ANGULAR MOMENTUM IN GRAVITATIONAL AND ELECTROMAGNETIC SCATTERING

by

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Dedication

Dedicated to Pritha Adhikary, who will
always gravitationally scatter towards me.

Contents

List of Figures

Abstract

Conservation laws are of fundamental importance in physics. In special relativity, the well known conserved quantity of angular momentum is paired with a lesser known conserved quantity we call “mass moment”, which reduces to center of mass in the Newtonian limit. In general relativity, there are severe difficulties even defining angular momentum and mass moment, especially in the context of radiation. The goal of this thesis is to better understand the nature of these conserved quantities in relativistic physics.

To gain insight, we will study the specific problem of weak-field, two-body, perturbative scattering for both gravity and electromagnetism. This problem is simple enough to be analytically tractable but complex enough to feature the radiation of angular momentum and mass moment. Previous calculations show that angular momentum is radiated at a perturbative order before energy and momentum. We calculate the full trajectories and find that the change in mechanical angular momentum of the two-particle system exactly balances the radiated angular momentum. We also calculate the shift in mass moment (a “scoot”) for the first time. However, we find a naive mismatch between the mechanical and radiated mass moment, which is reconciled by ensuring consistent choices of time slicing and asymptotic reference frame. There results establish a firm foundation for understanding angular momentum transfer in relativistic scattering.

1 Background and Motivation

In Newtonian mechanics, there are a few quantities that are conserved in closed systems: energy E, momentum $\boldsymbol{p}$, angular momentum $\boldsymbol{L}$, and the time-derivative of center of mass $\boldsymbol{R}$. In a system of I interacting particles of constant mass with a potential V that depends only on interparticle distances, these conservation laws are given respectively by

$$\frac{d}{dt}E = \frac{d}{dt}\left(V + \sum_I \frac{1}{2} m_I v_I^2\right) = 0 \tag{1.1a}$$

$$\frac{d}{dt}\boldsymbol{p} = \frac{d}{dt}\left(\sum_I m_I \boldsymbol{v}_I\right) = 0 \tag{1.1b}$$

$$\frac{d}{dt}\boldsymbol{L} = \frac{d}{dt}\left(\sum_I \boldsymbol{r}_I \times (m_I \boldsymbol{v}_I)\right) = 0 \tag{1.1c}$$

$$\frac{d^2}{dt^2}\boldsymbol{R} = \frac{d^2}{dt^2}\left(\frac{\sum_I m_I \boldsymbol{r}_I}{\sum_I m_I}\right) = 0. \tag{1.1d}$$

where m_I, $\boldsymbol{v}_I$, and $\boldsymbol{r}_I$ are the mass, velocity, and position of each particle, respectively. For gravitational interactions, V will be the gravitational potential energy, given by $V = \frac{Gm_1m_2}{r}$ for two particles, and for electromagnetic interactions, V will be the Coulomb potential energy, given by $V = \frac{kq_1q_2}{r}$ for two particles.

The last equation, (1.1d), for center of mass conservation, is different than the others since it involves a second derivative rather than a first. Additionally, the energy, momentum, and angular momentum of particles are additive while center of mass is not. To make the quantity additive, we can multiply by the total mass to get

$$\frac{d^2}{dt^2}\left(\sum_I m_I \boldsymbol{r}_I\right) = 0. \tag{1.2}$$

To write it in terms of a first derivative, parallel to the other equations, we can integrate both sides to get

$$\frac{d}{dt}\left(\sum_I m_I \boldsymbol{r}_I\right) = \boldsymbol{C}, \tag{1.3}$$

where $\boldsymbol{C}$ is some constant, which we recognize is the total momentum, giving us

$$\frac{d}{dt}\left(\sum_I m_I \boldsymbol{r}_I\right) - \boldsymbol{p} = 0. \tag{1.4}$$

Since the total momentum of the system is conserved, we can write

$$\frac{d}{dt}\left(\left(\sum_I m_I \boldsymbol{r}_I\right) - \boldsymbol{p}t\right) = 0, \tag{1.5}$$

and finally we can plug in our equation of momentum from (1.1b) to get

$$\frac{d}{dt}\left(\sum_I (m_I \boldsymbol{r}_I - m_I \boldsymbol{v}_I t)\right) = 0. \tag{1.6}$$

We define the "mass moment" of each particle to be

$$\boldsymbol{N}_I = m_I \boldsymbol{r}_I - m_I \boldsymbol{v}_I t = m_I \boldsymbol{r}_I - \boldsymbol{p}_I t \tag{1.7}$$

and equation (1.6) shows that the total mass moment of the system is conserved.

