\ p - (\frac{\alpha_1}{\alpha_0})^{(0)}E^p + \tilde{\varepsilon}_{21}^{(0)} & (a_2\alpha_0)^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} -\left(\frac{a_1p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)^{(0)} E^{k_0-p} + \frac{1}{p}\delta_{11}^{(0)} \\ p - (\frac{\alpha_1}{\alpha_0})^{(0)}E^p & (a_2\alpha_0)^{(0)} + \tilde{\delta}_{22}^{(0)} \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mathcal{A}_5^{(1)} = \begin{cases} \begin{pmatrix} 1+\tilde{\varepsilon}_{11}^{(1)} & \frac{1}{p}(a_2\alpha_0)^{(1)} + \frac{1}{p}\varepsilon_{22}^{(1)} \\ \tilde{\varepsilon}_{21}^{(1)} & E^{k_1}a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{S} \\ \begin{pmatrix} E^{k_1}a_1^{(1)} + \tilde{\delta}_{11}^{(1)} & \frac{1}{p}\delta_{12}^{(1)} \\ p(a_2\alpha_0)^{(1)} + \tilde{\delta}_{21}^{(1)} & 1+\delta_{22}^{(1)} \end{pmatrix} & \text{if } 1 \in \mathcal{T} \end{cases}$$

- Each of $\tilde{\varepsilon}_{11}^{(1)}, \tilde{\varepsilon}_{21}^{(1)}, \tilde{\delta}_{11}^{(1)}, \tilde{\delta}_{21}^{(1)}$ result from adding a $-\varphi(\frac{a_2\alpha_1^2}{a_1\alpha_0 p})^{(0)}\varphi(E)^{2p-k_0}$ multiple of the second columns of $\mathcal{A}_4^{(1)}$ to the first column. Since $\varphi(E)^{2p-k_0} \in p^4 S_F$ due to the fact that $2p-k_0 \geq 4$, then all of the described terms lie in $\varpi p^3 S_F \subset I$.
- $\tilde{\varepsilon}_{21}^{(0)} = p\varepsilon_{21}^{(0)} - \tilde{\varepsilon}_{11}^{(0)}(\frac{a_2\alpha_1^2}{a_1\alpha_0 p})^{(0)}E^{2p-k_0}$. Since $\tilde{\varepsilon}_{11}^{(0)} = \varepsilon_{11}^{(0)} - \varepsilon_{21}^{(0)}\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)}E^{k_0-p}$ then we need only be concerned with the resulting $\varepsilon_{11}^{(0)}(\frac{a_2\alpha_1^2}{a_1\alpha_0 p})^{(0)}E^{2p-k_0}$-term. However, recalling that $\varepsilon_{11}^{(0)} \in E^{k_0}I$ will imply that this term lies in $E^{2p}p^{-1}I$ but $p \mid E^{2p}$ in S_F so $\tilde{\varepsilon}_{21}^{(0)} \in I$.
- $\tilde{\delta}_{22}^{(0)} = \delta_{21}^{(0)} - \frac{1}{p}\delta_{11}^{(0)}(\frac{a_2\alpha_1^2}{a_1\alpha_0 p})^{(0)}E^{2p-k_0}$. Since $\delta_{11}^{(0)} = \delta_{21}^{(0)}\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)}E^{k_0-p}$ and $\delta_{21}^{(0)} \in pI$ due to $0 \in \mathcal{W}$, then it is easy to see that $\tilde{\delta}_{22}^{(0)} \in I$.

As a result, we have yet another case of satisfied descent assumptions.

Proposition 8.2.6: *Suppose $\mathcal{M}(\mathcal{A})$ is a rank-two Kisin module over S_F with labeled heights $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_i \leq p-3$ for $i \neq 0$. If $|\mathcal{S}|$ is odd and $\nu_p(a_2^{(i)}) \geq 1$ for all $i \in \mathcal{W}$, then the Frobenius matrix $(\mathcal{A})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0^{(0)} = \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)}E^{k_0-p} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)^{(0)}E^{k_0-p} \\ p - (\frac{\alpha_1}{\alpha_0})^{(0)}E^p & (a_2\alpha_0)^{(0)} \end{pmatrix}$$

$$\mathfrak{A}_0^{(i)} = \begin{cases} \begin{pmatrix} 1 & \frac{1}{p}(a_2\alpha_0)^{(i)} \\ 0 & E^{k_i}a_1^{(i)} \end{pmatrix} & \textit{if } 1 \leq i \textit{ and } i \in \mathcal{W} \\ \begin{pmatrix} E^{k_i}a_1^{(i)} & 0 \\ p(a_2\alpha_0)^{(i)} & 1 \end{pmatrix} & \textit{if } 1 \leq i \textit{ and } i \notin \mathcal{W}. \end{cases}$$

8.3 Complete Description in $f = 2$

Recall that when $f = 2$, we obtained in Section 6.3 an explicit description of finite height, rank-two Kisin modules $\mathcal{M}(\mathcal{A}, \mathcal{B})$ over S_F in terms of the three Type combinations given below:

- Type (I, I):

$$\mathcal{A} = \begin{pmatrix} 0 & E^{k_0}a_1 \\ 1 & a_2\lambda_2^{k_1-k_0\varphi} \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 & E^{k_1}b_1 \\ 1 & b_2\lambda_2^{k_0-k_1\varphi} \end{pmatrix}$$

  with $a_1, b_1 \in \mathscr{O}_F^*$ and $a_2 \cdot b_2 \in \varpi\mathscr{O}_F$.
- Type (I, II):

$$\mathcal{A} = \begin{pmatrix} 0 & E^{k_0}a_1 \\ 1 & a_2\lambda_4^{(1-\varphi^2)(k_1+k_0\varphi)} \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} E^{k_1}b_1 & 0 \\ b_2\lambda_4^{(1-\varphi^2)(k_1\varphi-k_0)} & 1 \end{pmatrix}$$

  with $a_1, b_1 \in \mathscr{O}_F^*$ and $a_2, b_2 \in \varpi\mathscr{O}_F$.

- Type $(\mathrm{II}, \mathrm{I})$:

$$\mathcal{A} = \begin{pmatrix} E^{k_0} a_1 & 0 \\ a_2 \lambda_4^{(1-\varphi^2)(k_0\varphi - k_1)} & 1 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 & E^{k_1} b_1 \\ 1 & b_2 \lambda_4^{(1-\varphi^2)(k_1\varphi + k_0)} \end{pmatrix}$$

with $a_1, b_1 \in \mathscr{O}_F^*$ and $a_2, b_2 \in \varpi\mathscr{O}_F$.

Here we will continue to assume that we have labeled heights in the range $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_1 \leq p-3$. It is then clear that the results of Sections 8.1 and 8.2 will provide satisfied descent assumptions outside of two cases. The first being Type (I, I) with $0 \leq \nu_p(b_2) < 1 - \nu_p(a_2)$ and the second being Type $(\mathrm{II}, \mathrm{I})$ with $0 < \nu_p(b_2) < 1$. Note that Type $(\mathrm{I}, \mathrm{II})$ is covered by Proposition 8.2.6 as $\mathcal{W} = \emptyset$ in this Type combination. Hence, to get a full description in the case of $f = 2$, we need to treat these two cases separately.

8.3.1 Type (I, I) with $0 \leq \nu_p(a_2) < 1 - \nu_p(b_2)$

Let us recall the following Type I matrices $\mathcal{A}_3^{(i)}$ found in Section 8.2.3,

$$\mathcal{A}_3^{(0)} = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} + \tilde{\varepsilon}_{11}^{(0)} \\ (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)} \frac{E^p}{p} & 1 + \varepsilon_{21}^{(0)} \end{pmatrix}$$

$$\mathcal{A}_3^{(1)} = \begin{pmatrix} 1 + \varepsilon_{21}^{(1)} & (a_2\alpha_0)^{(1)} + \varepsilon_{22}^{(1)} \\ \varepsilon_{11}^{(1)} & E^{k_1} a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix}.$$

We will change notation to ease the decomposition superscripts in the following manner: Set $\mathcal{A}_3 = \mathcal{A}_3^{(0)}$ and drop all of the superscripts (0). Also set $\mathcal{B}_3 = \mathcal{A}_3^{(1)}$ and make the replacements $b_j = a_j^{(1)}$, $\beta_j = \alpha_j^{(1)}$ and $\delta_{jk} = \varepsilon_{jk}^{(1)}$. Hence after performing this change, we are left with

$$\mathcal{A}_3 = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right) E^{k_0-p} & -\left(\frac{a_1 p}{a_2\alpha_1}\right) E^{k_0-p} + \tilde{\varepsilon}_{11} \\ a_2\alpha_0 + (a_2\alpha_1) \frac{E^p}{p} & 1 + \varepsilon_{21} \end{pmatrix}$$

$$\mathcal{B}_3 = \begin{pmatrix} 1 + \delta_{21} & b_2\beta_0 + \delta_{22} \\ \delta_{11} & E^{k_1} b_1 + \delta_{12} \end{pmatrix}.$$

Since we know that the product $a_2 b_2 \in \varpi\mathscr{O}_F$ by Proposition 4.2.1 then we can write $\delta_{2j}, \delta_{12}, \varepsilon_{21} \in pI$ and $\delta_{11}, \tilde{\varepsilon}_{11} \in I$.

We will now stray away from our normal strategy to instead apply a row operation on $\mathcal{B}_3$ intending to kill the $E^{k_1} b_1$-term. The reason for this deviation is that the p-adic valuation of b_2 is no longer large enough to absorb a division by p. As a result, we need to create another term that can be divided by p. To prepare for this however, we first scale the anti-diagonals of both matrices by $b_2 \in \mathscr{O}_F$ using the following base change over $\mathrm{GL}_2(F)$

given by $X_4 = Y_4 = \mathrm{Diag}(1, b_2)$. The result of $(X_4, Y_4) *_\varphi (\mathcal{A}_3, \mathcal{B}_3)$ will then be given by

$$\mathcal{A}_4 = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right) E^{k_0-p} & -\left(\frac{a_1 p}{a_2'\alpha_1}\right) E^{k_0-p} + \frac{1}{b_2}\tilde{\varepsilon}_{11} \\ a_2'\beta_0 + (a_2'\alpha_1)\frac{E^p}{p} & 1+\varepsilon_{21} \end{pmatrix}$$

$$\mathcal{B}_4 = \begin{pmatrix} 1+\delta_{21} & \beta_0 + \frac{1}{b_2}\delta_{22} \\ b_2\delta_{11} & E^{k_1}b_1 + \delta_{12} \end{pmatrix}$$

- $a_2' := a_2 b_2 \in \varpi\mathscr{O}_F$ so that $p/a_2' \in \varpi\mathscr{O}_F$ under our assumptions;
- $\frac{1}{b_2}\delta_{22} \in I$ since $\delta_{22} \in pI$;
- $\frac{1}{b_2}\tilde{\varepsilon}_{11} = \frac{1}{b_2}(\varepsilon_{11}^{(0)} - \varepsilon_{21}^{(0)}\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p})$ using the notation as in Section 8.2. Here we have that in our new notation, $\varepsilon_{11}^{(0)} \in E^{k_0}b_2p^2S_F$ by Lemma 8.1.1. Let us write $E^{k_0}b_2p^2S_F = E^{k_0-p}(E^p)b_2p^2S_F$ and observe that $(pE^p)S_F \in p\mathfrak{S}_F + p^2\mathrm{Fil}^{2p}S_F$. Hence, $\varepsilon_{11}^{(0)} \in pb_2E^{k_0-p}(p\mathfrak{S}_F + p^2\mathrm{Fil}^{2p}S_F) \subset b_2(p^2\mathfrak{S}_F + p^3E\mathrm{Fil}^{2p}S_F) \subset b_2(p^2\mathfrak{S}_F + p^3I)$. Moreover, since $p/(a_2b_2) \in \varpi\mathscr{O}_F$ then we see that $\frac{1}{b_2}\tilde{\varepsilon}_{11} \in p^2\mathfrak{S}_F + pI$ after recalling that $\varepsilon_{21}^{(0)} \in pI$.

Let us now kill the E^{k_1}-term in $\mathcal{B}_4$ via the following base change over $\mathrm{GL}_2(\mathfrak{S}_F)$,

$$X_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Y_5 = \begin{pmatrix} 1 & 0 \\ -\frac{b_1}{\beta_0}E^{k_1} & 1 \end{pmatrix}.$$

The result of $(X_5, Y_5) *_\varphi (\mathcal{A}_4, \mathcal{B}_4)$ will then be given by

$$\mathcal{A}_5 = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right) E^{k_0-p} + \varepsilon_{11}' & -\left(\frac{a_1 p}{a_2'\alpha_1}\right) E^{k_0-p} + \frac{1}{b_2}\tilde{\varepsilon}_{11} \\ a_2'\alpha_0 + (a_2'\alpha_1)\frac{E^p}{p} + \varepsilon_{21}' & 1+\varepsilon_{21} \end{pmatrix}$$

$$\mathcal{B}_5 = \begin{pmatrix} 1+\delta_{21} & \beta_0 + \frac{1}{b_2}\delta_{22} \\ -\frac{b_1}{\beta_0}E^{k_1} + \delta_{12}' & \delta_{22}' \end{pmatrix}$$

- $\delta_{12}' = b_2\delta_{11} - \delta_{21}\frac{b_1}{\beta_0}E^{k_1} \in I$.
- $\delta_{22}' = \delta_{12} - \frac{1}{b_2}\delta_{22}\frac{b_1}{\beta_0}E^{k_1} \in p^2\mathfrak{S}_F + pI$.
- $\varepsilon_{j1}' \in I$ since they both result from adding a $\varphi(E)^{k_1} \in p^2S_F \subset I$ multiple of the second column of $\mathcal{A}_3$ to first column.

Remark 8.3.2: When $f > 2$, this step of killing the E^{k_i}-term can be done as we have performed it here. However, we would at some point be required to scale by $a_2^{(i)}$ for all $i \in \mathcal{W}$ to clear denominators which leads to a proliferation of subcases depending on the valuations of each $a_2^{(i)}$ in relation to one another. It is for this reason that we restrict ourselves to $f = 2$ when computing these very small valuation cases.

We will now return to the strategy employed in Section 8.2.5. In particular, let us swap the columns of $\mathcal{A}_5$ which swaps the rows of $\mathcal{B}_5$ and scale the anti-diagonals of both matrices by p via $X_6 = Y_6 = \mathrm{Diag}(1,p)$. The result is

$$\mathcal{A}_6 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2'\alpha_1}\right)E^{k_0-p} + \frac{1}{b_2}\tilde{\varepsilon}_{11} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)E^{k_0-p} + \frac{1}{p}\varepsilon_{11}' \\ p + p\varepsilon_{21} & a_2'\alpha_0 + (a_2'\alpha_1)\frac{E^p}{p} + \varepsilon_{21}' \end{pmatrix}$$
$$\mathcal{B}_6 = \begin{pmatrix} -\frac{b_1}{\beta_0}E^{k_1} + \delta_{12}' & \frac{1}{p}\delta_{22}' \\ p + p\delta_{21} & \beta_0 + \frac{1}{b_2}\delta_{22} \end{pmatrix}$$

- $\frac{1}{p}\delta_{22}' \in p\mathfrak{S}_F + I$ since $\delta_{22}' \in p^2\mathfrak{S}_F + pI$;
- $\frac{1}{p}\varepsilon_{11}' = \frac{1}{p}(-(\frac{a_1 p}{a_2'\alpha_1})E^{k_0-p} + \frac{1}{b_2}\tilde{\varepsilon}_{11})\frac{b_1}{\varphi(\beta_0)}\varphi(E)^{k_1} \in I$ since $\varphi(E)^{k_1} \in p^2 S_F$ and $p/a_2' \in \varpi\mathscr{O}_F$.

Our final operation will be to kill the E^p/p-term in $\mathcal{A}_6$ via an operation analogous to $X_5^{(i)}$ in Section 8.2.5. Indeed, consider the base change over $\mathrm{GL}_2(S_F)$ given by

$$X_7 = \begin{pmatrix} 1 & \frac{a_2'\alpha_1^2}{a_1\alpha_0 p}E^{2p-k_0} \\ 0 & 1 \end{pmatrix} \qquad Y_7 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the result of $(X_7, Y_7) *_\varphi (\mathcal{A}_6, \mathcal{B}_6)$ will be given by

$$\mathcal{A}_6 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2'\alpha_1}\right)E^{k_0-p} + \frac{1}{b_2}\tilde{\varepsilon}_{11} & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)E^{k_0-p} + \frac{1}{p}\varepsilon_{11}' \\ p - \frac{\alpha_1}{\alpha_0}E^p + \varepsilon_{21}'' & a_2'\alpha_0 + \varepsilon_{22} \end{pmatrix}$$
$$\mathcal{B}_6 = \begin{pmatrix} -\frac{b_1}{\beta_0}E^{k_1} + \delta_{12}' & \delta_{12}'' \\ p + p\delta_{21} & \beta_0 + \delta_{22}'' \end{pmatrix}$$

- Both δ_{j2}'' result from adding a $-a_2'/p\varphi(E)^{2p-k_0}$ multiple of the first columns of $\mathcal{B}_5$ to the second column. Since $\varphi(E)^{2p-k_0} \in p^4 S_F$ then $\delta_{22}'' \in I$ and $\delta_{12}'' \in p\mathfrak{S}_F + I$;
- $\varepsilon_{21}'' = p\varepsilon_{21} + \frac{1}{b_2}\tilde{\varepsilon}_{11}\frac{a_2'\alpha_1^2}{a_1\alpha_0 p}E^{2p-k_0} \in I$ after recalling that $\frac{1}{b_2}\tilde{\varepsilon}_{11} \in p^2\mathfrak{S}_F + pI$ and observing that $\varpi p\mathfrak{S}_F \subset I$;
- $\varepsilon_{22} = \varepsilon_{21}' + \frac{1}{p}\varepsilon_{11}'\frac{a_2'\alpha_1^2}{a_1\alpha_0 p}E^{2p-k_0}$. By the discussion of $\frac{1}{p}\varepsilon_{11}'$ above, the only possible complication could arise from the product $\left(\frac{a_1 p}{a_2'\alpha_1}\right)E^{k_0-p}\frac{a_2'\alpha_1^2}{a_1\alpha_0 p}E^{2p-k_0} = \frac{\alpha_1}{\alpha_0}E^p$ which is divisible by p in S_F so that $\varepsilon_{22} \in I$.

