The Annals of Applied Probability
2023, Vol. 33, No. 1, 200–237
https://doi.org/10.1214/22-AAP1813

ON A NONLINEAR SPDE DERIVED FROM A HYDRODYNAMIC LIMIT IN A SINAI-TYPE RANDOM ENVIRONMENT

BY CLAUDIO LANDIM[1,a], CARLOS G. PACHECO[2,b], SUNDER SETHURAMAN[3,c] AND JIANFEI XUE[4,d]

[1]*IMPA*, [a]*landim@impa.br*
[2]*Department of Mathematics, CINVESTAV-IPN*, [b]*cpacheco@math.cinvestav.mx*
[3]*Department of Mathematics, University of Arizona*, [c]*sethuram@math.arizona.edu*
[4]*Department of Mathematics, University of Missouri*, [d]*jxue@missouri.edu*

With the recent developments on nonlinear SPDEs, where smoothing of rough noises is needed, one is naturally led to study interacting particle systems whose macroscopic evolution is described by these equations and which possess an in-built smoothing. In this article, our main results are to derive regularized versions of the ill-posed one-dimensional SPDE

$$\partial_t \rho = \frac{1}{2}\Delta\Phi(\rho) - 2\nabla\big(W'\Phi(\rho)\big),$$

where the spatial white noise W' is replaced by a regularization W'_ε, as quenched and annealed hydrodynamic limits of zero-range interacting particle systems in ε-regularized Sinai-type random environments. Some computations are also made about annealed mean hydrodynamic limits in unregularized Sinai-type random environments with respect to independent particles.

1. Introduction. Given recent developments in paracontrolled and regularity structures analysis of nonlinear SPDEs, where smoothing of noises is involved, it is natural to investigate microscopic interacting particle systems whose macroscopic hydrodynamic limit already captures the smoothing of the noise. In this article, we are interested in interacting particle systems whose hydrodynamic evolution is given by a smoothing of the ill-posed one-dimensional nonlinear SPDE

$$\partial_t \rho = \frac{1}{2}\Delta\Phi(\rho) - 2\nabla\big(W'\Phi(\rho)\big), \tag{1.1}$$

where the spatial white noise W' is replaced by a regularization W'_ε, and $\Phi \in C^1$. Such an equation might be considered a "mass conserved" generalized parabolic Anderson equation, noting that the parabolic Anderson equation is when $\Phi(v) \equiv v$ and ∇ is not present.

More specifically, with respect to a zero-range particle system in a common Sinai-type ε-regularized random environment on a one-dimensional discrete lattice, we derive "quenched" and "annealed" hydrodynamic limits, which feature a regularized noise W'_ε, emerging from a scaling of the random environment. Some computations are also made on the annealed mean hydrodynamic limit in unregularized Sinai-type random environments with respect to independent particles. We remark, as part of a two-scale approach, these results connect to the work Funaki et al. [15], where the limit of the hydrodynamic density as $\varepsilon \downarrow 0$ is shown to satisfy a paracontrolled version of the SPDE (1.1).

In the next subsections, we introduce the random environments studied, summarize the main results and their proofs, review some previous literature and discuss directions for future work.

Received December 2020; revised January 2022.
MSC2020 subject classifications. 60K35, 60K37, 60L50.
Key words and phrases. Sinai random environment, Brox diffusion, SPDE, interacting particle system, zero-range, hydrodynamic, quasilinear, inhomogeneous, annealed, quenched, regularization.

1.1. *"Sinai" random environments.* To introduce elements of the environments studied in the article, we recall that a "Sinai" random environment on $\mathbb{Z}$ is a sequence of independent and identically distributed (i.i.d.) random variables $\{u_i\}_{i\in\mathbb{Z}}$, indexed over vertices, with the property that $c \le u_0 \le 1-c$ for some constant $0 < c < 1/2$ and $E[\log(u_0/(1-u_0))] = 0$. Define $\sigma^2 = E[(\log(u_0/(1-u_0)))^2]$. Let U_n be the position of a discrete-time random walk in this random environment (RWRE):

$$P\big(U_{n+1} = U_n + 1 | U_n, \{u_i\}\big) = 1 - P\big(U_{n+1} = U_n - 1 | U_n, \{u_i\}\big) = u_{U_n}$$

for $n \ge 1$ and $U_0 = 0$. When $\sigma^2 > 0$, Sinai [36] showed that $\sigma^2 U_n/(\log(n))^2$ converges in the annealed sense weakly to a nontrivial random variable U_∞, whose law was identified in Kesten [24] and Golosov [16]. A functional limit theorem to a nontrivial process is problematic however as $\sigma^2 U_{\lfloor nt\rfloor}/(\log(n))^2 \Rightarrow U_\infty$, the limit process here being constant in time t.

However, a continuous analog X_t of the Sinai RWRE on $\mathbb{R}$ was introduced in Brox [6]: Formally,

$$dX_t = dB_t - \frac{1}{2} W'(X_t)\,dt$$

and $X_0 = 0$ or equivalently $\mathscr{L}_{\text{Brox}}$ takes form

$$\mathscr{L}_{\text{Brox}} = \frac{1}{2} e^{W(x)} \frac{d}{dx}\left(e^{-W(x)} \frac{d}{dx}\right).$$

Here, B is a standard Brownian motion and W is a two-sided Brownian motion on $\mathbb{R}$: $W(0) = 0$, $W(x) = \sigma W_+(x)$ for $x > 0$, and $W(x) = \sigma W_-(-x)$ for $x < 0$, where $W_\pm$ are independent standard Brownian motions. This description is only shorthand, as W is not differentiable a.s. More carefully, this "σ-Brox" diffusion is defined in terms of speed and scale measures:

$$X_t = A^{-1}(B_{T^{-1}(t)}) \quad \text{where } A(y) = \int_0^y e^{W(z)}\,dz \text{ and } T(t) = \int_0^t e^{-2W(A^{-1}(B_s))}\,ds$$

for $y \in \mathbb{R}$ and $t \ge 0$. Brox [6] showed that $X_t/(\log t)^2 \Rightarrow U_\infty$, the same limit as for the discrete Sinai RWRE convergence.

To connect the two models, related random environments $\{u_i^N\}_{i\in\mathbb{Z}}$, in terms of a scaling parameter N, were introduced in Seignourel [35]. An example is where $u_i^N = 1/2 + r_i/\sqrt{\sigma^2 N}$ where $\{r_i\}_{i\in\mathbb{Z}}$ is an i.i.d. sequence of bounded random variables, which are mean-zero and with positive variance σ^2. Let U_n^N be the corresponding RWRE with respect to the scaled environment $\{u_i^N\}$. Seignourel [35] showed that $\{N^{-1} U^N_{\lfloor N^2 t\rfloor} : t \ge 0\}$ converges weakly in the annealed sense to the 4-Brox diffusion $\{X_t : t \ge 0\}$, formally given by $dX_t = dB_t - (1/2)W'(X_t)\,dt$ with $\sigma = 4$. See also Andriopolous [2] and Pacheco [32] for extensions and variations of this convergence, and Hu, Le and Mytnik [18], Matzavinos, Roitershtein and Seol [28] and references therein in this context.

1.2. *Sketch of results.* To specify the "regularized" Sinai-type environment that we consider and the associated hydrodynamic limits of the zero-range process in this environment, we discuss first what might hold with respect to an "unregularized" Seignourel environment $\{u_i^N = 1/2 + r_i/\sqrt{\sigma^2 N}\}$ on a discrete torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$. Informally, in the zero-range process, a particle at site i, with k particles, will jump with rate $g(k)/k$ and move to a neighbor $j = i \pm 1$ with probability $p(i,j)$. We point out the case $g(k) \equiv k$ is when the random walks are all independent. When a random environment is imposed, $p(i,j) = u_i^N$ when $j = i+1$ and $= 1 - u_i^N$ when $j = i-1$. Since a random walker in such an environment experiences many traps and moves slowly, one expects the hydrodynamic limit to involve "drift" terms reflecting the environment.

In principle, since infinitesimally the rate of change of the number of particles $\eta_i(t)$ at site i at time t varies according to the generator action,

$$L\eta_i = \frac{1}{2}\{g(\eta_{i+1}) + g(\eta_{i-1}) - 2g(\eta_i)\} \\ + \{g(\eta_{i-1})(r_{i-1}/\sqrt{\sigma^2 N}) - g(\eta_{i+1})(r_{i+1}/\sqrt{\sigma^2 N})\},$$

and $r_i/\sqrt{\sigma^2 N} = [S_i - S_{i-1}]/\sqrt{N}$ where $S.$ is the partial sum of $\{r_i\}$, one might expect scaling space and time diffusively, that the hydrodynamic evolution of the mass satisfies the SPDE (1.1), where Φ is a homogenized version of the jump rate g. Of course, since W is not differentiable, the spatial white noise W' in (1.1) has to be interpreted as a distribution, and the equation is ill-posed. But, in a sense, an equation like this should represent the hydrodynamic equation, the term W' reflecting the random environment, even after some averaging in the scaling limit has been taken.

As a way to obtain the singular hydrodynamic equation (1.1), we consider a two-scale approach: We show here a hydrodynamic limit with respect to more regular random environments, those which average $\{u_i^N\}$ in small macroscopic blocks of width ε, in which the noise W' is replaced by an ε-smoothing W'_ε. Then later we connect to work on the $\varepsilon \downarrow 0$ limit of the associated hydrodynamic densities.

Let $\psi : [-1, 1] \mapsto \mathbb{R}$ be a continuously differentiable function such that $\int_{-1}^1 \psi(x)\,dx = 1$. Define $\{\bar{u}_i^N = 1/2 + r_i^N/\sqrt{N}\}$ where $r_i^N = (\sigma N\varepsilon)^{-1}\sum_{|j-i|\le\lfloor N\varepsilon\rfloor} r_j\psi((j-i)/(N\varepsilon))$ and $\varepsilon > 0$ is fixed. In this formula, $\lfloor a\rfloor$ stands for the integer part of $a \in \mathbb{R}$. Summing by parts,

$$r_i^N = \frac{1}{\sigma N\varepsilon}\sum_{j=-\lfloor N\varepsilon\rfloor}^{\lfloor N\varepsilon\rfloor-1} S_{i+j}\left[\psi\left(\frac{j}{N\varepsilon}\right) - \psi\left(\frac{j+1}{N\varepsilon}\right)\right] \\ + \frac{1}{\sigma N\varepsilon} S_{i+\lfloor N\varepsilon\rfloor}\psi\left(\frac{\lfloor N\varepsilon\rfloor}{N\varepsilon}\right) - \frac{1}{\sigma N\varepsilon} S_{i-1-\lfloor N\varepsilon\rfloor}\psi\left(-\frac{\lfloor N\varepsilon\rfloor}{N\varepsilon}\right).$$

In part (I) of Theorem 3.5, under a fixed random environment with the same distribution as $\{\bar{u}_i^N\}$, that is, in a "quenched" environment afforded by Skorokhod representation, we obtain the hydrodynamic equation

$$\partial_t\rho^{(\varepsilon)}(t,x) = \frac{1}{2}\Delta\Phi(\rho^{(\varepsilon)}(t,x)) - 2\nabla(W'_\varepsilon(x)\Phi(\rho^{(\varepsilon)}(t,x))), \tag{1.2}$$

where

$$W'_\varepsilon(v) := -\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} W(v+u)\psi'\left(\frac{u}{\varepsilon}\right)du + \frac{1}{\varepsilon}\{W(v+\varepsilon)\psi(1) - W(v-\varepsilon)\psi(-1)\}$$

is a form of ε-smoothing of W'. Observe that $W'_\varepsilon \in \bigcap_{0<\epsilon<1/2} C^{1/2-\epsilon}$ when either $\psi(1)$ or $\psi(-1)$ does not vanish. If $\psi(\pm 1) = 0$ and $\psi' \in C^\gamma$ for some $\gamma \ge 1/2$, then $W'_\varepsilon \in C^\gamma$. In contrast, if $\psi' \in C^\gamma$ for some $\gamma < 1/2$, then $W'_\varepsilon \in \bigcap_{0<\epsilon<1/2} C^{1/2-\epsilon}$. A natural case is $\psi(x) = (1/2)\mathbb{1}_{[-1,1]}(x)$ where $W'_\varepsilon(x) = (2\varepsilon)^{-1}\{W(x+\varepsilon) - W(x-\varepsilon)\} \in \bigcap_{0<\epsilon<1/2} C^{1/2-\epsilon}$.

In part (II) of Theorem 3.5, with respect to the random environments $\{\bar{u}_i^N\}$, we obtain that the annealed hydrodynamic limit of the space–time mass distributions is given by the distribution of $\rho^{(\varepsilon)}$ satisfying (1.2).

At this point, we see that the quenched versions of $\{\bar{u}_i^N\}$ fit into the general form $\{\hat{u}_i^N = 1/2 + \alpha_i^N/N\}$ where $\{\alpha_i^N : i \in \mathbb{T}_N\}_{N\ge 1}$ is a deterministic array such that piecewise continuous interpolations of $\{\alpha_i^N : i \in \mathbb{T}_N\}$ converge uniformly, as $N \uparrow \infty$, to a (necessarily) continuous function $\alpha(\cdot)$ on the unit torus $\mathbb{T}$: Indeed, we may take $\alpha_i^N = \sqrt{N} r_i^N$ and note that $\alpha_i^N \to \alpha(x) = W_\varepsilon'(x)$ when $i/N \to x$. In Theorem 3.3, we state that a hydrodynamic limit holds for this spatially inhomogeneous weakly asymmetric system with the hydrodynamic equation

$$\partial_t \rho = \frac{1}{2}\Delta\Phi(\rho) - 2\nabla\big(\alpha\Phi(\rho)\big). \tag{1.3}$$

Via Theorem 3.3, we deduce parts (I) and (II) of Theorem 3.5. We remark that Theorem 3.3 is a generalization to $\alpha \in C$ of the hydrodynamic limit mentioned in Benois, Kipnis and Landim [4] for weakly asymmetric zero-range processes with respect to smooth $\alpha \in C^2$ as a step toward proving lower bound large deviations for the symmetric process.

We comment, as the regularization vanishes in the SPDE (1.2), that one might expect some form of (1.1) should emerge. Indeed, as noted earlier, in the companion work Funaki et al. [15], the limit of $\rho^{(\varepsilon)}$ as $\varepsilon \downarrow 0$ is considered and shown to solve (1.1) in the "paracontrolled" sense.

In this respect, we discuss a computation, Theorem 3.6, later in the context of Section 1.5, on the form of the annealed mean hydrodynamic limit in unregularized random environments for independent particles.

1.3. *Idea of the proofs*. The proof of Theorem 3.3 broadly employs the "entropy" method of Guo–Papanicoloau–Varadhan (GPV) (cf. Kipnis and Landim [25]), by applying Itô's formula to the empirical measure of particle mass with respect to the zero-range evolution. However, the weak asymmetry α is not homogeneous and in particular is not translation-invariant, a key feature of the "GPV" technique to homogenize resulting nonlinear terms of the process in a replacement scheme. Instead of pursuing approximations with respect to the "GPV" technique for "gradient" systems detailed in [25], as is done for smooth weakly asymmetric perturbations in [4], we introduce "local" averages to piece together the "global" average in the hydrodynamic limit. Such a procedure works well in our setting when α is only assumed to be continuous, and may be of help later in more refined investigations.

To derive the homogenization, we make use of some averaging in time, afforded by spectral gap or mixing estimates of localized processes, to perform "local" 1 and 2-block replacements, leading to more "global" replacements (cf. Jara, Landim and Sethuraman [19, 23], and Fatkullin, Sethuraman and Xue [14] for other applications). Here, in our context, estimates on the random environment play a significant role in making the "local" replacements work in the context of the spatial inhomogeneity. We remark that in this work we do not assume the process rate g is an increasing function, as it is in [14, 19], that is, that the process is "attractive" (cf. Ch. II in [27]), a technical condition, which would allow the use of "basic" particle couplings.

The limiting equation (1.2), a case of equation (1.3), is not a standard one as the factor W_ε' may not be smooth. To complete the proof of the hydrodynamic limit in Theorem 3.3, we need to show uniqueness of weak solutions of (1.3). We can derive a certain continuum energy estimate by considering the microscopic particle system, namely that $\partial_x\Phi(\rho(t,x))$ can be defined in a weak sense. Uniqueness of weak solutions to (1.3) in an appropriate class, with respect to bounded, measurable $\alpha(\cdot)$, is then shown in Theorem 10.1 by a self-contained argument, perhaps of separate interest.

1.4. *Hydrodynamics in random environments*. To put our results in context, we remark that, with respect to systems of continuous-time random walks in a common random environment of different types, a "quenched" deterministic hydrodynamic limit for the bulk space time mass density of the walks has been shown in some models when there is enough averaging with respect to the random environment. For the exclusion process with random conductances, see Faggionato [10, 11], Jara and Landim [22] and Nagy [31], and with site disorder, see Quastel [34]. For independent random walks in a ballistic random environment, see Peterson [33]. For zero-range process, see Bahadoran et al. [3], Faggionato [12] and Gonçalves and Jara [17].

When the random environment does not allow sufficient averaging, a quenched hydrodynamic may involve random terms. In Jara, Landim and Teixera [20], the hydrodynamic limit is shown for a system of symmetric independent random walks in a common scaled "trap" environment on a torus, that one of the particles at a site i jumps at rate ξ_i^N to a nearest neighbor, where ξ_i^N is random and heavy-tailed; here, the limit equation involves a heavy-tailed subordinator arising from the random environment. In Faggionato, Jara and Landim [13], the hydrodynamic limit is shown for the symmetric simple exclusion processes on a torus, with heavy-tailed random conductances on the bonds, which also involves a heavy-tailed subordinator coming from the random environment. In Jara and Peterson [21], the hydrodynamic limit is shown for independent random walks in a random environment on $\mathbb{Z}$, where a single particle is transient but not ballistic, which incorporates a random term arising from the environment.

1.5. *Discussion and open problems.* Finally, given the development with respect to the regularized environments, we comment on two natural directions for future work with respect to the unregularized Seignourel environments. First, instead of using the two-scale approach, one might expect directly, say in a diagonal argument as both $N \uparrow \infty$ and $\varepsilon \downarrow 0$, that the equation (1.1), interpreted now in the paracontrolled sense, would govern the annealed hydrodynamic limit of the zero-range process with respect to the unregularized Seignourel environment $\{u_i^N = 1/2 + r_i/\sqrt{\sigma^2 N}\}$ as $N \uparrow \infty$.

Indeed, one can make some computations, which give a form, at least with respect to the annealed mean hydrodynamics, when the zero-range process consists of independent motions on say $\mathbb{Z}$, that is, when $g(k) \equiv k$ and so $\Phi(\rho) \equiv \rho$, and the initial distribution of particles on $\mathbb{Z}$ is a product measure with Poisson marginals with parameters $\{\beta(i/N)\}$ where $\beta : \mathbb{R} \to [0, \infty)$ is a bounded, continuous function. In part (I) of Theorem 3.6, we state that the law of the configuration $\{\eta(i) : i \in \mathbb{Z}\}$ at time $N^2 t$ is a product of Poisson marginals with quenched mean parameters $\{\sum_{x\in\mathbb{Z}} \beta(x/N) P_N^{\omega}(R_{N^2t}^{N,x} = i) : i \in \mathbb{Z}\}$ where $R_{N^2t}^{N,i}$ is the position at time $N^2 t$ of a single particle, starting at i, moving in a fixed Seignourel environment $\{u_i^N(\omega) : i \in \mathbb{Z}\}$. In part (II), we show that the annealed mean with respect to $\{u_i^N\}$ of the hydrodynamic limit mass density equals $\mathscr{E}[\beta(u - X_t)]$ where X_t is the 4-Brox diffusion starting from the origin and say $\mathscr{E}$ is the associated annealed expectation.

Moreover, consider the following formal argument: By the proof of Theorem 3.6, one can also compute the annealed hydrodynamic limit mean density with respect to the regularized environments $\{\bar{u}_i^N\}$ on $\mathbb{T}_N$ with $\varepsilon > 0$. Note that the quenched diffusively scaled path of a single particle $R_{N^2t}^{N,\varepsilon}/N$, starting at 0, in the regularized environment converges weakly to a diffusion X_t^{ε} with generator $(1/2)\Delta + 2W_{\varepsilon}'\nabla$ (cf. Chapter 7 of Ethier and Kurtz [7]). Then the annealed hydrodynamic limit mean density equals $\mathscr{E}[\beta(u - X_t^{\varepsilon})]$, which by our Theorem 3.5, equals the annealed mean of $\rho^{(\varepsilon)}(t, u)$. Since we expect, in the annealed sense, that $X_{\cdot}^{\varepsilon} \Rightarrow X_{\cdot}$ as $\varepsilon \downarrow 0$, one can link formally the annealed mean limit $\mathscr{E}[\beta(u - X_t)]$ to the $\varepsilon \downarrow 0$ limit of $\rho^{(\varepsilon)}$.

For the second direction, one might follow a tagged particle in the interacting system to understand relations with Brox diffusion. Let $x^N(t)$ be the position at time t of a tagged particle initially at the origin. Then, from Itô's formula,

$$x^N(t) = \frac{2}{\sqrt{N}} \int_0^t \bar{u}_{x^N(s)}^N \frac{g(\eta_s(x^N(s)))}{\eta_s(x^N(s))} \, ds + M^N(t).$$

One can compute that the quadratic variation of $M^N(t)$ equals $\int_0^t \frac{g(\eta_s(x^N(s)))}{\eta_s(x^N(s))}\,ds$. Formally, given the hydrodynamic limit (1.2) and following the scheme in [23], one might expect the quenched limit points v_t of $v_t^N = x^N_{N^2 t}/N$ to satisfy the equation

$$v_t = 2\int_0^t W'_\varepsilon(v_s)\frac{\Phi(\rho^{(\varepsilon)}(s,v_s))}{\rho^{(\varepsilon)}(s,v_s)}\,ds + M(t),$$

where M is a time-changed Brownian motion $B(T^{-1}(t))$ and $T(t) = \int_0^t \frac{\Phi(\rho^{(\epsilon)}(s,v_s))}{\rho^{(\epsilon)}(s,v_s)}\,ds$. Again, considering independent motions, when $g(k)\equiv k$ and so $\Phi(u)\equiv u$ and $M = B$, one would recover the diffusion discussed earlier with the drift $2W'_\varepsilon$. Of course, it would be of interest, in this direction also, to understand the limits as $\varepsilon \downarrow 0$.