To understand this quantity better, we can write the total mass moment as

$$\boldsymbol{N} = \left(\sum_I m_I \boldsymbol{r}_I\right) - \boldsymbol{p}t \tag{1.8}$$

where $\boldsymbol{p}$ is the total momentum of the system. The mass moment can be thought of as "subtracting" off the momentum of the system to get the center of mass times total mass in the center of mass frame, which is conserved.

When we turn to special relativity, we find that there is no useful generalization of center of mass. Center of relativistic energy, a naive attempt to find an analog, defined by $\boldsymbol{C} = \frac{\sum_I E_I \boldsymbol{r}_I}{\sum_I E_I}$ is also not a conserved quantity, even in the Newtonian limit. However, there is a useful generalization of mass moment, just as there is for energy, momentum, and angular momentum.

The mechanical contribution to relativistic energy, momentum, and angular momentum, and mass moment of a system of particles of mass m_I and velocity $\boldsymbol{v}_I$ are defined by

$$E_{\text{mech}} = \sum_I \frac{m_I}{\sqrt{1 - v_I^2}} \tag{1.9a}$$

$$\boldsymbol{p}_{\text{mech}} = \sum_I \frac{m_I \boldsymbol{v}_I}{\sqrt{1 - v_I^2}} \tag{1.9b}$$

$$\boldsymbol{L}_{\text{mech}} = \sum_I \boldsymbol{r}_I \times \boldsymbol{p}_I \tag{1.9c}$$

$$\boldsymbol{N}_{\text{mech}} = \sum_I E_I \boldsymbol{r}_I - \boldsymbol{p}_I t, \tag{1.9d}$$

where we use units with $c = 1$ and the subscript "mech" refers to only the particles' contribution. When we include the contributions from the interactions terms, in the same way that we include potential energy in Newtonian mechanics, then these quantities will be conserved.

Since energy had one component while momentum, angular momentum, and mass moment all have three components, there are ten total conserved quantities. Energy and momentum together form a four-vector, known as four-momentum, given by

$$p^\mu = (E, \boldsymbol{p}), \tag{1.10}$$

while angular momentum and mass moment together form an anti-symmetric rank two tensor, often just called the angular momentum tensor, given by

$$M^{\mu\nu} = \begin{pmatrix} 0 & -N_x & -N_y & -N_z \\ N_x & 0 & L_z & -L_y \\ N_y & -L_z & 0 & L_x \\ N_z & L_y & -L_x & 0 \end{pmatrix}. \tag{1.11}$$

A four-vector has four free components while an anti-symmetric rank-two tensor has six free components, for a total of ten components. The fact that both p^μ and $M^{\mu\nu}$ transform like a Lorentz tensor shows that angular momentum is fundamentally related to mass moment the same way that momentum is fundamentally related to energy.

While equations (1.9) define the conserved quantities only for the particles themselves, we also must consider interaction terms. In this thesis we only consider electromagnetic and gravitational interactions between particles. Starting with electromagnetism, the conserved quantities carried by the electromagnetic fields are given in Gaussian units by

$$E_{EM} = \frac{1}{8\pi}\int d^3x(E^2 + B^2) \tag{1.12a}$$

$$\boldsymbol{p}_{EM} = \frac{1}{4\pi}\int d^3x(\boldsymbol{E} \times \boldsymbol{B}) \tag{1.12b}$$

$$\boldsymbol{L}_{EM} = \frac{1}{4\pi}\int d^3x\boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}) \tag{1.12c}$$

$$\boldsymbol{N}_{EM} = \frac{1}{8\pi}\int d^3x\left((E^2 + B^2)\boldsymbol{r} - 2(\boldsymbol{E} \times \boldsymbol{B})t\right), \tag{1.12d}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields. In the Newtonian limit for two particles, we find that the field contribution for momentum and angular momentum falls off to zero, the energy integral will reproduce the Coulomb interaction term found in Newtonian mechanics, and the mass moment will depend on our choice of origin but be conserved through the process.[1] Semantically, we just give them different names – in special relativity, we call the interaction term the "field contribution" and in Newtonian mechanics, we call it the "potential energy".

To calculate the interaction term in gravity, there are two fundamental, unique challenges that we must overcome. First, due to the equivalence principle, there is no definition of the conserved quantities from the gravitational fields analogous to (1.12). Mathematically, there is no tensor we

[1]Because the $\boldsymbol{B}$ field vanishes in the low velocity limit, momentum and angular momentum will also vanish. In a two particle system, to find the energy and the mass moment, we integrate the electric field squared and consider only the cross terms between the two particles' fields (we drop the self energy terms for the same reason that in Newtonian mechanics we don't consider the rest mass contribution to energy). To find the energy, we go to the center of energy and momentum frame, define the z-axis as the axis that connects the two particles, and use integration via cylindrical coordinates to reproduce the Coulomb potential. Finally the mass moment N_{EM} will be the potential energy multiplied by the position of the midway point of the two particles.

can construct from the first derivative of a metric.[2] Second, the special relativity formulas for angular momentum and mass moment depend on the position vector, and since position vectors are ill-defined in general relativity, there is no obvious generalization for the two quantities.