With this, we have achieved the penultimate case of satisfied descent assumptions.

Proposition 8.3.3: *Suppose $\mathcal{M}(\mathcal{A},\mathcal{B})$ is a rank-two Kisin module over S_F of Type* (I, I) *with labeled heights $p+2 \le k_0 \le 2p-4$ and $2 \le k_1 \le p-3$. If $\nu_p(b_2) < 1 - \nu_p(a_2)$, then the Frobenius matrix $(\mathcal{A},\mathcal{B})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2'\alpha_1}\right)E^{k_0-p} + p\mathfrak{S}_F & -\left(\frac{a_1\alpha_0}{\alpha_1}\right)E^{k_0-p} \\ p - \frac{\alpha_1}{\alpha_0}E^p & a_2'\beta_0 \end{pmatrix}$$

$$\mathfrak{B}_0 = \begin{pmatrix} -\frac{b_1}{\beta_0}E^{k_1} & p\mathfrak{S}_F \\ p & \beta_0 \end{pmatrix}$$

where $p\mathfrak{S}_F$ denotes an arbitrary element in said ring.

8.3.4 Type $(\mathrm{II}, \mathrm{I})$ with $0 < \nu_p(b_2) < 1$

Let us recall the following matrices $\mathcal{A}_3^{(i)}$ from section 8.2.5,

$$\mathcal{A}_3^{(0)} = \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right)^{(0)} E^{k_0-p} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right)^{(0)} E^{k_0-p} + \delta_{11}^{(0)} \\ 1 & (a_2\alpha_0)^{(0)} + (a_2\alpha_1)^{(0)}\frac{E^p}{p} + \delta_{21}^{(0)} \end{pmatrix}$$
$$\mathcal{A}_3^{(1)} = \begin{pmatrix} 1+\varepsilon_{21}^{(1)} & (a_2\alpha_0)^{(1)} + \varepsilon_{22}^{(1)} \\ \varepsilon_{11}^{(1)} & E^{k_1}a_1^{(1)} + \varepsilon_{12}^{(1)} \end{pmatrix}$$

which are of Type II and Type I respectively. As we did before, we will change notation in the following way: Set $\mathcal{A}_3 = \mathcal{A}_3^{(0)}$ and drop all of the superscripts (0). Also set $\mathcal{B}_3 = \mathcal{A}_3^{(1)}$ and make the replacements $b_j = a_j^{(1)}$, $\beta_j = \alpha_j^{(1)}$ and $\varepsilon_{jk} = \varepsilon_{jk}^{(1)}$. Hence, after performing this replacements we are left with

$$\mathcal{A}_3 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right) E^{k_0-p} & -\left(\frac{a_1\alpha_0 p}{\alpha_1}\right) E^{k_0-p} + \delta_{11} \\ 1 & a_2\alpha_0 + (a_2\alpha_1)\frac{E^p}{p} + \delta_{21} \end{pmatrix}$$
$$\mathcal{B}_3 = \begin{pmatrix} 1+\varepsilon_{21} & b_2\beta_0 + \varepsilon_{22} \\ \varepsilon_{11} & E^{k_1}b_1 + \varepsilon_{12} \end{pmatrix}.$$

Since we know that both $a_2, b_2 \in \varpi\mathscr{O}_F$ by Proposition 4.2.1 then we can write that all of $\varepsilon_{jk}, \delta_{j1} \in pI$.

We follow much of the same processes that we undertook in the Type (I, I) case 8.3.1. Indeed, since $p/b_2 \in \varpi\mathscr{O}_F$, then our first step will be to scale by b_2 along the anti-diagonals of both matrices via $X_4 = Y_4 = \mathrm{Diag}(1, b_2) \in \mathrm{GL}_2(F)$ so that

$$\mathcal{A}_4 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right) E^{k_0-p} & -\left(\frac{a_1\alpha_0 p}{b_2\alpha_1}\right) E^{k_0-p} + \frac{1}{b_2}\delta_{11} \\ b_2 & a_2\alpha_0 + (a_2\alpha_1)\frac{E^p}{p} + \delta_{21} \end{pmatrix}$$
$$\mathcal{B}_4 = \begin{pmatrix} 1+\varepsilon_{21} & \beta_0 + \frac{1}{b_2}\varepsilon_{22} \\ b_2\varepsilon_{11} & E^{k_1}b_1 + \varepsilon_{12} \end{pmatrix}$$

where both $\frac{1}{a_2}\varepsilon_{22}, \frac{1}{a_2}\delta_{11} \in I$ since $\varepsilon_{22}, \delta_{11} \in pI$.

As before, let us now kill the $E^{k_1}b_1$ term in $\mathcal{B}_4$ via the following base change over $\mathrm{GL}_2(\mathfrak{S}_F)$

$$X_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad Y_5 = \begin{pmatrix} 1 & 0 \\ -\frac{b_1}{\beta_0}E^{k_1} & 1 \end{pmatrix}.$$

The result of $(X_5, Y_5) *_\varphi (\mathcal{A}_4, \mathcal{B}_4)$ will then be given by

$$\mathcal{A}_5 = \begin{pmatrix} -\left(\frac{a_1 p}{a_2\alpha_1}\right) E^{k_0-p} + \delta'_{11} & -\left(\frac{a_1\alpha_0 p}{b_2\alpha_1}\right) E^{k_0-p} + \frac{1}{b_2}\delta_{11} \\ b_2 + \delta'_{21} & a_2\alpha_0 + (a_2\alpha_1)\frac{E^p}{p} + \delta_{21} \end{pmatrix}$$

$$\mathcal{B}_5 = \begin{pmatrix} 1 + \varepsilon_{21} & \beta_0 + \frac{1}{b_2}\varepsilon_{22} \\ -\frac{b_1}{\beta_0}E^{k_1} + \varepsilon'_{21} & \varepsilon'_{22} \end{pmatrix}$$

- $\varepsilon'_{21} = b_2\varepsilon_{11} - \varepsilon_{21}\frac{b_1}{\beta_0}E^{k_1} \in I$;
- $\varepsilon'_{22} = \varepsilon_{12} - \frac{1}{b_2}\varepsilon_{22}\frac{b_1}{\beta_0}E^{k_1} \in I$;
- Both δ'_{11} and δ'_{21} arise from adding a $\frac{b_1}{\varphi(\beta_0)}\varphi(E)^{k_1}$-multiple of the second columns of $\mathcal{A}_3$ to the first column. Since $b_1\varphi(E)^{k_1} \in b_1p^2S_F \subset \varpi p S_F \subset I$ then both $\delta'_{11}, \delta'_{21} \in I$.

We will now swap the columns of $\mathcal{A}_5$ which swaps the rows of $\mathcal{B}_5$ and scale the anti-diagonals of the resulting matrices by $p/a_2 \in \varpi\mathscr{O}_F$ via $X_6 = Y_6 = \mathrm{Diag}(1, p/a_2)$ so that

$$\mathcal{A}_6 = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{b_2\alpha_1}\right) E^{k_0-p} + \frac{1}{b_2}\delta_{11} & -\left(\frac{a_1}{\alpha_1}\right) E^{k_0-p} + \frac{a_2}{p}\delta'_{11} \\ p\alpha_0 + \alpha_1 E^p + \frac{p}{a_2}\delta_{21} & b_2 + \delta'_{21} \end{pmatrix}$$

$$\mathcal{B}_6 = \begin{pmatrix} -\frac{b_1}{\beta_0}E^{k_1} + \varepsilon'_{21} & \frac{a_2}{p}\varepsilon'_{22} \\ \frac{p}{a_2} + \frac{p}{a_2}\varepsilon_{21} & \beta_0 + \frac{1}{b_2}\varepsilon_{22} \end{pmatrix}$$

- $\frac{a_2}{p}\varepsilon'_{22} = \frac{a_2}{p}(\varepsilon_{12} - \frac{1}{b_2}\varepsilon_{22}\frac{b_1}{\beta_0}E^{k_1})$. Since $\varepsilon_{12} \in pI$ then we need only concern ourselves with $\frac{a_2}{p}(\frac{1}{b_2}\varepsilon_{22}\frac{b_1}{\beta_0}E^{k_1})$. However, we may recall that $\varepsilon_{22} = \varepsilon_{22}^{(1)}$ results as a multiple of $\frac{p}{a_2}\varphi(E)^{k_0-p}$ so that because $\varphi(E)^{k_0-p} \in p^2S_F$ then $\frac{a_2}{p}(\frac{1}{b_2}\varepsilon_{22}\frac{b_1}{\beta_0}E^{k_1}) \in \frac{p^2}{b_2}S_F \subset \varpi p S_F \subset I$. Hence, $\frac{a_2}{p}\varepsilon'_{22} \in I$.
- $\frac{a_2}{p}\delta'_{11} \in I$ since δ'_{11} is divisible by $\varphi(E)^{k_1} \in p^2S_F$ so that $\frac{a_2}{p}\varphi(E)^{k_1} \in \varpi p S_F \subset I$.

At long last, we have reached our final case of satisfied descent assumptions.

Proposition 8.3.5: *Suppose $\mathcal{M}(\mathcal{A}, \mathcal{B})$ is a rank-two Kisin module over S_F of Type* (II, I) *with labeled heights $p+2 \le k_0 \le 2p-4$ and $2 \le k_1 \le p-3$. If $\nu_p(b_2) < 1$, then the Frobenius matrix $(\mathcal{A}, \mathcal{B})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0 = \begin{pmatrix} -\left(\frac{a_1\alpha_0 p}{b_2\alpha_1}\right) E^{k_0-p} & -\left(\frac{a_1}{\alpha_1}\right) E^{k_0-p} \\ p\alpha_0 + \alpha_1 E^p & b_2 \end{pmatrix}$$

$$\mathfrak{B}_0 = \begin{pmatrix} -\frac{b_1}{\beta_0}E^{k_1} & 0 \\ \frac{p}{a_2} & \beta_0 \end{pmatrix}.$$

Chapter 9

Computing Explicit Reductions

We are now ready to explicitly compute the reductions of irreducible, two-dimensional crystalline representations $V(A)$ of G_K which are associated to the Kisin modules $\mathcal{M}(\mathcal{A})$ from which the Descent Assumptions 7.2.2 have been shown to be satisfied in Chapter 8. This is done in each of the cases detailed in the previous chapter.

We begin in Section 9.1 by detailing how the descent algorithm of Chapter 7 produces a Kisin module $\mathfrak{M}(\mathfrak{A})$ which is canonically associated to $V(A)$. We also display how to use $\mathfrak{M}(\mathfrak{A})$ to compute the reduction of $V(A)$ along with various results that assist us in doing so. Sections 9.2, 9.3 and 9.4 then us these results to compute reductions in each of the cases from Chapter 8. We then use Section 9.5 to summarize the results of the previous sections in a concise manner for the readers reference.

9.1 From Descent to Reductions

Let us first summarize many of the results we have proven so far. Let V be any irreducible, two-dimensional crystalline F-representation of G_K. By Theorem 4.2.3, there exists an f-tuple of matrices $(A) = (A^{(i)}) \in \mathrm{GL}_2(\mathscr{O}_F)^f$ such that $V \cong V(A)$ where $V(A) := \mathbf{V}^*_{\mathrm{cris}}(D(A))$ and $D(A)$ is an irreducible, weakly admissible filtered φ-module over $F \otimes_{\mathbb{Q}_p} K$ admitting a decomposition $D(A) = \prod_i D(A)^{(i)}$ with i-th partial Frobenius action given by the matrix $A^{(i)}\Delta_{\mathbf{k}^{(i+1)}}$ in a basis adapted to the filtration structure by Proposition 3.1.6.

By Proposition 6.1.2, we have constructed a finite height Kisin module $\mathcal{M}(\mathcal{A})$ over $S_F = \mathfrak{S}_F[\![E^p/p]\!]$ from $D(A)$ with i-th partial Frobenius $\mathcal{A}^{(i)}$ given in terms of one of two Types originating from Proposition 6.2.2:

- Type I:
$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} \end{pmatrix};$$

- Type II:
$$\mathcal{A}^{(i)} = \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} & 1 \end{pmatrix}.$$

Supposing that we are able to show that the associated Frobenius matrix $(\mathcal{A}^{(i)})$ satisfies the Descent Assumptions 7.2.2, then the results of Theorem 7.2.3 implies the existence of a base change $(Y^{(i)})$ over $R = \mathfrak{S}_F[\![\varpi E/p]\!]$ such that for all $i \in \mathbb{Z}/f\mathbb{Z}$, we have

$$Y^{(i)} *_\varphi \mathcal{A}^{(i)} = \mathfrak{A}^{(i)} \in \mathrm{Mat}_2^{k_i}(\mathfrak{S}_F).$$

In particular, we will have found a basis $\{\eta_1, \eta_2\}$ of $\mathcal{M} \otimes_{S_F[1/p]} R[1/p]$ such that the $\mathfrak{S}_F$-submodule generated by $\{\eta_1, \eta_2\}$ has Frobenius action given by $(\mathfrak{A}^{(i)})$. We denote this integral submodule by $\mathfrak{M}(\mathfrak{A})$ and observe that it is a rank-two Kisin module of labeled height $\leq 2p-4$ over $\mathfrak{S}_F$. In other words, $\mathfrak{M}(\mathfrak{A})$ is a descent of $\mathcal{M}(\mathcal{A})$ to $\mathfrak{S}_F$ in the sense of Definition 5.1.3.

In Section 5.3 we displayed how one may use a Kisin module such as $\mathfrak{M}(\mathfrak{A})$ to compute reductions of $V(A)$ *if* we can canonically associated it to some G_∞-stable $\mathscr{O}_F$-lattice $T \subset V(A)$. The following proposition shows that this is indeed the case.

Proposition 9.1.1: *Using the notations from above, there exists a G_∞-stable $\mathscr{O}_F$ lattice $T \subset V(A)$ which is canonically associated to $\mathfrak{M}(\mathfrak{A})$.*

Proof. Let us set $\mathcal{M}_R = \mathfrak{M}(\mathfrak{A}) \otimes_{\mathfrak{S}_F} R[1/p]$. To display the proposition, we need only justify why we may apply Proposition 5.3.2 to $\mathcal{M}_R$. By Theorem 7.2.3 we have that

$$\mathcal{M}_R \cong \mathbf{M}_{\mathrm{Kis}}(\mathbf{D}^*_{\mathrm{cris}}(V(A))) \otimes_{\Lambda_F} R[1/p]$$

so we must justify why we may fit $R[1/p] = \mathfrak{S}_F[\![u, \frac{\varpi E}{p}]\!][1/p]$ inside of $\Lambda_{F,s}$ for some $1/p < s < 1$. However, this is clear once we notice that $E/p \in \Lambda_{F,1}$ so that $\varpi E/p \in \Lambda_{F,s}$ for some $s < 1$.

To be completely accurate, we have not shown that $1/p < s$ and indeed the inequality may not be true. To recover the situation, notice that if $s < 1/p$ then there exists an $1/p < s'$ such that $\Lambda_{F,s} \subset \Lambda_{F,s'}$ since $p^{-s'} < p^{-s}$. Hence, if $s < 1/p$, we may extend scalars to $\Lambda_{F,s'}$ to apply Proposition 5.3.2. □

Note that as a result of the above proposition, we have implicitly restricted the labeled Hodge-Tate weights of $V(A)$ to be $k_i \leq 2p-4$ as well. By setting $\overline{\mathfrak{M}(\mathfrak{A})} := \mathfrak{M}(\mathfrak{A})/\varpi\mathfrak{M}(\mathfrak{A})$, Proposition 5.3.5 says that the reduction $\overline{V(A)}|_{G_\infty} = (T/\varpi T)^{\mathrm{ss}}|_{G_\infty}$ is completely determined by the k_F-representation of G_∞ given by

$$\overline{V(A)}|_{G_\infty} \cong \mathbf{V}^*_{k_F}\left(\overline{\mathfrak{M}(\mathfrak{A})}\left[\frac{1}{u}\right]\right).$$

The object $\overline{\mathfrak{M}(\mathfrak{A})}[\frac{1}{u}]$ is an étale φ-module over $k_F(\!(u)\!)$ as described in Definition 5.3.3. In order to compute $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}(\mathfrak{A})}[\frac{1}{u}])$, let us recall some facts from Sections 1.2 and 2.2.