Organization of the paper. We define the microscopic model precisely in Section 2, before stating results, Theorems 3.3, 3.5 and Theorem 3.6, in Section 3. After preliminaries on basic martingales in Section 4, we give the main proof outline in Section 5. In Sections 6 and 7, hydrodynamic 1 and 2-block replacement estimates are proved. Tightness and useful properties of limit measures are given in Sections 8 and 9. Finally, uniqueness of weak solutions to the limit equations in a certain class, Theorem 10.1, is shown in Section 10.

2. Model description. We first introduce the random environments considered, before specifying the zero-range process in this random environment. Let $\{r_x\}_{x\in\mathbb{N}}$ be a sequence of i.i.d. random variables with mean 0 and variance $0 < \sigma^2 < \infty$.

Let $s_0 = 0$ and for $n \geq 1$, $s_n = \sum_{k=1}^n r_k$. For $0 \leq u \leq 1$, let

$$X_u^N = \frac{1}{\sigma\sqrt{N}} s_{\lfloor Nu\rfloor} + \frac{Nu - \lfloor Nu\rfloor}{\sigma\sqrt{N}} r_{\lfloor Nu\rfloor + 1}, \tag{2.1}$$

where recall $\lfloor a\rfloor$, $a\in\mathbb{R}$, stands for the integer part of a. It is standard that the random functions $\{X_u^N : 0 \leq u \leq 1\}$ converge in distribution as $N\uparrow\infty$ to the Brownian motion on $[0,1]$. By Skorokhod's representation theorem, we may find a probability space $(\Omega, \mathscr{P})$ and $\{W_u^N : 0\leq u\leq 1\}$, $N\in\mathbb{N}$, mappings from Ω to $C[0,1]$, such that, for all $N\in\mathbb{N}$,

$$\{X_u^N : 0\leq u\leq 1\} = \{W_u^N : 0\leq u\leq 1\}$$

in distribution and, moreover, $\{W_u^N : 0\leq u\leq 1\}$ converges uniformly almost surely to the standard Brownian motion $\{W_u, 0\leq u\leq 1\}$.

Quenched formulation. We will now fix an $\omega\in\Omega$ such that $\{W_u^N(\omega) : 0\leq u\leq 1\}$ converges (uniformly) to a Brownian path $\{W_u(\omega) : 0\leq u\leq 1\}$.

We comment that of course "quenched" with respect to the original paths $X_\cdot^N$ would not make sense in diffusive scale given the law of the iterated logarithm for the partial sums s_n.

Recall the discrete torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ for $N\in\mathbb{N}$. Throughout this article, we will identify $\mathbb{T}_N$ with $\{1,2,\dots,N\}$ and also identify the unit torus $\mathbb{T}$ with $(0,1]$.

It will be convenient to extend W_t^N as well as W_t to $t\in[-1,2]$: with $\tilde{W}_u$ representing either W_u^N or W_u, define

$$\tilde{W}_u = \begin{cases} \tilde{W}_{u+1} - \tilde{W}_1, & u\in[-1,0), \\ \tilde{W}_{u-1} + \tilde{W}_1, & u\in(1,2]. \end{cases}$$

Here, in the time intervals $[-1,0)$ and $(1,2]$, the trajectory $\tilde{W}$ starts respectively from $-\tilde{W}_1$ and $\tilde{W}_1$, then displacing according to the trajectory $\tilde{W}$ in $[0,1]$. Importantly, the increments of $\tilde{W}$ are periodic in $\mathbb{T}$. Throughout, to simplify notation, we will drop the tilde in the notation for $\tilde{W}$.

Let $\varepsilon > 0$ be a parameter, fixed throughout the paper. Let also $\psi : [-1, 1] \mapsto \mathbb{R}$ be a continuously differentiable function with $\int_{-1}^{1} \psi(x)\, dx = 1$. For each $N \in \mathbb{N}$ and $k \in \mathbb{T}_N$, consider an ε-regularization of local environments such that

$$q_k^N \stackrel{d}{=} \frac{1}{\sigma N} \sum_{|j-k| \le \lfloor N\varepsilon \rfloor} r_j \left[\frac{1}{\varepsilon} \psi\left(\frac{j-k}{N\varepsilon} \right) \right],$$

where

$$\begin{aligned} q_k^N := {} & \frac{1}{\sqrt{N}\varepsilon} \sum_{j=k-\lfloor N\varepsilon \rfloor}^{k+\lfloor N\varepsilon \rfloor - 1} W_j^N \left[\psi\left(\frac{j-k}{N\varepsilon} \right) - \psi\left(\frac{j-k+1}{N\varepsilon} \right) \right] \\ & + \frac{1}{\sqrt{N}\varepsilon} W_{k+N\varepsilon}^N \psi\left(\frac{\lfloor N\varepsilon \rfloor}{N\varepsilon} \right) - \frac{1}{\sqrt{N}\varepsilon} W_{k-1-N\varepsilon}^N \psi\left(-\frac{\lfloor N\varepsilon \rfloor}{N\varepsilon} \right). \end{aligned} \tag{2.2}$$

In particular, when $k/N \to x \in \mathbb{T}$ as $N \uparrow \infty$, we have $\sqrt{N} q_k^N$ converges to

$$\begin{aligned} & -\frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} W(u) \psi'\left(\frac{u-x}{\varepsilon} \right) du + \frac{1}{\varepsilon} \{ W(x+\varepsilon)\psi(1) - W(x-\varepsilon)\psi(-1) \} \\ & := W_\varepsilon'(x). \end{aligned} \tag{2.3}$$

As $W_\cdot^N$ converges uniformly to $W_\cdot$, by the continuity of W, and the properties of ψ, in view of (2.2), there exists a constant $C = C(\omega) < \infty$ such that

$$\limsup_{N\to\infty} \max_{1 \le k \le N} \{ \sqrt{N} |q_k^N| \} \le C. \tag{2.4}$$

As remarked in the Introduction, W_ε' is a smoothing of W'. In particular, $W_\varepsilon' \in C^\gamma$ when $\psi' \in C^\gamma$ for $\gamma \ge 1/2$ and $\psi(\pm 1) = 0$. When one of $\psi(1)$ or $\psi(-1)$ does not vanish, then $W_\varepsilon' \in \bigcap_{0<\epsilon<1/2} C^{1/2-\epsilon}$. Also, when $\psi' \in C^\gamma$ for $\gamma < 1/2$, then $W_\varepsilon' \in \bigcap_{0<\epsilon<1/2} C^{1/2-\epsilon}$. A natural case is when $\psi(x) = (1/2)\mathbb{1}_{[-1,1]}(x)$ for which $W_\varepsilon'(x) = (2\varepsilon)^{-1}[W(x+\varepsilon) - W(x-\varepsilon)]$.

Zero-range process in an abstract inhomogeneous environment. We now introduce the zero-range process in an abstract deterministic environment $\{\alpha_k^N : k \in \mathbb{T}_N\}$ where the linear interpolations for $u \in \mathbb{T}$,

$$Y_u^N = \alpha_{\lfloor Nu \rfloor}^N + (Nu - \lfloor Nu \rfloor) \alpha_{\lfloor Nu \rfloor + 1}^N,$$

converge uniformly to $\alpha(u)$ where $\alpha(\cdot)$ is a continuous function on $\mathbb{T}$. In the above quenched setting, an example is $\alpha_k^N \equiv \sqrt{N} q_k^N$ and $\alpha(u) \equiv W_\varepsilon'(u)$.

Set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and let $\Sigma_N = \mathbb{N}_0^{\mathbb{T}_N}$ be the configuration space. Elements of Σ_N are represented by the Greek letter ξ. Thus, $\xi(k)$, $k \in \mathbb{T}_N$, stands for the number of particles at site k for the configuration ξ.

Fix a function $g : \mathbb{N}_0 \to \mathbb{R}_+$. Denote by $(\xi_t : t \ge 0)$ the continuous-time Markov chain whose evolution can be informally described as follows. Since $\max_{1\le k\le N} |\alpha_k^N|$ is uniformly bounded in N, take N sufficiently large so that $|\alpha_k^N|/N < 1/2$ for all $k \in \mathbb{T}_N$. At rate $g(\xi(k))[(1/2) \pm (\alpha_k^N/N)]$, a particle jumps from k to $k \pm 1$.

More precisely, the process $\{\xi_t : t \ge 0\}$ is the Markov process, well defined on the countable space Σ_N, with generator L given by

$$\begin{aligned} Lf(\xi) = \sum_{k=1}^{N} \bigg\{ & g(\xi(k)) \left(\frac{1}{2} + \frac{\alpha_k^N}{N} \right) (f(\xi^{k,k+1}) - f(\xi)) \\ & + g(\xi(k)) \left(\frac{1}{2} - \frac{\alpha_k^N}{N} \right) (f(\xi^{k,k-1}) - f(\xi)) \bigg\}. \end{aligned} \tag{2.5}$$

Here, $\xi^{j,k}$ is the configuration obtained from ξ by moving a particle from j to k, that is,

$$\xi^{j,k}(\ell) = \begin{cases} \xi(j) - 1, & \ell = j, \\ \xi(k) + 1, & \ell = k, \\ \xi(\ell), & \ell \neq j, k. \end{cases}$$

To avoid degeneracies, we suppose that $g(0) = 0$ and $g(k) > 0$ for $k \geq 1$. With $g(0)! := 1$ and $g(n)! := \prod_{j=1}^{n} g(j)$ for $n \geq 1$, we define the partition function $Z(\phi)$:

$$Z(\phi) := \sum_{n=0}^{\infty} \frac{\phi^n}{g(n)!}.$$

We assume, further, that $g(\cdot)$ satisfies:

1. $\sup_{k \in \mathbb{N}} |g(k+1) - g(k)| \leq g^* < \infty$;
2. $Z(\phi) < \infty$ for all $0 \leq \phi < \infty$.

These properties are standard and used in several places to make estimates. In particular, the condition $Z(\phi) < \infty$ for all $\phi \geq 0$, usually referred as "FEM" (cf. page 69, [25]), is used for particle truncation in the 1 and 2-blocks estimates presented in Lemmas 6.3 and 7.2.

We note that the case $g(k) \equiv k$ is when all the particles in the system are independent, and construction of the process even on $\mathbb{Z}$ is immediate.

2.1. *Invariant measure.* The building block for the invariant measures are $\{\mathcal{P}_\phi\}$, a family of Poisson-like distributions indexed by "fugacities" $\phi \geq 0$. For each ϕ, $\mathcal{P}_\phi$ is defined by

$$\mathcal{P}_\phi(n) = \frac{1}{Z(\phi)} \frac{\phi^n}{g(n)!} \quad \text{for } n \geq 0.$$

Let $R(\phi) = E_{\mathcal{P}_\phi}[X]$, where $X(n) = n$, be the mean of the distribution $\mathcal{P}_\phi$. A direct computation yields that $R'(\phi) > 0$, $R(0) = 0$, $\lim_{\phi \to \infty} R(\phi) = \infty$. Since R is strictly increasing, it has an inverse, denoted by $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, and we may parametrize the family of distributions $\mathcal{P}_\phi$ by its mean. For $\rho \geq 0$, let $\mathcal{Q}_\rho = \mathcal{P}_{\Phi(\rho)}$, so that $E_{\mathcal{Q}_\rho}[X] = E_{\mathcal{P}_{\Phi(\rho)}}[X] = R(\Phi(\rho)) = \rho$.

A straightforward computation yields that $E_{\mathcal{P}_\phi}[g(X)] = \phi$ for $\phi \geq 0$. Thus,

$$\Phi(\rho) = E_{\mathcal{P}_{\Phi(\rho)}}\big[g(X)\big] = E_{\mathcal{Q}_\rho}\big[g(X)\big], \quad \rho \geq 0. \tag{2.6}$$

As $g(k) \leq g^* k$, we have that $\Phi(\rho) \leq g^* \rho$. A simple computation yields that $\Phi'(\rho) = \Phi(\rho)/\sigma^2(\rho)$ where $\sigma^2(\rho)$ is the variance of X under $\mathcal{Q}_\rho$. Under our assumptions on g, in fact, it holds that $0 \leq \Phi'(\rho) \leq g^*$ for all $\rho \geq 0$ (cf. page 33, [25]). In particular, $\Phi \in C^1[0, \infty)$ is an increasing function with a uniformly bounded derivative.

We note, in the case $g(k) \equiv k$, that $\Phi(\rho) \equiv \rho$ and $\mathcal{P}_\phi$ is a Poisson measure with mean ϕ.

Fix a vector $(\phi_{k,N} : k \in \mathbb{T}_N)$ of nonnegative real numbers. Denote by $\mathscr{R}_N$ the product measure on $\mathbb{N}_0^{\mathbb{T}_N}$ whose marginals are given by

$$\mathscr{R}_N\big(\xi(k) = n\big) = \mathcal{P}_{\phi_{k,N}}(n) \quad \text{for } k \in \mathbb{T}_N, n \geq 0. \tag{2.7}$$

It is straightforward (cf. [1, 9]) to check that $\mathscr{R}_N$ is invariant with respect to the generator L in (2.5) as long as the fugacities $\{\phi_{k,N}\}_{k \in \mathbb{T}_N}$ satisfy

$$\left(\frac{1}{2} + \frac{\alpha_{k-1}^N}{N}\right)\phi_{k-1,N} + \left(\frac{1}{2} - \frac{\alpha_{k+1}^N}{N}\right)\phi_{k+1,N} = \phi_{k,N}, \quad k = 1, 2, \ldots, N. \tag{2.8}$$

Notice that $\{c\phi_{k,N}\}_{k \in \mathbb{T}_N}$, $c \in \mathbb{R}$ is a solution of (2.8) if $\{\phi_{k,N}\}_{k \in \mathbb{T}_N}$ is a solution. In particular, any solution gives rise to a one-parameter family of solutions. Although standard, to introduce useful notation, we show that (2.8) admits a solution.

LEMMA 2.1. *The equation* (2.8) *admits a solution, unique up to a multiplicative constant. Moreover, the solution is either strictly positive or strictly negative or identically equal to* 0.

PROOF. Let $\mathfrak{r}_k = (1/2) + (\alpha_k^N/N)$, $\mathfrak{l}_k = (1/2) - (\alpha_k^N/N)$, $k \in \mathbb{T}_N$. With this notation, equation (2.8) becomes

$$\mathfrak{r}_{k-1}\phi_{k-1,N} + \mathfrak{l}_{k+1}\phi_{k+1,N} = \phi_{k,N}.$$

Since $\mathfrak{r}_k + \mathfrak{l}_k = 1$, we have that

$$\mathfrak{r}_{k-1}\phi_{k-1,N} - \mathfrak{l}_k\phi_{k,N} = \mathfrak{r}_k\phi_{k,N} - \mathfrak{l}_{k+1}\phi_{k+1,N}, \quad k \in \mathbb{T}_N.$$

Denote by γ be the common value of $\mathfrak{r}_k\phi_{k,N} - \mathfrak{l}_{k+1}\phi_{k+1,N}$, to get the recursive equation

$$\phi_{k+1,N} = \frac{\mathfrak{r}_k\phi_{k,N} - \gamma}{\mathfrak{l}_{k+1}}. \tag{2.9}$$

With the convention that $\mathfrak{r}_{N+1} = \mathfrak{r}_1$ and $\mathfrak{l}_{N+1} = \mathfrak{l}_1$, let

$$\mathfrak{R}_{j,k} = \prod_{i=j}^{k} \mathfrak{r}_i, \qquad \mathfrak{L}_{j,k} = \prod_{i=j}^{k} \mathfrak{l}_i, \quad 1 \le j \le k \le N+1.$$

We extend the definition to indices $j > k$ by setting $\mathfrak{R}_{j,k} = \mathfrak{L}_{j,k} = 1$ if $j > k$. Solving the recursive equation yields that

$$\phi_{k,N} = \frac{\mathfrak{R}_{1,k-1}}{\mathfrak{L}_{2,k}}\phi_{1,N} - \frac{\mathfrak{S}_k}{\mathfrak{L}_{2,k}}\gamma, \quad 2 \le k \le N+1.$$

In this formula,

$$\mathfrak{S}_k = \sum_{j=2}^{k} \mathfrak{L}_{2,j-1}\mathfrak{R}_{j,k-1},$$

with the convention, adopted above, that $\mathfrak{L}_{2,1} = \mathfrak{R}_{k,k-1} = 1$.

Since $\phi_{N+1,N} = \phi_{1,N}$, we have that

$$\gamma = \frac{\mathfrak{R}_{1,N} - \mathfrak{L}_{2,N+1}}{\mathfrak{S}_{N+1}}\phi_{1,N}. \tag{2.10}$$

Reporting this value in the equation for $\phi_{k,N}$ yields that

$$\phi_{k,N} = \left\{\frac{\mathfrak{R}_{1,k-1}}{\mathfrak{L}_{2,k}} - \frac{\mathfrak{S}_k}{\mathfrak{S}_{N+1}}\frac{\mathfrak{R}_{1,N} - \mathfrak{L}_{2,N+1}}{\mathfrak{L}_{2,k}}\right\}\phi_{1,N}, \quad 2 \le k \le N. \tag{2.11}$$

Therefore, for each $\phi_{1,N} \in \mathbb{R}$, the solution of the difference equation (2.8) is given by (2.11). This proves existence and uniqueness up to a multiplicative constant. Moreover, it is not difficult to check that $\mathfrak{S}_k\mathfrak{R}_{1,N} \le \mathfrak{S}_{N+1}\mathfrak{R}_{1,k-1}$. Therefore, as each variable $\mathfrak{r}_i$, $\mathfrak{l}_j$ is strictly positive for sufficiently large N, the solution is strictly positive if $\phi_{1,N} > 0$. □

Let $\phi_{\max,N} = \max_{1\le k\le N}\{\phi_{k,N}\}$ and $\phi_{\min,N} = \min_{1\le k\le N}\{\phi_{k,N}\}$.

LEMMA 2.2. *Let $\phi_{k,N}$ be a solution of* (2.8). *Then there exist constants $C_1, C_2 < \infty$ such that for all $N \in \mathbb{N}$,*

$$1 \le \frac{\phi_{\max,N}}{\phi_{\min,N}} \le C_1 \quad \textit{and} \quad \max_{1\le k\le N} |\phi_{k,N} - \phi_{k+1,N}| \le \frac{C_2}{N}\phi_{\max,N}.$$

PROOF. By the definition of α_k^N, there is a finite constant C such that $\max_{1\le k\le N}\alpha_k^N \le C$. Therefore, for each $j,k\in\mathbb{T}_N$, using the notation of Lemma 2.1 $|(\mathfrak{r}_j/\mathfrak{l}_k)-1|$ and $|(\mathfrak{l}_k/\mathfrak{r}_j)-1|$ are bounded from above by

$$2\frac{\max_\ell|\alpha_\ell^N|}{N}\frac{1}{\frac{1}{2}-\frac{\max_\ell|\alpha_\ell^N|}{N}}\le\frac{8C}{N}\quad\text{for } N\ge 4C. \tag{2.12}$$

Without loss of generality, we may assume that $\phi_{1,N}=\phi_{\min,N}$, $\phi_{m,N}=\phi_{\max,N}$. Hence, by (2.11) and since all terms are positive and $\mathfrak{L}_{2,N+1}=\mathfrak{L}_{2,m}\mathfrak{L}_{m+1,N+1}$, $\phi_{\max,N}/\phi_{\min,N}$ is bounded by

$$\frac{\mathfrak{R}_{1,m-1}}{\mathfrak{L}_{2,m}}+\frac{\mathfrak{S}_m\mathfrak{L}_{m+1,N+1}}{\mathfrak{S}_{N+1}}.$$

By (2.12), the first term is bounded by $[1+(8C/N)]^N$. We further show that the second term is also bounded by $[1+(8C/N)]^N$ by rewriting $\mathfrak{L}_{m+1,N+1}$ as $\mathfrak{R}_{m,N}[\mathfrak{L}_{m+1,N+1}/\mathfrak{R}_{m,N}]$ and using $\mathfrak{S}_m\mathfrak{R}_{m,N}\le\mathfrak{S}_{N+1}$. This proves the first assertion of the lemma.

We turn to the second assertion. By (2.9),

$$\phi_{m+1,N}-\phi_{m,N}=\left(\frac{\mathfrak{r}_m}{\mathfrak{l}_{m+1}}-1\right)\phi_{m,N}-\frac{\gamma}{\mathfrak{l}_{m+1}}.$$

By (2.12), the absolute value of the first term on the right-hand side is bounded by $(8C/N)\max_k\phi_{k,N}$. By (2.10), and since $\mathfrak{L}_{2,N+1}=\mathfrak{L}_{1,N}$, the second one is equal to

$$\frac{\phi_{1,N}}{\mathfrak{l}_{m+1}}\frac{1}{\mathfrak{S}_{N+1}}\sum_{j=0}^{N-1}\{\mathfrak{L}_{1,j}\mathfrak{R}_{j+1,N}-\mathfrak{L}_{1,j+1}\mathfrak{R}_{j+2,N}\},$$

where we used the convention that $\mathfrak{L}_{i,j}=1$ and $\mathfrak{R}_{i,j}=1$ if $i>j$. Changing variables, this expression becomes

$$\frac{\mathfrak{l}_1}{\mathfrak{l}_{m+1}}\frac{\phi_{1,N}}{\mathfrak{S}_{N+1}}\sum_{j=2}^{N+1}\mathfrak{L}_{2,j-1}\mathfrak{R}_{j,N}\left(\frac{\mathfrak{r}_{j-1}}{\mathfrak{l}_{j-1}}-1\right),$$

There is $C_0>0$ such that $\mathfrak{l}_1/\mathfrak{l}_{m+1}\le C_0$. Also by (2.12), the absolute value of the expression inside the parenthesis is bounded by $8C/N$, uniformly over j. The remaining sum is bounded by $\mathfrak{S}_{N+1}$. This expression is, therefore, bounded by $C_0(8C/N)\phi_{1,N}$. To complete the proof of the second assertion of the lemma, it remains to recollect all previous estimates. □

3. Results.

We first specify the initial measures for the zero-range processes. These include the usual "local equilibrium" measures as well as others. We then state the main results of this work.