To overcome these challenges, we need a new approach to conserved quantities. In curved space-time, the symmetries of the spacetime metric determine the conserved quantities. If the space-time is asymptotically flat, we expect at least asymptotically to have the same conserved quantities as special relativity. However, Bondi-Metzner-Sachs (BMS) [2–4] surprisingly found that due to "supertranslations", there are an infinite number of asymptotic symmetries. Angular momentum and mass moment in flat space-time depend on the choice of origin and are non-invariant to translations. In curved space-time, angular momentum and mass moment are also non-invariant for "supertranslations", so any attempt to choose which angular momentum is the "correct" one will be arbitrary.

Our goal in this thesis is to better understand the nature of angular momentum. We will analyze the regime that we know the most about – the weak-field limit. Using perturbation theory, we can calculate analytical expressions that we wouldn't be able to calculate in full general relativity. To make it simpler, our system will have exactly two particles in it – since the three-body problem even in Newtonian mechanics remains unsolved. Finally, we look specifically at two *unbound* particles, such that the particles will experience an asymptotically flat space-time at early and late times. This process is known as Post-Minkowskian scattering, which we define mathematically in the next section. To help us understand some of the mysteries in gravity, we also study an electromagnetic analog of the problem. We know that gravity and electromagnetism are highly analogous in the weak field, low velocity limit; for example, the gravitational and electromagnetic force both take the form of the inverse square law. Since angular momentum and mass moment are better understood in electromagnetism, looking at parallels will also help us shed some light on some of the subtleties of angular momentum in general relativity.

[2]While no tensor can be constructed to calculate the energy and momentum of the "gravitational field", there are "pseudotensors" that will act as the "effective stress-energy tensor". These pseudotensors do not transform like a tensor as they are not gauge invariant, but in certain situations, linearized gravity as an example, they can be useful for calculating the total energy flux [1].

The thesis is organized as follows. In chapter 2, we will set up the problem of two-body scattering. In chapter 3, we will discuss the conserved quantities and summarize the main results of the thesis. The remaining three chapters consist of coauthored papers. In chapter 4, we will calculate the trajectories of two-body gravitational scattering and the change of energy, momentum, angular momentum, and mass moment. In chapter 5, we will consider an EM analog to understand some unexpected characteristics of Post-Minkowskian Scattering. In chapter 6, we will discuss the corresponding radiation of angular momentum from electromagnetic and gravitational scattering, and tie everything together to ensure that every conserved quantity is truly conserved. Throughout this thesis, we use geometrized units where the speed of light c as well as the gravitational and electromagnetic coupling constants (G and k, respectively) are all set equal to one.

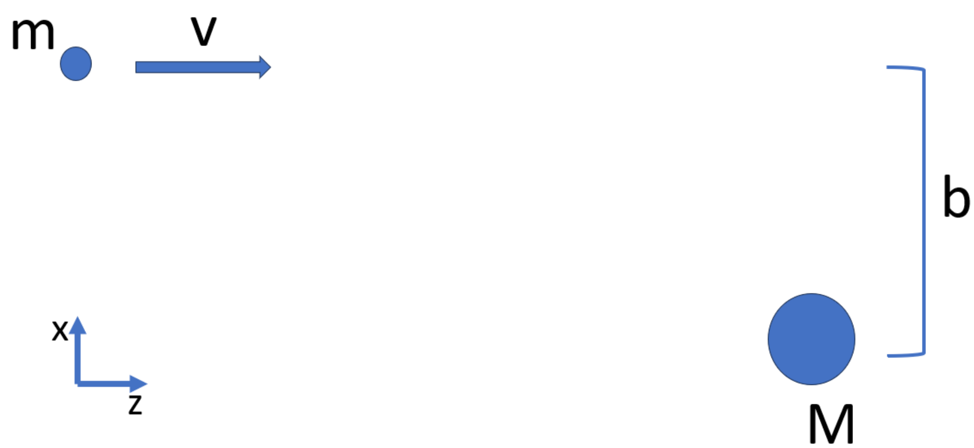

FIGURE 2.1: Two-particle Post-Minkowskian Scattering in the initial rest frame of a star of mass M. A particle of mass m approaches the star from infinity with impact parameter b and initial relative velocity v.

2 Post-Minkowskian Scattering

Gravitational scattering is a process where two particles approach each other from infinitely far away with impact parameter b (Figure 2.1). For the duration of this thesis, we will refer to the body with mass m as "the particle" and the body with mass M as "the star". We begin with the simplest case – where the star is much more massive than the particle ($M \gg m$), and we will increase the complexity of the problem throughout the chapter.