Recall that we have fixed a p-power compatible sequence $\pi = (\pi_0, \pi_1, \pi_2, \dots) \in \mathscr{O}^\flat_{\mathbb{C}_p}$. We may embed

$$k_F[\![u]\!] \hookrightarrow \mathscr{O}^\flat_{\mathbb{C}_p} \qquad \text{via} \qquad u \mapsto \pi$$

with this embedding being naturally compatible with respect to both φ and G_∞. This embedding lifts to $k_F(\!(u)\!) \hookrightarrow \mathbb{C}^\flat_p$ so that we may write

$$\mathbf{V}^*_{k_F}\left(\overline{\mathfrak{M}(\mathfrak{A})}\left[\frac{1}{u}\right]\right) := \mathrm{Hom}_{k_F(\!(u)\!),\varphi}\left(\overline{\mathfrak{M}(\mathfrak{A})}\left[\frac{1}{u}\right], \mathbb{C}^\flat_p\right).$$

Since we have computed $\overline{\mathfrak{M}(\mathfrak{A})}$ to be defined over $k_F[\![u]\!]$, then $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}(\mathfrak{A})}[\frac{1}{u}])$ is completely determined by

$$\mathbf{V}^*_{k_F}\left(\overline{\mathfrak{M}(\mathfrak{A})}\right) = \mathrm{Hom}_{k_F[\![u]\!],\varphi}\left(\overline{\mathfrak{M}(\mathfrak{A})}, \mathscr{O}^\flat_{\mathbb{C}_p}\right).$$

We summarize the above discussion in the following diagram:

$$\begin{tikzcd}
V(A) \arrow[r, "\mathbf{D}^*_{\mathrm{cris}}"] \arrow[d] & D(A) \arrow[r, "\text{Prop 6.1.2}"] & \mathcal{M}(\mathcal{A}) \arrow[d, "\text{Thm 7.2.3}"] \\
T|_{G_\infty} \arrow[rr, leftrightarrow, "\text{Prop 9.1.1}"] \arrow[d] & & \mathfrak{M}(\mathfrak{A}) \arrow[d, "\text{mod } \varpi"] \\
\overline{V(A)}|_{G_\infty} & & \overline{\mathfrak{M}(\mathfrak{A})} \arrow[ll, "\mathbf{V}^*_{k_F}"']
\end{tikzcd}$$

When $K = \mathbb{Q}_p$, one normally computes the Hom space of $\mathbf{V}^*_{k_F}$ by considering the action of the semi-simple Frobenius matrix on the basis elements. However, in our case of $K = \mathbb{Q}_{p^f}$, we instead deal with f-many partial Frobenius matrices, one in each idempotent piece. It is therefore convenient to reduce to looking at a single Frobenius matrix in the following construction from [CDM18, §2].

As an étale φ-module, we may perform the usual idempotent decomposition via the equivalence functor of Theorem 5.3.4. Hence, $\overline{\mathfrak{M}(\mathfrak{A})} = \prod_i \overline{\mathfrak{M}(\mathfrak{A})}^{(i)}$ so that each $\overline{\mathfrak{M}(\mathfrak{A})}^{(i)}$ is a φ-module with Frobenius action given by $\overline{\mathfrak{A}}^{(i)} := \mathfrak{A}^{(i)} \pmod{\varpi}$. We may endow the $k_F[\![u]\!]$-module $\overline{\mathfrak{M}(\mathfrak{A})}^{(0)}$ with the structure of an étale φ^f-module by defining its Frobenius action to be given by the product

$$\prod_{i \in \mathbb{Z}/f\mathbb{Z}} \mathfrak{A}^{(i)} = \overline{\mathfrak{A}}^{(0)} \varphi\left(\overline{\mathfrak{A}}^{(1)}\right) \varphi^2\left(\overline{\mathfrak{A}}^{(2)}\right) \cdots \varphi^{f-1}\left(\overline{\mathfrak{A}}^{(f-1)}\right).$$

Such objects are defined just as in Definition 5.3.3 after replacing φ-linearity with φ^f-linearity. The associated category would then be denoted by $\mathrm{Mod}^{\varphi^f,\text{ét}}_{k_F(\!(u)\!)}$ and there is an evident functor

$$\varepsilon_0 : \mathrm{Mod}^{\varphi,\text{ét}}_{k_F(\!(u)\!)} \to \mathrm{Mod}^{\varphi^f,\text{ét}}_{k_F(\!(u)\!)} \qquad\qquad (M,\varphi) \mapsto (M^{(0)}, \varphi^f).$$

The following lemma shows that is suffices to compute the functor $\mathbf{V}^*_{k_F}$ by only looking at this associated étale φ^f-module via ε_0.

Lemma 9.1.2: *Using the notation from above, let $(M^{(0)}, \varphi^f)$ be the étale φ^f-module over $k_F(\!(u)\!)$ associated to an étale φ-module (M,φ) over $k_F(\!(u)\!)$ via ε_0. Then*

$$\mathbf{V}^*_{k_F}(M) \cong \mathrm{Hom}_{k_F(\!(u)\!),\varphi^f}\left(M^{(0)}, \mathscr{O}^\flat_{\mathbb{C}_p}\right).$$

Proof. As detailed in [CDM18, Thm 2.1.6] there is an equivalence of categories

$$\mathbf{M}^{(0)}_{k_F} : \mathrm{Rep}_{\mathrm{cris}/k_F}(G_\infty) \to \mathrm{Mod}^{\varphi^f,\text{ét}}_{k_F(\!(u)\!)}.$$

with quasi-inverse functor $\mathbf{V}^{(0)}_{k_F}$. However, [CDM18, Prop 2.1.7] implies that for any $V \in \mathrm{Rep}_{\mathrm{cris}/k_F}(G_\infty)$, there is a natural isomorphism $\mathbf{M}^{(0)*}_{k_F}(V) \cong \mathbf{M}^*_{k_F}(V)^{(0)}$in $\mathrm{Mod}^{\varphi^f,\text{ét}}_{k_F(\!(u)\!)}$ where $\mathbf{M}^*_{k_F}(V)^{(0)}$ is the image of $\mathbf{M}^*_{k_F}(V)$ under the functor ε_0. It follows that $\mathbf{V}^*_{k_F}(M) \cong \mathbf{V}^{(0)*}_{k_F}(M^{(0)})$ where $\mathbf{V}^{(0)*}_{k_F}$ is defined as in [CDM18] and the above discussion by $\mathbf{V}^{(0)*}_{k_F}(M^{(0)}) = \mathrm{Hom}_{k_F(\!(u)\!),\varphi^f}(M^{(0)}, \mathscr{O}^\flat_{\mathbb{C}_p})$. □

We are now ready to explorer the representation theory necessary to compute the reduction $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}(\mathfrak{A})})$. Let I_K denote the inertial subgroup of G_K and choose for any $r \geq 1$ a $p^r - 1$-th root of $\pi_K = -p$ denoted $\xi_r = (-p)^{1/p^r-1} \in \overline{K}$. This choice of ξ_r defines a character $\omega_{\xi_r} : I_K \to \mathscr{O}^*_K$. By composing with our fixed embedding $\tau_0 : \mathscr{O}_K \hookrightarrow \mathscr{O}_F$, we give rise to a *fundamental character of niveau r*

$$\widetilde{\omega}_{\xi_r} : I_K \to \mathscr{O}^*_F.$$

The residual character of $\widetilde{\omega}_{\xi_r}$, denoted ω_r, is then a *fundamental character of level r* such that the its action on any $g \in G_\infty$ is given by $\omega_r(g) = g(\xi_r)/\xi_r$. The following proposition does the job of computing the Hom-space described in Lemma 9.1.2. We follow the ideas found in the proof of [Liu21, Prop 5.1] which itself is a special case of [LHL19, Prop 3.1.2].

Proposition 9.1.3: *Let $\overline{\mathfrak{M}} = \prod_i \overline{\mathfrak{M}}^{(i)}$ be a finite height Kisin module over $k_F[\![u]\!]$ with basis $\{e^{(i)}_1, \ldots, e^{(i)}_d\}_{i\in\mathbb{Z}/f\mathbb{Z}}$. Suppose that $\varphi^f(e^{(0)}_j) = u^{a_j} e^{(0)}_{j+1}$ with $e^{(0)}_{d+1} = e^{(0)}_1$ and $0 \leq a_j \leq p^f - 1$. Then*

$$\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}})|_{G_\infty} = \mathrm{Ind}^{G_{\mathbb{Q}_{p^f}}}_{G_{\mathbb{Q}_{p^{df}}}} \omega_{df}^{\sum_{j=1}^d p^{d-j} a_j}.$$

Proof. As discussed above, we may view $\overline{\mathfrak{M}}$ as an étale φ-module over $k_F[\![u]\!]$ and by Lemma

9.1.2 we have reduced to computing

$$\mathbf{V}^*_{k_F}\left(\overline{\mathfrak{M}}\right) \cong \mathrm{Hom}_{k_F[\![u]\!],\varphi^f}\left(\overline{\mathfrak{M}}^{(0)}, \mathscr{O}^\flat_{\mathbb{C}_p}\right).$$

Consider a general homomorphism $X = (x_1, \dots, x_d) \in \mathrm{Hom}_{k_F[\![u]\!],\varphi^f}(\overline{\mathfrak{M}}^{(0)}, \mathscr{O}^\flat_{\mathbb{C}_p})$. We may see that φ^f-compatibility necessarily implies that $\varphi^f(x_j) = \pi^{a_j}x_{j+1}$ via the embedding $k_F[\![u]\!] \hookrightarrow \mathscr{O}^\flat_{\mathbb{C}_p}$ sending $u \mapsto \pi$ where π is a p-power compatible sequence in $\mathscr{O}^\flat_{\mathbb{C}_p}$ starting at $\pi_K = -p$. It follows that

$$(\varphi^f)^d(x_j) = \pi^{\sum_{\ell=1}^d p^{d-\ell}a_\ell}(x_j)$$

and hence, that x_1 is a solution to the equation

$$x^{p^{df}} = \pi^{\sum_{j=1}^d p^{d-j}a_j}(x).$$

Recall that $\xi_r = (-p)^{1/p^r-1}$ for any $r \geq 1$. Let us pick $\chi = (\chi_j)_{j\geq 0} \in \mathscr{O}^\flat_{\mathbb{C}_p}$ such that $\chi_0 = \xi_{df}$ and $\chi^{p^{df}-1} = \pi$. By setting

$$x_1 = \chi^{\sum_{j=1}^d p^{d-j}a_j}$$

then x_1 is a solution to the above equation and all other roots may be attained by multiplication with a $p^{df}-1$-th root of unity in $\overline{\mathbb{F}}^*_p$. Hence, all other x_j are determined by our choice of x_1 so that X is determined by x_1. As a result, for any $g \in G_\infty = \mathrm{Gal}(K_\infty/K)$, the action of $g(X)$ is determined by the action of $g(x_1)$ so that if we set H to be the splitting field of $Y^{p^{df}-1} + p$ in $\overline{\mathbb{Q}}_p$, then the G_∞ action on $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}})$ factors through $\mathrm{Gal}(H/K)$.

Since $\mathrm{Gal}(H/K)$ is finite, then we have a short exact sequence

$$0 \to I_{H/K} \to \mathrm{Gal}(H/K) \to \mathrm{Gal}(k_H/\mathbb{F}_{p^f}) \to 0$$

where $I_{H/K}$ denotes the inertia subgroup of $\mathrm{Gal}(H/K)$. As H is totally ramified over K, then the Schur–Zassenhaus theorem [Rot95, Theorem 7.41] implies that $\mathrm{Gal}(H/K)$ is the semi-direct product $\mathrm{Gal}(H/K) \cong \mathrm{Gal}(k_H/\mathbb{F}_{p^f}) \rtimes I_{H/K}$. Hence, upon restricting to inertia, we see that for any $g \in I_{H/K}$ we may write

$$g(X) = \omega_{df}^{\sum_{j=0}^d p^{d-j}a_j}(g)X$$

due to the fact that $g(\chi) = \omega_{df}(g)\chi$ by the definition of χ and the fundamental character.

The proposition then follows after observing the following fact from [Dou10a, Lemma 6.1]. Let $W = \omega_{df}^{\sum_{j=0}^d p^{d-j}a_j}$ be interpreted as the restriction of $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}})$ to $I_{H/K}$. By the definition of H, we may view W as a representation of $\mathrm{Gal}(\mathbb{Q}_{p^\infty}/\mathbb{Q}_{p^{df}})$ which has finite index

d in G_∞. Since $\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}})$ is irreducible then Frobenius reciprocity will imply an isomorphism

$$\mathbf{V}^*_{k_F}(\overline{\mathfrak{M}})|_{G_\infty} \cong \mathrm{Ind}_{G_{\mathbb{Q}_{p^{df}}}}^{G_{\mathbb{Q}_{p^f}}} \omega_{df}^{\sum_{j=0}^{d} p^{d-j} a_j}.$$

□

With this, we have our primary method of computing the k_F-representation associated to a finite height Kisin module. However, we will soon see that the condition that $\varphi^f(e_j^{(0)}) = u^{a_j} e_{j+1}^{(0)}$ is often not satisfied out of the box. In order to find the basis where this behavior holds, we rely on the following lemma based on [Liu21, Lemma 5.2]. The idea is that if a row operation is equivalent to identity modulo u^2, then we may 'straightened' the φ-twisted conjugation to simply be the single row operation.

Lemma 9.1.4: *For $i \in \mathbb{Z}/f\mathbb{Z}$, suppose $\mathfrak{A}^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(k_F[\![u]\!])$ for $h^{(i)} \leq 2p-3$. If $(X^{(i)}) \in \mathrm{GL}_d(k_F[\![u]\!])^f$ is such that $X^{(i)} \equiv I_d \pmod{u^2}$ for all i, then there exists $(Y^{(i)}) \in \mathrm{GL}_d(k_F[\![u]\!])$ such that*

$$Y^{(i)} *_\varphi \mathfrak{A}^{(i)} = X^{(i)}\mathfrak{A}^{(i)}.$$

Proof. Since $X^{(i)} \equiv I_d \pmod{u^2}$ for all $i \in \mathbb{Z}/f\mathbb{Z}$, then

$$X^{(i)} *_\varphi \mathfrak{A}^{(i)} = X^{(i)}\mathfrak{A}^{(i)}\varphi(X^{(i-1)})^{-1} = X^{(i)}\mathfrak{A}^{(i)} + u^{2p}Z^{(i)}$$

for some $Z^{(i)} \in \mathrm{Mat}_d(k_F[\![u]\!])$. Since each $\mathfrak{A}^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(k_F[\![u]\!])$ then there exists a matrix $B^{(i)} \in \mathrm{Mat}_d^{h^{(i)}}(k_F[\![u]\!])$ such that $\mathfrak{A}^{(i)}B^{(i)} = B^{(i)}\mathfrak{A}^{(i)} = u^{h^{(i)}}I_d$. Hence, we may write

$$\begin{aligned} X^{(i)} *_\varphi \mathfrak{A}^{(i)} &= X^{(i)}\mathfrak{A}^{(i)} + u^{2p-h^{(i)}}Z^{(i)}B^{(i)}(X^{(i)})^{-1}X^{(i)}\mathfrak{A}^{(i)} \\ &= (I_d + u^{h_1^{(i)}}Z_1^{(i)})X^{(i)}\mathfrak{A}^{(i)} \end{aligned}$$

where $h_1^{(i)} = 2p - h^{(i)} \geq 3$ and $Z_1^{(i)} = Z^{(i)}B^{(i)}(X^{(i)})^{-1}$. We set $Y_1^{(i)} = (I_d + u^{h_1^{(i)}}Z_1^{(i)})^{-1} \in \mathrm{GL}_d(k_F[\![u]\!])$ so that by the same reasoning as the above arguments, we have

$$\begin{aligned} (Y_1X)^{(i)} *_\varphi \mathfrak{A}^{(i)} &= Y_1^{(i)}X^{(i)}\mathfrak{A}^{(i)}\varphi\left((X^{(i-1)})^{-1}(Y_1^{(i-1)})^{-1}\right) \\ &= Y_1^{(i)}(Y_1^{(i)})^{-1}X^{(i)}\mathfrak{A}^{(i)}\varphi(Y_1^{(i-1)})^{-1} \\ &= X^{(i)}\mathfrak{A}^{(i)} + u^{ph_1^{(i-1)}}Z'^{(i)} \\ &= X^{(i)}\mathfrak{A}^{(i)} + u^{ph_1^{(i-1)}-h^{(i)}}Z'^{(i)}B^{(i)}(X^{(i)})^{-1}X^{(i)}\mathfrak{A}^{(i)} \\ &= (I_d + u^{h_2^{(i)}}Z_1^{(i)})X^{(i)}\mathfrak{A}^{(i)} \end{aligned}$$

where $h_2^{(i)} = ph_1^{(i-1)} - h^{(i)} \geq p+3$ and $Z_2^{(i)} = Z'^{(i)}B^{(i)}(X^{(i)})^{-1}$. Repeating this process will give us a base change

$$(Y_nY_{n-1}\cdots Y_1X)^{(i)} *_\varphi \mathfrak{A}^{(i)} = (I_d + u^{h_n^{(i)}} Z_n^{(i)})X^{(i)}\mathfrak{A}^{(i)}$$

with $h_n^{(i)} = ph_{n-1}^{(i-1)} - h^{(i)} \geq 3 + \sum_{j=1}^n p^j$. Hence we see that $h_n^{(i)} \to \infty$ and by setting $Y^{(i)} = \left(\prod_{n=1}^\infty Y_n\right)^{(i)} X^{(i)} \in \mathrm{GL}_d(k_F[\![u]\!])$ then our result follows. $\square$

One complication may stand out to the reader at this point. That is, how are we to compute the product

$$\prod_{i\in\mathbb{Z}/f\mathbb{Z}} \mathfrak{A}^{(i)} = \overline{\mathfrak{A}}^{(0)}\varphi\left(\overline{\mathfrak{A}}^{(1)}\right)\varphi^2\left(\overline{\mathfrak{A}}^{(2)}\right)\cdots\varphi^{f-1}\left(\overline{\mathfrak{A}}^{(f-1)}\right)$$

in an explicit way so as to use Proposition 9.1.3 when f is allowed to be arbitrarily large? Indeed, this is impractical but let us introduce some notation to recover the situation somewhat.