In the following, with respect to a given probability measure μ, we denote by E_μ and Var_μ its expectation and variance.

3.1. *Initial measures.* We consider an initial macroscopic density profile $\rho_0(\cdot)\in L^1(\mathbb{T})$, and an initial microscopic measure satisfying the following condition. Denote by $\mathscr{R}_N$ (cf. (2.7)) the invariant measure chosen so that $\phi_{\max,N}=\max_{k\in\mathbb{T}_N}\{\phi_{k,N}\}=1$. Such a choice (which could be another constant) makes definite the normalization of $\phi_{k,N}$ to reduce notation in later estimates.

CONDITION 3.1. *Let $\{\mu^N\}_{N\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{N}_0^{\mathbb{T}_N}$ such that*

(a) $\{\mu^N\}_{N\in\mathbb{N}}$ *is associated with profile* ρ_0 *in the sense that for any* $G \in C(\mathbb{T})$ *and* $\delta > 0$,

$$\lim_{N\to\infty} \mu^N\left[\left|\frac{1}{N}\sum_{k=1}^N G\left(\frac{k}{N}\right)\xi(k) - \int_{\mathbb{T}} G(x)\rho_0(x)\,dx\right| > \delta\right] = 0.$$

(b) *The relative entropy of* μ^N *with respect to* $\mathscr{R}_N$ *is of order* N. *That is, there exists a finite constant* C_0 *such that* $H(\mu^N|\mathscr{R}_N) := \int f_0 \ln f_0\, d\mathscr{R}_N \le C_0 N$ *for all* $N \ge 1$, *where* $f_0 = d\mu^N/d\mathscr{R}_N$.

A useful consequence of the relative entropy bound in part (b) of Condition 3.1 and the bounds on the fugacities of the invariant measure $\mathscr{R}_N$ in Lemma 2.2 is that the expected number of particles under μ^N is of order N. Indeed, by the entropy inequality (cf. [25]),

$$E_{\mu^N}\left[\sum_{k\in\mathbb{T}_N} \xi(k)\right] \le \frac{1}{\gamma} H\big(\mu^N|\mathscr{R}_N\big) + \frac{1}{\gamma}\log E_{\mathscr{R}_N}\big[e^{\gamma\sum_{x\in\mathbb{T}_N}\xi(k)}\big] \tag{3.1}$$

for all $\gamma > 0$. By Condition 3.1.b and by definition of $\mathscr{R}_N$, this expression is bounded by

$$\frac{C_0}{\gamma}N + \frac{1}{\gamma}\sum_{k=1}^N \log\frac{Z(e^\gamma\phi_{k,N})}{Z(\phi_{k,N})}.$$

Since $\max_k \phi_{k,N} = 1$, $\min_k \phi_{k,N} \ge c_0 > 0$ and Z is an increasing function defined on $\mathbb{R}_+$, choosing, say, $\gamma = 1$, yields that the previous expression is bounded by $C_0 N$ for some finite constant C_0.

Condition 3.1 is satisfied, for example, by "local equilibrium" measures $\{\mu^N_{\text{le}}\}_{N\in\mathbb{N}}$ associated to macroscopic profiles ρ_0 in $L^\infty(\mathbb{T})$. For each $N\in\mathbb{N}$, let μ^N_{le} be the product measure on $\mathbb{N}_0^{\mathbb{T}_N}$ with marginals given by

$$\mu^N_{\text{le}}\big(\xi(k) = n\big) = \mathcal{P}_{\tilde\phi_{k,N}}(n) \quad \text{for } k\in\mathbb{T}_N, n\ge 0,$$

where the parameters $\{\tilde\phi_{k,N} : 1 \le k \le N\}$ are such that $E_{\mu^N_{\text{le}}}[\xi(k)] = \rho_{k,N}$ for

$$\rho_{k,N} = N\int_{(k-1)/N}^{k/N} \rho_0(x)\,dx.$$

Indeed, Condition 3.1(a), for the product measure μ^N_{le}, holds straightforwardly as G is uniformly continuous by Chebychev and triangle inequalities. The next lemma asserts that Condition 3.1(b) holds.

LEMMA 3.2. *There exists* $C_0 > 0$ *such that* $H(\mu^N_{\text{le}}|\mathscr{R}_N) \le C_0 N$ *for all* $N\in\mathbb{N}$.

PROOF. Write

$$\begin{aligned} H\big(\mu^N_{\text{le}}|\mathscr{R}_N\big) &= \sum_{k=1}^N E_{\mu^N_{\text{le}}}\left\{\ln\left[\left(\frac{\tilde\phi_{k,N}}{\phi_{k,N}}\right)^{\xi(k)}\right] + \ln\frac{Z(\phi_{k,N})}{Z(\tilde\phi_{k,N})}\right\} \\ &= \sum_{k=1}^N \rho_{k,N}\ln\frac{\tilde\phi_{k,N}}{\phi_{k,N}} + \sum_{k=1}^N \ln\frac{Z(\phi_{k,N})}{Z(\tilde\phi_{k,N})}. \end{aligned}$$

Then the lemma follows as (a) $\tilde\phi_{k,N} \le g^*\rho_{k,N}$ is uniformly bounded by the fugacity bounds after (2.6) as $\|\rho_0\|_\infty < \infty$, (b) $Z(0) = 1$ and (c) $0 < c \le \phi_{k,N} \le 1$ for all $1 \le k \le N$ by Lemma 2.2. □

3.2. *Main results*. For each N, we will observe the evolution of the zero-range process speeded up by N^2, and consider in the sequel the process $\eta_t := \xi_{N^2 t}$, generated by $N^2 L$ (cf. (2.5)), for times $0 \le t \le T$, where $T > 0$ refers to a fixed time horizon. We will access the space–time structure of the process through the scaled mass empirical measure:

$$\pi_t^N(dx) := \frac{1}{N}\sum_{k=1}^{N} \eta_t(k)\delta_{k/N}(dx),$$

where δ_x, $x \in \mathbb{T}$ stands for the Dirac mass at x.

Let $\mathcal{M}$ be the space of finite nonnegative measures on $\mathbb{T}$, and observe that $\pi_t^N \in \mathcal{M}$. We will place a metric $d(\cdot,\cdot)$ on $\mathcal{M}$, which realizes the dual topology of $C(\mathbb{T})$ (see [25], page 49, for a definitive choice). Here, the trajectories $\{\pi_t^N : 0 \le t \le T\}$ are elements of the Skorokhod space $D([0,T],\mathcal{M})$, endowed with the associated Skorokhod topology.

In the following, for $G \in C(\mathbb{T})$ and $\pi \in \mathcal{M}$, denote $\langle G, \pi\rangle = \int_0^1 G(u)\pi(du)$. Also, the process measure and associated expectation governing $\eta_\cdot$ starting from μ will be denoted by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$. When the process starts from $\{\mu^N\}_{N\in\mathbb{N}}$, in the class satisfying Condition 3.1, we will denote by $\mathbb{P}_N := \mathbb{P}_{\mu^N}$ and $\mathbb{E}_N := \mathbb{E}_{\mu^N}$, the associated process measure and expectation.

We first state a hydrodynamic limit (HDL) with respect to the abstract environments $\{\alpha_k^N\}$. The proof of Theorem 3.3 is outlined in Section 5.

THEOREM 3.3. *For initial distributions μ^N satisfying Condition 3.1, consider the speeded up process η_t as above with generator N^2L. Then, for any $t \ge 0$, test function $G \in C^\infty(\mathbb{T})$, and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}_N\left[\left|\langle G, \pi_t^N\rangle - \int_{\mathbb{T}} G(x)\rho(t,x)\,dx\right| > \delta\right] = 0,$$

where $\rho(t,x)$ is the unique weak solution of

$$\begin{cases} \partial_t\rho(t,x) = \frac{1}{2}\partial_{xx}\Phi(\rho(t,x)) - 2\partial_x(\alpha(x)\Phi(\rho(t,x))), \\ \rho(0,x) = \rho_0(x), \end{cases} \tag{3.2}$$

in the class of "good" weak solutions given in Definition 3.4 below with $\Psi = \Phi/2$ and $K = -2\alpha$.

Consider the PDE

$$\begin{cases} \partial_t\rho(t,x) = \partial_{xx}\Psi(\rho(t,x)) + \partial_x(K(x)\Psi(\rho(t,x))), \quad x \in \mathbb{T}, t \ge 0 \\ \rho(0,x) = \rho_0(x), \end{cases} \tag{3.3}$$

where $\Psi(\cdot) \in C^1[0,\infty)$ satisfies $0 \le \Psi'(\cdot) \le C_\Psi$, the function $K(\cdot)$ is measurable and in $L^\infty(\mathbb{T})$, and $\rho_0 \in L^1(\mathbb{T})$ is nonnegative.

In the context of Theorem 3.3, Φ is certainly continuously differentiable with Φ' positive and bounded above (cf. below (2.6)), and α is a continuous function.

DEFINITION 3.4. We say $\rho(t,x) = \rho(t,x;\Psi,K) : [0,T] \times \mathbb{T} \mapsto [0,\infty)$ is a good weak solution to (3.3) if:

1. $\int_{\mathbb{T}} \rho(t,x)\,dx = \int_{\mathbb{T}} \rho_0(x)\,dx$ for all $t \in [0,T]$.
2. $\rho(t,\cdot)$ is weakly continuous, that is, for all $G \in C(\mathbb{T})$, $\int_{\mathbb{T}} G(x)\rho(t,x)\,dx$ is a continuous function in t.

3. There exists an $L^1([0,T]\times\mathbb{T})$ function denoted by $\partial_x\Psi(\rho(s,x))$ such that, for all $G(s,x)\in C^{0,1}([0,T]\times\mathbb{T})$, it holds

$$\int_0^T\int_{\mathbb{T}}\partial_x G(s,x)\Psi\big(\rho(s,x)\big)\,dx\,ds=-\int_0^T\int_{\mathbb{T}}G(s,x)\partial_x\Psi\big(\rho(s,x)\big)\,dx\,ds. \tag{3.4}$$

4. For all $G(s,x)\in C_c^\infty([0,T)\times\mathbb{T})$,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}}&\partial_s G(s,x)\rho(s,x)\,dx\,ds+\int_{\mathbb{T}}G(0,x)\rho_0(x)\,dx\\&=\int_0^T\int_{\mathbb{T}}\big[-\partial_{xx}G(s,x)\Psi\big(\rho(s,x)\big)+\partial_x G(s,x)\big[K(x)\Psi\big(\rho(s,x)\big)\big]\big]\,dx\,ds.\end{aligned}$$

Later, in Theorem 10.1 in Section 10, we show that the PDE (3.3) has at most one good weak solution.

As a consequence of Theorem 3.3, we observe that both quenched and annealed hydrodynamic limits follow with respect to random environment:

$$\left\{r_k^N:=\frac{1}{\sigma N\varepsilon}\sum_{|j-k|\le\lfloor N\varepsilon\rfloor}r_j\psi\left(\frac{j-k}{\varepsilon}\right):k\in\mathbb{T}_N\right\}.$$

Let $E_P\mathbb{P}_N^\omega$ be the annealed probability measure, where $P(d\omega)$ governs the random environment $\omega=\{r_x\}_{x\in\mathbb{N}}$ and $\mathbb{P}_N^\omega=\mathbb{P}_N$ is the process measure of the speeded up zero-range process η, with single particle jump rates $(1/2)\pm r_k^N/\sqrt{N}$ to the left and right from location $k\in\mathbb{T}_N$, conditioned on the environment. Recall that $\{q_k^N\}\overset{d}{=}\{r_k^N\}$ and that a.s. $\sqrt{N}q_k^N\to W'_\varepsilon(x)$ when $k/N\to x$ (cf. (2.2), (2.3)).

THEOREM 3.5. *With respect to the random environments, we have the hydrodynamic limits*:

(*I*) (*Quenched HDL*) *For almost all realizations ω, the statement of Theorem 3.3 holds with respect to $\alpha_\cdot^N=\sqrt{N}q_\cdot^N$ and $\alpha=W'_\varepsilon$.*

(*II*) (*Annealed HDL*) *The law of $\pi_\cdot^N$, under $E_P\mathbb{P}_N^\omega$, converges weakly to the law of $\rho(\cdot,x)\,dx=\rho(\cdot,x;\Phi,W'_\varepsilon)\,dx$ with respect to the distribution of Brownian motion W.*

PROOF. Part (I) follows directly from Theorem 3.3. For part (II), let Q_ω^N be the distribution of $\pi_\cdot^N\in D([0,T],\mathcal{M})$ conditioned on the environment realization ω. Then, averaging over the environment, the distribution of $\pi_\cdot^N$ is $\int Q_\omega^N P(d\omega)$. From part (I), we observe that $\rho(\cdot,x;\Phi,W'_\varepsilon(\omega))\,dx$, noting the dependence on ω, is the quenched hydrodynamic limit conditioned on the environment. Averaging over the environment, the distribution of $\rho(\cdot,x)\,dx$ is $\int\delta_{\rho(\cdot,dx;\Phi,W'_\varepsilon(\omega))\,dx}P(d\omega)$. Here, the law of $W'_\varepsilon(\omega)$ under $P(d\omega)$ is that of $\frac{-1}{\varepsilon}\int W(u)\psi'(\frac{u-x}{\varepsilon})\,du+\frac{1}{\varepsilon}\{W(x+\varepsilon)\psi(1)-W(x-\varepsilon)\psi(-1)\}$ (cf. (2.3)).

Let f be a bounded continuous function on $D([0,T],\mathcal{M})$. By the proof of Theorem 3.3 (see Step 5 in the proof outline in Section 5), applied to the quenched environments, we have that Q_ω^N, the law of the trajectory $\pi_\cdot^N$, converges weakly to $\delta_{\rho(\cdot,x;\Phi,W'_\varepsilon(\omega))\,dx}$ for almost all environments ω. Hence, $\int f(\pi)Q_\omega^N(d\pi)$ converges for almost all environments ω to $\int f(\pi)\delta_{\rho(\cdot,x;\Phi,W'_\varepsilon(\omega))\,dx}(d\pi)$. Then, by bounded convergence, as $N\uparrow\infty$,

$$\int\int f(\pi)Q_\omega^N(d\pi)P(d\omega)\to\int\int f(\pi)\delta_{\rho(\cdot,x;\Phi,W'_\varepsilon(\omega))\,dx}(d\pi)P(d\omega),$$

and part (II) follows. $\square$

Finally, as remarked in Section 1.5 in the Introduction, we now consider a system of independent particles, seen in diffusive scale and starting from a local equilibrium measure, in a common unregularized Seignourel environment on $\mathbb{Z}$: That is, when $g(k) \equiv k$, and a single particle has rates of moving from k to $k \pm 1$ are $\{1/2 \pm r_k/\sqrt{\sigma^2 N} : k \in \mathbb{Z}\}$, where $\omega = \{r_i\}$ are i.i.d. bounded, mean 0 random variables with variance $\sigma^2 > 0$. Let P_N^ω and E_N^ω refer to the quenched process measure and expectation with respect to a random walk moving in this environment.

THEOREM 3.6. *Consider a system of independent particles on $\mathbb{Z}$ moving in a common Seignourel environment, with initial product Poisson distribution $\nu_{le}^N = \prod_{k\in\mathbb{Z}} \mathcal{P}_{\beta(k/N)}$ associated to a bounded, continuous profile $\beta : \mathbb{R} \to [0, \infty)$.*

(*I*) (*Quenched law at time t*) *For fixed ω, the law of the configuration $\{\eta_k(t) : k \in \mathbb{Z}\}$ is a product Poisson measure $\prod_{k\in\mathbb{Z}} \mathcal{P}_{\beta_t(k,N)}$ where $\beta_t(k, N) = \sum_{x\in\mathbb{Z}} \beta(x/N) P_N^\omega(R_{N^2t}^{x,N} = k)$ for $k \in \mathbb{Z}$, and $R_s^{k,N}$ is the position at time s of a random walk starting at k in this environment.*

(*II*) (*Annealed mean HDL*) *Let $G \in C^\infty(\mathbb{R})$ be a compactly supported test function. Then*

$$\lim_{N\to\infty} E_P \mathbb{E}_N^\omega[\langle G, \pi_t^N\rangle] = \int_{\mathbb{R}} G(u)\mathscr{E}[\beta(u - X_t)]\,du,$$

where $\mathscr{E}$ is the annealed expectation with respect to the 4-Brox diffusion $X_\cdot$.

PROOF. We show part (I) by computing explicitly the moment generating function of the law of the configuration $\{\eta_t(x) : x \in \mathbb{Z}\}$ via the method of Chapter 1 [25]. Let $\{\lambda(i) : i \in \mathbb{Z}\}$ be real parameters, where $\lambda(i) \neq 0$ only for a finite number of $i \in \mathbb{Z}$. With respect to a fixed random environment ω, decompose $\eta_t(x) = \sum_{z\in\mathbb{Z}} \sum_{j=1}^{\eta_0(z)} \mathbb{1}(R_{N^2t}^{z,j,N} = x)$ in terms of the initial configuration and the positions $R_{N^2t}^{z,j,N}$ of the jth particle at time N^2t, which is initially at z. Here, we use the convention that empty sums vanish.

Hence, $\sum_{x\in\mathbb{Z}} \lambda(x)\eta_t(x) = \sum_{z\in\mathbb{Z}} \sum_{j=1}^{\eta_0(z)} \lambda(R_{N^2t}^{z,j,N})$. Since, for each z, $\{R_{N^2t}^{z,j,N} : 1 \le j \le \eta_0(z)\}$ are i.i.d., and η_0 is distributed according to ν_{le}^N, we have

$$\mathbb{E}_N^\omega\biggl[\exp\biggl\{\sum_{x\in\mathbb{Z}} \lambda(x)\eta_t(x)\biggr\}\biggr] = \prod_{z\in\mathbb{Z}} e^{\beta(z/N)\{E_N^\omega[e^{\lambda(R_{N^2t}^{z,N})}]-1\}}.$$

Since $E_N^\omega[e^{\lambda(R_{N^2t}^{z,N})}] = \sum_{x\in\mathbb{Z}} e^{\lambda(x)} P_N^\omega(R_{N^2t}^{z,N} = x)$, the last display equals, as desired,

$$\prod_{x\in\mathbb{Z}} \exp\biggl\{\biggl[\sum_{z\in\mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2t}^{z,N} = x)\biggr](e^{\lambda(x)} - 1)\biggr\}.$$

To show part (II), as a consequence of part (I), we find that the quenched mean of $N^{-1}\sum_{k\in\mathbb{Z}} G(k/N)\eta_t(k)$ equals

$$\frac{1}{N}\sum_{k\in\mathbb{Z}} G(k/N) \sum_{z\in\mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2t}^{z,N} = k).$$

By translation-invariance of ω under $P(d\omega)$, $E_P[P_N^\omega(R_{N^2t}^{z,N} = k)] = E_P[P_N^\omega(R_{N^2t}^{0,N} + z = k)]$. Hence, the annealed mean of $N^{-1}\sum_{k\in\mathbb{Z}} G(k/N)\eta_{N^2t}(k)$, equals

$$\begin{aligned}
&\int \frac{1}{N}\sum_{k\in\mathbb{Z}} G(k/N) \sum_{z\in\mathbb{Z}} \beta(z/N) P_N^\omega(R_{N^2t}^{0,N} + z = k) P(d\omega)\\
&\quad = \int \frac{1}{N}\sum_{k\in\mathbb{Z}} G(k/N) E_N^\omega[\beta(k/N - R_{N^2t}^{0,N}/N)] P(d\omega).
\end{aligned}$$

Now, from Seignourel's annealed limit, we have $R^{0,N}_{N^2 t}/N \Rightarrow X_t$. Since β is bounded and continuous and $G \in C^\infty(\mathbb{R})$ has compact support, we conclude that the previous display converges, as $N \uparrow \infty$, to $\int_{\mathbb{R}} G(u)\mathscr{E}[\beta(u - X_t)]\,du$ as desired. $\square$

4. Stochastic differentials and martingales. To analyze $\langle G, \pi_t^N\rangle$, we compute its stochastic differential in terms of certain martingales. Let G be a smooth function on $[0, T] \times \mathbb{T}$, and let us write $G_t(x) := G(t, x)$. Then

$$M_t^{N,G} = \langle G_t, \pi_t^N\rangle - \langle G_0, \pi_0^N\rangle - \int_0^t \{\langle \partial_s G_s, \pi_s^N\rangle + N^2 L\langle G_s, \pi_s^N\rangle\}\,ds$$

is a mean zero martingale. Denote the discrete Laplacian Δ_N and discrete gradient ∇_N by

$$\Delta_N G\Big(\frac{k}{N}\Big) := N^2\Big(G\Big(\frac{k+1}{N}\Big) + G\Big(\frac{k-1}{N}\Big) - 2G\Big(\frac{k}{N}\Big)\Big),$$
$$\nabla_N G\Big(\frac{k}{N}\Big) := \frac{N}{2}\Big(G\Big(\frac{k+1}{N}\Big) - G\Big(\frac{k-1}{N}\Big)\Big),$$

and write

$$N^2 L\langle G, \pi_s^N\rangle = \frac{1}{N}\sum_{1\le k\le N}\Big(\frac{1}{2}\Delta_N G_s\Big(\frac{k}{N}\Big)g(\eta_s(k)) + 2\nabla_N G_s\Big(\frac{k}{N}\Big)g(\eta_s(k))\alpha_k^N\Big). \tag{4.1}$$

We will define

$$D_{N,k}^{G,s} := \frac{1}{2}\Delta_N G_s\Big(\frac{k}{N}\Big) + 2\nabla_N G_s\Big(\frac{k}{N}\Big)\alpha_k^N. \tag{4.2}$$

As $|\alpha_k^N|$ is uniformly bounded from above by a constant C, given uniform convergence to α on the torus $\mathbb{T}$, we have

$$|D_{N,k}^{G,s}| \le \|\partial_{xx}G\|_\infty + 2C\|\partial_x G\|_\infty. \tag{4.3}$$

The quadratic variation of $M_t^{N,G}$ is given by

$$\langle M^{N,G}\rangle_t = \int_0^t \{N^2 L(\langle G_s, \pi_s^N\rangle^2) - 2\langle G_s, \pi_s^N\rangle N^2 L\langle G_s, \pi_s^N\rangle\}\,ds.$$

Standard calculation shows that

$$\langle M^{N,G}\rangle_t = \int_0^t \sum_{k=1}^N \Big\{g(\eta_s(k))\Big(\frac{1}{2} + \frac{\alpha_k^N}{N}\Big)\Big(G_s\Big(\frac{k+1}{N}\Big) - G_s\Big(\frac{k}{N}\Big)\Big)^2 + g(\eta_s(k))\Big(\frac{1}{2} - \frac{\alpha_k^N}{N}\Big)\Big(G_s\Big(\frac{k-1}{N}\Big) - G_s\Big(\frac{k}{N}\Big)\Big)^2\Big\}\,ds.$$

This variation may be bounded as follows.