2.1 Unbound Geodesic Orbits

In the case where the star is much more massive than the particle, to leading order, the particle moves on a geodesic in the background metric of the star. This background metric is well known and called the Schwarzschild metric. In Schwarzschild coordinates, setting $c = G = 1$, the metric is given by

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + d\theta^2 + \sin^2\theta d\phi^2. \tag{2.1}$$

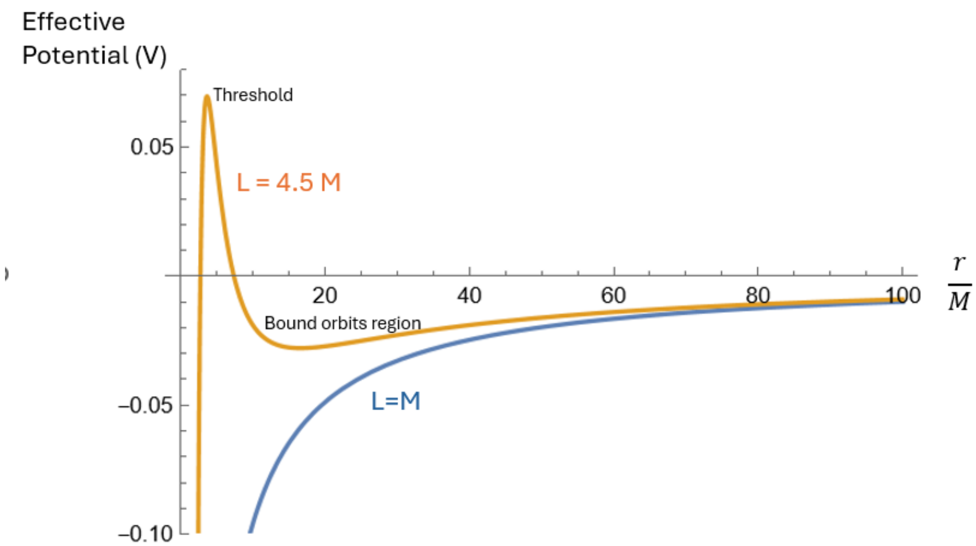

FIGURE 2.2: Effective potential for $L = M$ and $L = 4.5M$. In the former case, there are no unbound orbits and all particles would fall into the black hole. In the latter case, there are two circular orbits, one stable circular orbit at the local minimum in the bound orbits region and one unstable circular orbit at the local maximum at the threshold.

The particle traveling in this metric will follow the equations of motion given by [5]

$$\frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V(R) = \frac{1}{2}(E^2 - 1) \tag{2.2}$$

where E is the energy of the particle per mass given by $E = \gamma$ and τ is the proper time of the particle. The effective potential is given by

$$V(r) = -\frac{M}{R} + \frac{L^2}{2R^2} - \frac{ML^2}{R^3}, \tag{2.3}$$

where the first two terms are the Newtonian effective potential, and the third term is the relativistic correction. Here, L is defined as the angular momentum per unit mass, which is conserved throughout the process. In scattering, it is given by

$$L = \gamma v b, \tag{2.4}$$

where v is the initial velocity, and $\gamma = (1 - v^2)^{-1/2}$.

Figure 2.2 shows the potential for $L = M$ and $L = 4.5M$. There are three possible types of geodesics. One type is bound orbits, which occur when the particle is in the potential well, where the total energy is negative. The second type is the unbound scattering, which occurs if the initial energy is positive but below the peak value of the effective potential (threshold). The third type occurs when the energy is above the threshold and the particle will simply fall into the black hole. To determine whether the particle will scatter or fall into the black hole, we find the radius of unstable circular orbit where the effective potential will be maximized (R_*), plug in our initial angular momentum $L = \gamma v b$, and set $E = V(R_*)$. Solving the equation for velocity, we find that the threshold satisfies

$$v_* = \left(\frac{2MR_*^2}{-R_*^3 + 2MR_*^2 + b^2 R_* - 2Mb^2} \right)^{1/2}, \tag{2.5}$$

where the unstable circular orbit radius has the exact value

$$R_* = \frac{L}{M} \frac{L - \sqrt{L^2 - 12M^2}}{2}, \tag{2.6}$$

for $L > \sqrt{12}M$. If the velocity is lower than this threshold, it will fall into the black hole. If the velocity is greater than this threshold, it will scatter.[1]

Unfortunately, if we plugged in $L = \gamma_* v_* b$, into (2.6), we only have an implicit equation involving v_* and R_* which is difficult to solve. If $L > \sqrt{12}M$, then the value of R_* will always be in the range.

$$3M < R_* < 6M, \tag{2.7}$$

which means that the radius of closest approach is of order M. If $L < \sqrt{12}M$ (see $L = M$ plot in figure 2.2), then there will be no circular orbit and all particles will fall into the black hole.