Let $I = I_2$ denote the identity matrix and let $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For a pair $\lambda_i = (n_i, m_i) \in \mathbb{Z}^2$ with each $0 \leq n_i, m_i \leq p-1$, we will set $u^{\lambda_i} = \mathrm{Diag}(u^{n_i}, u^{m_i})$. In the coming sections, we will describe a $k_F[\![u]\!]$-basis such that each $\overline{\mathfrak{A}}^{(i)}$ is either Iu^{λ_i} or Su^{λ_i} up to scalars in k_F. We will then summarize this data in the form of a map $\overline{\mathfrak{M}(\mathfrak{A})} \mapsto \mu = (\mu_i)_{i\in\mathbb{Z}/f\mathbb{Z}}$ where each $\mu_i = (M_i, \lambda_i)$ with $M_i \in \{I, S\}$ and λ_i is as defined above. Let us also decompose $i \in \mathbb{Z}/f\mathbb{Z}$ into two disjoint subsets $\mathbb{Z}/f\mathbb{Z} = \mathcal{I} \sqcup \mathcal{P}$ where $\mathcal{I}$ denotes the set of i such that $A_i = I$ and $\mathcal{P}$ denotes the set of i with $A_i = S$. In practice, A_i tells us if the partial Frobenius $\overline{\mathfrak{A}}^{(i)}$ is reducible or irreducible and λ_i encodes its elementary divisors in $k_F[\![u]\!]$.

Definition 9.1.5: We will call the f-tuple of pairs $\mu = (\mu_i) = (M_i, \lambda_i)_i$ derived from $\overline{\mathfrak{M}(\mathfrak{A})}$ to be its *reduction data.*

As the name suggests, this data suffices to compute the reduction $\overline{V(A)}|_{G_\infty}$ explicitly given specific data, up to an unramified twist in the irreducible case or restriction to inertia in the reducible case. The following Proposition lays out an algorithm to compute the reduction given such data.

Proposition 9.1.6: *Using the notation from above, let $\overline{\mathfrak{M}(\mathfrak{A})}$ be a finite height, rank-two Kisin module with reduction data $\mu = (\mu_i)$ canonically associated to $T \subset V(A)$. Then there exists v_i, w_i with $0 \leq v_i, w_i \leq p-1$ such that one of two cases hold:*

(a) If the order of the set $|\mathcal{P}|$ for μ is even or zero, then we have reducible reduction

$$\overline{V(A)}|_{I_K} = \omega_f^{\sum_{j=0}^{f-1} p^j v_j} \oplus \omega_f^{\sum_{j=0}^{f-1} p^j w_j};$$

(b) If the order of the set $|\mathcal{P}|$ is odd, then we have an irreducible reduction up to unramified twist

$$\overline{V(A)}|_{G_\infty} = \mathrm{Ind}_{G_{\mathbb{Q}_{p^{2f}}}}^{G_{\mathbb{Q}_{p^f}}} \omega_{2f}^{\sum_{j=0}^{f-1} p^{j+1} w_j + \sum p^j v_j}.$$

Proof. We will construct the required v_i and w_i from the powers of the elementary divisors n_i and m_i of the reduction data μ. A sum of these elements will then form the powers of u in the product $\prod \overline{\mathfrak{A}}^{(i)}$ so that we may apply Proposition 9.1.3 to get a reduction. Since $\mathcal{P}$ denotes the set of $i \in \mathbb{Z}/f\mathbb{Z}$ associated to anti-diagonal matrices, then it is easy to see that the order $|\mathcal{P}|$ being odd will result in an irreducible product and an even order will result in a reducible product.

Let us first suppose that the set $\mathcal{P} = \emptyset$ so that the reduction data is given by $\mu_i = (I, \lambda_i)$ for all $i \in \mathbb{Z}/f\mathbb{Z}$. Hence, the product is easily seen to be

$$\prod_{i \in \mathbb{Z}/f\mathbb{Z}} \overline{\mathfrak{A}}^{(i)} = \begin{pmatrix} u^{\sum_{j=0}^{f-1} p^j n_j} & 0 \\ 0 & u^{\sum_{j=0}^{f-1} p^j m_j} \end{pmatrix}.$$

We may then set $v_i = n_i$ and $w_i = m_i$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and apply Proposition 9.1.3 to get a reduction in the form of (a) after noticing that since each $v_i, w_i \leq p-1$ then both the sums $\sum_{j=0}^{f-1} p^j n_j$ and $\sum_{j=0}^{f-1} p^j m_j$ are necessarily smaller than $p^f - 1$.

Now let us tackle the more complicated case of $\mathcal{P} \neq \emptyset$. We may enumerate the set $\mathcal{P}$ by $\mathcal{P} = \{r_0, r_1, \ldots, r_d\}$ where $r_{j-1} < r_j$ and $0 \leq d \leq f-1$. For $\ell \geq 0$, let us assign:

- For $0 \leq i < r_0$ and $r_0 \neq 0$, set $v_i = n_i$ and $w_i = m_i$;
- For $r_0 \leq i < r_1$, set $v_i = m_i$ and $w_i = n_i$;
- Continue this process in that for ℓ odd we set $v_i = n_i$ and $w_i = m_i$ for $r_\ell \leq i < r_{\ell+1}$ and $v_i = m_i$ and $w_i = n_i$ for $r_{\ell+1} \leq i < r_{\ell+2}$.
- When we reach r_d, repeat the pattern for $r_d \leq i < f-1$.

The properties of matrix multiplication between diagonal and anti-diagonal matrices means that we will have one of two cases depending on the order $|\mathcal{P}|$. Namely, if $|\mathcal{P}|$ is even then d is odd and the product will be the reducible matrix

$$\prod_{i \in \mathbb{Z}/f\mathbb{Z}} \overline{\mathfrak{A}}^{(i)} = \begin{pmatrix} u^{\sum_{j=0}^{f-1} p^j v_j} & 0 \\ 0 & u^{\sum_{j=0}^{f-1} p^j w_j} \end{pmatrix}.$$

As before, both sums are guaranteed to be at least $p^f - 1$ so we may apply Proposition 9.1.3 to get a reduction in the form of (a). Finally, if $|\mathcal{P}|$ is odd then d is even and the product will be an irreducible matrix in the form

$$\prod_{i \in \mathbb{Z}/f\mathbb{Z}} \overline{\mathfrak{A}}^{(i)} = \begin{pmatrix} 0 & u^{\sum_{j=0}^{f-1} p^j v_j} \\ u^{\sum_{j=0}^{f-1} p^j w_j} & 0 \end{pmatrix}.$$

We may then apply Proposition 9.1.3 to get a reduction in the form of (b). □

9.2 Reductions for the Large Valuation Case

Let $\mathcal{M}(\mathcal{A})$ be a Kisin module over S_F with labeled heights in the range

$$p+2 \leq k_0 \leq 2p-4 \qquad\qquad 2 \leq k_i \leq p-3 \ \text{ for all } \ 1 \leq i \leq f-1$$

and $\nu_p(a_2^{(0)}) \geq 1$ as in Section 8.1. The results of Proposition 8.1.2 combined with Theorem 7.2.3 allow use to identify a descent of $\mathcal{M}(\mathcal{A})$ to $\mathfrak{S}_F$ denoted $\mathfrak{M}(\mathfrak{A})$. The reduction of which $\overline{\mathfrak{M}(\mathfrak{A})}$ necessarily has partial Frobenius matrices given by

$$\overline{\mathfrak{A}}^{(0)} = \begin{cases} \begin{pmatrix} 0 & u^{k_0}\overline{a}_1^{(0)} \\ 1 & \overline{\left(a_2^{(0)}/p\right)}\,\overline{a}_1^{(0)}u^p \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} u^{k_0}\overline{a}_1^{(0)} & 0 \\ \overline{\left(a_2^{(0)}/p\right)}\,\overline{a}_1^{(0)}u^p & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{T} \end{cases}$$

due to the fact that $E = u+p$, $a_2^{(i)} \in \varpi\mathcal{O}_F$ for all $i \in \mathcal{T}$ and the following lemma regarding the reduction of the λ_b-coefficients $\alpha_0^{(i)}$ and $\alpha_1^{(i)}$.

Lemma 9.2.1: *Let $\gamma = \varphi(E)/p$ and $\lambda_b^{f(\varphi)} = \prod_{n\geq 0}\varphi^{bn}(\gamma)^{f(\varphi)} = \sum_{j\geq 0}\alpha_j\left(\frac{E^p}{p}\right)^j$. We have the following estimates:*

(a) $\gamma \equiv \frac{E^p}{p} + 1 \pmod{p}$*;*
(b) $\alpha_0 \equiv 1 \pmod{p}$*;*
(c) $\overline{\alpha}_1 \equiv f(0)^{\pm 1} \pmod{p}$*.*

Proof. (a) follows from the fact that $\frac{E^p}{p} \equiv \frac{u^p}{p} \pmod{p}$ and that $\gamma = 1 + \frac{u^p}{p}$. For (b) and (c), write $\lambda_b^{f(\varphi)} = \gamma^{f(0)}\lambda_b^{f(\varphi)-f(0)}$. Here we have $\lambda_b^{f(\varphi)-f(0)} = 1 + \sum_{r\geq 2,\ell\geq 1} a_\ell(\frac{u^{rp}}{p})^\ell$ where the sum only contributes to coefficients of $\left(\frac{E^p}{p}\right)^j$ for $j \geq 2$. Hence, α_0 and α_1 only depend on $\gamma^{f(0)}$ and the result follows from (a) after performing a binomial expansion on $\gamma^{f(0)} \equiv (\frac{E^p}{p}+1)^{f(0)}$. □

In preparation, let us define subsets of $\mathcal{S}$ by

$$\mathcal{V}^{=} := \{i \in \mathcal{S} : \nu_p(a_2^{(i)}) = 0\} \qquad \mathcal{V}^{>} := \{i \in \mathcal{S} : \nu_p(a_2^{(i)}) > 0\}.$$

That is, we have decomposed the set $\mathcal{S} = \mathcal{V}^{>} \sqcup \mathcal{V}^{=}$ so that $\mathcal{V}^{>}$ gives the $i \in \mathcal{S}$ with partial

Frobenius $\overline{\mathfrak{A}}^{(i)}$ being anti-diagonal. We will repeatedly see that the elements of these sets dictate the shape of the reduction data.

9.2.2 Reduction Data for $\nu_p(a_2^{(0)}) > 1$

Let us begin with the case that $\nu_p(a_2^{(0)}) > 1$ so that we can take $0 \in \mathcal{V}^{>}$ if $0 \in \mathcal{S}$. Hence, we can simplify our reduced partial Frobenius matrices to

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^{=} \\ \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & 0 \end{pmatrix} & \text{if } i \in \mathcal{V}^{>} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \in \mathcal{T}. \end{cases}$$

We have two subcases depending on whether or not the set $\mathcal{V}^{=}$ is empty.

Subcase: $\mathcal{V}^{=}$ Empty

If $\mathcal{V}^{=}$ is empty then all the matrices are diagonal or anti-diagonal so our only obstacle to displaying the reduction data is the fact that the powers of u in $\overline{\mathfrak{A}}^{(0)}$ are not smaller than p. To fix this, let us apply the base change over $k_F(\!(u)\!)$ given by $X^{(f-1)} = \mathrm{Diag}(1,u)$ if $0 \in \mathcal{S}$ and $X^{(f-1)} = \mathrm{Diag}(u,1)$ if $0 \in \mathcal{T}$ and $X^{(i)} = I_2$ everywhere else. The result of $X^{(i)} *_{\varphi} \overline{\mathfrak{A}}^{(i)}$ will leave $\overline{\mathfrak{A}}^{(i)}$ unaffected for $0 < i < f-1$ and provide

$$\overline{\mathfrak{A}}^{(0)} = \begin{cases} \begin{pmatrix} 0 & u^{k_0-p}\overline{a}_1^{(0)} \\ 1 & 0 \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} u^{k_0-p}\overline{a}_1^{(0)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\overline{\mathfrak{A}}^{(f-1)} = \begin{cases} \begin{pmatrix} 0 & u^{k_{f-1}}\overline{a}_1^{(f-1)} \\ u & 0 \end{pmatrix} & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{S} \\ \begin{pmatrix} 0 & u^{k_{f-1}+1}\overline{a}_1^{(f-1)} \\ 1 & 0 \end{pmatrix} & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{T} \\ \begin{pmatrix} u^{k_{f-1}}\overline{a}_1^{(f-1)} & 0 \\ 0 & u \end{pmatrix} & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{S} \\ \begin{pmatrix} u^{k_{f-1}+1}\overline{a}_1^{(f-1)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{T}. \end{cases}$$

Up to an unramified twist or restriction to inertia, the reduction does not depend on the k_F-elements $\overline{a}_1^{(i)}$. Hence, in the case that $\nu_p(a_2^{(f-1)}) > 1$ and $\nu_p(a_2^{(i)}) > 0$ for all $i < f-1$, we have the reduction data $\mu = (\mu_i)$ where

$$\mu_0 = \begin{cases} (S, (0, k_0 - p)) & \text{if } 0 \in \mathcal{S} \\ (I, (k_0 - p, 0)) & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mu_i = \begin{cases} (S, (0, k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{S} \\ (I, (k_i, 0)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{T} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (S, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{S} \\ (S, (0, k_{f-1}+1)) & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{T} \\ (I, (k_{f-1}, 1)) & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{S} \\ (I, (k_{f-1}+1, 0)) & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{T}. \end{cases}$$

Notice that the reduction given by Proposition 9.1.6 will be irreducible if and only if $|\mathcal{S}|$ is odd.

Subcase: $\mathcal{V}^=$ Nonempty

Now let us return ourselves to the original setting of $\nu_p(a_2^{(0)}) > 1$ and suppose $\mathcal{V}^=$ is nonempty so we have reduced partial Frobenius given by

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^= \\ \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & 0 \end{pmatrix} & \text{if } i \in \mathcal{V}^> \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \in \mathcal{T}. \end{cases}$$

We will need to permute rows and columns so as to make each matrix lower triangular in order to perform a semi-simplification. To do this, we define a subset $\mathcal{X} \subset \mathbb{Z}/f\mathbb{Z}$ in a similar manner to how we defined $\mathcal{W}$ in Definition 8.2.1.

Definition 9.2.3: We define a subset $\mathcal{X} \subset \mathbb{Z}/f\mathbb{Z}$ with the following steps:

1. Label the elements of $\mathcal{S} = \{s_1, \ldots, s_d\}$ and assign $s_j = r_j$ whenever $s_j \in \mathcal{V}^>$ so that $\mathcal{V}^> = \{r_j\}_j$. If $\mathcal{V}^> = \emptyset$ then set $\mathcal{X} = \emptyset$.
2. Choose any element $s_m \in \mathcal{V}^=$. Let $s_\ell = r_\ell$ denote the smallest element of $\mathcal{V}^>$ with $s_m < r_\ell$. Include $i \in \mathcal{X}$ for all $r_\ell \leq i < s_{\ell+1}$.
3. Now pick r_k to be the smallest element of $\mathcal{V}^>$ with $s_{\ell+1} < r_k$. Include $i \in \mathcal{X}$ for all $r_k \leq i < s_{k+1}$.
4. Repeat this process, possibly taking the labels modulo f. After a finite amount of steps, we will return to s_m and the process will terminate as $s_m \notin \mathcal{V}^>$.

We will now permute the rows of $\overline{\mathfrak{A}}^{(i)}$ according to the subset $\mathcal{X}$ by making $X^{(i)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $i \in \mathcal{X}$ and $X^{(i)} = I_2$ for all $i \notin \mathcal{X}$. The result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will be

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ \overline{a}_2^{(i)} & 1 \end{pmatrix} & \text{if } i \in \mathcal{V}^= \text{ and } i-1 \in \mathcal{X} \\ \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^= \text{ and } i-1 \notin \mathcal{X} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } i \notin \mathcal{V}^= \text{ and } i \in \mathcal{X} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \notin \mathcal{V}^= \text{ and } i \notin \mathcal{X}. \end{cases}$$

We can now invoke Lemma 9.1.4 by base changing over $k_F[\![u]\!]$ via

$$X^{(i)} = \begin{pmatrix} 1 & -\frac{\overline{a}_1^{(i)}}{\overline{a}_2^{(i)}} u^{k_i} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k_F[\![u]\!])$$

whenever $i \in \mathcal{V}^=$ *and* $i - 1 \notin \mathcal{X}$ and setting all other $X^{(i)}$ to be the identity. The result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will be

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ \overline{a}_2^{(i)} & 1 \end{pmatrix} & \text{if } i \in \mathcal{V}^= \text{ and } i-1 \in \mathcal{X} \\ \begin{pmatrix} -\frac{\overline{a}_1^{(i)}}{\overline{a}_2^{(i)}} u^{k_i} & 0 \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^= \text{ and } i-1 \notin \mathcal{X} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } i \notin \mathcal{V}^= \text{ and } i \in \mathcal{X} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \notin \mathcal{V}^= \text{ and } i \notin \mathcal{X}. \end{cases}$$

Since $\overline{a}_2^{(i)} \in k_F^*$ for all $i \in \mathcal{V}^=$, then we may semi-simplify along the φ-stable subspace given by $\varphi^{(i)}(e_1^{(i-1)})$ to disregard the lower left entry in the $\overline{\mathfrak{A}}^{(i)}$ corresponding to $i \in \mathcal{V}^=$. As before, we may also disregard the k_F-elements $\overline{a}_1^{(i)}$ and $\overline{a}_2^{(i)}$ when computing reductions up to unramified twists so that

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \mathrm{Diag}\left(1, u^{k_i}\right) & \text{if } i \in \mathcal{X} \\ \mathrm{Diag}\left(u^{k_i}, 1\right) & \text{if } i \notin \mathcal{X} \end{cases}$$

due to the fact that any element of $\mathcal{V}^{=}$ cannot lie in $\mathcal{X}$.