LEMMA 4.1. *For smooth functions G on $[0, T] \times \mathbb{T}$, there is a constant C_G such that for large N,*

$$\sup_{0\le t\le T} \mathbb{E}_N\langle M^{N,G}\rangle_t \le g^* C_G T N^{-1}.$$

PROOF. For N large, we may assume that $1/2+|\alpha_k^N/N|\le 1$. Also, since G is smooth and $g(\cdot)$ grows at most linearly, we obtain

$$\begin{aligned}\mathbb{E}_N\langle M^{N,G}\rangle_t &\le 2g^*\|\partial_x G\|_\infty^2 N^{-1}\mathbb{E}_N\left[\int_0^t \frac{1}{N}\sum_{k=1}^N \eta_s(k)\,ds\right]\\ &= 2g^*\|\partial_x G\|_\infty^2 N^{-1} t\,\mathbb{E}_N\left[\frac{1}{N}\sum_{k=1}^N \eta_0(k)\right].\end{aligned}$$

We have used that total number of particles is conserved in the last equality. By (3.1), we have that $\sup_N E_N[N^{-1}\sum_{k=1}^N \eta_0(k)]<\infty$, and the result follows. □

5. Proof outline. We now give a detailed outline of the proof of Theorem 3.3. Let Q^N be the probability measure on the trajectory space $D([0,T],\mathcal{M})$ governing $\pi_\cdot^N$ when the process starts from μ^N. By Lemma 8.1, the family of measures $\{Q^N\}_{N\in\mathbb{N}}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology.

Let now Q be any limit measure. We will show that Q is supported on a class of weak solutions to the nonlinear PDE (3.2).

Step 1. Let G be smooth on $[0,T]\times\mathbb{T}$. Recall the martingale $M_t^{N,G}$ and its quadratic variation $\langle M^{N,G}\rangle_t$ in the last section. By Lemma 4.1, we have $\mathbb{E}_N(M_T^{N,G})^2=\mathbb{E}_N\langle M^{N,G}\rangle_T$ vanishes as $N\to\infty$. By Doob's inequality, for each $\delta>0$,

$$\begin{aligned}&\mathbb{P}_N\left[\sup_{0\le t\le T}\left|\langle G_t,\pi_t^N\rangle-\langle G_0,\pi_0^N\rangle-\int_0^t\left(\langle\partial_s G_s,\pi_s^N\rangle+N^2L\langle G_s,\pi_s^N\rangle\right)ds\right|>\delta\right]\\ &\quad\le\frac{4}{\delta^2}\mathbb{E}_N\langle M^{N,G}\rangle_T \;\to\; 0\text{as }\; N\to\infty.\end{aligned}$$

Recall the evaluation of $N^2L\langle G_s,\pi_s^N\rangle$ in (4.1). Then

$$\begin{aligned}&\lim_{N\to\infty}\mathbb{P}_N\left[\sup_{0\le t\le T}\left|\langle G_t,\pi_t^N\rangle-\langle G_0,\pi_0^N\rangle-\int_0^t\left\{\langle\partial_s G_s,\pi_s^N\rangle\right.\right.\right.\\ &\quad\left.\left.\left.+\frac{1}{N}\sum_{1\le k\le N}\left(\frac{1}{2}\Delta_N G_s\left(\frac{k}{N}\right)g(\eta_s(k))+2\nabla_N G_s\left(\frac{k}{N}\right)g(\eta_s(k))\alpha_k^N\right)\right\}ds\right|>\delta\right]=0.\end{aligned}\tag{5.1}$$

Step 2. We now replace the nonlinear term $g(\eta_s(k))$ by a function of the empirical density of particles. To be precise, let $\eta^l(x)=\frac{1}{2l+1}\sum_{|y-x|\le l}\eta(y)$, that is, the average density of particles in the box centered at x with length $2l+1$.

Recall the coefficient $D_{N,k}^{G,s}$ in (4.2). By the triangle inequality, the 1- and 2-block estimates (Lemmas 6.3 and 7.2) imply the following replacement lemma.

LEMMA 5.1 (Replacement Lemma). *For each $\delta>0$, we have*

$$\limsup_{\theta\to0}\limsup_{N\to\infty}\mathbb{P}_N\left[\left|\frac{1}{N}\sum_{1\le k\le N}\int_0^T D_{N,k}^{G,t}\left(g(\eta_t(k))-\Phi(\eta_t^{\theta N}(k))\right)dt\right|\ge\delta\right]=0.$$

Step 3. For each $\theta>0$, take $\iota_\theta=(2\theta)^{-1}\mathbb{1}_{[-\theta,\theta]}$. The average density $\eta_t^{\theta N}(k)$ is written as a function of the empirical measure π_t^N,

$$\eta_t^{\theta N}(k)=\frac{2\theta N}{2\theta N+1}\langle\iota_\theta(\cdot-k/N),\pi_t^N\rangle.$$

Then, noting the form of $D^{G,s}_{N,k}$ and the uniform convergence of $\alpha^N_{\cdot}$ to $\bar{\alpha}$, we may replace ∇_N, Δ_N and α^N_k by ∂_x, ∂_{xx} and $\alpha(\frac{k}{N})$, respectively, and also the sum by an integral. Hence, we get from (5.1) in terms of the induced distribution Q^N that

$$\begin{aligned}\limsup_{\theta\to 0}\limsup_{N\to\infty} Q^N\Big[\Big|\langle G_T,\pi^N_T\rangle - \langle G_0,\pi^N_0\rangle - \int_0^T\Big\{\langle\partial_s G_s,\pi^N_s\rangle\\ + \int_{\mathbb{T}}\Big(\frac{1}{2}\partial_{xx}G_s(x) + 2\partial_x G_s(x)\alpha(x)\Big)\Phi\big(\langle\iota_\theta(\cdot - x),\pi^N_s\rangle\big)\,dx\Big\}\,ds\Big| > \delta\Big] = 0.\end{aligned} \tag{5.2}$$

Taking $N\to\infty$, along a subsequence, as the set of trajectories in (5.2) is open with respect to the uniform topology, we obtain

$$\begin{aligned}\limsup_{\theta\to 0} Q\Big[\Big|\langle G_T,\pi_T\rangle - \langle G_0,\pi_0\rangle - \int_0^T\Big\{\langle\partial_s G_s,\pi_s\rangle\\ + \int_{\mathbb{T}}\Big(\frac{1}{2}\partial_{xx}G_s(x) + 2\partial_x G_s(x)\alpha(x)\Big)\Phi\big(\langle\iota_\theta(\cdot - x),\pi_s\rangle\big)\,dx\Big\}\,ds\Big| > \delta\Big] = 0.\end{aligned}$$

Step 4. We show in Lemma 9.1 that Q is supported on trajectories $\pi_s(dx) = \rho(s,x)\,dx$ where $\rho\in L^1([0,T]\times\mathbb{T})$. To replace $\langle\iota_\theta(\cdot - x),\pi_s\rangle$ by $\rho(s,x)$, it is enough to show, for all $\delta > 0$, that

$$\limsup_{\theta\to 0} Q\Big[\Big|\int_0^T\int_{\mathbb{T}} D_{G,s}\big(\Phi\big(\langle\iota_\theta(\cdot - x),\pi_s\rangle\big) - \Phi\big(\rho(s,x)\big)\big)\,dx\,ds\Big| > \delta\Big] = 0,$$

where $D_{G,s} = \frac{1}{2}\partial_{xx}G_s(x) + 2\partial_x G_s(x)\alpha(x)$. By considering the Lebesgue points of ρ, almost surely with respect to Q (cf. [25]), we have

$$\lim_{\theta\to 0}\int_0^T\int_{\mathbb{T}} D_{G,s}\Phi\big(\langle\iota_\theta(\cdot - x),\pi_s\rangle\big)\,dx\,ds = \int_0^T\int_{\mathbb{T}} D_{G,s}\Phi\big(\rho(s,x)\big)\,dx\,ds.$$

Now, we have

$$\begin{aligned}Q\Big[\big\langle G_T,\rho(T,x)\big\rangle - \big\langle G_0,\rho(0,x)\big\rangle - \int_0^T\Big\{\big\langle\partial_s G_s,\rho(s,x)\big\rangle\\ + \int_{\mathbb{T}}\Big(\frac{1}{2}\partial_{xx}G_s(x) + 2\partial_x G_s(x)\alpha(x)\Big)\Phi\big(\rho(s,x)\big)\,dx\Big\}\,ds = 0\Big] = 1.\end{aligned}$$

Step 5. Now, each $\rho(t,x)$ solves weakly the equation

$$\partial_t\rho = \frac{1}{2}\partial_{xx}\Phi(\rho) - 2\partial_x\big(\alpha(x)\Phi(\rho)\big).$$

As a consequence of Lemma 9.1, ρ satisfies conservation of mass: $\int_{\mathbb{T}}\rho(t,x)\,dx = \int_{\mathbb{T}}\rho_0(x)\,dx$. Moreover, the initial condition $\rho(0,x) = \rho_0(x)$ holds by Condition 3.1. From convergence of Q^N to Q with respect to the uniform topology, ρ is weakly continuous in time, namely for each test function $G\in C(\mathbb{T})$, the map $t\mapsto\int_{\mathbb{T}}G(x)\rho(t,x)\,dx$ is continuous. In addition, in Proposition 9.2, we show an energy estimate, which defines a weak spatial derivative of $\Phi(\rho(t,x))$. Therefore, ρ is a good weak solution to (3.2).

As mentioned after Theorem 3.3, in Section 10, we show that there is at most one good weak solution ρ to (3.2) (cf. Definition 3.4). We conclude that the sequence of Q^N converges weakly to the Dirac measure on $\rho(\cdot,x)\,dx$. Finally, as Q^N converges to Q with respect to the uniform topology, we have for each $0\le t\le T$ that $\langle G,\pi^N_t\rangle$ weakly converges to the constant $\int_{\mathbb{T}}G(x)\rho(t,x)\,dx$ and, therefore, convergence in probability as stated in Theorem 3.3.

6. 1-block estimate. The "1-block" estimate is obtained by using a Rayleigh-type estimation of a variational eigenvalue expression derived from a Feynman–Kac bound. A spectral gap bound plays an important role in this step. There are differences here with the scheme of [14] and [23] in the context of the inhomogeneous environment that we detail.

Recall the generator L (cf. (2.5)), and the invariant measure $\mathscr{R}_N$ (cf. (2.7), where $\phi_{k,N}$ is taken so that $\max_k \phi_{k,N} = 1$). As $\mathscr{R}_N$ is not reversible with respect to L, we will work with S, the symmetric part of L:

$$Sf(\eta) = \frac{1}{2}\sum_{k=1}^{N}\{g(\eta(k))\mathfrak{p}^N_{k,+}(f(\eta^{k,k+1}) - f(\eta)) + g(\eta(k))\mathfrak{p}^N_{k,-}(f(\eta^{k,k-1}) - f(\eta))\},$$

where

$$\mathfrak{p}^N_{k,+} := \left(\frac{1}{2} + \frac{\alpha^N_k}{N}\right) + \frac{\phi_{k+1,N}}{\phi_{k,N}}\left(\frac{1}{2} - \frac{\alpha^N_{k+1}}{N}\right),$$
$$\mathfrak{p}^N_{k,-} := \left(\frac{1}{2} - \frac{\alpha^N_k}{N}\right) + \frac{\phi_{k-1,N}}{\phi_{k,N}}\left(\frac{1}{2} + \frac{\alpha^N_{k-1}}{N}\right). \tag{6.1}$$

Then $\mathscr{R}_N$ is reversible under the generator S. The Dirichlet form is

$$\begin{aligned} E_{\mathscr{R}_N}[f(-Sf)] &= \frac{1}{4}\sum_{k=1}^{N} E_{\mathscr{R}_N}[g(\eta(k))\mathfrak{p}^N_{k,+}(f(\eta^{k,k+1}) - f(\eta))^2] \\ &\quad + \frac{1}{4}\sum_{k=1}^{N} E_{\mathscr{R}_N}[g(\eta(k))\mathfrak{p}^N_{k,-}(f(\eta^{k,k-1}) - f(\eta))^2] \\ &= \frac{1}{2}\sum_{k=1}^{N} E_{\mathscr{R}_N}[g(\eta(k))\mathfrak{p}^N_{k,+}(f(\eta^{k,k+1}) - f(\eta))^2]. \end{aligned} \tag{6.2}$$

In the following, to reduce notation, we will use the formulation in the last line of (6.2), and related expressions for other Dirichlet forms defined subsequently.

6.1. *Spectral gap bound for* 1*-block estimate.* For $k \in \mathbb{T}_N$ and $l \geq 1$, define the set $\Lambda_{k,l} = \{k-l, k-l+1, \dots, k+l\} \subset \mathbb{T}_N$. Consider the process restricted to $\Lambda_{k,l}$ generated by $S_{k,l}$ where

$$\begin{aligned} S_{k,l}f(\eta) = \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,l}} &\{g(\eta(x))\mathfrak{p}^N_{x,+}(f(\eta^{x,x+1}) - f(\eta)) \\ &+ g(\eta(x+1))\mathfrak{p}^N_{x+1,-}(f(\eta^{x+1,x}) - f(\eta))\}. \end{aligned} \tag{6.3}$$

Let $\Omega_{k,l} = \mathbb{N}_0^{\Lambda_{k,l}}$ be the state space of configurations restricted on sites $\Lambda_{k,l}$. For each $\eta \in \Omega_{k,l}$, define $\kappa_{k,l}(\eta) = \prod_{x\in\Lambda_{k,l}} \mathcal{P}_{\phi_{x,N}}(\eta(x))$, that is, $\kappa_{k,l}$ is the product measure $\mathscr{R}_N$ restricted to $\Omega_{k,l}$. Define the state space of configurations with exactly j particles on the sites $\Lambda_{k,l}$:

$$\Omega_{k,l,j} = \left\{\eta \in \Omega_{k,l} : \sum_{x\in\Lambda_{k,l}} \eta(x) = j\right\}.$$

Let $\kappa_{k,l,j}$ be the associated reversible canonical measure obtained by conditioning $\kappa_{k,l}$ on $\Omega_{k,l}$. The corresponding Dirichlet form is

$$E_{\kappa_{k,l,j}}[f(-S_{k,l}f)] = \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,l}} E_{\kappa_{k,l,j}}[g(\eta(x))\mathfrak{p}^N_{x,+}(f(\eta^{x,x+1}) - f(\eta))^2]. \tag{6.4}$$

We will obtain the spectral gap estimate corresponding to the localized inhomogeneous process by comparison with the spectral gap for the standard translation-invariant localized process. Consider the generator L_l on $\Omega_{k,l}$ given by

$$L_l f(\eta) = \sum_{x,x+1\in\Lambda_{k,l}} \frac{1}{2}\{g(\eta(x))[f(\eta^{x,x+1}) - f(\eta)] + g(\eta(x+1))[f(\eta^{x+1,x}) - f(\eta)]\}.$$

For any $\rho > 0$, let ν_ρ be the product measure on $\Omega = \mathbb{N}_0^{\mathbb{T}_N}$ with common marginal $\mathcal{P}_\rho = \mathcal{P}_{\Phi(\phi)}$ on each site $k \in \mathbb{T}_N$ with mean ρ, and let $\nu_l^\rho = \nu_{k,l}^\rho$ be its restriction to $\Omega_{k,l}$.

Consider $\nu_{l,j} = \nu_{k,l,j}$, the associated canonical measure on $\Omega_{k,l,j}$, with respect to j particles in $\Lambda_{k,l}$. Notice that $\nu_{l,j}$ does not depend on ρ. It is well known that both ν_l^ρ and $\nu_{l,j}$ are invariant measures with respect to the localized generator L_l (cf. [1]). The corresponding Dirichlet form is given by

$$E_{\nu_{l,j}}[f(-L_l f)] = \frac{1}{2}\sum_{x,x+1\in\Lambda_{k,l}} E_{\nu_{l,j}}[g(\eta(x))(f(\eta^{x,x+1}) - f(\eta))^2]. \tag{6.5}$$

For $j \geq 1$, let $b_{l,j}$ be the spectral gap of $-L_l$ on $\Omega_{k,l,j}$ (cf. page 374, [25]):

$$b_{l,j} := \inf_f \frac{E_{\nu_{l,j}}[f(-L_l f)]}{\mathrm{Var}_{\nu_{l,j}}(f)}. \tag{6.6}$$

As $\Omega_{k,l,j}$ is a finite space, the infimum in the above formula is taken over all functions f from $\Omega_{k,l,j}$ to $\mathbb{R}$. For all $l, j \geq 1$, we have $b_{l,j} > 0$. We remark that even though, for a large class of $g(\cdot)$'s, sharp estimates of $b_{l,j}$ are available (see, for instance. [26, 29, 30]), we will only need that $b_{l,j}$ is strictly positive for all $l, j \geq 1$.

We are now ready to state the lemma for the spectral gap bounds. Recall $\mathfrak{p}_{k,+}^N$ from (6.1). Let $r_{k,l,N}^{-1} := \min_{x\in\Lambda_{k,l}}\{\mathfrak{p}_{x,+}^N\}$.

LEMMA 6.1. *We have the following estimates*:

1. *Uniform bound*: *For all $\eta \in \Omega_{k,l,j}$, we have*

$$\left(\frac{\phi_{\min,k,l}}{\phi_{\max,k,l}}\right)^j \leq \frac{\kappa_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} \leq \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j, \tag{6.7}$$

where $\phi_{\min,k,l} = \min_{x\in\Lambda_{k,l}} \phi_{x,N}$ and $\phi_{\max,k,l} = \max_{x\in\Lambda_{k,l}} \phi_{x,N}$.

2. *Poincaré inequality*: *We have for $j \geq 1$,*

$$\mathrm{Var}_{\kappa_{k,l,j}}(f) \leq C_{k,l,j} E_{\kappa_{k,l,j}}[f(-S_{k,l} f)],$$

where $C_{k,l,j} := b_{l,j}^{-1} r_{k,l,N} (\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}})^{2j}$ bounds the inverse of the spectral gap of $-S_{k,l}$ on $\Omega_{k,l,j}$.

3. *For each l fixed*, *we have*

$$\lim_{N\uparrow\infty} \sup_{1\leq k\leq N} \frac{\phi_{\max,k,l}}{\phi_{\min,k,l}} = 1, \lim_{N\uparrow\infty} \sup_{1\leq k\leq N} r_{k,l,N} = 1,$$

and hence, *for fixed l and $j \geq 1$, $\sup_{N\geq 1}\sup_{1\leq k\leq N} C_{k,l,j} < \infty$.*

PROOF. Fix an arbitrary $\rho > 0$. By the definitions of conditioned measures $\kappa_{k,l,j}$ and $\nu_{l,j}$, we have for $\eta \in \Omega_{k,l,j}$,

$$\frac{\kappa_{k,l,j}(\eta)}{\nu_{l,j}(\eta)} = \frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} \frac{\nu_l^\rho(\Omega_{k,l,j})}{\kappa_{k,l}(\Omega_{k,l,j})}. \tag{6.8}$$

The product structure of $\kappa_{k,l}$ and ν_l^ρ allows a direct computation

$$\frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} = \frac{\prod_{x\in\Lambda_{k,l}}\{(\phi_{x,N})^{\eta(x)}/Z(\phi_{x,N})\}}{\prod_{x\in\Lambda_{k,l}}\{(\phi_0)^{\eta(x)}/Z(\phi_0)\}}, \tag{6.9}$$

where ϕ_0 is the common fugacity for (the marginals of) ν_ρ. Recalling that $\phi_{\min,k,l} = \min_{x\in\Lambda_{k,l}}\phi_{x,N}$ and $\phi_{\max,k,l} = \max_{x\in\Lambda_{k,l}}\phi_{x,N}$, for $\eta\in\Omega_{k,l,j}$, we can estimate $\kappa_{k,l}(\eta)/\nu_l^\rho(\eta)$ by

$$\left(\frac{\phi_{\min,k,l}}{\phi_0}\right)^j \prod_{x\in\Lambda_{k,l}}\frac{Z(\phi_{x,N})}{Z(\phi_0)} \le \frac{\kappa_{k,l}(\eta)}{\nu_l^\rho(\eta)} \le \left(\frac{\phi_{\max,k,l}}{\phi_0}\right)^j \prod_{x\in\Lambda_{k,l}}\frac{Z(\phi_{x,N})}{Z(\phi_0)}. \tag{6.10}$$

Note that $\kappa_{k,l}(\Omega_{k,l,j}) = \sum_{\eta\in\Omega_{k,l,j}}[\kappa_{k,l}(\eta)/\nu_l^\rho(\eta)]\nu_l^\rho(\eta)$. Then $\kappa_{k,l}(\Omega_{k,l,j})/\nu_l^\rho(\Omega_{k,l,j})$ is estimated by the same bounds as in (6.10). Then, rearranging these estimates, (6.7) follows from (6.8).