We are mainly interested in perturbative scattering, where the deviation to the particle's trajectory due to curvature of space-time is small compared to the background motion. A necessary condition for perturbative scattering is $\frac{M}{b} \ll 1$, as if this criteria isn't met, then the particle wouldn't remain in the weak field at closest approach. From (2.7), $R_* \sim M$, so we can expand our threshold

[1] It is somewhat counter-intuitive that for the energy to below a certain amount the velocity must be above a certain amount, but we remind the reader that we are currently considering fixed angular momentum.

velocity from (2.5) to get

$$v_* \approx \sqrt{\frac{2MR_*^2}{b^2(R_* - 2M)}} \sim \frac{M}{b}. \tag{2.8}$$

If the velocity is much lower than the critical value, it will fall quickly into the black hole. If the velocity is slightly less than the critical value, it will approach the unstable circular orbit and eventually fall into the black hole. If the velocity is slightly higher than the critical value, it will approach the unstable circular orbit and eventually scatter back to infinity. Thus, if we want the particle to stay in the weak-field regime, the velocity must be much bigger than $\frac{M}{b}$,

$$v \gg \frac{M}{b}. \tag{2.9}$$

However, this is not enough to guarantee small angle scattering, as even in weak-field Newtonian mechanics, there can be large angle scattering. The other condition that needs to be met for perturbative scattering is for trajectories to be close to the background motion. One way to calculate the condition is to compare the radius of closest approach to the impact parameter. We can find the radius of closest approach by setting the effective potential equal to the energy, setting $\frac{dR}{d\tau} = 0$ in equation (2.2), and setting the energy equal to $E = \gamma$. Solving the cubic equation and then perturbing it at small $\frac{M}{b}$ gives us

$$R_0 = b - \frac{M}{v^2} + O\left(\frac{M^2}{b^2}\right) = b\left(1 - \frac{M}{bv^2} + O\left(\frac{M^2}{b^2}\right)\right). \tag{2.10}$$

It is interesting to note that the radius of closest approach in Newtonian mechanics is also given by $R_0 = b\left(1 - \frac{M}{bv^2}\right)$. In order for the radius of closest approach to be close to the impact parameter, the following condition must be met:

$$\frac{M}{bv^2} \ll 1. \tag{2.11}$$

If this condition is not met, meaning $\frac{M}{bv^2} \sim 1$, but the particle remains in the weak field $\frac{M}{b} \ll 1$, we are in the Post-Newtonian regime, where we also expand at low velocities, given $v^2 = O(\frac{M}{b})$.

To summarize, assuming $\frac{M}{b} \ll 1$, there are a few different types of motion that are possible:

$$v \lesssim \frac{M}{b} \qquad \text{strong field geodesics} \tag{2.12a}$$

$$v \sim \left(\frac{M}{b}\right)^{1/2} \qquad \text{Post-Newtonian large angle scattering} \tag{2.12b}$$

$$v \gg \left(\frac{M}{b}\right)^{1/2} \qquad \text{Post-Minkowskian small angle scattering} \tag{2.12c}$$

In the Post-Minkowskian (PM) approximation, we satisfy the condition given in (2.12c) by taking velocity to be order constant,

$$v^2 \sim 1, \quad \frac{M}{b} \ll 1. \tag{2.13}$$

2.2 Geodesic Trajectories

Although Schwarzschild coordinates work fine in the extreme mass ratio limit, when we move away from the extreme mass ratio limit, we will need to use a coordinate system that satisfies certain gauge conditions (Lorenz Gauge, see chapter 4 section 6). One set of coordinates that satisfies the Lorenz gauge condition are the isotropic, Cartesian coordinates. The metric in those coordinates is given by

$$ds^2 = -\left(\frac{1-\frac{M}{2r}}{1+\frac{M}{2r}}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 (dx^2 + dy^2 + dz^2), \tag{2.14}$$

where

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{2.15}$$

We will notice that for the Schwarzschild metric, in either coordinate system, as we take the distance from the star to infinity, the metric reduces to the Minkowski metric. This is known as asymptotic flatness. In isotropic coordinates, the metric will approach the Minkowski metric in Cartesian coordinates. The full trajectory of the particle, outside of the extreme mass ratio limit, will include the motion in the Schwarzschild metric described by isotropic coordinates, and so we outline that calculation in this section.