Our final step is the same as the $\mathcal{V}^{=}$ empty case. In particular, we take $X^{(f-1)} = \mathrm{Diag}(1,u)$ if $0 \in \mathcal{X}$ and $X^{(f-1)} = \mathrm{Diag}(u,1)$ if $0 \notin \mathcal{X}$ with $X^{(i)} = I_2$ for all $i \neq 0$. The result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will leave $\overline{\mathfrak{A}}^{(i)}$ unaffected for $0 < i < f-1$ and give

$$\overline{\mathfrak{A}}^{(0)} = \begin{cases} \mathrm{Diag}\left(1, u^{k_0 - p}\right) & \text{if } 0 \in \mathcal{X} \\ \mathrm{Diag}\left(u^{k_0-p}, 1\right) & \text{if } 0 \notin \mathcal{X} \end{cases}$$
$$\overline{\mathfrak{A}}^{(f-1)} = \begin{cases} \mathrm{Diag}\left(1, u^{k_{f-1}+1}\right) & \text{if } 0, f-1 \in \mathcal{X} \\ \mathrm{Diag}\left(u, u^{k_{f-1}}\right) & \text{if } f-1 \in \mathcal{X} \text{ and } 0 \notin \mathcal{X} \\ \mathrm{Diag}\left(u^{k_{f-1}}, u\right) & \text{if } f-1 \notin \mathcal{X} \text{ and } 0 \in \mathcal{X} \\ \mathrm{Diag}\left(u^{k_{f-1}+1}, 1\right) & \text{if } 0, f-1 \notin \mathcal{X}. \end{cases}$$

Hence, we get reduction data up to restriction to inertia given by $\mu = (\mu_i)$ with

$$\mu_0 = \begin{cases} (I, (0, k_0 - p)) & \text{if } 0 \in \mathcal{X} \\ (I, (k_0 - p, 0)) & \text{if } 0 \notin \mathcal{X} \end{cases}$$
$$\mu_i = \begin{cases} (I, (0, k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{X} \\ (I, (k_i, 0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{X} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I, (0, k_{f-1}+1)) & \text{if } 0, f-1 \in \mathcal{X} \\ (I, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{X} \text{ and } 0 \notin \mathcal{X} \\ (I, (k_{f-1}, 1)) & \text{if } f-1 \in \mathcal{X} \text{ and } 0 \in \mathcal{X} \\ (I, (k_{f-1}0+1, 0)) & \text{if } 0, f-1 \notin \mathcal{X}. \end{cases}$$

Note that the reduction given by Proposition 9.1.6 will necessarily be reducible.

Remark 9.2.4: Observe that the above arguments would provide reduction data in the case that all weights small minus the need to scale by u to reduce the power of the single large weight, regardless of the p-adic valuations of any $a_2^{(i)}$ parameters. Hence, whenever $2 \leq k_i \leq p-3$ for all $i \in \mathbb{Z}/f\mathbb{Z}$, the reduction data for $\overline{\mathfrak{M}(\mathfrak{A})}$ is given by

- If $\nu_p(a_2^{(i)}) > 0$ for all $i \in \mathcal{S}$:

$$\mu_i = \begin{cases} (S, (0, k_i)) & \text{if } i \in \mathcal{S} \\ (I, (k_i, 0)) & \text{if } i \in \mathcal{T}; \end{cases}$$

- If there exists $r \in \mathbb{Z}/f\mathbb{Z}$ with $\nu_p(a_2^{(r)}) = 0$:

$$\mu_i = \begin{cases} (I, (0, k_i)) & \text{if } i \in \mathcal{X} \\ (I, (k_i, 0)) & \text{if } i \notin \mathcal{X}. \end{cases}$$

Such reductions are already known by Fontaine-Laffaille methods.

9.2.5 Reduction Data for $\nu_p(a_2^{(0)}) = 1$

Recall that we have reduced partial Frobenius given by

$$\overline{\mathfrak{A}}^{(0)} = \begin{cases} \begin{pmatrix} 0 & u^{k_0}\overline{a}_1^{(0)} \\ 1 & \left(\overline{a_2^{(0)}/p}\right)\overline{\alpha}_1^{(0)}u^p \end{pmatrix} & \text{if } 0 \in \mathcal{S} \\ \begin{pmatrix} u^{k_0}\overline{a}_1^{(0)} & 0 \\ \left(\overline{a_2^{(0)}/p}\right)\overline{\alpha}_1^{(0)}u^p & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \neq 0 \text{ and } i \in \mathcal{T}. \end{cases}$$

We will again consider the sets $\mathcal{V}^{>}$ and $\mathcal{V}^{=}$ but will thankfully not have to split into subcases depending on the emptiness of these sets. Unfortunately, we will have to define a new subset $\mathcal{Y} \subset \mathbb{Z}/f\mathbb{Z}$ that is slightly different from the $\mathcal{X}$ we defined in Definition 9.2.3.

Definition 9.2.6: We define a subset $\mathcal{Y} \subset \mathbb{Z}/f\mathbb{Z}$ with the following steps:

1. Label the elements of $\mathcal{S} = \{s_1, \ldots, s_d\}$ and assign $s_j = r_j$ whenever $s_j \in \mathcal{V}^{>}$ so that $\mathcal{V}^{>} = \{r_j\}_j$. If $\mathcal{V}^{>} = \emptyset$ then set $\mathcal{Y} = \emptyset$.
2. Let r_ℓ denote the smallest element of $\mathcal{V}^{>}$. Include $i \in \mathcal{Y}$ for all $r_\ell \leq i < s_{\ell+1}$.
3. Now pick r_k to be the smallest element of $\mathcal{V}^{>}$ with $s_{\ell+1} < r_k$. Include $i \in \mathcal{Y}$ for all $r_k \leq i < s_{k+1}$.
4. Repeat this process until we run out of elements of $\mathcal{V}^{>}$. If $r_d = s_d$ exists, include $r_d \leq i \leq f-1$.

As before, we swap the rows of $\overline{\mathfrak{A}}^{(i)}$ whenever $i \in \mathcal{Y}$ via $X^{(i)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $i \in \mathcal{Y}$ and $X^{(i)}$ being the identity otherwise. For any $i \in \mathcal{V}^{=}$ with $i-1 \notin \mathcal{Y}$, we can invoke Lemma 9.1.4 by setting

$$X^{(i)} = \begin{pmatrix} 1 & -\frac{\overline{a}_1^{(i)}}{\overline{a}_2^{(i)}}u^{k_i} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k_F[\![u]\!])$$

whenever $i \in \mathcal{V}^{=}$ and $i-1 \in\notin \mathcal{Y}$ and setting all other $X^{(i)}$ to be the identity. The result of all of this will be

$$\overline{\mathfrak{A}}^{(0)} = \begin{cases} \begin{pmatrix} 0 & u^{k_0}\overline{a}_1^{(0)} \\ 1 & \tilde{a}_2^{(0)}\overline{\alpha}_1^{(0)}u^p \end{pmatrix} & \text{if } 0 \in \mathcal{S} \text{ and } f-1 \notin \mathcal{Y} \text{ or } 0 \in \mathcal{T} \text{ and } f-1 \in \mathcal{Y} \\ \begin{pmatrix} u^{k_0}\overline{a}_1^{(0)} & 0 \\ \tilde{a}_2^{(0)}\overline{\alpha}_1^{(0)}u^p & 1 \end{pmatrix} & \text{if } 0 \in \mathcal{T} \text{ and } f-1 \notin \mathcal{Y} \text{ or } 0 \in \mathcal{S} \text{ and } f-1 \in \mathcal{Y} \end{cases}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ \overline{a}_2^{(i)} & 1 \end{pmatrix} & \text{if } i \in \mathcal{V}^{=} \text{ and } i-1 \in \mathcal{Y} \\ \begin{pmatrix} -\frac{\overline{a}_1^{(i)}}{\overline{a}_2^{(i)}}u^{k_i} & 0 \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^{=} \text{ and } i-1 \notin \mathcal{Y} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } i \notin \mathcal{V}^{=} \text{ and } i \in \mathcal{Y} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \notin \mathcal{V}^{=} \text{ and } i \notin \mathcal{Y} \end{cases}$$

where we begin to write $\tilde{a}_2^{(0)} = \overline{(a_2^{(0)}/p)}$. At this point, we are required to diverge into two separate subcases depending on the shape of the $\overline{\mathfrak{A}}^{(0)}$ matrix.

Subcase: $0 \in \mathcal{S}$ and $f-1 \notin \mathcal{Y}$ or $0 \in \mathcal{T}$ and $f-1 \in \mathcal{Y}$

We begin by using Lemma 9.1.4 with $k_0 - p \geq 4$ to kill the u^{k_0}-term in $\overline{\mathfrak{A}}^{(0)}$. Indeed, we set

$$X^{(0)} = \begin{pmatrix} 1 & -\left(\frac{\overline{a}_1}{\tilde{a}_2\overline{\alpha}_1}\right)^{(0)} u^{k_0-p} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k_F[\![u]\!])$$

and $X^{(i)} = I_2$ for all $i \neq 0$. The result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will leave $\overline{\mathfrak{A}}^{(i)}$ unaffected for $1 \leq i \leq f-1$ and give

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} -\left(\frac{\overline{a}_1}{\tilde{a}_2\overline{\alpha}_1}\right)^{(0)} u^{k_0-p} & 0 \\ 1 & \tilde{a}_2^{(0)}\overline{\alpha}_1^{(0)}u^p \end{pmatrix}.$$

Hence, we may semi-simplify to disregard the lower left entries of each matrix. Our last operation will be to scale by u via $X^{(f-1)} = \mathrm{Diag}(1, u)$ so that up restriction to inertia, our reduction data will be given by $\mu = (\mu_i)$ with

$$\mu_0 = (I, (k_0 - p, 0)).$$

$$\mu_i = \begin{cases} (I, (k_i, 0)) & \text{if } 1 \leq i < f-1 \text{ and } i \notin \mathcal{Y} \\ (I, (0, k_i)) & \text{if } 1 \leq i < f-1 \text{ and } i \in \mathcal{Y} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (I, (k_{f-1}, 1)) & \text{if } f-1 \notin \mathcal{Y} \\ (I, (0, k_{f-1}+1)) & \text{if } f-1 \in \mathcal{Y}. \end{cases}$$

Note that this case will necessarily result in a reducible reduction by Proposition 9.1.6.

Subcase: $0 \in \mathcal{T}$ **and** $f-1 \notin \mathcal{Y}$ **or** $0 \in \mathcal{S}$ **and** $f-1 \in \mathcal{Y}$

Recall that we have reduced partial Frobenius given by

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} u^{k_0}\overline{a}_1^{(0)} & 0 \\ \tilde{a}_2^{(0)}\overline{\alpha}_1^{(0)}u^p & 1 \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ \overline{a}_2^{(i)} & 1 \end{pmatrix} & \text{if } i \in \mathcal{V}^{=} \text{ and } i-1 \in \mathcal{Y} \\ \begin{pmatrix} -\frac{\overline{a}_1^{(i)}}{\overline{a}_2^{(i)}}u^{k_i} & 0 \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{V}^{=} \text{ and } i-1 \notin \mathcal{Y} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } i \notin \mathcal{V}^{=} \text{ and } i \in \mathcal{Y} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \notin \mathcal{V}^{=} \text{ and } i \notin \mathcal{Y}. \end{cases}$$

Let us scale by u via $X^{(f-1)} = \mathrm{Diag}(u, 1) \in \mathrm{GL}_2(k_F((u)))$ and $X^{(i)} = I_2$ for $i \neq f-1$ so that $X^{(i)} *_{\varphi} \overline{\mathfrak{A}}^{(i)}$ leaves $\overline{\mathfrak{A}}^{(i)}$ unaffected for $1 \leq i < f-1$ and

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} u^{k_0-p}\overline{a}_1^{(0)} & 0 \\ \tilde{a}_2^{(0)}\overline{\alpha}_1^{(0)} & 1 \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(f-1)} = \begin{cases} \begin{pmatrix} u^{k_{f-1}+1}\overline{a}_1^{(f-1)} & 0 \\ \overline{a}_2^{(f-1)} & 1 \end{pmatrix} & \text{if } f-1 \in \mathcal{V}^{=} \text{ and } f-2 \in \mathcal{Y} \\ \begin{pmatrix} -\frac{\overline{a}_1^{(f-1)}}{\overline{a}_2^{(f-10)}}u^{k_{f-1}+1} & 0 \\ 1 & \overline{a}_2^{(f-1)} \end{pmatrix} & \text{if } f-1 \in \mathcal{V}^{=} \text{ and } f-2 \notin \mathcal{Y} \\ \begin{pmatrix} u & 0 \\ 0 & u^{k_{f-1}}\overline{a}_1^{(f-1)} \end{pmatrix} & \text{if } f-1 \notin \mathcal{V}^{=} \text{ and } f-1 \in \mathcal{Y} \\ \begin{pmatrix} u^{k_{f-1}+1}\overline{a}_1^{(f-1)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } f-1 \notin \mathcal{V}^{=} \text{ and } f-1 \notin \mathcal{Y}. \end{cases}$$

Hence, up to semi-simplification and restriction to inertia, we will have reduction data given by

$$\mu_0 = (I, (k_0 - p, 0))$$

$$\mu_i = \begin{cases} (I,(k_i,0)) & \text{if } 1 \le i < f-1 \text{ and } i \notin \mathcal{Y} \\ (I,(0,k_i)) & \text{if } 1 \le i < f-1 \text{ and } i \in \mathcal{Y} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{Y} \\ (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{Y}. \end{cases}$$

Note that this case will necessarily result in a reducible reduction by Proposition 9.1.6.

9.3 Reductions for the Small Valuation Case

Let $\mathcal{M}(\mathcal{A})$ be a Kisin module over S_F with labeled heights in the range

$$p+2 \le k_0 \le 2p-4 \qquad\qquad 2 \le k_i \le p-3 \;\text{ for all }\; 1 \le i \le f-1$$

and $0 \le \nu_p(a_2^{(0)}) < 1$ as in Section 8.2. We recall that we encountered a dichotomy in behavior with respect to the even or oddness of the number of Type I matrices present. Indeed, we will need to take these two cases separately.

9.3.1 Reduction Data for $|\mathcal{S}|$ Even

Under the assumption that $\nu_p(a_2^{(i)}) \ge 1 - \nu_p(a_2^{(0)})$ for all $i \in \mathcal{W}$, the results of Proposition 8.2.4 combined with Theorem 7.2.3 allow use to identify a descent of $\mathcal{M}(\mathcal{A})$ to $\mathfrak{S}_F$ denoted $\mathfrak{M}(\mathfrak{A})$. The reduction of which $\overline{\mathfrak{M}(\mathfrak{A})}$ necessarily has partial Frobenius matrices given by

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})^{(0)}u^{k_0-p} \\ \overline{\alpha}_1^{(0)}u^p & 1 \end{pmatrix}$$
$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 1 & (\overline{a_2^{(0)}a_2^{(i)}/p}) \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } 1 \le i \le f-1 \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 1 \le i \le f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$

since $p/a_2^{(0)} \in \varpi\mathscr{O}_F$ and applying Lemma 9.2.1. As before, let us redefine a subset $\mathcal{V} \subset \mathcal{W}$ by

$$\mathcal{V} = \{i \in \mathcal{W} : \nu_p(a_2^{(i)}) = 1 - \nu_p(a_2^{(0)})\}.$$

That is, the set of $i \in \mathcal{W}$ such that the upper right hand entry of $\overline{\mathfrak{A}}^{(i)}$ is nonzero. Just as before, we have two subcases depending on whether or not the set $\mathcal{V}$ is empty.

Subcase: $\mathcal{V}$ Empty

In the case that the set $\mathcal{V}$ is empty then our only obstruction to displaying the reduction data is $\overline{\mathfrak{A}}^{(0)}$. Since $k_0 - p \geq 2$ then we may evoke Lemma 9.1.4 with

$$X^{(0)} = \begin{pmatrix} 1 & \overline{(a_1/\alpha_1)}^{(0)} u^{k_0-p} \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(k_F[\![u]\!])$$

and $X^{(i)} = I_2$ for all $i \neq 0$. The result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will leave $\overline{\mathfrak{A}}^{(i)}$ unaffected for $1 \leq i \leq f-1$ and give

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} \overline{a}_1^{(0)} u^{k_0} & 0 \\ \overline{\alpha}_1^{(0)} u^p & 1 \end{pmatrix}.$$

We will now scale by u via $X^{(f-1)} = \mathrm{Diag}(u, 1)$ and $X^{(i)} = I_2$ for $i \neq f-1$ so that the result of $X^{(i)} *_\varphi \overline{\mathfrak{A}}^{(i)}$ will leave $\overline{\mathfrak{A}}^{(i)}$ unaffected for $0 < i < f-1$ and give

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} \overline{a}_1^{(0)} u^{k_0-p} & 0 \\ \overline{\alpha}_1^{(0)} & 1 \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(f-1)} = \begin{cases} \begin{pmatrix} u & 0 \\ 0 & u^{k_{f-1}} \overline{a}_1^{(f-1)} \end{pmatrix} & \text{if } f-1 \in \mathcal{W} \\ \begin{pmatrix} u^{k_{f-1}+1} \overline{a}_1^{(f-1)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

Since $\overline{\alpha}_1^{(0)} \in k_F^*$ by Lemma 9.2.1, then up to semi-simplification, the lower right hand entry of $\overline{\mathfrak{A}}^{(0)}$ may be disregarded to get reduction data given by $\mu = (\mu_i)$ with

$$\mu_0 = (I, (k_0 - p, 0))$$
$$\mu_i = \begin{cases} (I, (0, k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{W} \\ (I, (k_i, 0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I, (k_{f-1}+1, 0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

Note that this case will necessarily result in a reducible reduction by Proposition 9.1.6.