Turning now to the Poincaré inequality, the proof relies on the spectral gap given above (cf. (6.6)): For all $l, j\ge 1$, we have

$$\mathrm{Var}_{\nu_{l,j}}(f) \le b_{l,j}^{-1} E_{\nu_{l,j}}[f(-L_l f)]. \tag{6.11}$$

To get an estimate with respect to $-S_{k,l}$, from (6.5) and (6.4), using (6.7), we have

$$E_{\nu_{l,j}}[f(-L_l f)] \le r_{k,l,N}\left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j E_{\kappa_{k,l,j}}[f(-S_{k,l}f)]. \tag{6.12}$$

Now, since

$$\begin{aligned}\mathrm{Var}_{\kappa_{k,l,j}}(f) &= \inf_a E_{\kappa_{k,l,j}}[(f-a)^2]\\ &\le \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j \inf_a E_{\nu_{l,j}}[(f-a)^2] = \left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^j \mathrm{Var}_{\nu_{l,j}}(f),\end{aligned}$$

the desired Poincaré inequality follows from (6.11) and (6.12).

The last item (3) follows from Lemma 2.2 and that $\sup_N \sup_{1\le x\le N}\alpha_x^N < C$. $\square$

6.2. *Relative entropy.* For $t>0$, let μ_t^N be the distribution of η_t. As the entropy production is negative (cf. page 340, [25]), we have $H(\mu_t^N|\mathscr{R}_N) \le H(\mu^N|\mathscr{R}_N) \le C_0 N$. Furthermore, the relative entropy of μ_t^N with respect to the homogeneous invariant measures ν_ρ is also of order $O(N)$, which will be useful in the sequel.

LEMMA 6.2. *For any fixed $\rho>0$, there is a constant $C = C(\omega)$ such that $H(\mu_t^N|\nu_\rho)\le CN$.*

PROOF. Write

$$H(\mu_t^N|\nu_\rho) = \int \ln\left(\frac{d\mu_t^N}{d\nu_\rho}\right)d\mu_t^N = \int \ln\left(\frac{d\mu_t^N}{d\mathscr{R}_N}\right)d\mu_t^N + \int \ln\left(\frac{d\mathscr{R}_N}{d\nu_\rho}\right)d\mu_t^N.$$

The first term on the right-hand side is exactly $H(\mu_t^N|\mathscr{R}_N) = O(N)$ by part (2) of Condition 3.1. The integrand in the second term equals

$$\ln\frac{d\mathscr{R}_N}{d\nu_\rho}(\eta) = \ln\frac{\mathscr{R}_N(\eta)}{\nu_\rho(\eta)} = \sum_{k=1}^N \eta(k)\ln\frac{\phi_{k,N}}{\phi_0} + \sum_{k=1}^N \ln\frac{Z(\phi_0)}{Z(\phi_{k,N})}.$$

The desired estimate now follows these observations: $0<c\le\phi_{k,N}\le 1$ by Lemma 2.2, and the mean expected number of particles, $\int\sum_{k\in\mathbb{T}_N}\eta(k)\,d\mu_t^N = O(N)$ by (3.1). $\square$

6.3. 1-*block estimate*. We prove the 1-block estimate.

LEMMA 6.3 (1-block estimate). *For every $T > 0$,*

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\mathbb{E}_N\left[\frac{1}{N}\sum_{1\le k\le N}\left|\int_0^T V_{k,l}(s,\eta_s)\,ds\right|\right]=0,$$

where $V_{k,l}(s,\eta) := D^{G,s}_{N,k}(g(\eta(k)) - \Phi(\eta^l(k)))$ and $D^{G,s}_{N,k}$ is as in (4.2).

PROOF. We first introduce a cutoff of large densities. Fix $A > 0$, and let

$$\tilde V_{k,l,A}(s,\eta) := V_{k,l}(s,\eta)\mathbb{1}_{\{\eta^l(k)\le A\}}.$$

Notice that $g(\cdot)$ considered here satisfies the "FEM" assumption in [25]. By Lemma 6.2, one can replace $V_{k,l}$ by $\tilde V_{k,l,A}$, following the argument in Lemma 4.2 on page 90 [25].

It now remains to prove that for every $A > 0$, $T > 0$,

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{1\le k\le N}\mathbb{E}_N\left[\left|\int_0^T \tilde V_{k,l,A}(s,\eta_s)\,ds\right|\right]=0.$$

Define $\Lambda_{k,l}(\eta)$ as the number of particles in $\Lambda_{k,l}$, that is, $\Lambda_{k,l}(\eta) := (2l+1)\eta^l(k)$. As in [14], we will replace $\tilde V_{k,l,A}(s,\eta)$ by its "centering":

$$V_{k,l,A}(s,\eta) := D^{G,s}_{N,k}\{g(\eta(k)) - E_{\kappa_{k,l,\Lambda_{k,l}(\eta)}}[g(\eta(k))]\}\mathbb{1}_{\{\eta^l(k)\le A\}}.$$

Note that $E_{\kappa_{k,l,j}}V_{k,l,A} = 0$ for all k, l, j, which will be used in the Rayleigh-type estimation. The error introduced by such a replacement, noting (4.3), is less than or equal to

$$C(G)\mathbb{E}_N\left[\int_0^T \mathbb{1}_{\{0<\eta^l_s(k)\le A\}}\left|E_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}}[g(\eta(k))] - \Phi(\eta^l_s(k))\right|ds\right]. \tag{6.13}$$

Note that $\Phi(\eta^l_s(k)) = E_{\nu_{\eta^l_s(k)}}[g]$. By the triangle inequality, the expectation in (6.13) is bounded by

$$\begin{aligned}&\mathbb{E}_N\left[\int_0^T \mathbb{1}_{\{0<\eta^l_s(k)\le A\}}\left|E_{\kappa_{k,l,\Lambda_{k,l}(\eta_s)}}[g(\eta(k))] - E_{\nu_{k,l,\Lambda_{k,l}(\eta_s)}}[g(\eta(k))]\right|ds\right]\\ &\quad+\mathbb{E}_N\left[\int_0^T \mathbb{1}_{\{0<\eta^l_s(k)\le A\}}\left|E_{\nu_{k,l,\Lambda_{k,l}(\eta_s)}}[g(\eta(k))] - E_{\nu_{\eta^l_s(k)}}[g]\right|ds\right] =: I_1 + I_2.\end{aligned}$$

Using (6.7) and then $g(k) \le g^*k$, the term I_1 is bounded by

$$\begin{aligned}&\mathbb{E}_N\left[\int_0^T \mathbb{1}_{\{0<\eta^l_s(k)\le A\}}E_{\nu_{k,l,\Lambda_{k,l}(\eta_s)}}[g(\eta(k))]\left[\left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^{\Lambda_{k,l}(\eta_s)} - 1\right]ds\right]\\ &\quad\le Tg^*(2l+1)A\left[\left(\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}}\right)^{(2l+1)A} - 1\right].\end{aligned}$$

Notice that $\sup_{1\le k\le N}\frac{\phi_{\max,k,l}}{\phi_{\min,k,l}} \to 1$ by Lemma 6.1. Then, for each fixed l and A, the term $\sup_{1\le k\le N} I_1$ vanishes as $N\uparrow\infty$.

Now, we turn to estimate the term I_2. By the equivalence of ensembles (cf. page 355, [25]), the absolute value in I_2 vanishes as $l\uparrow\infty$, uniformly in k. Therefore, the term I_2 vanishes as soon as we take $N\uparrow\infty$, $l\uparrow\infty$ in order.

To prove the lemma, it now remains to show

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{0\le k\le N}\mathbb{E}_N\left[\left|\int_0^T V_{k,l,A}(s,\eta_s)\,ds\right|\right]=0.$$

By the entropy inequality (cf. page 338 [25]) and the assumption $H(\mu^N|\mathscr{R}_N) \le C_0 N$, we have for $\gamma > 0$ that

$$\begin{aligned} &\mathbb{E}_N\left[\left|\int_0^T V_{k,l,A}(s,\eta_s)\,ds\right|\right] \\ &\quad\le \frac{C_0}{\gamma} + \frac{1}{\gamma N}\ln \mathbb{E}_{\mathscr{R}_N}\left[\exp\left\{\gamma N\left|\int_0^T V_{k,l,A}(s,\eta_s)\,ds\right|\right\}\right]. \end{aligned} \tag{6.14}$$

By the Feynman–Kac formula (cf. Lemma 7.20 [25]), since $e^{|x|} \le e^x + e^{-x}$, to estimate (6.14), it will be enough to bound

$$\frac{C_0}{\gamma} + \frac{1}{\gamma N}\int_0^T \lambda_{N,l}(s)\,ds, \tag{6.15}$$

where $\lambda_{N,l}(s)$ is the largest eigenvalue of $N^2 S + \gamma N V_{k,l,A}(s,\eta)$, and a similar expression where $V_{k,l,A}$ is replaced by $-V_{k,l,A}$. As the argument to bound (6.15) will also hold for the expression with respect to $-V_{k,l,A}$, we concentrate on the following in the display (6.15).

Now, fix $s \in [0,T]$ and omit the argument s to simplify notation. To estimate the eigenvalue $\lambda_{N,l}(s)$, we make use of the variational formula:

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f\{E_{\mathscr{R}_N}[V_{k,l,A} f] - \gamma^{-1} N E_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})]\},$$

where the supremum is over all f, which are densities with respect to $\mathscr{R}_N$.

Recall that $\kappa_{k,l}$ is the restriction of $\mathscr{R}_N$ to $\Lambda_{k,l}$, and that $S_{k,l}$ is the localized generator. For any density f, we consider its restriction with respect to configuration sites $\Lambda_{k,l}$, that is, we define $f_{k,l} = E_{\mathscr{R}_N}[f|\Omega_{k,l}]$. Notice that $E_{\mathscr{R}_N}[\sqrt{f}(-S_{k,l}\sqrt{f})] \le E_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})]$. By convexity of the Dirichlet form, we have

$$(\gamma N)^{-1}\lambda_{N,l} \le \sup_{f_{k,l}}\{E_{\kappa_{k,l}}[V_{k,l,A} f_{k,l}] - \gamma^{-1} N E_{\kappa_{k,l}}[\sqrt{f_{k,l}}(-S_{k,l}\sqrt{f_{k,l}})]\}.$$

We now write $f_{k,l}\,d\kappa_{k,l}$ with respect to sets $\Omega_{k,l,j}$ of configurations with total particle number j on $\Lambda_{k,l}$:

$$E_{\kappa_{k,l}}[V_{k,l,A} f_{k,l}] = \sum_{j\ge 0} c_{k,l,j}(f)\int V_{k,l,A} f_{k,l,j}\,d\kappa_{k,l,j}, \tag{6.16}$$

where $c_{k,l,j}(f) = \int_{\Omega_{k,l,j}} f_{k,l}\,d\kappa_{k,l}$, $f_{k,l,j} = c_{k,l,j}(f)^{-1}\kappa_{k,l}(\Omega_{k,l,j}) f_{k,l}$, $\sum_{j\ge 0} c_{k,l,j} = 1$ and $f_{k,l,j}$ is a density with respect to $\kappa_{k,l,j}$.

Then, on $\Omega_{k,l,j}$, we have $\frac{S_{k,l}\sqrt{f_{k,l}}}{\sqrt{f_{k,l}}} = \frac{S_{k,l}\sqrt{f_{k,l,j}}}{\sqrt{f_{k,l,j}}}$. By (6.16), we write

$$E_{\kappa_{k,l}}[\sqrt{f_{k,l}}(-S_{k,l}\sqrt{f_{k,l}})] = \sum_{j\ge 0} c_{k,l,j}(f) E_{\kappa_{k,l,j}}[\sqrt{f_{k,l,j}}(-S_{k,l}\sqrt{f_{k,l,j}})].$$

Then we have

$$(\gamma N)^{-1}\lambda_{N,l} \le \sup_{0\le j\le A(2l+1)}\ \sup_f\{E_{\kappa_{k,l,j}}[V_{k,l,A} f] - \gamma^{-1} N E_{\kappa_{k,l,j}}[\sqrt{f}(-S_{k,l}\sqrt{f})]\},$$

where the inside supremum is on densities f with respect to $\kappa_{k,l,j}$. When $j = 0$, the system is empty and this supremum vanishes since $V_{k,l,A} = E_{\kappa_{k,l,j}}[\sqrt{f}(-S_{k,l}\sqrt{f})] = 0$.

By Lemma 6.1, for $j \ge 1$, we have $C_{k,l,j}$ is the inverse spectral gap estimate of $S_{k,l}$. Note also that $\| V_{k,l,A}\|_\infty \le C(A,G)$. Using the Rayleigh estimate (cf. page 377, [25]), we have

$$\begin{aligned}
&E_{\kappa_{k,l,j}}[V_{k,l,A} f] - \gamma^{-1} N E_{\kappa_{k,l,j}}[\sqrt{f}(-S_{k,l}\sqrt{f})] \\
&\quad \le \frac{\gamma N^{-1}}{1-2C(A,G)C_{k,l,j}\gamma N^{-1}} E_{\kappa_{k,l,j}}[V_{k,l,A}(-S_{k,l})^{-1} V_{k,l,A}].
\end{aligned}$$

As remarked in the beginning of the proof, $E_{\kappa_{k,l,j}} V_{k,l,A} = 0$. Observe that the spectral gap estimate of $S_{k,l}$ in Lemma 6.1 also implies that $\|S_{k,l}^{-1}\|_2$, the $L^2(\kappa_{k,l,j})$ norm of the operator $S_{k,l}^{-1}$ on mean zero functions is less than or equal to $C_{k,l,j}$. Thus, by Cauchy–Schwarz, we have

$$E_{\kappa_{k,l,j}}[V_{k,l,A}(-S_{k,l})^{-1} V_{k,l,A}] \le C_{k,l,j} E_{\kappa_{k,l,j}}[V_{k,l,A}^2].$$

Retracing the steps, we obtain

$$\mathbb{E}_N\left[\left|\int_0^T V_{k,l,A}(\eta_s)\,ds\right|\right] \le \frac{C_0}{\gamma} + \sup_{1\le j\le A(2l+1)} \frac{T\gamma N^{-1} C_{k,l,j}}{1-2C(A,G)C_{k,l,j}\gamma N^{-1}} E_{\kappa_{k,l,j}}[V_{k,l,A}^2].$$

The second term in the last expression vanishes uniformly as $N\to\infty$ for $1\le k\le N$ and $j \le A(2l+1)$. The lemma now is proved by letting $\gamma\to\infty$. $\square$

7. 2-block estimate. We now detail the 2-block estimate, in the context of the inhomogeneous environment, following the outline of the 1-block estimate. Recall the notation $\Lambda_{k,l}$ from the 1-block estimate.

For $l \ge 1$ and k, k' such that $|k-k'| > 2l$ and $k+l \le k'-l$, let $\Lambda_{k,k',l} = \Lambda_{k,l} \cup \Lambda_{k',l}$. We introduce the following localized generator $S_{k,k',l}$ governing a process on $\Omega_{k,k',l} = \mathbb{N}_0^{\Lambda_{k,k',l}}$. Inside each block, the process moves as before, but we add an extra bond interaction between sites $k+l$ and $k'-l$:

$$\begin{aligned}
S_{k,k',l} f(\eta) &= S_{k,l} f(\eta) + S_{k',l} f(\eta) + \frac{1}{2} g(\eta(k+l))\mathfrak{p}^N_{k+l,k'-l}(f(\eta^{k+l,k'-l}) - f(\eta)) \\
&\quad + \frac{1}{2} g(\eta(k'-l))\mathfrak{p}^N_{k'-l,k+l}(f(\eta^{k'-l,k+l}) - f(\eta)),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{p}^N_{k+l,k'-l} &= \frac{1}{2} + \frac{\alpha^N_{k+l}}{N} + \frac{\phi_{k'-l,N}}{\phi_{k+l,N}}\left(\frac{1}{2} - \frac{\alpha^N_{k'-l}}{N}\right), \\
\mathfrak{p}^N_{k'-l,k+l} &= \frac{1}{2} - \frac{\alpha^N_{k'-l}}{N} + \frac{\phi_{k+l,N}}{\phi_{k'-l,N}}\left(\frac{1}{2} + \frac{\alpha^N_{k+l}}{N}\right).
\end{aligned}$$

As before, consider the localized product measure $\kappa_{k,k',l} = \prod_{x\in\Lambda_{k,k',l}} \mathcal{P}_{\phi_{x,N}}$, namely $\kappa = \mathscr{R}_N$ limited to sites in $\Lambda_{k,k',l}$. Define as well as the canonical measure $\kappa_{k,k',l,j}$ on $\Omega_{k,k',l,j} := \{\eta \in \Omega_{k,k',l} : \sum_{x\in\Lambda_{k,k',l}} \eta(x) = j\}$, that is, $\kappa_{k,k',l}$ is conditioned so that there are exactly j particles counted in $\Omega_{k,k',l}$. With the choice of extra rates $\mathfrak{p}^N_{k+l,k'-l}$ and $\mathfrak{p}^N_{k'-l,k+l}$, both measures are invariant and reversible for the dynamics with Markov generator $S_{k,k',l}$.

The corresponding Dirichlet form, with measure $\tilde\kappa$ given by $\kappa_{k,k',l}$ or $\kappa_{k,k',l,j}$, is given by

$$\begin{aligned}
&E_{\tilde\kappa}[f(-S_{k,k',l} f)] \\
&\quad = \frac{1}{2} \sum_{x,x+1\in\Lambda_{k,k',l}} E_{\tilde\kappa}[g(\eta(k))\mathfrak{p}^N_{k,+}(f(\eta^{x,x+1}) - f(\eta))^2] \qquad (7.1)\\
&\qquad + \frac{1}{2} E_{\tilde\kappa}[g(\eta(k+l))\mathfrak{p}^N_{k+l,k'-l}(f(\eta^{k+l,k'-l}) - f(\eta))^2].
\end{aligned}$$

Recall also the generator of symmetric zero-range L_l with respect to $\Lambda_{k,l}$ (cf. below (6.4)). Let L_l' be the corresponding generator with respect to $\Lambda_{k',l}$. Define the generator $L_{l,l}$ with respect to $\Lambda_{k,k',l}$ by

$$L_{l,l}f(\eta) = L_l f(\eta) + L_l' f(\eta) + \frac{1}{2}[f(\eta^{k+l,k'-l}) - f(\eta)]g(\eta(k+l))$$
$$+ \frac{1}{2}[f(\eta^{k'-l,k+l}) - f(\eta)]g(\eta(k'-l)).$$

When $|k-k'|$ is large, the process governed by $L_{l,l}$ is the same as if the blocks were adjacent, with a connecting bond, and so no longer depends on k, k' but only on the width l.

Corresponding to the set-up of the gap bound Lemma 6.1, let $\nu_{l,l}^{\rho}$ be the product of $4l+2$ distributions with common marginal ρ. One may inspect that $\nu_{l,l}^{\rho}$ is invariant to the dynamics generated by $L_{l,l}$. Let now $\nu_{l,l,j}$ be $\nu_{l,l}^{\rho}$ conditioned on the total number of particles in the $4l+2$ sites being j. Note that $\nu_{l,l,j}$ is independent of ρ. This canonical measure $\nu_{l,l,j}$ is also invariant to the dynamics. The corresponding Dirichlet form is given by

$$E_{\nu_{l,l,j}}[f(-L_{l,l}f)] = \sum_{x,x+1\in\Lambda_{k,k',l}} E_{\nu_{l,l,j}}\left[\frac{1}{2}g(\eta(x))[f(\eta^{x,x+1}) - f(\eta)]^2\right]$$
$$+ E_{\nu_{l,l,j}}\left[\frac{1}{2}g(\eta(k'-l))[f(\eta^{k'-l,k+l}) - f(\eta)]^2\right].$$

Similar to (6.6), for each l and $j \geq 1$, we let $b_{l,l,j} > 0$ be the spectral gap of $-L_{l,l}$ on $\Omega_{k,k',l,j}$:

$$b_{l,l,j} := \inf_f \frac{E_{\nu_{l,l,j}}[f(-L_l f)]}{\mathrm{Var}_{\nu_{l,l,j}}(f)}. \tag{7.2}$$

Let $r_{k,k',l,N}^{-1} := \min\{\mathfrak{p}_{k+l,k'-l}^N, \min_{x,x+1\in\Lambda_{k,k',l}}\{\mathfrak{p}_{x,+}^N\}\}$.

LEMMA 7.1. *We have the following estimates*:

1. *Uniform bound*: *For all* $\eta \in \Omega_{k,k',l,j}$, *we have*

$$\left(\frac{\phi_{\min,k,k',l}}{\phi_{\max,k,k',l}}\right)^j \leq \frac{\kappa_{k,k',l,j}(\eta)}{\nu_{l,l,j}(\eta)} \leq \left(\frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}}\right)^j, \tag{7.3}$$

where $\phi_{\min,k,k',l} = \min_{x\in\Lambda_{k,k',l}} \phi_{x,N}$ *and* $\phi_{\max,k,k',l} = \max_{x\in\Lambda_{k,k',l}} \phi_{x,N}$.