We can then set up the scattering process by defining the x-direction to be the direction of the impact parameter and the z-direction to be the direction of the initial velocity. The particle's background four-position is given by

$$z^\mu(t) = (t, b, 0, vt) + O\left(\frac{M}{b}\right). \tag{2.16}$$

One of our conditions for scattering that we discussed in the previous section is that the radius of closest approach is close to the impact parameter. Thus $\frac{M}{r} \lesssim \frac{M}{b}$. So, to second order in the Post-Minkowskian approximation, we can write the metric at the position of the particle as

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{2M}{r}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{M^2}{r^2}\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix} + O\left(\frac{M}{b}\right)^3. \tag{2.17}$$

Using this metric, we can calculate the geodesic of the particle. There are many ways to solve the geodesic equation, but we will use the Lagrangian from equation 6.46 of [6],

$$L = 1 - \sqrt{-g_{\mu\nu}\dot{z}^\mu \dot{z}^\nu}, \tag{2.18}$$

where $\dot{z}^\mu = \frac{d}{dt}z^\mu$, to solve the Euler-Lagrange equations perturbatively, order by order. The full geodesic trajectories up to 2PM are given in Appendix B of chapter 4. From these trajectories, we get a deflection angle of

$$\chi_{\text{grav}} = \frac{2M}{bv^2}(1+v^2) + \frac{M^2}{b^2v^2}\frac{3\pi}{4}(v^2+4) + O\left(\frac{M^3}{b^3}\right). \tag{2.19}$$

From the scattering angle, our condition in (2.11) for perturbative scattering is confirmed.

During this geodesic motion, energy and angular momentum are completely conserved. The direction of the momentum changes due to the deflection angle, but the magnitude remains the same. This will be the case not just to second order in the Post-Minkowskian expansion but to any higher order as well. So long as we fix the star in place and only calculate the trajectories of the particle in the background metric, no energy or momentum will ever be radiated away via gravitational waves. In order to see radiation of energy, momentum, angular momentum, and mass moment, we will have to consider more than just the geodesic motion by dropping the extreme mass ratio assumption.

2.3 Self-Force and Matter-Mediated Force

In the previous section, we considered a toy model with an extreme mass ratio $M \gg m$ to calculate the trajectory of the particle to zeroth order in $\frac{m}{M}$. For the rest of this thesis, we will relax that assumption and treat the two masses as the same order, $M \sim m$. This means that $\frac{M}{b} \ll 1$ also implies $\frac{m}{b} \ll 1$ and they can be treated symmetrically. Henceforth, as a matter of terminology, we say each of these parameters contributes 1PM order. So, to second order in PM (2PM), our metric will take the form

$$g = g_{0PM} + g_{1PM} + g_{2PM} + O(3\text{PM}) \tag{2.20a}$$

$$g_{0\text{PM}} = \eta \tag{2.20b}$$

$$g_{1\text{PM}} = \frac{M}{b} h_M + \frac{m}{b} h_m \tag{2.20c}$$

$$g_{2\text{PM}} = \frac{M^2}{b^2} h_{M^2} + \frac{Mm}{b^2} h_{Mm} + \frac{m^2}{b^2} h_{m^2}, \tag{2.20d}$$

where η is the Minkowski metric and h are each rank-two tensors that are order constant in the Post-Minkowskian approximation. The terms $\frac{M}{b} h_M$ and $\frac{M^2}{b^2} h_{M^2}$ will just be the Schwarzschild metric in isotropic coordinates given by (2.17), as we can see by taking the limit m goes to zero. Similarly, $\frac{m}{b} h_m$ and $\frac{m^2}{b^2} h_{m^2}$ also must reduce to the Schwarzschild metric of mass m boosted by the particle velocity $\boldsymbol{v}$.

To calculate the term h_{Mm} is a more difficult task. There are two contributions to this term to consider.

The first contribution comes from the perturbation to the star's stress energy tensor due to the presence of the particle. There is no way to hold the star fixed since the star itself must move in the background metric of the particle. The perturbation to the star due to the particle will be order m, and since the star's leading order stress energy is order M, the metric due to this term will have the coefficient Mm. The force of that perturbed metric on the particle is what [6] called the "matter-mediated force". This will be discussed at length in section five of chapter 4.

The other contribution to h_{Mm} comes from the self-force effects of the particle moving through curved space time. The presence of the star perturbs not just the particle but also the particle's field itself, and the particle's perturbed field can affect the particle. Since the particle's field is infinite at the position of the particle, to calculate the "force" from this metric requires regularization. After regularization, the resulting force is known as the "self-force" [7]. In flat-space time, the self-force is necessarily zero to ensure Newton's first law. In curved space-time, when the particle's field is modified by the curvature, the field is no longer isotropic around the particle, so the self-force can be non-zero.