Subcase: $\mathcal{V}$ Nonempty

Let us return ourselves to the original setting with the assumption that the set $\mathcal{V}$ is nonempty,

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)}^{(0)} u^{k_0-p} \\ \overline{\alpha}_1^{(0)} u^p & 1 \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 1 & \tilde{a}_2^{(i)} \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } 1 \leq i \leq f-1 \text{ and } i \in \mathcal{V} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } 1 \leq i \leq f-1 \text{ and } i \notin \mathcal{V} \text{ but } i \in \mathcal{W} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 1 \leq i \leq f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$

where we write $\tilde{a}_2^{(i)} = \overline{(a_2^{(0)} a_2^{(i)}/p)}$. Let r denote the minimal element in $\mathcal{V}$. Since $k_r \geq 2$ and $\tilde{a}_2^{(r)} \in k_F^*$ then we may evoke Lemma 9.1.4 with

$$X^{(r)} = \begin{pmatrix} 1 & 0 \\ -\overline{(a_1/\tilde{a}_2)}^{(r)} u^{k_r} & 1 \end{pmatrix} \in \mathrm{GL}_2(k_F[\![u]\!])$$

and $X^{(i)} = I_2$ for $i \neq r$. The result will leave all $\overline{\mathfrak{A}}^{(i)}$ the same except for $i = r$ where we have

$$\overline{\mathfrak{A}}^{(r)} = \begin{pmatrix} 1 & \tilde{a}_2^{(r)} \\ -\overline{(a_1/\tilde{a}_2)}^{(r)} u^{k_r} & 0 \end{pmatrix}.$$

Now let us swap rows and columns by $X^{(i)} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $0 \leq i < r$ and $X^{(j)} = I_2$ for all $r \leq j \leq f-1$. The result will be

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} \overline{\alpha}_1^{(0)} u^p & 1 \\ 0 & -\overline{(a_1/\alpha_1)}^{(0)} u^{k_0-p} \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(j)} = \begin{cases} \begin{pmatrix} u^{k_j}\overline{a}_1^{(j)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 < j < r \text{ and } j \in \mathcal{W} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_j}\overline{a}_1^{(j)} \end{pmatrix} & \text{if } 0 < j < r \text{ and } j \notin \mathcal{W} \end{cases}$$

$$\overline{\mathfrak{A}}^{(r)} = \begin{pmatrix} \tilde{a}_2^{(r)} & 1 \\ 0 & -\overline{(a_1/\tilde{a}_2)}^{(r)} u^{k_r} \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 1 & \tilde{a}_2^{(i)} \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } r < i \leq f-1 \text{ and } i \in \mathcal{V} \\ \begin{pmatrix} 1 & 0 \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } r < i \leq f-1 \text{ and } i \notin \mathcal{V} \text{ but } i \in \mathcal{W} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } r < i \leq f-1 \text{ and } i \notin \mathcal{W}. \end{cases}$$

Since $\tilde{a}_2^{(i)} \in k_F^*$ for all $i \in \mathcal{V}$ then up to semi-simplification we may disregard all upper right entries. Our last operation is to scale by u via $X^{(f-1)} = \mathrm{Diag}(u, 1)$ so we get reduction data given by $\mu = (\mu_i)$ with

$$\begin{aligned} \mu_0 &= (I, (0, k_0 - p)) \\ \mu_j &= \begin{cases} (I, (k_j, 0)) & \text{if } 0 < j < r \text{ and } j \in \mathcal{W} \\ (I, (0, k_j)) & \text{if } 0 < j < r \text{ and } j \notin \mathcal{W} \end{cases} \\ \mu_i &= \begin{cases} (I, (0, k_i)) & \text{if } r \leq i < f-1 \text{ and } i \in \mathcal{W} \\ (I, (k_i, 0)) & \text{if } r \leq i < f-1 \text{ and } i \notin \mathcal{W} \end{cases} \\ \mu_{f-1} &= \begin{cases} (I, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I, (k_{f-1}+1, 0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases} \end{aligned}$$

Note that this case will necessarily result in a reducible reduction by Proposition 9.1.6.

9.3.2 Reduction Data for $|\mathcal{S}|$ Odd

Under the assumption that $\nu_p(a_2^{(i)}) \geq 1$ for all $i \in \mathcal{W}$, the results of Proposition 8.2.6 combined with Theorem 7.2.3 allow use to identify a descent of $\mathcal{M}(\mathcal{A})$ to $\mathfrak{S}_F$ denoted $\mathfrak{M}(\mathfrak{A})$. The reduction of which $\overline{\mathfrak{M}(\mathfrak{A})}$ necessarily has partial Frobenius matrices given by

$$\overline{\mathfrak{A}}^{(0)} = \begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)}^{(0)} u^{k_0-p} \\ -\overline{\alpha}_1^{(0)} u^p & \overline{a}_2^{(0)} \end{pmatrix}$$

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 1 & \overline{(a_2^{(i)}/p)} \\ 0 & u^{k_i}\overline{a}_1^{(i)} \end{pmatrix} & \text{if } 1 \leq i \leq f-1 \text{ and } i \in \mathcal{W} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 1 \leq i \leq f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$

since $p/a_2^{(0)} \in \varpi\mathscr{O}_F$ and applying Lemma 9.2.1.

We follow very similar ideas as in Section 9.3.1 with the only exception being the possibility that $\nu_p(a_2^{(0)}) > 0$ so that $\overline{\mathfrak{A}^{(0)}}$ is anti-diagonal. For this reason, we will leave out the details in the completely similar cases. As before, let us redefine a subset $\mathcal{V} \subset \mathcal{W}$ by

$$\mathcal{V} = \{i \in \mathcal{W} : \nu_p(a_2^{(i)}) = 1\}.$$

Subcase: $\mathcal{V}$ Empty and $\nu_p(a_2^{(0)}) > 0$

Here we have each reduced partial Frobenius matrix is either diagonal or anti-diagonal so we can write down the reduction data right away after scaling by u via $X^{(f-1)} = \mathrm{Diag}(u, 1)$ so that

$$\begin{aligned}
\mu_0 &= (S, (0, k_0 - p)) \\
\mu_i &= \begin{cases} (I, (0, k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{W} \\ (I, (k_i, 0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{W} \end{cases} \\
\mu_{f-1} &= \begin{cases} (I, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I, (k_{f-1}+1, 0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}
\end{aligned}$$

Note that Proposition 9.1.6 will necessarily provide an irreducible reduction from this data.

Subcase: $\mathcal{V}$ Empty and $\nu_p(a_2^{(0)}) = 0$

This case proceeds precisely as in the $\mathcal{V}$ empty case of Section 9.3.1 after making the necessary coefficient adjustments to account for changing 1 into $\overline{a}_2^{(0)}$ and $(\overline{a_2^{(0)} a_2^{(i)}/p})$ into $(\overline{a_2^{(i)}/p})$. Up to restriction to inertia, these coefficients do not change the reduction so we may copy the reduction data exactly as it is displayed in that case.

$$\begin{aligned}
\mu_0 &= (I, (k_0 - p, 0)) \\
\mu_i &= \begin{cases} (I, (0, k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{W} \\ (I, (k_i, 0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{W} \end{cases} \\
\mu_{f-1} &= \begin{cases} (I, (1, k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I, (k_{f-1}+1, 0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}
\end{aligned}$$

Note that Proposition 9.1.6 will necessarily provide a reducible reduction from this data.

Subcase: $\mathcal{V}$ Nonempty

Just as before, this case is completely similar to the $\mathcal{V}$ nonempty case of Section 9.3.1 after making the necessary adjustments to various coefficients. Hence, up to restriction to inertia we have our reduction data given by

$$\mu_0 = (I, (0, k_0 - p))$$

$$\mu_j = \begin{cases} (I,(k_j,0)) & \text{if } 0<j<r \text{ and } j \in \mathcal{W} \\ (I,(0,k_j)) & \text{if } 0<j<r \text{ and } j \notin \mathcal{W} \end{cases}$$
$$\mu_i = \begin{cases} (I,(0,k_i)) & \text{if } r \le i < f-1 \text{ and } i \in \mathcal{W} \\ (I,(k_i,0)) & \text{if } r \le i < f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

where $r \in \mathcal{W}$ is the largest integer with $\nu_p(a_2^{(i)}) = 1$. Note that Proposition 9.1.6 will necessarily provide a reducible reduction from this data.

9.4 Explicit Reductions for the Complete $f = 2$ Case

Type	Conditions on valuations	Reduced Frobenius $(\overline{\mathfrak{A}}, \overline{\mathfrak{B}})$
(I,I)	$\nu_p(a_2) \ge 1$	$\begin{pmatrix} 0 & \overline{a}_1 u^{k_0} \\ 1 & \overline{(a_2/p)}\overline{\alpha}_1 u^p \end{pmatrix}, \begin{pmatrix} 0 & \overline{b}_1 u^{k_1} \\ 1 & \overline{b}_2 \end{pmatrix}$
(I,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) \ge 1 - \nu_p(a_2)$	$\begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ \overline{\alpha}_1 u^p & 1 \end{pmatrix}, \begin{pmatrix} 1 & \overline{(a_2 b_2/p)} \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix}$
(I,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) < 1 - \nu_p(a_2)$	$\begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ -\overline{\alpha}_1 u^p & 0 \end{pmatrix}, \begin{pmatrix} -\overline{b}_1 u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}$
(II,I)	$\nu_p(a_2) \ge 1$	$\begin{pmatrix} \overline{a}_1 u^{k_0} & 0 \\ \overline{(a_2/p)}\overline{\alpha}_1 u^p & 1 \end{pmatrix}, \begin{pmatrix} 0 & \overline{b}_1 u^{k_1} \\ 1 & 0 \end{pmatrix}$
(II,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) \ge 1$	$\begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ -\overline{\alpha}_1 u^p & 0 \end{pmatrix}, \begin{pmatrix} 1 & \overline{(b_2/p)} \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix}$
(II,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) < 1$	$\begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ \overline{\alpha}_1 u^p & 0 \end{pmatrix}, \begin{pmatrix} -\overline{b}_1 u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}$
(I,II)	$\nu_p(a_2) \ge 1$	$\begin{pmatrix} 0 & \overline{a}_1 u^{k_0} \\ 1 & \overline{(a_2/p)}\overline{\alpha}_1 u^p \end{pmatrix}, \begin{pmatrix} \overline{b}_1 u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}$
(I,II)	$\nu_p(a_2) < 1$	$\begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ -\overline{\alpha}_1 u^p & 0 \end{pmatrix}, \begin{pmatrix} \overline{b}_1 u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}$

Figure 9.1: Frobenius in $f = 2$ over $k_F[\![u]\!]$ depending on Type and valuations.

As opposed to the general $f > 2$ case, the simplicity of the $f = 2$ case allows us to explicitly compute reductions in all cases as well as describe the unramified characters resulting from the k_F-elements in the determinant.

Using the notation from Section 8.3, we congregate all the results of Theorem 7.2.3 and Propositions 8.1.2, 8.2.4, 8.2.6, 8.3.3, and 8.3.5 above in terms of the three irreducible Type combinations (I,I), (I,II) and (II,I). Let us recall that when we are of Type (I,I), the product $a_2b_2 \in \varpi\mathscr{O}_F$ and in the other two Type combinations, we have both $a_2, b_2 \in \varpi\mathscr{O}_F$. Let $\overline{\mathcal{A}} := \mathcal{A} \pmod{\varpi}$ and $\overline{\mathcal{B}} := \mathcal{B} \pmod{\varpi}$. Let us make the observation that all cases where $\nu_p(b_2)$ and $\nu_p(a_2)$ are less than one have the same shape.

For the rest of this section, let us denote $\chi(c)$ to be the unramified character sending Frobenius to $c \in \overline{\mathbb{F}}_p$.

9.4.1 Explicit Reductions for Type (I, I)

We begin by making the simplifying observation that the five cases in the Type (I, I) can be simplified to three cases depending on $\nu_p(a_2b_2)$ when computing reductions. Indeed, we will suppose each subcase below and show that the cases are equivalent.

Subcase: $\nu_p(a_2b_2) > 1$

The bound $\nu_p(a_2b_2) > 1$ is satisfied three times in Figure 9.1.

(a) $\nu_p(a_2) > 1$ and $\nu_p(b_2) \geq 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \bar{a}_1 u^{k_0} \\ 1 & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \bar{b}_1 u^{k_1} \\ 1 & \bar{b}_2 \end{pmatrix};$$

(b) $\nu_p(a_2) = 1$ and $\nu_p(b_2) > 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \bar{a}_1 u^{k_0} \\ 1 & \overline{(a_2/p)\alpha_1} u^p \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \bar{b}_1 u^{k_1} \\ 1 & 0 \end{pmatrix};$$

(c) $\nu_p(a_2) < 1$ and $\nu_p(b_2) > 1 - \nu_p(a_2)$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0 - p} \\ \bar{\alpha}_1 u^p & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}}^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{b}_1 u^{k_1} \end{pmatrix}.$$

In (a) we follow the the methods of Section 9.2.2, in (b) we follow Section 9.2.5 and for (c) we utilize Section 9.3.1. The result will be the pair of reduced matrices in each case given by

(a) $\nu_p(a_2) >$ and $\nu_p(b_2) \geq 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{a}_1 u^{k_0 - p} \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1 u^{k_1} & 0 \\ 0 & u \end{pmatrix};$$

(b) $\nu_p(a_2) = 1$ and $\nu_p(b_2) > 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} \bar{a}_1 u^{k_0 - p} & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} u & 0 \\ 0 & \bar{b}_1 u^{k_1} \end{pmatrix};$$

(c) $\nu_p(a_2) < 1$ and $\nu_p(b_2) > 1 - \nu_p(a_2)$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} \overline{a}_1 u^{k_0-p} & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} u & 0 \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix};$$

It is then easy to see that all three cases above will result in a product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} \overline{a}_1 u^{k_0} & 0 \\ 0 & \overline{b}_1 u^{pk_1} \end{pmatrix}.$$

Hence, we can use Proposition 9.1.3 to compute the reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi(\overline{a}_1)\omega_2^{k_0} \oplus \chi(\overline{b}_1)\omega_2^{pk_1}.$$

Subcase: $\nu_p(a_2b_2) = 1$

The bound $\nu_p(a_2b_2) = 1$ is satisfied twice in Figure 9.1.

(a) $\nu_p(a_2) = 1$ and $\nu_p(b_2) = 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \overline{a}_1 u^{k_0} \\ 1 & \overline{(a_2/p)}\overline{\alpha}_1 u^p \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \overline{b}_1 u^{k_1} \\ 1 & \overline{b}_2 \end{pmatrix};$$

(b) $\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1 - \nu_p(a_2)$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -\overline{(a_1/\alpha_1)} u^{k_0-p} \\ \overline{\alpha}_1 u^p & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 1 & \overline{(a_2b_2/p)} \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix}.$$

In (a) we follow Section 9.2.5 and for (b) we follow Section 9.3.1. The result will be

(a) $\nu_p(a_2) = 1$ and $\nu_p(b_2) = 0$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} -\frac{\overline{a}_1}{\overline{(a_2/p)}\overline{\alpha}_1} u^{k_0-p} & 0 \\ 0 & \overline{(a_2/p)}\overline{\alpha}_1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} -\overline{(b_1/b_2)} u^{k_1} & 0 \\ 0 & \overline{b}_2 u \end{pmatrix};$$

(b) $\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1 - \nu_p(a_2)$:

$$\overline{\mathfrak{A}} = \begin{pmatrix} \overline{\alpha}_1 & 0 \\ 0 & -(\overline{a}_1/\overline{\alpha}_1) u^{k_0-p} \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \overline{(a_2b_2/p)} u & 0 \\ 0 & -\frac{\overline{b}_1}{\overline{(a_2b_2/p)}} u^{k_1} \end{pmatrix}.$$

Hence, in both cases we will have a product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} (\overline{\frac{a_2b_2\alpha_1}{p}}) u^p & 0 \\ 0 & (\overline{\frac{a_1b_1p}{a_2b_2\alpha_1}}) u^{(k_0-p)+pk_1} \end{pmatrix}.$$

We may now use Proposition 9.1.3 to get a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi_2\omega_2^p \oplus \chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1}$$

where $\chi_1 = \chi(\overline{a_1 b_1})$ and $\chi_2 = \chi(\overline{a_2 b_2 \alpha_1/p})$.

Subcase: $\nu_p(a_2 b_2) < 1$

Under the bounds $\nu_p(a_2 b_2) < 1$ we have reduced partial Frobenius from Figure 9.1 after scaling by u via $X^{(f-1)} = \mathrm{Diag}(u,1)$,

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ -\overline{\alpha}_1 & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} -\bar{b}_1 u^{k_1+1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thankfully we can take the product immediately so that

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ (\overline{b_1\alpha_1})u^{p(k_1+1)} & 0 \end{pmatrix}.$$

For the purposes of identifying the unramified character, consider the following base change over $\overline{\mathbb{F}}_p$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{(\overline{-a_1 b_1})^{1/2}}{\overline{b_1\alpha_1}} \end{pmatrix} *_\varphi \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ (\overline{b_1\alpha_1})u^{p(k_1+1)} & 0 \end{pmatrix} = (\overline{-a_1 b_1})^{1/2} \begin{pmatrix} 0 & u^{k_0-p} \\ u^{p(k_1+1)} & 0 \end{pmatrix}.$$

This allows us to invoke Proposition 9.1.3 to compute the irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{-a_1 b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right).$$

9.4.2 Explicit Reductions for Type $(\mathrm{II}, \mathrm{I})$

Let us begin with the case of $\nu_p(a_2) > 1$ so that we have reduced partial Frobenius from Figure 9.1 given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} \bar{a}_1 u^{k_0} & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \bar{b}_1 u^{k_1} \\ 1 & 0 \end{pmatrix}.$$

Proceeding as in Section 9.2.2, we obtain

$$\overline{\mathfrak{A}} = \begin{pmatrix} \bar{a}_1 u^{k_0-p} & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \bar{b}_1 u^{k_1+1} \\ 1 & 0 \end{pmatrix}.$$

The product will then be given by

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} 0 & (\overline{a_1 b_1})u^{(k_0-p)+p(k_1+1)} \\ 1 & 0 \end{pmatrix}.$$

Scaling by $(\overline{a_1 b_1})^{1/2}$ will make

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = (\overline{a_1 b_1})^{1/2} \begin{pmatrix} 0 & u^{(k_0-p)+p(k_1+1)} \\ 1 & 0 \end{pmatrix}.$$

Hence, we may apply Proposition 9.1.3 to get an irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{a_1 b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}} \left(\omega_4^{(k_0-p)+p(k_1+1)}\right) = \chi((\overline{a_1 b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}} \left(\omega_4^{k_0+pk_1}\right).$$

Next, we have $\nu_p(a_2) = 1$ so that we get reduced partial Frobenius from Figure 9.1 given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} \bar{a}_1 u^{k_0} & 0 \\ \overline{(a_2/p)}\bar{\alpha}_1 u^p & 1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \bar{b}_1 u^{k_1} \\ 1 & 0 \end{pmatrix}.$$

We proceed as in Section 9.2.5 to get

$$\overline{\mathfrak{A}} = \begin{pmatrix} -(\overline{\frac{a_1 p}{a_2 \alpha_1}})u^{k_0-p} & 0 \\ 0 & \overline{(a_2/p)}\bar{\alpha}_1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{b}_1 u^{k_1+1} \end{pmatrix}.$$

We can then easily see the product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} -(\overline{\frac{a_1 p}{a_2 \alpha_1}})u^{k_0-p} & 0 \\ 0 & (\overline{\frac{b_1 a_2 \alpha_1}{p}})u^{p(k_1+1)} \end{pmatrix}.$$

We can now use Proposition 9.1.3 to get a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi_1 \chi_3^{-1} \omega_2^{k_0-p} \oplus \chi_2 \chi_3 \omega_2^{p(k_1+1)}$$

where $\chi_1 = \chi(\bar{a}_1)$, $\chi_2 = \chi(\bar{b}_1)$ and $\chi_3 = \chi(\overline{a_2\alpha_1/p})$.