2. *Poincaré inequality*: *For fixed* $j \geq 1$ *and* k, k' *such that* $|k-k'| > 2l+1$, *we have*

$$\mathrm{Var}_{\kappa_{k,k',l,j}}(f) \leq C_{k,k',l,j} E_{\kappa_{k,k',l,j}}[f(-S_{k,k',l}f)], \tag{7.4}$$

where $C_{k,k',l,j} = b_{l,l,j}^{-1} r_{k,k',l,N} (\frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}})^{2j}$.

3. *For each l fixed*, *there exists a constant* C_0 *such that*

$$\sup_{k,k',N} \frac{\phi_{\max,k,k',l}}{\phi_{\min,k,k',l}} \leq C_0 \limsup_{N\uparrow\infty} \sup_{k',k} r_{k,k',l,N} \leq C_0.$$

Hence, *for fixed l and* $j \geq 1$, *we have* $\limsup_{\theta\downarrow 0} \limsup_{N\uparrow\infty} \sup_{2l+1\leq|k'-k|\leq\theta N} C_{k,k',l,j} < \infty$.

PROOF. The argument follows closely the proof of Lemma 6.1, by comparing $\kappa_{k,k',l,j}$ with $\nu_{l,l,j}$ and making use of the spectral gap estimate (7.2). To be brief, we omit these details. □

We now state a 2-block estimate.

LEMMA 7.2 (2-block estimate). *We have*

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty} \mathbb{E}_N\left[\frac{1}{N}\sum_{1\le k\le N}\int_0^T D^{G,s}_{N,k}\big(\Phi(\eta^l_s(k))-\Phi(\eta^{\theta N}_s(k))\big)\,ds\right]=0. \tag{7.5}$$

PROOF. We separate the argument into steps.

Step 1. Since $\Phi(\cdot)$ is Lipschitz on $\mathbb{R}^+$ and $D^{G,s}_{N,k}$ is bounded (cf. (4.3)), it is enough to show

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\mathbb{E}_N\left[\frac{1}{N}\sum_{1\le k\le N}\int_0^T\big|\eta^{\theta N}_s(k)-\eta^l_s(k)\big|\,ds\right]=0.$$

By the triangle inequality, it will be enough to show that, as $N\uparrow\infty$, $\theta\downarrow 0$ and $l\uparrow\infty$,

$$\begin{aligned}&\mathbb{E}_N\left[\int_0^T\frac{1}{N}\sum_{1\le k\le N}\left|\eta^{\theta N}_s(k)-\frac{1}{2\theta N+1}\sum_{|x-k|\le\theta N}\eta^l_s(x)\right|ds\right]\to 0\quad\text{and}\\ &\mathbb{E}_N\left[\int_0^T\frac{1}{N}\sum_{1\le k\le N}\left|\frac{1}{2\theta N+1}\sum_{|x-k|\le\theta N}\eta^l_s(x)-\eta^l_s(k)\right|ds\right]\to 0.\end{aligned}\tag{7.6}$$

Step 2. Note that

$$\begin{aligned}\left|\eta^{\theta N}(k)-\frac{1}{2\theta N+1}\sum_{|x-k|\le\theta N}\eta^l(x)\right| &\le\frac{1}{2\theta N+1}\sum_{\substack{|x-k-\theta N|\le l\\ \text{or }|x-k+\theta N|\le l}}\eta(x)\\ &=\frac{2l+1}{2\theta N+1}\big(\eta^l(k-\theta N)+\eta^l(k+\theta N)\big).\end{aligned}$$

Then the first limit in (7.6), as $\sum_k\eta^l_s(k\pm\theta N)=\sum_k\eta_0(k)$, vanishes as $N\to\infty$ for fixed l, by the estimate (3.1)). By a similar argument, we can restrain the x in the summation of the second limit in (7.6) to k' such that $2l+1\le|k'-k|\le\theta N$. Then the second limit will follow if we show that

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\mathbb{E}_N\left[\int_0^T\frac{1}{N}\sum_{1\le k\le N}\left|\frac{1}{2\theta N+1}\sum_{2l+1\le|x-k|\le\theta N}\eta^l_s(x)-\eta^l_s(k)\right|ds\right]=0.$$

Step 3. We will apply a cutoff of large densities first. Let

$$\eta^l_s(k,k')=\eta^l_s(k)+\eta^l_s(k').$$

The same argument as for the cutoff in Lemma 6.3 (cf. page 92, [25]) gives, as $A\uparrow\infty$,

$$\limsup_{l\uparrow\infty}\limsup_{\theta\downarrow 0}\limsup_{N\uparrow\infty}\sup_{2l+1\le|y|\le\theta N}\mathbb{E}_N\left[\int_0^T\frac{1}{N}\sum_{1\le k\le N}\eta^l_s(k,k+y)\mathbb{1}_{\eta^l_s(k,k+y)>A}\,ds\right]\to 0.$$

Hence, it remains to show that, for any fixed A,

$$\sup_{\substack{1\le k<k'\le N\\ 2l+1\le|k'-k|\le\theta N}}\mathbb{E}_N\left[\int_0^T|\eta^l_s(k)-\eta^l_s(k')|\mathbb{1}_{\{\eta^l_s(k,k')\le A\}}\,ds\right]$$

vanishes as we take $N\to\infty$, $\theta\to 0$, and then $l\to\infty$.

Step 4. Let

$$V_{k,k',l,A}(\eta):=|\eta^l(k)-\eta^l(k')|\mathbb{1}_{\{\eta^l(k,k')\le A\}}.$$

Following the proof of Lemma 6.3, for fixed l, θ, N, k, k', in order to estimate

$$\mathbb{E}_N\left[\int_0^T V_{k,k',l,A}(\eta_s)\,ds\right]$$

it suffices to bound

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f \{E_{\mathscr{R}_N}[V_{k,k',l,A}f] - \gamma^{-1}NE_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})]\}, \tag{7.7}$$

where the supremum is over all f which are densities with respect to $\mathscr{R}_N$.

Step 5. Recall the generator $S_{k,k',l}$ and its Dirichlet form defined in the beginning of this subsection. Recall also $\kappa_{k,k',l}$ is the restriction of $\kappa = \mathscr{R}_N$ to $\Lambda_{k,k',l}$. The Dirichlet form with respect to the full generator S under $\mathscr{R}_N$ is given by

$$\begin{aligned}
&E_{\mathscr{R}_N}[f(-Sf)]\\
&\quad = \sum_{1\le k\le N} E_{\mathscr{R}_N}\left[g(\eta(k))\frac{1}{2}\left[\left(\frac{1}{2}+\frac{\alpha_k^N}{N}\right)+\frac{\phi_{k+1,N}}{\phi_{k,N}}\left(\frac{1}{2}-\frac{\alpha_{k+1}^N}{N}\right)\right]\left(f(\eta^{k,k+1})-f(\eta)\right)^2\right].
\end{aligned}$$

We now argue the following Dirichlet form inequality for some C:

$$E_{\kappa_{k,k',l}}[\sqrt{f}(-S_{k,k',l}\sqrt{f})] \le C(1+\theta N)E_{\kappa_{k,k',l}}[\sqrt{f}(-S\sqrt{f})]. \tag{7.8}$$

First, writing out the Dirichlet form in (7.1), in terms of the product measure $\mathscr{R}_N$, we have

$$\begin{aligned}
&E_{\kappa_{k,k',l}}[f(-S_{k,k',l}f)]\\
&\quad = \sum_{x,x+1\in\Lambda_{k,k',l}} E_{\mathscr{R}_N}\left[g(\eta(x))\frac{1}{2}\left[\left(\frac{1}{2}+\frac{\alpha_x^N}{N}\right)+\frac{\phi_{x+1,N}}{\phi_{x,N}}\left(\frac{1}{2}-\frac{\alpha_{x+1}^N}{N}\right)\right]\right.\\
&\qquad \left.\times\left(f(\eta^{x,x+1})-f(\eta)\right)^2\right]\\
&\qquad + E_{\mathscr{R}_N}\left[g(\eta(k+l))\frac{1}{2}\left[\left(\frac{1}{2}+\frac{\alpha_{k+l}^N}{N}\right)+\frac{\phi_{k'-l,N}}{\phi_{k+l,N}}\left(\frac{1}{2}-\frac{\alpha_{k'-l}^N}{N}\right)\right]\left(f(\eta^{k+l,k'-l})-f(\eta)\right)^2\right].
\end{aligned}$$

Next, by adding and subtracting at most θN terms, we have

$$\begin{aligned}
&[f(\eta^{k+l,k'-l})-f(\eta)]^2\\
&\quad \le (k'-k-2l)\sum_{q=0}^{k'-k-2l-1}[f(\eta^{k+l,k+l+q+1})-f(\eta^{k+l,k+l+q})]^2.
\end{aligned}$$

By the change of variables $\xi = \eta^{k+l,k+l+q}$, which takes away a particle at $k+l$ and adds one at $k+l+q$, we have that

$$\mathscr{R}_N(\eta) = \frac{\phi_{k+l,N}}{\phi_{k+l+q,N}}\frac{g(\eta(k+l+q)+1)}{g(\eta(k+l))}\mathscr{R}_N(\xi).$$

Also, as $\frac{\phi_{\max,N}}{\phi_{\min,N}}$ is uniformly bounded from above and below, and $\sup_N \sup_{1\le x\le N}\alpha_x^N < \infty$, we may find a constant C such that

$$\begin{aligned}
\mathfrak{p}_{k+l,k'-l}^N &= \left(\frac{1}{2}+\frac{\alpha_{k+l}^N}{N}\right)+\frac{\phi_{k'-l,N}}{\phi_{k+l,N}}\left(\frac{1}{2}-\frac{\alpha_{k'-l}^N}{N}\right)\\
&\le C\left\{\left(\frac{1}{2}+\frac{\alpha_{k+l+q}^N}{N}\right)+\frac{\phi_{k+l+q+1,N}}{\phi_{k+l+q,N}}\left(\frac{1}{2}-\frac{\alpha_{k+l+q+1}^N}{N}\right)\right\}.
\end{aligned}$$

Then, taking into account the above relations, we have in terms of another constant C that

$$
\begin{aligned}
&E_{\mathscr{R}_N}\Big[g\big(\eta(k+l)\big)\frac{1}{2}\mathfrak{p}^N_{k+l,k'-l}\big[f\big(\eta^{k+l,k+l+q+1}\big)-f\big(\eta^{k+l,k+l+q}\big)\big]^2\Big]\\
&\quad=\sum_{\xi}\mathscr{R}_N(\eta)g\big(\eta(k+l)\big)\frac{1}{2}\mathfrak{p}^N_{k+l,k'-l}\big[f\big(\xi^{k+l+q,k+l+q+1}\big)-f(\xi)\big]^2\\
&\quad\le C E_{\mathscr{R}_N}\Big[g\big(\eta(k+l+q)\big)\frac{1}{2}\Big\{\Big(\frac{1}{2}+\frac{\alpha^N_{k+l+q}}{N}\Big)+\frac{\phi_{k+l+q+1,N}}{\phi_{k+l+q,N}}\Big(\frac{1}{2}-\frac{\alpha^N_{k+l+q+1}}{N}\Big)\Big\}\\
&\qquad\times\big[f\big(\eta^{k+l+q,k+l+q+1}\big)-f(\eta)\big]^2\Big].
\end{aligned}
$$

From these observations, (7.8) follows.

Step 6. Inputting (7.8) into (7.7), and considering the conditional expectation of f with respect to $\Omega_{k,k',l}$ as in the 1-block estimate proof, for N large, we have

$$
\begin{aligned}
&(\gamma N)^{-1}\lambda_{N,l}\\
&\quad\le\sup_{f_{k,k',l}}\Big\{E_{\kappa_{k,k',l}}[V_{k,k',l,A}f_{k,k',l}]-\frac{1}{2\theta\gamma}E_{\kappa_{k,k',l}}\big[\sqrt{f_{k,k',l}}\big(-S_{k,k',l}\sqrt{f_{k,k',l}}\big)\big]\Big\},
\end{aligned}
$$

where the supremum is over densities $f_{k,k',l}$ with respect to $\kappa_{k,k',l}$.

Again, as in the proof of the 1-block estimate, decomposing $f_{k,k',l}\,d\mu_{k,k',l}$ along configurations with common total number j, we need only to bound

$$
\sup_{0\le j\le A(2l+1)}\sup_{f}\Big\{E_{\kappa_{k,k',l,j}}[V_{k,k',l,A}f]-\frac{1}{2\theta\gamma}E_{\kappa_{k,k',l,j}}\big[\sqrt{f}\big(-S_{k,k',l}\sqrt{f}\big)\big]\Big\},
$$

where the supremum is over densities f with respect to $\kappa_{k,k',l,j}$. When $j=0$, again the supremum vanishes.

Step 7. Consider the centered object

$$
\widehat{V}_{k,k',l,A}=V_{k,k',l,A}-E_{\kappa_{k,k',l,j}}[V_{k,k',l,A}].
$$

For $j\ge 1$, using the Rayleigh expansion (cf. page 375, [25]) where the inverse spectral gap is bounded by $C_{k,k',l,j}$ of $S_{k,k',l}$ (Lemma 7.1), and $\|\widehat{V}_{k,k',l,A}\|_\infty\le A$, we have

$$
\begin{aligned}
&E_{\kappa_{k,k',l,j}}[\widehat{V}_{k,k',l,A}f]-\frac{1}{2\theta\gamma}E_{\kappa_{k,k',l,j}}\big[\sqrt{f}\big(-S_{k,k',l}\sqrt{f}\big)\big]\\
&\quad\le\frac{2\theta\gamma}{1-4AC_{k,k',l,j}\theta\gamma}E_{\mu_{k,k;,l,j}}\big[\widehat{V}_{k,k',l,A}(-S_{k,k',l})^{-1}\widehat{V}_{k,k',l,A}\big]\\
&\quad\le\frac{2\theta\gamma C_{k,k',l,j}}{1-4AC_{k,k',l,j}\theta\gamma}E_{\kappa_{k,k',l,j}}\big[\widehat{V}^2_{k,k',l,A}\big]\to 0\text{ as }\theta\to 0.
\end{aligned}
$$

Step 8. For completion, we still need to estimate the centering term $E_{\kappa_{k,k',l,j}}[V_{k,k',l,A}]$. By Lemma 7.1,

$$
E_{\kappa_{k,k',l,j}}[V_{k,k',l,A}]\le {C_0}^jE_{\nu_{l,l,j}}[V_{k,k',l,A}].
$$

Note that this bound of $E_{\nu_{l,l,j}}[V_{k,k',l,A}]$ does not depend on N or θ. By adding and subtracting $j/(2(2l+1))$, we need only bound $E_{\nu_{l,l,j}}[|\eta^l(k)-j/(2(2l+1))|]$. By exchangeability and an equivalence of ensemble estimate (cf. page 355 [25]), the canonical variance

$$
E_{\nu_{l,l,j}}\big[\big|\eta^l(k)-j/(2(2l+1))\big|^2\big]
$$

$$
\begin{aligned}
&= O(l^{-1}) E_{\nu_{l,l,j}}\big[\big(\eta(k) - j/\big(2(2l+1)\big)\big)^2\big] \\
&\quad + O(1) E_{\nu_{l,l,j}}\big[\big(\eta(k) - j/\big(2(2l+1)\big)\big)\big(\eta(k+1) - j/\big(2(2l+1)\big)\big)\big]
\end{aligned}
$$

and further bounded by $C(A)\mathrm{Var}_{\nu_{l,l}^{j/(2(2l+1))}}(\eta^l(k))$ (recall $\nu_{l,l}^{\rho}$ defined before Lemma 7.1). This variance is of order $O(l^{-1})$, since the single site variance $\mathrm{Var}_{\nu_{l,l}^{j/(2(2l+1))}}(\eta(k))$ is uniformly bounded for $j/(2(2l+1)) \le A$. Hence, $\limsup_{N\uparrow\infty,\theta\downarrow 0} E_{\nu_{l,l,j}}[V_{k,k',l,A}]$ is of order $O(l^{-1/2})$, vanishing as $l \uparrow \infty$. This completes the proof. $\square$

8. Tightness of limit measures. In this section, we obtain tightness of the family of probability measures $\{Q^N\}_{N\in\mathbb{N}}$ on the trajectory space $D([0,T],\mathcal{M})$. We show that $\{Q^N\}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology on $D([0,T],\mathcal{M})$.

LEMMA 8.1. *$\{Q^N\}_{N\in\mathbb{N}}$ is relatively compact with respect to the uniform topology. As a consequence, all limit points Q are supported on weakly continuous trajectories π, that is, for $G \in C^\infty(\mathbb{T})$ we have $t \in [0,T] \mapsto \langle G, \pi_t\rangle$ is continuous.*

PROOF. To deduce that $\{Q^N\}$ is relatively compact with respect to uniform topology, we show the following items (cf. Theorem 15.5 in [5]):

1. For each $t \in [0,T]$, $\epsilon > 0$, there exists a compact set $K_{t,\epsilon} \subset \mathcal{M}$ such that

$$
\sup_N Q^N[\pi_\cdot^N : \pi_t^N \notin K_{t,\epsilon}] \le \epsilon. \tag{8.1}
$$

2. For every $\epsilon > 0$,

$$
\lim_{\gamma\to 0}\limsup_{N\to\infty} Q^N\Big[\pi_\cdot^N : \sup_{|t-s|<\gamma} d(\pi_t^N, \pi_s^N) > \epsilon\Big] = 0. \tag{8.2}
$$

We now consider (8.1). Notice that, for any $A > 0$, the set $\{\mu \in \mathcal{M} : \langle 1, \mu\rangle \le A\}$ is compact in $\mathcal{M}$. Since the total number of particles is conserved, we have $Q^N[\langle 1, \pi_t^N\rangle > A] = Q^N[\langle 1, \pi_0^N\rangle > A] \le \frac{1}{A}\mathbb{E}_N[\frac{1}{N}\sum_{k=1}^N \eta_0(k)]$. By (3.1), we have $\mathbb{E}_N[\frac{1}{N}\sum_{k=1}^N \eta_0(k)] \le C$ for some constant $C < \infty$ independent of N and A. Then the first condition (8.1) is checked by taking A large.

To verify the second condition (8.2), it is enough to show a counterpart of the condition for the distributions of $\langle G, \pi_\cdot^N\rangle$ where G is any smooth test function on $\mathbb{T}$ (cf. page 54, [25]). In other words, we need to show, for every $\epsilon > 0$,

$$
\lim_{\gamma\to 0}\lim_{N\to\infty} Q^N\Big[\pi_\cdot^N : \sup_{|t-s|<\gamma} |\langle G, \pi_t^N\rangle - \langle G, \pi_s^N\rangle| > \epsilon\Big] = 0. \tag{8.3}
$$

To this end, notice that $\langle G, \pi_t^N\rangle = \langle G, \pi_0^N\rangle + \int_0^t N^2 L\langle G, \pi_s^N\rangle\, ds + M_t^{N,G}$, then we only need to consider the oscillations of $\int_0^t N^2 L\langle G, \pi_s^N\rangle\, ds$ and $M_t^{N,G}$, respectively.

Recall the generator computation (4.1) and the notation $D_{N,k}^{G,s}$ in (4.2). As g grows at most linearly and $D_{N,k}^{G,s}$ is bounded (cf. (4.3)), we have

$$
\begin{aligned}
&\sup_{|t-s|<\gamma}\left|\int_s^t N^2 L\langle G, \pi_\tau^N\rangle\, d\tau\right| \\
&\quad = \sup_{|t-s|<\gamma}\left|\int_s^t \frac{1}{N}\sum_{1\le k\le N} D_{N,k}^{G,s} g\big(\eta_\tau(k)\big)\, d\tau\right| \\
&\quad \le g^* C_G \sup_{|t-s|<\gamma}\int_s^t \left\{\frac{1}{N}\sum_{1\le k\le N}\eta_\tau(k)\right\} d\tau = g^* C_G \gamma \frac{1}{N}\sum_{1\le k\le N}\eta_0(k).
\end{aligned}
$$

Recall that $\mathbb{E}_N[\frac{1}{N}\sum_{k=1}^N \eta_0(k)]$ is uniformly bounded in N. Then, by Markov inequality, we conclude that $Q^N[\sup_{|t-s|<\gamma} |\int_s^t N^2 L\langle G, \pi_\tau^N\rangle\, d\tau| > \epsilon]$ vanishes as $N \uparrow \infty$ and $\gamma \downarrow 0$.

We turn to the martingale $M_t^{N,G}$. By $|M_t^{N,G} - M_s^{N,G}| \le |M_t^{N,G}| + |M_s^{N,G}|$, we have $\mathbb{P}_N[\sup_{|t-s|<\gamma} |M_t^{N,G} - M_s^{N,G}| > \epsilon] \le 2\mathbb{P}_N[\sup_{0\le t\le T} |M_t^{N,G}| > \epsilon/2]$. Using Chebyshev and Doob's inequality, we further bound it by

$$\frac{8}{\epsilon^2}\mathbb{E}_N\Big[\Big(\sup_{0\le t\le T}|M_t^{N,G}|\Big)^2\Big] \le \frac{32}{\epsilon^2}\mathbb{E}_N\big[(M_T^{N,G})^2\big] = \frac{32}{\epsilon^2}\mathbb{E}_N\langle M^{N,G}\rangle_T.$$

By Lemma 4.1, $\mathbb{E}_N\langle M^{N,G}\rangle_T = O(N^{-1})$. Then we conclude

$$\lim_{\gamma\to 0}\lim_{N\to\infty}\mathbb{P}_N\Big[\sup_{|t-s|<\gamma}|M_t^{N,G} - M_s^{N,G}| > \epsilon\Big] = 0. \qquad \square$$

9. Properties of limit measures. By Lemma 8.1, the sequence $\{Q^N\}$ is relatively compact with respect to the uniform topology. Consider any convergent subsequence of Q^N and relabel so that $Q^N \Rightarrow Q$. We now consider absolute continuity and an energy estimate for trajectories under Q.