By calculating the self-force and matter mediated force, we can still get an equation for the full trajectories. Similar to the metric, the 2PM trajectories of the particle can be given by

$$z = z_{0\text{PM}} + z_{1\text{PM}} + z_{2\text{PM}} + O(3\text{PM}) \tag{2.21a}$$

$$z_{0\text{PM}} = z_0 \tag{2.21b}$$

$$z_{1\text{PM}} = \frac{M}{b} z_{g1} \tag{2.21c}$$

$$z_{2\text{PM}} = \frac{M^2}{b^2} z_{g2} + \frac{Mm}{b^2}(z_{\text{sf}} + z_{\text{mm}}), \tag{2.21d}$$

where z is the trajectory of the particle (we present without the index for simplicity), z_0 (background), z_{g1} (1PM geodesic), z_{g2} (2PM geodesic), and z_{sf} (self-force), z_{mm} (matter mediated force) are all terms with dimensions of length that are order constant in the Post-Minkowskian expansion. Note, that z_m and z_{m^2} must be zero to maintain Newton's first law (gravitational self-force in flat space-time must equal zero). Since z_{g1} and z_{g2} don't depend on m, they are just the geodesic terms in the isotropic Schwarzschild metric, and this confirms that the 1PM motion of the particle is just its geodesic motion. In the extreme mass ratio limit, which is often done in the self-force literature, the second term is small compared to the first one in (2.21d). However, we do not make that assumption, so while we still separate the above terms for convenience, both terms can be considered of equal size. The full trajectories of the geodesic, self-force, and matter mediated force are given in Appendix B of chapter 4.

When we add the self-force and matter-mediated force trajectories, the scattering angle in (2.19) now becomes

$$\chi_{\text{grav}} = \frac{M}{bv^2}\left(2(1+v^2) + \frac{M+m}{b}\frac{3\pi}{4}(v^2+4)\right) + O(3\text{PM}). \tag{2.22}$$

Notice that for the 2PM terms, the coefficient for the M^2 term, calculated from the geodesic equation alone, and the Mm term, calculated by integrating the self force and matter mediated force, are identical. We will see in the next section that this means that total mechanical momentum of the system is conserved at 2PM.

2.4 Center of Energy and Momentum Frame

So far, all of our calculations have been done in the initial, asymptotic rest frame of the star.[2] This is a useful frame for calculations and for comparison with self-force literature using the extreme mass ratio limit. However, our ultimate goal for this problem is to have all of our solutions, for the trajectories and later the observables, completely symmetric in both masses. While the star's frame is useful for performing calculations, the center of energy and momentum (CEM) frame will be useful for understanding the results and seeing the symmetry between the star and the particle. We will define the CEM frame as the frame where the initial total momentum and mass moment are equal to zero.

[2] Even the star's initial position cannot be placed at the origin properly due to the logarithmically divergent trajectories of the star at early and late times, which will be discussed in chapters 4 and 5. To be more precise, we should say the background (0PM) position of the star is at the origin and the star approaches zero velocity at early times at any PM perturbative order.

To shift from the star's frame to the CEM frame, we perform a Lorentz boost and translation.[3] The Lorentz boost matrix will take the form,

$$\Lambda = \begin{pmatrix} \frac{M+\gamma m}{E} & 0 & 0 & -\frac{\gamma v m}{E} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v m}{E} & 0 & 0 & \frac{M+\gamma m}{E} \end{pmatrix}, \tag{2.23}$$

where γ and v refer to the relative velocity (or the particle velocity in the initial star's frame) and E refers to the total initial energy of the two particles in the CEM frame, given by

$$E = \sqrt{M^2 + m^2 + 2\gamma M m}. \tag{2.24}$$

As a consistency check, one can take the extreme mass ratio limit to check that the Lorentz boost matrix reduces to the identity matrix. In the initial CEM frame, the initial velocities of the star and particle are given by

$$\boldsymbol{v}_p = \frac{\gamma v M}{m + \gamma M}\hat{\boldsymbol{z}} \tag{2.25}$$

$$\boldsymbol{v}_s = -\frac{\gamma v m}{M + \gamma m}\hat{\boldsymbol{z}}. \tag{2.26}$$

After the boost, we are in a center of momentum frame, but this is not the CEM frame, as there is an asymmetry in the x position. To get into the CEM frame, we must also do a translation

$$x \to x - b\frac{m(\gamma M + m)}{E^2}. \tag{2.27}$$

[3] In chapter 4, we choose conditions such that this 0PM boost and translation is enough to put us in the CEM frame at all orders.