Now we consider the case of $\nu_p(a_2) < 1$ but $\nu_p(b_2) > 1$ so that by Figure 9.1 we have partial reduced Frobenius given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ -\bar{\alpha}_1 u^p & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{b}_1 u^{k_1} \end{pmatrix}.$$

Following 9.3.2, we have

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ -\overline{\alpha}_1 & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} u & 0 \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix}.$$

We may immediately compute the product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} 0 & -(\overline{a_1 b_1/\alpha_1})u^{(k_0-p)+pk_1} \\ -(\overline{\alpha}_1)u^p & 0 \end{pmatrix}.$$

Scaling by $(-(\overline{a_1 b_1})^{1/2}/\overline{\alpha}_1) \in \overline{\mathbb{F}}_p$ will give

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = (\overline{a_1 b_1})^{1/2} \begin{pmatrix} 0 & u^{(k_0-p)+pk_1} \\ u^p & 0 \end{pmatrix}.$$

Hence we may use Proposition 9.1.3 to get an irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{a_1 b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}} \left(\omega_4^{(k_0-p)+pk_1+p^2}\right).$$

The penultimate case is when $\nu_p(a_2) < 1$ but $\nu_p(b_2) = 1$ so that we have reduced partial Frobenius given by Figure 9.1

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \\ -\overline{\alpha}_1 u^p & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 1 & (\overline{b_2/p}) \\ 0 & \overline{b}_1 u^{k_1} \end{pmatrix}.$$

Proceeding as in Section 9.3.2 will result in

$$\overline{\mathfrak{A}} = \begin{pmatrix} -\overline{\alpha}_1 & 0 \\ 0 & -(\overline{a_1/\alpha_1})u^{k_0-p} \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} (\overline{b_2/p})u & 0 \\ 0 & -(\overline{\frac{b_1 p}{b_2}})u^{k_1} \end{pmatrix}.$$

Hence, we have a product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} -(\overline{\frac{b_2\alpha_1}{p}})u^p & 0 \\ 0 & (\overline{\frac{a_1 b_1 p}{b_2\alpha_1}})u^{(k_0-p)+pk_1} \end{pmatrix}.$$

We may now invoke Proposition 9.1.3 to get a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi_2\omega_2^p \oplus \chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1}$$

where $\chi_1 = (-\overline{a_1 b_1})$ and $\chi_2 = \chi(-\overline{b_2\alpha_1/p})$. Notice that up to restriction to inertia, this is exactly the same reduction that we got in the Type (I, I) case with $\nu_p(a_2 b_2) = 1$.

Our final case is when both $\nu_p(b_2), \nu_p(a_2) < 1$. Thankfully, we have already noticed that the reduced partial Frobenius from Figure 9.1 is exactly the same, up to a scaling operation, to the case of Type (I, I) with $\nu_p(a_2b_2) < 1$. As a result, we have an irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{a_1b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right).$$

9.4.3 Explicit Reductions for Type (I, II)

For our final Type combination, we recall the shape of the reduced partial Frobenius from Figure 9.1 depend only on $\nu_p(a_2)$ as $\mathcal{W} = \emptyset$ in this case. Hence, we begin as before with the case of $\nu_p(a_2) > 1$ so that we have partial reduced Frobenius given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \bar{a}_1u^{k_0} \\ 1 & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proceeding as in 9.2.2, we will get

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \bar{a}_1u^{k_0-p} \\ 1 & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1u^{k_1} & 0 \\ 0 & u \end{pmatrix}.$$

Hence, we may take the product to be

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} 0 & (\bar{a}_1)u^{(k_0-p)+p} \\ (\bar{b}_1)u^{pk_1} & 0 \end{pmatrix}.$$

Scaling the anti-diagonal of the above product by $((\overline{a_1b_1})^{1/2}/\bar{b}_1) \in \overline{\mathbb{F}}_p$ will make

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = (\overline{a_1b_1})^{1/2}\begin{pmatrix} 0 & u^{(k_0-p)+p} \\ u^{pk_1} & 0 \end{pmatrix}.$$

We may now use Proposition 9.1.3 to get an irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{a_1b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}}\left(\omega_4^{(k_0-p)+p^2k_1+p}\right) = \chi((\overline{a_1b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}}\left(\omega_4^{k_0+p^2k_1}\right).$$

We now consider the case of $\nu_p(a_2) = 1$ so that by Figure 9.1 we have reduced partial Frobenius given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \bar{a}_1u^{k_0} \\ 1 & \overline{(a_2/p)}\overline{\alpha}_1u^p \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1u^{k_1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Proceeding as in Section 9.2.5, we will receive

$$\overline{\mathfrak{A}} = \begin{pmatrix} -(\overline{\frac{a_1p}{a_2\alpha_1}})u^{k_0-p} & 0 \\ 0 & \overline{(a_2/p)}\overline{\alpha}_1 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1u^{k_1} & 0 \\ 0 & u \end{pmatrix}.$$

We may now take the product

$$\overline{\mathfrak{A}}\varphi(\overline{\mathfrak{B}}) = \begin{pmatrix} -(\overline{\frac{a_1 b_1 p}{a_2\alpha_1}})u^{(k_0-p)+pk_1} & 0 \\ 0 & (\overline{\frac{a_2\alpha_1}{p}})u^p \end{pmatrix}.$$

Hence, we may invoke Proposition 9.1 to get a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi_2\omega_2^p \oplus \chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1}.$$

We may notice that the this reduction has already happened twice; The first time in Type (I,I) with $\nu_p(a_2b_2) = 1$ and the second in Type (II,I) with $\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1$.

Our final case is when $\nu_p(a_2) < 1$. Thankfully, we have already noticed that the reduced partial Frobenius from Figure 9.1 is exactly the same, up to a scaling operation, to the case of Type (I,I) with $\nu_p(a_2b_2) < 1$ and Type (II,I) with $\nu_p(a_2), \nu_p(b_2) < 1$. As a result, we have an irreducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi((\overline{a_1b_1})^{1/2})\mathrm{Ind}_{G_{\mathbb{Q}_{p^4}}}^{G_{\mathbb{Q}_{p^2}}}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right).$$

9.5 Summary of Reductions

Above we have found, up to an unramified twist or restriction to inertia, the reduction data in the sense of Definition 9.1.5 for Kisin modules $\overline{\mathfrak{M}(\mathfrak{A})}$ over $k_F[\![u]\!]$ with labeled heights in the range $p+2 \leq k_0 \leq 2p-4$ and $2 \leq k_i \leq p-3$ for $1 \leq i \leq f-1$. By the discussion found in Section 9.1, this data suffices to compute the reduction of the associated crystalline representations $V(A)$. Moreover, such representations cover all possible irreducible, two-dimensional crystalline representations of G_K with labeled Hodge-Tate weights in the same range and under certain bounds on the p-adic valuations of $a_2^{(i)}$. As a result, we have computed all such reductions and we summarize our results below in terms of the p-adic valuations of the $a_2^{(i)}$ parameters originating from the matrices $A = (A^{(i)})$ and the Type combination of said matrices.

Large Valuation $\nu_p(a_2^{(0)}) > 1$:

Recall that $\mathcal{S}$ denotes the set of $i \in \mathbb{Z}/f\mathbb{Z}$ corresponding to Type I Frobenius and $\mathcal{T}$ is the set of $i \in \mathbb{Z}/f\mathbb{Z}$ corresponding to Type II Frobenius.

- $\nu_p(a_2^{(i)}) > 0$ for all $i \in \mathcal{S}$:

$$\mu_0 = \begin{cases} (S,(0,k_0-p)) & \text{if } 0 \in \mathcal{S} \\ (I,(k_0-p,0)) & \text{if } 0 \in \mathcal{T} \end{cases}$$

$$\mu_i = \begin{cases} (S,(0,k_i)) & \text{if } 0<i<f-1 \text{ and } i \in \mathcal{S} \\ (I,(k_i,0)) & \text{if } 0<i<f-1 \text{ and } i \in \mathcal{T} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (S,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{S} \\ (S,(0,k_{f-1}+1)) & \text{if } f-1 \in \mathcal{S} \text{ and } 0 \in \mathcal{T} \\ (I,(k_{f-1},1)) & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{S} \\ (I,(k_{f-1}+1,0)) & \text{if } f-1 \in \mathcal{T} \text{ and } 0 \in \mathcal{T}. \end{cases}$$

Note that this case is reducible if and only if $|\mathcal{S}|$ is even and it is irreducible otherwise.

- $\nu_p(a_2^{(i)}) = 0$ for some $i \in \mathcal{S}$:
  Recall that the subset $\mathcal{X} \subset \mathbb{Z}/f\mathbb{Z}$ is as defined in Definition 9.2.3.

$$\mu_0 = \begin{cases} (I,(0,k_0-p)) & \text{if } 0 \in \mathcal{X} \\ (I,(k_0-p,0)) & \text{if } 0 \notin \mathcal{X} \end{cases}$$

$$\mu_i = \begin{cases} (I,(0,k_i)) & \text{if } 0<i<f-1 \text{ and } i \in \mathcal{X} \\ (I,(k_i,0)) & \text{if } 0<i<f-1 \text{ and } i \notin \mathcal{X} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (I,(0,k_{f-1}+1)) & \text{if } 0, f-1 \in \mathcal{X} \\ (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{X} \text{ and } 0 \notin \mathcal{X} \\ (I,(k_{f-1},1)) & \text{if } f-1 \in \mathcal{X} \text{ and } 0 \in \mathcal{X} \\ (I,(k_{f-1}0+1,0)) & \text{if } 0, f-1 \notin \mathcal{X}. \end{cases}$$

Note that this case is necessarily reducible.

Integer Valuation $\nu_p(a_2^{(0)}) = 1$:

Recall that the subset $\mathcal{Y} \subset \mathbb{Z}/f\mathbb{Z}$ is defined as in Definition 9.2.6. We encourage the reader to notice the intersections of reductions between multiple Type combinations.

- $0 \in \mathcal{S}$ and $f-1 \notin \mathcal{Y}$ **OR** $0 \in \mathcal{T}$ and $f-1 \in \mathcal{Y}$:

$$\mu_0 = (I,(k_0-p,0)).$$

$$\mu_i = \begin{cases} (I,(k_i,0)) & \text{if } 1 \le i < f-1 \text{ and } i \notin \mathcal{Y} \\ (I,(0,k_i)) & \text{if } 1 \le i < f-1 \text{ and } i \in \mathcal{Y} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (I,(k_{f-1},1)) & \text{if } f-1 \notin \mathcal{Y} \\ (I,(0,k_{f-1}+1)) & \text{if } f-1 \in \mathcal{Y}. \end{cases}$$

Note that this case is necessarily reducible.

- $0 \in \mathcal{T}$ and $f-1 \notin \mathcal{Y}$ **OR** $0 \in \mathcal{S}$ and $f-1 \in \mathcal{Y}$:

$$\mu_0 = (I,(k_0-p,0))$$

$$\mu_i = \begin{cases} (I,(k_i,0)) & \text{if } 1 \le i < f-1 \text{ and } i \notin \mathcal{Y} \\ (I,(0,k_i)) & \text{if } 1 \le i < f-1 \text{ and } i \in \mathcal{Y} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{Y} \\ (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{Y}. \end{cases}$$

Note that this case is necessarily reducible.

Small Valuation $\nu_p(a_2^{(0)}) < 1$:

Recall that the subset $\mathcal{W} \subset \mathbb{Z}/f\mathbb{Z}$ is defined in Definition 8.2.1. We encourage the reader to notice the intersections of reductions between multiple Type combinations.

- $|\mathcal{S}|$ even and $\nu_p(a_2^{(i)}) > 1 - \nu_p(a_2^{(0)})$ for all $i \in \mathcal{W}$ **OR** $|\mathcal{S}|$ odd and and $\nu_p(a_2^{(i)}) > 1$ for all $i \in \mathcal{W}$ with $\nu_p(a_2^{(0)}) = 0$:

$$\mu_0 = (I,(k_0-p,0))$$
$$\mu_i = \begin{cases} (I,(0,k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{W} \\ (I,(k_i,0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

  Note that this case is necessarily reducible.
- $|\mathcal{S}|$ even and $r \in \mathcal{W}$ is the largest integer with $\nu_p(a_2^{(r)}) = 1 - \nu_p(a_2^{(0)})$ **OR** $|\mathcal{S}|$ odd and $r \in \mathcal{W}$ is the largest integer with $\nu_p(a_2^{(r)}) = 1$:

$$\mu_0 = (I,(0,k_0-p))$$
$$\mu_j = \begin{cases} (I,(k_j,0)) & \text{if } 0 < j < r \text{ and } j \in \mathcal{W} \\ (I,(0,k_j)) & \text{if } 0 < j < r \text{ and } j \notin \mathcal{W} \end{cases}$$
$$\mu_i = \begin{cases} (I,(0,k_i)) & \text{if } r \le i < f-1 \text{ and } i \in \mathcal{W} \\ (I,(k_i,0)) & \text{if } r \le i < f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$
$$\mu_{f-1} = \begin{cases} (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

  Note that this case is necessarily reducible.
- $|\mathcal{S}|$ odd and $\nu_p(a_2^{(i)}) > 1$ for all $i \in \mathcal{W}$ with $\nu_p(a_2^{(0)}) \neq 0$:

$$\mu_0 = (S,(0,k_0-p))$$
$$\mu_i = \begin{cases} (I,(0,k_i)) & \text{if } 0 < i < f-1 \text{ and } i \in \mathcal{W} \\ (I,(k_i,0)) & \text{if } 0 < i < f-1 \text{ and } i \notin \mathcal{W} \end{cases}$$

$$\mu_{f-1} = \begin{cases} (I,(1,k_{f-1})) & \text{if } f-1 \in \mathcal{W} \\ (I,(k_{f-1}+1,0)) & \text{if } f-1 \notin \mathcal{W}. \end{cases}$$

Note that this case is necessarily irreducible.

Reductions for $f = 2$:

Type	Conditions on valuations	Reduction $\overline{V(A,B)}\|_{G_\infty}$	Unramified Characters
(I,I)	$\nu_p(a_2b_2) > 1$	$\chi_1\omega_2^{k_0} \oplus \chi_2\omega_2^{pk_1}$	$\chi_1 = \chi(\bar{a}_1)$ $\chi_2 = \chi(\bar{b}_1)$
(I,I)	$\nu_p(a_2b_2) = 1$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1} \oplus \chi_2\omega_2^{p}$	$\chi_1 = \chi(\overline{a_1b_1})$ $\chi_2 = \chi(\overline{a_2b_2\alpha_1/p})$
(I,I)	$\nu_p(a_2b_2) < 1$	$\chi\mathrm{Ind}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right)$	$\chi = \chi((-\overline{a_1b_1})^{1/2})$
(II,I)	$\nu_p(a_2) > 1$	$\chi\mathrm{Ind}\left(\omega_4^{k_0+pk_1}\right)$	$\chi = \chi((\overline{a_1b_1})^{1/2})$
(II,I)	$\nu_p(a_2) = 1$	$\chi_1\chi_3^{-1}\omega_2^{k_0-p} \oplus \chi_2\chi_3\omega_2^{p(k_1+1)}$	$\chi_1 = \chi(\bar{a}_1)$ $\chi_2 = \chi(\bar{b}_1)$ $\chi_3 = \chi(\overline{a_2\alpha_1/p})$
(II,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) > 1$	$\chi\mathrm{Ind}\left(\omega_4^{(k_0-p)+p(k_1+p)}\right)$	$\chi = \chi((\overline{a_1b_1})^{1/2})$
(II,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1} \oplus \chi_2\omega_2^{p}$	$\chi_1 = \chi(-\overline{a_1b_1})$ $\chi_2 = \chi(-\overline{b_2\alpha_1/p})$
(II,I)	$\nu_p(a_2) < 1$ and $\nu_p(b_2) < 1$	$\chi\mathrm{Ind}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right)$	$\chi = \chi((\overline{a_1b_1})^{1/2})$
(I,II)	$\nu_p(a_2) > 1$	$\chi\mathrm{Ind}\left(\omega_4^{k_0+p^2k_1}\right)$	$\chi = \chi((\overline{a_1b_1})^{1/2})$
(I,II)	$\nu_p(a_2) = 1$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p)+pk_1} \oplus \chi_2\omega_2^{p}$	$\chi_1 = \chi(-\overline{a_1b_1})$ $\chi_2 = \chi(\overline{a_2\alpha_1/p})$
(I,II)	$\nu_p(a_2) < 1$	$\chi\mathrm{Ind}\left(\omega_4^{(k_0-p)+p^2(k_1+1)}\right)$	$\chi = \chi((\overline{a_1b_1})^{1/2})$

Figure 9.2: Reductions in $f = 2$ depending on Type and p-adic valuations of a_2 and b_2.