9.1. *Absolute continuity.* We now address absolute continuity and conservation of mass properties under Q.

LEMMA 9.1. *Q is supported on absolutely continuous trajectories: We have Q-a.s., for all $0 \le t \le T$, that $\pi_t(dx) = \rho(t,x)\,dx$ where $\int_{\mathbb{T}}\rho(t,x)\,dx = \int_{\mathbb{T}}\rho(0,x)\,dx$ with respect to a measurable, nonnegative ρ.*

PROOF. A standard proof, namely that of Lemma 1.6, page 73, [25], shows the first statement. The second follows directly from the weak convergence of Q^N to Q and the conservation of mass $\sum_{x\in\mathbb{T}_N}\eta_t(x) = \sum_{x\in\mathbb{T}_N}\eta_0(x)$. $\square$

9.2. *Energy estimate.* We now state an important "energy estimate" for the paths on which Q is supported. We follow the framework presented in Section 5.7 of [25]; however, there are major differences due to the inhomogeneous random environment. Previous bounds on the random environment developed in Section 2.1 will be useful in the argument.

PROPOSITION 9.2. *Q is supported on paths $\rho(t,x)\,dx$ with the property that there exists an $L^1([0,T]\times\mathbb{T})$ function denoted by $\partial_x\Phi(\rho(s,x))$ such that*

$$\int_0^T\int_{\mathbb{T}}\partial_x G(s,x)\Phi\big(\rho(s,x)\big)\,dx\,ds = -\int_0^T\int_{\mathbb{T}} G(s,x)\partial_x\Phi\big(\rho(s,x)\big)\,dx\,ds$$

for all G smooth on $[0,T]\times\mathbb{T}$.

A main ingredient for the proof of Proposition 9.2 is the following lemma. For $\epsilon > 0$, $\delta > 0$, $H(\cdot) \in C^1(\mathbb{T})$ and $N \in \mathbb{N}$, we define

$$\begin{aligned} W_N(\epsilon,\delta,H,\eta) := {} & \sum_{1\le x\le N}\frac{H(x/N)}{\epsilon N}\big[\Phi\big(\eta^{\delta N}(x)\big) - \Phi\big(\eta^{\delta N}(x+\epsilon N)\big)\big] \\ & - \frac{4}{c^2 N}\sum_{1\le x\le N}\frac{H^2(x/N)}{\epsilon N}\sum_{0\le k\le \epsilon N}\Phi\big(\eta^{\delta N}(x+k)\big) \\ & - \sum_{1\le x\le N}\frac{CH(x/N)}{cN}\Phi\big(\eta^{\delta N}(x)\big). \end{aligned}$$

Here, the constants c and C, as we recall from Lemma 2.2, come from the inequalities

$$0 < c \le \min_k \phi_{k,N} \le \max_k \phi_{k,N} \le 1 \quad \text{and} \quad \max_k |\phi_{k,N} - \phi_{k+1,N}| \le \frac{C}{N}.$$

LEMMA 9.3. *Let $\{H_j\}_{j\in\mathbb{N}}$ be a dense sequence in $C^{0,1}([0,T]\times\mathbb{T})$. Then there exists a constant K_0 such that for any $m \ge 1$, and $\epsilon > 0$,*

$$\limsup_{\delta\to 0}\limsup_{N\to\infty} \mathbb{E}_N\Big[\max_{1\le j\le m}\Big\{\int_0^T W_N(\epsilon,\delta,H_j(s,\cdot),\eta_s)\,ds\Big\}\Big] \le K_0.$$

Before going to the proof of the lemma, we turn to Proposition 9.2.

PROOF OF PROPOSITION 9.2. It follows from Lemma 9.3 that

$$\begin{aligned}E_Q\Big[&\sup_{H\in C^{0,1}([0,T]\times\mathbb{T})}\Big\{\int_0^T\int_{\mathbb{T}} \partial_x H(s,x)\Phi(\rho(s,x))\,dx\,ds\\ &-\frac{4}{c^2}\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds - \frac{C}{c}\int_0^T\int_{\mathbb{T}} H(s,x)\Phi(\rho(s,x))\,dx\,ds\Big\}\Big] \le K_0;\end{aligned}$$

cf. page 103, Lemma 7.2 in [25]. As a result, for Q-a.e. path $\rho(s,u)\,du$, there exists $B = B(\rho)$ such that, for all $H \in C^{0,1}([0,T]\times\mathbb{T})$,

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}} \partial_x H(s,x)\Phi(\rho(s,x))\,dx\,ds - \frac{4}{c^2}\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds\\ &\quad - \frac{C}{c}\int_0^T\int_{\mathbb{T}} H(s,x)\Phi(\rho(s,x))\,dx\,ds \le B.\end{aligned}$$

Notice that

$$\begin{aligned}&\int_0^T\int_{\mathbb{T}} H(s,x)\Phi(\rho(s,x))\,dx\,ds\\ &\quad\le \frac{1}{2}\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds + \frac{1}{2}\int_0^T\int_{\mathbb{T}} \Phi(\rho(s,x))\,dx\,ds\\ &\quad\le \frac{1}{2}\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds + \frac{g^*}{2}\int_0^T\int_{\mathbb{T}} \rho(s,x)\,dx\,ds\\ &\quad= \frac{1}{2}\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds + \frac{g^*T}{2}\int_{\mathbb{T}} \rho(0,x)\,dx.\end{aligned}$$

We obtain

$$\int_0^T\int_{\mathbb{T}} \partial_x H(s,x)\Phi(\rho(s,x))\,dx\,ds - C'\int_0^T\int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))\,dx\,ds \le B',$$

where $C' = \frac{4}{c^2} + \frac{C}{2c}$ and $B' = \frac{Cg^*T\int_{\mathbb{T}}\rho(0,x)\,dx}{2c} + B$. Now, the proof follows exactly from proof of Theorem 7.1, page 105 [25]. $\square$

We now return to the proof of Lemma 9.3.

PROOF OF LEMMA 9.3. By the replacement lemma (Lemma 5.1, and notice that $D^{G,t}_{N,k}$ can be replaced by any bounded function), it suffices to show that there exists constant K_0 such that for any $m \ge 1$ and $\epsilon > 0$,

$$\limsup_{N\to\infty} \mathbb{E}_N\Big[\max_{1\le j\le m}\Big\{\int_0^T \tilde{W}_N(\epsilon,H_j(s,\cdot),\eta_s)\,ds\Big\}\Big] \le K_0, \tag{9.1}$$

where

$$\tilde{W}_N(\epsilon, H(\cdot), \eta) := \sum_{1\le x\le N} \frac{H(x/N)}{\epsilon N}\big(g(\eta(x)) - g(\eta(x+\epsilon N))\big) - \frac{4}{c^2 N}\sum_{1\le x\le N}\frac{H^2(x/N)}{\epsilon N}\sum_{0\le k\le \epsilon N} g(\eta(x+k)) - \sum_{1\le x\le N}\frac{CH(x/N)}{cN} g(\eta(x)).$$

Let $\mathbb{1}_{A,N}(\eta) := \mathbb{1}_{\sum_{k=1}^N \eta(k)\le AN}$. Define $W_{A,N}(\epsilon, H(\cdot), \eta) := \tilde{W}_N(\epsilon, H(\cdot), \eta)\mathbb{1}_{A,N}(\eta)$.

As in the beginning of proof of the 1-block estimate, stated in Lemma 6.3, where we cut off high densities, assertion (9.1) holds provided we prove that

$$\limsup_{N\to\infty} \mathbb{E}_N\Big[\max_{1\le j\le m}\Big\{\int_0^T W_{A,N}(\epsilon, H_j(s,\cdot), \eta_s)\, ds\Big\}\Big] \le K_0. \tag{9.2}$$

To this end, by the entropy inequality, the expectation in (9.2) is bounded from above by

$$\frac{1}{N} H(\mu^N|\mathscr{R}_N) + \frac{1}{N}\ln \mathbb{E}_{\mathscr{R}_N}\Big[\exp\Big\{\max_{1\le j\le m}\Big\{N\int_0^T W_{A,N}(\epsilon, H_j(s,\cdot), \eta_s)\, ds\Big\}\Big\}\Big].$$

Since the relative entropy $H(\mu^N|\mathscr{R}_N) \le C_0 N$, we obtain the left-hand side of (9.2) is bounded from above by

$$C_0 + \max_{1\le j\le m}\limsup_{N\to\infty}\frac{1}{N}\ln \mathbb{E}_{\mathscr{R}_N}\Big[\exp\Big\{N\int_0^T W_{A,N}(\epsilon, H_j(s,\cdot), \eta_s)\, ds\Big\}\Big].$$

By the Feynman–Kac formula, for any fixed index j, the limsup term in previous expression is bounded from above by

$$\limsup_{N\to\infty}\int_0^T \sup_f\{E_{\mathscr{R}_N}[W_{A,N}(\epsilon, H_j(s,\cdot), \eta) f(\eta)] - N E_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})]\}\, ds,$$

where the supremum is over all f, which are densities with respect to $\mathscr{R}_N$. As $c \le \min_k \phi_{k,N}$ and α_k^N is bounded, the Dirichlet form $E_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})]$ (cf. (6.2)) is estimated as

$$\begin{aligned}\sum_{1\le x\le N}&\frac{1}{2}\Big(\phi_{x,N}\Big(\frac{1}{2}+\frac{\alpha_x^N}{N}\Big)+\phi_{x+1,N}\Big(\frac{1}{2}-\frac{\alpha_{x+1}^N}{N}\Big)\Big)\\ &\times E_{\mathscr{R}_N}[(\sqrt{f(\eta+\delta_x)}-\sqrt{f(\eta+\delta_{x+1})})^2]\\ &\ge \sum_{1\le x\le N} E_{\mathscr{R}_N}\Big[\frac{c}{4}(\sqrt{f(\eta+\delta_x)}-\sqrt{f(\eta+\delta_{x+1})})^2\Big].\end{aligned}\tag{9.3}$$

Here, we used, for each x, $E_{\mathscr{R}_N}[g(\eta(x)) f(\eta)] = \phi_{x,N} E_{\mathscr{R}_N}[f(\eta+\delta_x)]$, where δ_x stands for the configuration with the only particle at x; $\eta+\delta_x$ is the configuration obtaining from adding one particle at x to η.

It now remains to show, for all H in $C^{0,1}([0,T]\times\mathbb{T})$, that

$$E_{\mathscr{R}_N}[W_{A,N}(\epsilon, H(s,\cdot), \eta) f(\eta)] - N E_{\mathscr{R}_N}[\sqrt{f}(-S\sqrt{f})] \le 0. \tag{9.4}$$

We first compute that $E_{\mathscr{R}_N}[W_{A,N}(\epsilon, H(s,\cdot),\eta) f(\eta)]$ equals

$$
\begin{aligned}
&E_{\mathscr{R}_N}\Big[\sum_{1\le x\le N} \frac{H(x/N)}{\epsilon N}\big(g(\eta(x)) - g(\eta(x+\epsilon N))\big)\mathbb{1}_{A,N}(\eta) f(\eta)\Big] \\
&\quad - \frac{4}{c^2 N} E_{\mathscr{R}_N}\Big[\sum_{1\le x\le N} \frac{H^2(x/N)}{\epsilon N} \sum_{0\le k\le \epsilon N} g(\eta(x+k))\mathbb{1}_{A,N}(\eta)\Big] \\
&\quad - E_{\mathscr{R}_N}\Big[\sum_{1\le x\le N} \frac{CH(x/N)}{cN} g(\eta(x))\mathbb{1}_{A,N}(\eta)\Big] = I_1 + I_2 + I_3.
\end{aligned}
\tag{9.5}
$$

Notice that

$$
\begin{aligned}
E_{\mathscr{R}_N}\big[g(\eta(x))\mathbb{1}_{A,N}(\eta) f(\eta)\big] &= E_{\mathscr{R}_N}\big[\phi_{x,N}\mathbb{1}_{A,N}(\eta+\delta_x) f(\eta+\delta_x)\big] \\
&= E_{\mathscr{R}_N}\big[\phi_{x,N}\mathbb{1}_{A-1/N,N}(\eta) f(\eta+\delta_x)\big].
\end{aligned}
$$

Then the first expectation I_1 in (9.5) is written as

$$
\begin{aligned}
&\sum_{x=1}^{N} \frac{H(x/N)}{\epsilon N} E_{\mathscr{R}_N}\big[\big(\phi_{x,N} f(\eta+\delta_x) - \phi_{x+\epsilon N,N} f(\eta+\delta_{x+\epsilon N})\big)\mathbb{1}_{A-1/N,N}(\eta)\big] \\
&\quad \le \sum_{x=1}^{N} \frac{H(x/N)}{\epsilon N} E_{\mathscr{R}_N}\big[\big(f(\eta+\delta_x)(\phi_{x,N} - \phi_{x+\epsilon N,N})\big)\mathbb{1}_{A-1/N,N}(\eta)\big] \\
&\qquad + \sum_{x=1}^{N} \frac{H(x/N)}{\epsilon N} E_{\mathscr{R}_N}\big[\big(\phi_{x+\epsilon N,N}\big(f(\eta+\delta_x) - f(\eta+\delta_{x+\epsilon N})\big)\big)\mathbb{1}_{A-1/N,N}(\eta)\big] \\
&\quad = I_1^1 + I_1^2.
\end{aligned}
\tag{9.6}
$$

Using that

$$
0 < c \le \min_{1\le k\le N} \phi_{k,N} \le \max_k \phi_{k,N} \le 1 \quad \text{and} \quad \max_{1\le k\le N} |\phi_{k,N} - \phi_{k+1,N}| \le CN^{-1},
$$

the first sum I_1^1 on the right-hand side of (9.6) is bounded from above by

$$
\begin{aligned}
&\sum_{x=1}^{N} \frac{H(x/N)}{N} E_{\mathscr{R}_N}\Big[f(\eta+\delta_x)\frac{|\phi_{x,N} - \phi_{x+\epsilon N,N}|}{\epsilon}\mathbb{1}_{A-1/N,N}(\eta)\Big] \\
&\quad \le \sum_{x=1}^{N} \frac{CH(x/N)}{N} E_{\mathscr{R}_N}\big[f(\eta+\delta_x)\mathbb{1}_{A-1/N,N}(\eta)\big] \\
&\quad = \sum_{x=1}^{N} \frac{CH(x/N)}{N\phi_{x,N}} E_{\mathscr{R}_N}\big[g(\eta(x))\mathbb{1}_{A,N}(\eta) f(\eta)\big] \\
&\quad \le \sum_{x=1}^{N} \frac{CH(x/N)}{cN} E_{\mathscr{R}_N}\big[g(\eta(x))\mathbb{1}_{A,N}(\eta) f(\eta)\big].
\end{aligned}
\tag{9.7}
$$

Now, we proceed to the second sum I_2^1 in (9.6). Using $0 < c \le \min_k \phi_{k,N} \le \max_k \phi_{k,N} \le 1$, the sum is bounded from above by

$$
\sum_{x=1}^{N} \frac{H(x/N)}{\epsilon N} E_{\mathscr{R}_N}\big[\big(f(\eta+\delta_x) - f(\eta+\delta_{x+\epsilon N})\big)\mathbb{1}_{A-1/N,N}(\eta)\big],
$$

which is rewritten as

$$
\begin{aligned}
&E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\sum_{k=0}^{\epsilon N-1}\frac{H(x/N)}{\epsilon N}\big(f(\eta+\delta_{x+k})-f(\eta+\delta_{x+k+1})\big)\mathbb{1}_{A-1/N,N}(\eta)\Bigg]\\
&\quad= E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\sum_{k=0}^{\epsilon N-1}\frac{H(x/N)}{\epsilon N}\mathbb{1}_{A-1/N,N}(\eta)\big(\sqrt{f(\eta+\delta_{x+k})}+\sqrt{f(\eta+\delta_{x+k+1})}\big)\\
&\qquad\times\big(\sqrt{f(\eta+\delta_{x+k})}-\sqrt{f(\eta+\delta_{x+k+1})}\big)\Bigg].
\end{aligned}\tag{9.8}
$$

Using $2ab\le a^2+b^2$, for any $\beta>0$, (9.8) is bounded from above by

$$
\begin{aligned}
&E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\sum_{k=0}^{\epsilon N-1}\frac{H^2(x/N)}{2\epsilon N\beta}\mathbb{1}_{A-1/N,N}(\eta)\big(\sqrt{f(\eta+\delta_{x+k})}+\sqrt{f(\eta+\delta_{x+k+1})}\big)^2\Bigg]\\
&\quad+E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\sum_{k=0}^{\epsilon N-1}\frac{\beta}{2\epsilon N}\big(\sqrt{f(\eta+\delta_{x+k})}-\sqrt{f(\eta+\delta_{x+k+1})}\big)^2\Bigg].
\end{aligned}\tag{9.9}
$$

The first expectation in (9.9) is bounded from above by

$$
\begin{aligned}
&E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\sum_{k=0}^{\epsilon N-1}\frac{H^2(x/N)}{2\epsilon N\beta}2\big(f(\eta+\delta_{x+k})+f(\eta+\delta_{x+k+1})\big)\mathbb{1}_{A-1/N,N}(\eta)\Bigg]\\
&\quad=\sum_{x=1}^{N}\frac{H^2(x/N)}{\epsilon N\beta}\sum_{k=0}^{\epsilon N-1}E_{\mathscr{R}_N}\Bigg[\bigg(\frac{g(\eta(x+k))}{\phi_{x+k,N}}+\frac{g(\eta(x+k+1))}{\phi_{x+k+1,N}}\bigg)\mathbb{1}_{A,N}(\eta)f(\eta)\Bigg]\\
&\quad\le\sum_{x=1}^{N}\frac{2H^2(x/N)}{c\epsilon N\beta}\sum_{k=0}^{\epsilon N}E_{\mathscr{R}_N}\big[g\big(\eta(x+k)\big)\mathbb{1}_{A,N}(\eta)f(\eta)\big].
\end{aligned}\tag{9.10}
$$

The second expectation in (9.9) is rewritten and bounded, noting (9.3), as

$$
E_{\mathscr{R}_N}\Bigg[\sum_{x=1}^{N}\frac{\beta}{2}\big(\sqrt{f(\eta+\delta_x)}-\sqrt{f(\eta+\delta_{x+1})}\big)^2\Bigg]\le\frac{2\beta}{c}E_{\mathscr{R}_N}\big[\sqrt{f}(-S\sqrt{f})\big].\tag{9.11}
$$

Hence, collecting (9.7), (9.10) and (9.11), the sum $I_1=I_1^1+I_1^2$ is bounded by

$$
\begin{aligned}
&\sum_{x=1}^{N}\frac{CH(x/N)}{cN}E_{\mathscr{R}_N}\big[g\big(\eta(x)\big)\mathbb{1}_{A,N}(\eta)f(\eta)\big]\\
&\quad+\sum_{x=1}^{N}\frac{2H^2(x/N)}{c\epsilon N\beta}\sum_{k=0}^{\epsilon N}E_{\mathscr{R}_N}\big[g(x+k)\mathbb{1}_{A,N}(\eta)f(\eta)\big]+\frac{2\beta}{c}E_{\mathscr{R}_N}\big[\sqrt{f}(-S\sqrt{f})\big].
\end{aligned}
$$

Now, we set $\beta=cN/2$. Adding the last expression to I_2+I_3 in (9.5), we obtain (9.4). $\square$

10. Uniqueness of weak solutions. In this section, we present results on uniqueness of good weak solutions to the PDE (3.3). Recall Definition 3.4 of a "good" weak solution to (3.3).

THEOREM 10.1. *There exists at most one good weak solution to* (3.3).