When $f = 2$, we are able to explicitly compute the reductions along with the associated unramified characters. As before, the shape of these reductions turn out to depend on both the p-adic valuations of the parameters a_2 and b_2 along with the three distinct Type combinations. The table above summarizes these computations. Recall that $\chi(c)$ denotes the unramified character sending Frobenius to $c \in \overline{\mathbb{F}}_p$. All inductions are to be taken from $G_{\mathbb{Q}_{p^4}}$ to $G_{\mathbb{Q}_{p^2}}$.

Let us explicitly point out the intersections of reductions up to unramified characters.

- The reducible reduction $\omega_2^{(k_0-p)+pk_1} \oplus \omega_2^p$ occurs three times:
  - Type (I, I) with $\nu_p(a_2b_2) = 1$ which if we recall from Section 9.4 happened when $\nu_p(a_2) = 1$ and $\nu_p(b_2) = 0$ *or* $\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1 - \nu_p(a_2)$;
  - Type $(\mathrm{II}, \mathrm{I})$ with $\nu_p(a_2) < 1$ and $\nu_p(b_2) = 1$;
  - Type $(\mathrm{I}, \mathrm{II})$ with $\nu_p(a_2) = 1$.
- The irreducible reduction $\mathrm{Ind}\omega_4^{(k_0-p)+p^2(k_1+1)}$ occurs three times:
  - Type (I, I) with $\nu_p(a_2b_2) < 1$;
  - Type $(\mathrm{II}, \mathrm{I})$ with $\nu_p(a_2) < 1$ and $\nu_p(b_2) < 1$;
  - Type $(\mathrm{I}, \mathrm{II})$ with $\nu_p(a_2) < 1$.

Remark 9.5.1: By the classification of two dimensional, irreducible representations of $\mathrm{GL}_2(\mathbb{F}_{p^2})$, see [Hum05], any $\overline{V(A,B)}$ with Hodge-Tate weights $2 \leq k_0 < p$ and $p < k_1 < 2p$ necessarily lies in

$$\mathrm{Sym}^{k_0-1}(\mathbb{F}_{p^2}^2) \otimes_{\overline{\mathbb{F}}_p} \varphi(\mathrm{Sym}^{k_1-1}(\mathbb{F}_{p^2}^2)).$$

This space decomposes by the Steinberg tensor product theorem into eight irreducible components, one of which corresponds to when $V(A,B)$ is reducible. As a result, we should expect to get seven distinct reductions when $f = 2$ and a brief look at Figure 9.2 will confirm that we indeed get the required number of reductions.

Appendix A

Allowing More Large Weights

Beginning in Chapter 8, we restricted ourselves to only allow the k_0 labeled Hodge-Tate weight to exceed p, forcing all other weights to remain small. The purpose of this Appendix is to display the fact that some of the theory carries through if we allow more weights to exceed p as long as the p-adic valuations of the $a_2^{(i)}$ parameters are sufficiently large. Unsurprisingly, this case is more complicated and borderline unfeasible without heavy assumptions.

We follow the strategies laid out in Sections 8.1 and 9.2 heavily. As a result of these repeated arguments, we leave out many expository details.

A.1 Preparatory Computations

Recall that we have a finite height Kisin module $\mathcal{M}(\mathcal{A})$ over S_F with i-th partial Frobenius given by

- Type I:
$$\mathcal{A}^{(i)} = \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} \end{pmatrix};$$
- Type II:
$$\mathcal{A}^{(i)} = \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ a_2^{(i)} \lambda_b^{g^{(i)}(\varphi)} & 1 \end{pmatrix}.$$

Let us partition the set $\mathbb{Z}/f\mathbb{Z}$ into two disjoint subsets defined by

$$\mathcal{L} := \{i \in \mathbb{Z}/f\mathbb{Z} : 2 \leq k_i \leq p-3\} \qquad \mathcal{B} := \{i \in \mathbb{Z}/f\mathbb{Z} : p+2 \leq k_i \leq 2p-4\}.$$

We will assume that every $i \in \mathbb{Z}/f\mathbb{Z}$ lies in one of these two subsets. Let us set

$$x^{(i)} = \begin{cases} -\left(\frac{a_2}{a_1}\right)^{(i)} \sum_{j=1}^{\infty} \alpha_j^{(i)} \frac{E^{jp-k_i}}{p^j} & \text{if } i \in \mathcal{L} \\ -\left(\frac{a_2}{a_1}\right)^{(i)} \sum_{j=2}^{\infty} \alpha_j^{(i)} \frac{E^{jp-k_i}}{p^j} & \text{if } i \in \mathcal{B}. \end{cases}$$

Set $X_1^{(i)} = \left(\begin{smallmatrix} 1 & 0 \\ x^{(i)} & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(S_F[1/p])$ for all $i \in \mathbb{Z}/f\mathbb{Z}$ and observe that by setting $(X_1) *_\varphi (\mathcal{A}) = (\mathcal{A}_1)$ we will obtain

$$\mathcal{A}_1^{(i)} = \begin{cases} \begin{pmatrix} E^{k_i} a_1^{(i)} \varphi(x^{(i+1)}) & E^{k_i} a_1^{(i)} \\ 1 + (a_2\alpha_0)^{(i)} \varphi(x^{(i+1)}) & (a_2\alpha_0)^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{L} \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \varphi(x^{(i+1)}) & 1 \end{pmatrix} & \text{if } i \in \mathcal{L} \text{ and } i \in \mathcal{T} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} \varphi(x^{(i+1)}) & E^{k_i} a_1^{(i)} \\ 1 + ((a_2\alpha_0)^{(i)} + (a_2\alpha_1)^{(i)} \frac{E^p}{p}) \varphi(x^{(i+1)}) & (a_2\alpha_0)^{(i)} + (a_2\alpha_1)^{(i)} \frac{E^p}{p} \end{pmatrix} & \text{if } i \in \mathcal{B} \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + (a_2\alpha_1)^{(i)} \frac{E^p}{p} + \varphi(x^{(i+1)}) & 1 \end{pmatrix} & \text{if } i \in \mathcal{B} \text{ and } i \in \mathcal{T}. \end{cases}$$

By Lemma 8.1.1, each $\varphi(x^{(i)}) \in I$ so as long as $\nu_p(a_2^{(i)}) \geq 1$ for all $i \in \mathcal{B}$ then the Descent Assumptions 7.2.2 will be satisfied. Hence, we have

Proposition A.1.1: *Suppose $\mathcal{M}(\mathcal{A})$ is a rank-two Kisin module over S_F with labeled heights $2 \leq k_i \leq p-3$ for $i \in \mathcal{L}$ and $p+2 \leq k_i \leq 2p-4$ for $i \in \mathcal{B}$. If $\nu_p(a_2^{(i)}) \geq 1$ for all $i \in \mathcal{B}$, then the Frobenius matrix $(\mathcal{A})$ satisfies the Descent Assumptions 7.2.2 with*

$$\mathfrak{A}_0^{(i)} = \begin{cases} \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & (a_2\alpha_0)^{(i)} \end{pmatrix} & \textit{if } i \in \mathcal{L} \textit{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} & 1 \end{pmatrix} & \textit{if } i \in \mathcal{L} \textit{ and } i \in \mathcal{T} \\ \begin{pmatrix} 0 & E^{k_i} a_1^{(i)} \\ 1 & (a_2\alpha_0)^{(i)} + \left(\frac{a_2^{(i)}}{p}\right) \alpha_1^{(i)} E^p \end{pmatrix} & \textit{if } i \in \mathcal{B} \textit{ and } i \in \mathcal{S} \\ \begin{pmatrix} E^{k_i} a_1^{(i)} & 0 \\ (a_2\alpha_0)^{(i)} + \left(\frac{a_2^{(i)}}{p}\right) \alpha_1^{(i)} E^p & 1 \end{pmatrix} & \textit{if } i \in \mathcal{B} \textit{ and } i \in \mathcal{T}. \end{cases}$$

Remark A.1.2: The preparatory computations for when $\nu_p(a_2^{(i)}) < 1$ for $i \in \mathcal{B}$ would be substantially more complicated then those found in Section 8.2. In particular, it would require careful analysis of the $i-1$ matrix for every $i \in \mathcal{B}$. Moreover, it is not clear how one would deal with the $|\mathcal{S}|$ even case as our current methods in 8.2.3 would require a scaling by $p/a_2^{(i)}$ for every $i \in \mathcal{B}$.

A.2 Reductions in $f = 2$

Let $\mathcal{M}(\mathcal{A})$ be a Kisin module over S_F with labeled heights in the range $2 \leq k_i \leq p-3$ for $i \in \mathcal{L}$ and $p+2 \leq k_i \leq 2p-4$ for $i \in \mathcal{B}$. Supposing that $\nu_p(a_2^{(i)}) \geq 1$ for all $i \in \mathcal{B}$, then the results of Proposition A.1.1 combined with Theorem 7.2.3 allow use to identify a descent of $\mathcal{M}(\mathcal{A})$ to $\mathfrak{S}_F$ denoted $\mathfrak{M}(\mathfrak{A})$. The reduction of which $\overline{\mathfrak{M}(\mathfrak{A})}$ necessarily has i-th partial Frobenius matrices given by

$$\overline{\mathfrak{A}}^{(i)} = \begin{cases} \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \overline{a}_2^{(i)} \end{pmatrix} & \text{if } i \in \mathcal{L} \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i \in \mathcal{L} \text{ and } i \in \mathcal{T} \\ \begin{pmatrix} 0 & u^{k_i}\overline{a}_1^{(i)} \\ 1 & \left(\overline{a_2^{(i)}/p}\right)\overline{\alpha}_1^{(i)}u^p \end{pmatrix} & \text{if } i \in \mathcal{B} \text{ and } i \in \mathcal{S} \\ \begin{pmatrix} u^{k_i}\overline{a}_1^{(i)} & 0 \\ \left(\overline{a_2^{(i)}/p}\right)\overline{\alpha}_1^{(i)}u^p & 1 \end{pmatrix} & \text{if } i \in \mathcal{B} \text{ and } i \in \mathcal{T}. \end{cases}$$

Recalling our methods from Section 9.2, it quickly becomes apparent that computing the reduction data in the sense of Definition 9.1.5 will devolve into a deluge of subcases depending on the integrality of the p-adic valuation of each $a_2^{(i)}$ parameter. To lighten this load, let us restrict ourselves to the case of $f = 2$. Note that by Remark 8.0.1, we need only concern ourselves with the cases when $\mathbb{Z}/f\mathbb{Z} = \mathcal{B}$. We will treat the Type (I,I) case in full and leave out the details for the rest as the methods required are completely similar.

We have reduced partial Frobenius Type (I,I) matrices given by

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & u^{k_0}\overline{a}_1 \\ 1 & \left(\overline{a_2/p}\right)\overline{\alpha}_1 u^p \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & u^{k_1}\overline{b}_1 \\ 1 & \left(\overline{b_2/p}\right)\overline{\beta}_1 u^p \end{pmatrix}.$$

We have four subcases:

- Both $\nu_p(a_2), \nu_p(b_2) > 1$: We need only scale by u via $X = Y = \mathrm{Diag}(1, u)$ so that $(X, Y) *_\varphi (\overline{\mathfrak{A}}, \overline{\mathfrak{B}})$ gives

$$\overline{\mathfrak{A}} = \begin{pmatrix} 0 & \overline{a}_1 u^{k_0 - p} \\ u & 0 \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} 0 & \overline{b}_1 u^{k_1 - p} \\ u & 0 \end{pmatrix}.$$

Hence, Proposition 9.1.3 produces a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi(\bar{a}_1)\omega_2^{(k_0-p)+p} \oplus \chi(\bar{b}_1)\omega_2^{1+p(k_1-p)}.$$

- $\nu_p(a_2) = 1$ and $\nu_p(b_2) > 1$: Swapping the rows of $\overline{\mathfrak{A}}$ following by scaling by u via $X = I_2$ and $Y = \mathrm{Diag}(1, u)$ allows us to semi-simplify to make both matrices diagonal. Then we need only apply $X = \mathrm{Diag}(u, 1)$ and $Y = I_2$ to get

$$\overline{\mathfrak{A}} = \begin{pmatrix} u & 0 \\ 0 & \bar{a}_1 u^{k_0-p} \end{pmatrix} \qquad \overline{\mathfrak{B}} = \begin{pmatrix} \bar{b}_1 u^{k_1-p} & 0 \\ 0 & u \end{pmatrix}.$$

  Proposition 9.1.3 then gives a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi(\bar{a}_1)\omega_2^{(k_0-p)+p} \oplus \chi(\bar{b}_1)\omega_2^{1+p(k_1-p)}.$$

- $\nu_p(a_2) > 1$ and $\nu_p(b_2) = 1$: The methods are the same as the case after replacing X with Y and vice versa. Hence we get a reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi(\bar{a}_1)\omega_2^{(k_0-p)+p} \oplus \chi(\bar{b}_1)\omega_2^{1+p(k_1-p)}.$$

- Both $\nu_p(a_2), \nu_p(b_2) = 1$: We need only apply Lemma 9.1.4 top kill the u^{k_i} term in each matrix. This allows us to semi-simplify to achieve

$$\overline{\mathfrak{A}} = \begin{pmatrix} -\frac{\bar{a}_1}{\overline{(a_2/p)}\bar{\alpha}_1} u^{k_0-p} & \\ 0 & \left(\overline{a_2/p}\right)\bar{\alpha}_1 u^p \end{pmatrix} \quad \overline{\mathfrak{B}} = \begin{pmatrix} -\frac{\bar{b}_1}{\overline{(b_2/p)}\bar{\beta}_1} u^{k_1-p} & 0 \\ 0 & \left(\overline{b_2/p}\right)\bar{\beta}_1 u^p \end{pmatrix}.$$

  Scaling by u via $X = Y = \mathrm{Diag}(1, u)$ allows the use of Proposition 9.1.3 to get a reducible reduction

$$\overline{V(A,B)}|_{G_\infty} = \chi(\bar{a}_1\bar{b}_1)\chi(\bar{a}_2\bar{b}_2\bar{\alpha}_1\bar{\beta}_1/p^2)^{-1}\omega_2^{(k_0-p)+p(k_1-p)} \oplus \chi(\bar{a}_2\bar{b}_2\bar{\alpha}_1\bar{\beta}_1/p^2)\omega_2^{p+1}.$$

We summarize the rest of the reductions using similar methods above in the following table. All inductions are taken to be from $G_{\mathbb{Q}_{p^4}}$ to $G_{\mathbb{Q}_{p^2}}$.

Type	Conditions	Reduction $\overline{V(A,B)}\|_{G_\infty}$	Characters
(I,I)	$\nu_p(a_2b_2)>2$	$\chi_1\omega_2^{k_0}\oplus\chi_2\omega_2^{1+p(k_1-p)}$	$\chi_1=\chi(\overline{a}_1)$ $\chi_2=\chi(\overline{b}_1)$
(I,I)	$\nu_p(a_2b_2)=2$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p)+p(k_1-p)}\oplus\chi_2\omega_2^{p+1}$	$\chi_1=\chi(\overline{a_1b_1})$ $\chi_2=\chi(\overline{a_2b_2\alpha_1\beta_1/p^2})$
(II,I)	$\nu_p(a_2),\nu_p(b_2)>1$	$\chi\mathrm{Ind}\left(\omega_4^{(k_0-p)+p(k_1-p+1)}\right)$	$\chi=\chi((\overline{a_1b_1})^{1/2})$
(II,I)	$\nu_p(a_2)\geq 1$ & $\nu_p(b_2)=1$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p)+p(k_1-p+1)}\oplus\chi_2\omega_2$	$\chi_1=\chi(-\overline{a_1b_1})$ $\chi_2=\chi(\overline{b_2\beta_1/p})$
(II,I)	$\nu_p(a_2)=1$ & $\nu_p(b_2)>1$	$\chi_1\chi_3^{-1}\omega_2^{k_0-p}\oplus\chi_2\chi_3\omega_2^{1+p(k_1-p+1)}$	$\chi_1=\chi(-\overline{a}_1)$ $\chi_2=\chi(\overline{b}_1)$ $\chi_3=\chi(\overline{a_2\alpha_1/p})$
(I,II)	$\nu_p(a_2),\nu_p(b_2)>1$	$\chi\mathrm{Ind}\left(\omega_4^{p^2(k_0-p)+(k_0+1)}\right)$	$\chi=\chi((\overline{a_1b_1})^{1/2})$
(I,II)	$\nu_p(a_2)>1$ & $\nu_p(b_2)=1$	$\chi_1\chi_3\omega_2^{k_0+1}\oplus\chi_2\chi_3^{-1}\omega_2^{p(k_1-p)}$	$\chi_1=\chi(\overline{a}_1)$ $\chi_2=\chi(-\overline{b}_1)$ $\chi_3=\chi(\overline{b_2\beta_1/p})$
(I,II)	$\nu_p(a_2)=1$ & $\nu_p(b_2)\geq 1$	$\chi_1\chi_2^{-1}\omega_2^{(k_0-p+1)+p(k_1-p)}\oplus\chi_2\omega_2^{p}$	$\chi_1=\chi(-\overline{a_1b_1})$ $\chi_2=\chi(\overline{a_2\alpha_1/p})$

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