PROOF. Let ρ be a good weak solution to (3.3). Since $\partial_x \Psi(\rho(s,x))$ exists, for all $G(s,x) \in C_c^\infty((0,T)\times\mathbb{T})$, we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}} \partial_s G(s,x)\rho(s,x)\,dx\,ds \\
&\quad = \int_0^T \int_{\mathbb{T}} \partial_x G(s,x)\bigl[\partial_x \Psi\bigl(\rho(s,x)\bigr) + K(x)\Psi\bigl(\rho(s,x)\bigr)\bigr]\,dx\,ds.
\end{aligned} \tag{10.1}$$

Define $\varphi(s,x) = \int_0^x \rho(s,u)\,du$. For any $G(s,x) \in C_c^\infty((0,T)\times\mathbb{T})$, let

$$F(s,x) := \int_0^x G(s,x)\,dx - x\int_{\mathbb{T}} G(s,x)\,dx.$$

Note that $F(s,x)$ is also in the space $C_c^\infty((0,T)\times\mathbb{T})$. Therefore, we may apply (10.1) for $F(s,x)$. The left-hand side, after integration by parts, becomes

$$\int_0^T \bigl[\partial_s F(s,x)\varphi(s,x)\bigr]\big|_0^1\,ds - \int_0^T \int_0^1 \partial_s\Bigl(G(s,x) - \int_0^1 G(s,x)\,dx\Bigr)\varphi(s,x)\,dx\,ds. \tag{10.2}$$

The term $\int_0^T [\partial_s F(s,x)\varphi(s,x)]|_0^1\,ds$ vanishes as the total mass $\int_{\mathbb{T}} \rho(s,x)\,dx$ is conserved. Then (10.2) can be rewritten as

$$\begin{aligned}
&-\int_0^T \int_0^1 \partial_s\Bigl(G(s,x) - \int_0^1 G(s,x)\,dx\Bigr)\varphi(s,x)\,dx\,ds \\
&\quad = -\int_0^T \int_0^1 \partial_s G(s,x)\Bigl(\varphi(s,x) - \int_0^1 \varphi(s,x)\,dx\Bigr)\,dx\,ds.
\end{aligned}$$

Define $\phi(s,x) = \varphi(s,x) - \int_0^1 \varphi(s,u)\,du$. We now have

$$\int_0^T \int_0^1 \partial_s G(s,x)\phi(s,x)\,dx\,ds = -\int_0^T \int_0^1 G(s,x)h(s,x)\,dx\,ds, \tag{10.3}$$

where

$$h(s,x) = \partial_x \Psi\bigl(\rho(s,x)\bigr) + K(x)\Psi\bigl(\rho(s,x)\bigr) - \int_0^1 K(x)\Psi\bigl(\rho(s,x)\bigr)\,dx.$$

By straightforward approximation, we obtain from (10.3), for any $G(\cdot) \in C_c^\infty(0,T)$ and $q(\cdot) \in L^\infty[0,1]$,

$$\int_0^T \partial_s G(s)\Bigl[\int_0^1 q(x)\phi(s,x)\,dx\Bigr]\,ds = -\int_0^T G(s)\Bigl[\int_0^1 q(x)h(s,x)\,dx\Bigr]\,ds \tag{10.4}$$

As $\phi(s,\cdot)$ and $h(s,\cdot)$ are both in $L^1([0,T];L^1(\mathbb{T}))$, equation (10.4) implies that $\frac{d}{ds}\phi(s,\cdot)$, the weak derivative of $\phi(s,\cdot)$, exists and $\frac{d}{ds}\phi(s,\cdot) = h(s,\cdot)$. Moreover, in terms of the Bochner integral (cf. [8], page 302),

$$\phi(t,\cdot) = \int_0^t \frac{d}{ds}\phi(s,\cdot)\,ds + \phi(0,\cdot). \tag{10.5}$$

Now, assume there are two good weak solutions ρ_1, ρ_2 and corresponding quantities ϕ_1, ϕ_2. If we show that $\phi_1 = \phi_2$, then it follows $\varphi_1(s,x) - \varphi_2(s,x) = \int_0^1(\varphi_1(s,u) - \varphi_2(s,u))\,du$ for all s, x. By conservation of mass, it holds that $\varphi_1(s,1) - \varphi_2(s,1) = 0$ for all s. Then we conclude $\varphi_1 = \varphi_2$, and hence, $\rho_1 = \rho_2$ a.e.

To this end, let $\overline{\phi} = \phi_1 - \phi_2$ and $\overline{\Psi}_{s,x} = \Psi(\rho_1(s,x)) - \Psi(\rho_2(s,x))$. Therefore, by Lemma 10.2, we obtain

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 (\overline{\phi}(t,x))^2\,dx - \frac{1}{2}\int_0^1 (\overline{\phi}(0,x))^2\,dx \\
&\quad = \int_0^t\int_0^1 \overline{\phi}(s,x)\big[\partial_x\overline{\Psi}_{s,x} + K(x)\overline{\Psi}_{s,x}\big]\,dx\,ds \\
&\qquad - \int_0^t\int_0^1 \overline{\phi}(s,x)\Big[\int_0^1 K(u)\overline{\Psi}_{s,u}\,du\Big]\,dx\,ds \\
&\quad = \int_0^t\int_0^1 \overline{\phi}(s,x)\big[\partial_x\overline{\Psi}_{s,x} + K(x)\overline{\Psi}_{s,x}\big]\,dx\,ds
\end{aligned}
\tag{10.6}
$$

as $\int_0^1 \overline{\phi}(s,x)\,dx = 0$. Notice, from Lemma 10.3, that

$$
\begin{aligned}
&\int_0^t\int_0^1 \overline{\phi}(s,x)\partial_x\overline{\Psi}_{s,x}\,dx\,ds \\
&\quad = -\int_0^t\int_0^1 \partial_x\overline{\phi}(s,x)\overline{\Psi}_{s,x}\,dx\,ds = -\int_0^t\int_0^1 (\partial_x\overline{\phi}(s,x))^2\Psi'_{s,x}\,dx\,ds.
\end{aligned}
$$

Here, we have applied the mean value theorem so that $\overline{\Psi}_{s,x} = \Psi'_{s,x}\partial_x\overline{\phi}(s,x)$.

Let A be such that $|K(x)| \le A < \infty$. Note that $\Psi(\cdot)$ is an increasing function and $0 \le \Psi'(\cdot) \le C_\Psi$. We have

$$
\begin{aligned}
&\int_0^t\int_0^1 \overline{\phi}(s,x)K(x)\overline{\Psi}_{s,x}\,dx\,ds \\
&\quad \le \int_0^t\int_0^1 A|\overline{\phi}(s,x)||\partial_x\overline{\phi}(s,x)|\Psi'_{s,x}\,dxds \\
&\quad \le \int_0^t\int_0^1 \Big[\frac{A^2}{2}(\overline{\phi}(s,x))^2\Psi'_{s,x} + \frac{1}{2}(\partial_x\overline{\phi}(s,x))^2\Psi'_{s,x}\Big]\,dx\,ds \\
&\quad \le \frac{A^2 C_\Psi}{2}\int_0^t\int_0^1 (\overline{\phi}(s,x))^2\,dx\,ds + \frac{1}{2}\int_0^t\int_0^1 (\partial_x\overline{\phi}(s,x))^2\Psi'_{s,x}\,dx\,ds.
\end{aligned}
$$

Putting together the above, from equation (10.6), we get

$$
\int_0^1 (\overline{\phi}(t,x))^2\,dx - \int_0^1 (\overline{\phi}(0,x))^2\,dx \le \frac{A^2}{2}C_\Psi\int_0^t\int_0^1 (\overline{\phi}(s,x))^2\,dx\,ds.
$$

Notice that $\overline{\phi}(0,x) = 0$. Then it follows from Gronwall's inequality that $\phi_1 = \phi_2$ as desired. □

Recall the formulation of h near (10.2).

LEMMA 10.2. *Let $\overline{h}(s,\cdot) := \frac{d}{ds}\overline{\phi}(s,\cdot)$. We have*

$$
\frac{1}{2}\int_0^1 (\overline{\phi}(t,x))^2\,dx - \frac{1}{2}\int_0^1 (\overline{\phi}(0,x))^2\,dx = \int_0^t\int_0^1 \overline{\phi}(s,x)\overline{h}(s,x)\,dx\,ds.
$$

PROOF. For each $n \in \mathbb{N}$, let $t_{k,n} = \frac{kt}{n}$, $k = 0, 1, \ldots, n-1$. Then

$$
\frac{1}{2}\int_0^1 \big[(\overline{\phi}(t,x))^2 - (\overline{\phi}(0,x))^2\big]\,dx
$$

$$= \frac{1}{2}\int_0^1 \sum_{k=0}^{n-1}\big[\big(\overline{\phi}(t_{k+1,n},x)\big)^2 - \big(\overline{\phi}(t_{k,n},x)\big)^2\big]\,dx$$

$$= \int_0^1 \sum_{k=0}^{n-1}\Big[\frac{\overline{\phi}(t_{k+1,n},x)+\overline{\phi}(t_{k,n},x)}{2}\int_{t_k}^{t_{k+1}} \overline{h}(s,x)\,ds\Big]\,dx$$

$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_0^1 \frac{\overline{\phi}(t_{k+1,n},x)+\overline{\phi}(t_{k,n},x)}{2}\overline{h}(s,x)\,dx\,ds$$

Define $\overline{\phi}_n(s,x) := \frac{\overline{\phi}(t_{k+1,n},x)+\overline{\phi}(t_{k,n},x)}{2}$ if $s \in [t_{k,n}, t_{k+1,n})$. Then, by weak continuity of $\rho(t,x)$, we have that $\overline{\phi}_n(s,x)$ converges a.e. to $\overline{\phi}(s,x)$ on $[0,T]\times\mathbb{T}$. By dominated convergence, noting that $\overline{h}(s,x)$ belongs to $L^1([0,T],L^1(\mathbb{T}))$, we have as $n\uparrow\infty$, that

$$\tag{10.7}\begin{aligned}\frac{1}{2}\int_0^1\big[\big(\overline{\phi}(t,x)\big)^2 - \big(\overline{\phi}(0,x)\big)^2\big]\,dx &= \int_0^t\int_0^1 \overline{\phi}_n(s,x)\overline{h}(s,x)\,dx\,ds\\ &\to \int_0^t\int_0^1 \overline{\phi}(s,x)\overline{h}(s,x)\,dx\,ds.\end{aligned}$$

Since the left-hand side of equation (10.7) is independent of n, the lemma is proved. $\square$

LEMMA 10.3. *We have* $\int_0^t\int_0^1 \overline{\phi}(s,x)\partial_x\overline{\Psi}_{s,x}\,dx\,ds = -\int_0^t\int_0^1 \partial_x\overline{\phi}(s,x)\overline{\Psi}_{s,x}\,dx\,ds$.

PROOF. We first extend the class of test functions in (3.4). Assume that $F:[0,T]\times[0,1]\mapsto\mathbb{R}$ satisfies the following: (1) F is measurable; (2) for any fixed s, $F(s,\cdot)$ is absolutely continuous; (3) there is a constant $C<\infty$ such that $|\partial_x F(s,x)|\le C$ for almost all s,x; (4) $\int_0^1 \partial_x F(s,x)\,dx = 0$ for all s.

Let $\tau_\epsilon(x)$ be the standard mollifier supported on $[-\epsilon,\epsilon]$. Define

$$F_\epsilon(s,x) = \int_0^T\int_{\mathbb{T}} F(s,u)\tau_\epsilon(x-u)\tau_\epsilon(s-q)\,du\,dq$$

with F extended to be 0 for $s\notin[0,T]$. By (3.4),

$$\int_0^T\int_{\mathbb{T}} F_\epsilon(s,x)\partial_x\overline{\Psi}_{s,x}\,dx\,ds = -\int_0^T\int_{\mathbb{T}} \partial_x F_\epsilon(s,x)\overline{\Psi}_{s,x}\,dx\,ds.$$

Taking $\epsilon\to 0$, as $\partial_x F$ (and therefore F) is bounded and $\overline{\Psi}_{s,x}$ is integrable, dominated convergence gives

$$\tag{10.8}\int_0^T\int_{\mathbb{T}} F(s,x)\partial_x\overline{\Psi}_{s,x}\,dx\,ds = -\int_0^T\int_{\mathbb{T}} \partial_x F(s,x)\overline{\Psi}_{s,x}\,dx\,ds.$$

We now extend the admissible test functions further from F to $\overline{\phi}$ as claimed in the lemma. Introduce a truncation on $\partial_x\overline{\phi}$:

$$(\partial_x\overline{\phi})_{A,s,u} = \begin{cases}\partial_x\overline{\phi}(s,u), & -A\le\partial_x\overline{\phi}(s,u)\le A,\\ A, & \partial_x\overline{\phi}(s,u)>A,\\ -A, & \partial_x\overline{\phi}(s,u)<-A.\end{cases}$$

Apply (10.8) with $F(s,x) = [\int_0^x(\partial_x\overline{\phi})_{A,s,u}\,du - x\int_0^1(\partial_x\overline{\phi})_{A,s,u}\,du]\mathbb{1}_{[0,t]}(s)$ to get

$$\begin{aligned}&\int_0^t\int_0^1\Big[\int_0^x(\partial_x\overline{\phi})_{A,s,u}\,du - x\int_0^1(\partial_x\overline{\phi})_{A,s,u}\,du\Big]\partial_x\overline{\Psi}_{s,x}\,dxds\\ &\quad= -\int_0^t\int_0^1\Big[(\partial_x\overline{\phi})_{A,s,x} - \int_0^1(\partial_x\overline{\phi})_{A,s,u}\,du\Big]\overline{\Psi}_{s,x}\,dx\,ds.\end{aligned}$$

As $|(\partial_x\overline{\phi})_{A,s,x}| \le |\partial_x\overline{\phi}(s,x)| = |\rho_1(s,x) - \rho_2(s,x)|$, by conservation of mass, and that $\partial_x\overline{\Psi}_{s,x}$ is integrable, we have by dominated convergence that

$$\begin{aligned}
&\lim_{A\to\infty}\int_0^t\int_0^1\Big[\int_0^x(\partial_x\overline{\phi})_{A,s,u}\,du - x\int_0^1(\partial_x\overline{\phi})_{A,s,u}\,du\Big]\partial_x\overline{\Psi}_{s,x}\,dx\,ds \\
&\quad= \int_0^t\int_0^1\Big[\int_0^x\partial_x\overline{\phi}(s,u)\,du - x\int_0^1\partial_x\overline{\phi}(s,u)\,du\Big]\partial_x\overline{\Psi}_{s,x}\,dx\,ds \\
&\quad= \int_0^t\int_0^1\overline{\phi}(s,x)\partial_x\overline{\Psi}_{s,x}\,dx\,ds.
\end{aligned}$$

Here, we used $\int_0^1\partial_x\overline{\phi}(s,u)\,du = \int_0^1(\rho_1(s,u) - \rho_2(s,u))\,du = 0$. Similarly,

$$\lim_{A\to\infty}\int_0^t\int_0^1\Big[\int_0^1(\partial_x\overline{\phi})_{A,s,u}\,du\Big]\overline{\Psi}_{s,x}\,dxds = 0.$$

Finally, notice that $(\partial_x\overline{\phi})_{A,s,x}\overline{\Psi}_{s,x} = (\partial_x\overline{\phi})_{A,s,x}\partial_x\overline{\phi}(s,x)\Psi'_{s,x}$ increases in A since $\Psi'_{s,x} \ge 0$ (cf. (2.6)). By monotone convergence, we have

$$\lim_{A\to\infty}\int_0^t\int_0^1(\partial_x\overline{\phi})_{A,s,x}\overline{\Psi}_{s,x}\,dx\,ds = \int_0^t\int_0^1\partial_x\overline{\phi}(s,x)\overline{\Psi}_{s,x}\,dx\,ds,$$

completing the proof. □

Funding. C. L. has been partly supported by FAPERJ CNE E-26/201.207/2014, by CNPq Bolsa de Produtividade em Pesquisa PQ 303538/2014-7 and by ANR-15-CE40-0020-01 LSD of the French National Research Agency. S. S. was partly supported by ARO-W911NF-18-1-0311 and a Simons Foundations Sabbatical grant.

REFERENCES

[1] ANDJEL, E. D. (1982). Invariant measures for the zero range processes. *Ann. Probab.* **10** 525–547. MR0659526

[2] ANDRIOPOULOS, G. (2021). Invariance principles for random walks in random environment on trees. *Electron. J. Probab.* **26** Paper No. 115. MR4309973 https://doi.org/10.1214/21-ejp687

[3] BAHADORAN, C., MOUNTFORD, T. S., RAVISHANKAR, K. and SAADA, E. (2021). Zero-range process in random environment. In *From Particle Systems to Partial Differential Equations*. *Springer Proc. Math. Stat.* **352** 51–77. Springer, Cham. MR4376180 https://doi.org/10.1007/978-3-030-69784-6_4

[4] BENOIS, O., KIPNIS, C. and LANDIM, C. (1995). Large deviations from the hydrodynamical limit of mean zero asymmetric zero range processes. *Stochastic Process. Appl.* **55** 65–89. MR1312149 https://doi.org/10.1016/0304-4149(95)91543-A

[5] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[6] BROX, T. (1986). A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.* **14** 1206–1218. MR0866343

[7] ETHIER, S. and KURTZ, T. (2005). *Markov Processes: Characterization and Convergence*. Wiley-Interscience, Hoboken, NJ.

[8] EVANS, L. C. (2010). *Partial Differential Equations*, 2nd ed. *Graduate Studies in Mathematics* **19**. Amer. Math. Soc., Providence, RI. MR2597943 https://doi.org/10.1090/gsm/019

[9] EVANS, M. R. and HANNEY, T. (2005). Nonequilibrium statistical mechanics of the zero-range process and related models. *J. Phys. A* **38** R195–R240. MR2145800 https://doi.org/10.1088/0305-4470/38/19/R01

[10] FAGGIONATO, A. (2007). Bulk diffusion of 1D exclusion process with bond disorder. *Markov Process. Related Fields* **13** 519–542. MR2357386

[11] FAGGIONATO, A. (2008). Random walks and exclusion processes among random conductances on random infinite clusters: Homogenization and hydrodynamic limit. *Electron. J. Probab.* **13** 2217–2247. MR2469609 https://doi.org/10.1214/EJP.v13-591

[12] FAGGIONATO, A. (2010). Hydrodynamic limit of zero range processes among random conductances on the supercritical percolation cluster. *Electron. J. Probab.* **15** 259–291. MR2609588 https://doi.org/10.1214/EJP.v15-748

[13] FAGGIONATO, A., JARA, M. and LANDIM, C. (2009). Hydrodynamic behavior of 1D subdiffusive exclusion processes with random conductances. *Probab. Theory Related Fields* **144** 633–667. MR2496445 https://doi.org/10.1007/s00440-008-0157-7

[14] FATKULLIN, I., SETHURAMAN, S. and XUE, J. (2020). On hydrodynamic limits of Young diagrams. *Electron. J. Probab.* **25** Paper No. 58. MR4095054 https://doi.org/10.1214/20-ejp455

[15] FUNAKI, T., HOSHINO, M., SETHURAMAN, S. and XIE, B. (2021). Asymptotics of PDE in random environment by paracontrolled calculus. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1702–1735. MR4291459 https://doi.org/10.1214/20-aihp1129

[16] GOLOSOV, A. O. (1986). Limit distributions for a random walk in a critical one-dimensional random environment. *Russian Math. Surveys* **41** 199–200.

[17] GONÇALVES, P. and JARA, M. (2008). Scaling limits for gradient systems in random environment. *J. Stat. Phys.* **131** 691–716. MR2398949 https://doi.org/10.1007/s10955-008-9509-z

[18] HU, Y., LÊ, K. and MYTNIK, L. (2017). Stochastic differential equation for Brox diffusion. *Stochastic Process. Appl.* **127** 2281–2315. MR3652414 https://doi.org/10.1016/j.spa.2016.10.010

[19] JARA, M., LANDIM, C. and SETHURAMAN, S. (2013). Nonequilibrium fluctuations for a tagged particle in one-dimensional sublinear zero-range processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 611–637. MR3112428 https://doi.org/10.1214/12-AIHP478

[20] JARA, M., LANDIM, C. and TEIXEIRA, A. (2011). Quenched scaling limits of trap models. *Ann. Probab.* **39** 176–223. MR2778800 https://doi.org/10.1214/10-AOP554

[21] JARA, M. and PETERSON, J. (2017). Hydrodynamic limit for a system of independent, sub-ballistic random walks in a common random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1747–1792. MR3729634 https://doi.org/10.1214/16-AIHP770

[22] JARA, M. D. and LANDIM, C. (2006). Nonequilibrium central limit theorem for a tagged particle in symmetric simple exclusion. *Ann. Inst. Henri Poincaré Probab. Stat.* **42** 567–577. MR2259975 https://doi.org/10.1016/j.anihpb.2005.04.007

[23] JARA, M. D., LANDIM, C. and SETHURAMAN, S. (2009). Nonequilibrium fluctuations for a tagged particle in mean-zero one-dimensional zero-range processes. *Probab. Theory Related Fields* **145** 565–590. MR2529439 https://doi.org/10.1007/s00440-008-0178-2

[24] KESTEN, H. (1986). The limit distribution of Sinaĭ's random walk in random environment. *Phys. A* **138** 299–309. MR0865247 https://doi.org/10.1016/0378-4371(86)90186-X

[25] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[26] LANDIM, C., SETHURAMAN, S. and VARADHAN, S. (1996). Spectral gap for zero-range dynamics. *Ann. Probab.* **24** 1871–1902. MR1415232 https://doi.org/10.1214/aop/1041903209

[27] LIGGETT, T. M. (1985). *Interacting Particle Systems. Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **276**. Springer, New York. MR0776231 https://doi.org/10.1007/978-1-4613-8542-4

[28] MATZAVINOS, A., ROITERSHTEIN, A. and SEOL, Y. (2016). Random walks in a sparse random environment. *Electron. J. Probab.* **21** Paper No. 72. MR3592203 https://doi.org/10.1214/16-EJP16

[29] MORRIS, B. (2006). Spectral gap for the zero range process with constant rate. *Ann. Probab.* **34** 1645–1664. MR2271475 https://doi.org/10.1214/009117906000000304

[30] NAGAHATA, Y. (2010). Spectral gap for zero-range processes with jump rate $g(x) = x^\gamma$. *Stochastic Process. Appl.* **120** 949–958. MR2610333 https://doi.org/10.1016/j.spa.2010.01.019

[31] NAGY, K. (2002). Symmetric random walk in random environment in one dimension. *Period. Math. Hungar.* **45** 101–120. MR1955197 https://doi.org/10.1023/A:1022354131403

[32] PACHECO, C. G. (2018). From the Sinai's walk to the Brox diffusion using bilinear forms. Available at arXiv:1605.02826v1.

[33] PETERSON, J. (2010). Systems of one-dimensional random walks in a common random environment. *Electron. J. Probab.* **15** 1024–1040. MR2659756 https://doi.org/10.1214/EJP.v15-784

[34] QUASTEL, J. (2006). Bulk diffusion in a system with site disorder. *Ann. Probab.* **34** 1990–2036. MR2271489 https://doi.org/10.1214/009117906000000322

[35] SEIGNOUREL, P. (2000). Discrete schemes for processes in random media. *Probab. Theory Related Fields* **118** 293–322. MR1800534 https://doi.org/10.1007/PL00008743

[36] SINAĬ, Y. G. (1982). The limit behavior of a one-dimensional random walk in a random environment. *Teor. Veroyatn. Primen.* **27** 247–258. MR0657919