We can define the impact parameter as $b = |\boldsymbol{b}_p - \boldsymbol{b}_s|$ where

$$\boldsymbol{b}_p = \frac{M(M+\gamma m)}{E^2} b\hat{\boldsymbol{x}} \tag{2.28}$$

$$\boldsymbol{b}_s = -\frac{m(m+\gamma M)}{E^2} b\hat{\boldsymbol{x}}, \tag{2.29}$$

where $\boldsymbol{b}_p$ and $\boldsymbol{b}_s$ represent the initial x position of the particle and star, respectively. As a consistency check, if we take the extreme mass ratio, we can see that the particle will retain speed v with x-position b while the star's velocity and initial position go to zero.

We also notice in (2.25) and (2.26) that if we take the particle's velocity, switch the sign and exchange the masses, it equals the star's velocity. Here we see this symmetry at 0PM, and in our calculations it also holds at higher PM orders initially. Since there's nothing in this problem that can break the symmetry, we expect it to hold throughout the process.

Therefore, in the CEM frame, the trajectories are invariant under exchanging the masses of the particles ($M \to m$ and $m \to M$) and rotating by π radians about the y-axis. This is the nature of the CEM frame, and this symmetry will allow us to find the star's full 2PM trajectory as well. Thus, once we have the particle's full 2PM trajectory in the star's frame, we can first boost and translate the trajectories into the CEM frame and then apply this symmetry (exchanging the masses and rotating by π) to find the full 2PM trajectory of the star.

After that, we will be in the CEM frame that doesn't prefer one mass over the other, and we will have the trajectory of both the star and the particle. All that will be different about the two masses now is that we have arbitrarily chosen to call one the "star" and the other the "particle".

The scattering angle in the CEM frame is given by

$$\chi_{\text{grav}} = \frac{E}{bv^2}\left(2(1+v^2) + \frac{M+m}{b}\frac{3\pi}{4}(v^2+4)\right) + O(3\text{PM}). \tag{2.30}$$

Strictly speaking, this is the scattering angle of the particle in the CEM frame. However, we can see that the deflection angle is symmetric in the two masses. Thus, in the CEM frame, the deflection angle of the star will be the same. This confirms that the mechanical momentum is conserved and there is no radiation of momentum at 2PM.

Note that a condition for perturbative scattering is the deflection angle being small. From equation (2.30), we find that a physical condition for perturbative scattering is given by

$$\frac{E}{bv^2} \ll 1. \tag{2.31}$$

Notice that this parameter is symmetric in the masses, as we expect. From the definition of E (2.24) it is easy to see that $M < E$ and $m < E$ always. Thus, equation (2.31) is sufficient for our conditions $\frac{M}{bv^2} \ll 1$ and $\frac{m}{bv^2} \ll 1$.

2.5 The Electromagnetic Analog

The metric in general relativity is analogous to the electromagnetic four-potential except that it is a rank-two tensor rather than a rank-one tensor. There are two other differences between gravity and electromagnetism as it applies to our calculations. The first is that electromagnetism is a linear theory while gravity is non-linear – in a multiparticle system, the electromagnetic four-potential can be calculated by adding up the four-potentials of each particle, while the metric of space-time cannot be calculated by adding the "metric" of each particle.[4] Second, in electromagnetism, there is a clearly defined stress-energy tensor that we can calculate the energy, momentum, angular momentum, and mass moment of the electromagnetic fields with, whereas in gravity there is no such tensor (see chapter 1 and footnote 2). Since we have a better understanding of angular momentum and mass moment in electromagnetism, doing a comparison between the two theories can help us understand the lesser known gravitational angular momentum better.

If we start with Figure 2.1, we can define the particle with mass m to have charge q and the star with mass M to have charge Q. We assume that the charges are the same order as each other.

[4] In linearized gravity, the first order metric perturbations from each particle can be added together to get the total first order perturbation. However, this does not work for higher orders.

First, we deduce the conditions for perturbative scattering in electromagnetism. In the Post-Minkowskian approximation, our condition (2.31) can also be written as

$$\frac{Mm}{b^2}\frac{1}{m}\frac{Eb}{Mv^2} \ll 1. \tag{2.32}$$

Naively, the first fraction can be thought of as the maximum gravitational force (at closest approach). In the electromagnetic analog, the only thing that changes is that force, which will now be given by $\frac{Qq}{b^2}$. This will give us the parameter

$$\frac{Qq}{b^2}\frac{1}{m}\frac{Eb}{Mv^2} = \frac{Qq}{Ebv^2}\frac{E^2}{Mm} \ll 1. \tag{2.33}$$

Since the second fraction is order constant, the first fraction is the electromagnetic weak field parameter analogous to the Post-Minkowskian expansion parameter. Respectively, the Post-Minkowskian expansion and the electromagnetic PM analog expansion parameters can be given most simply by

$$\frac{E}{bv^2} \ll 1 \qquad \text{Gravity} \tag{2.34}$$

$$\frac{Qq}{Ebv^2} \ll 1 \qquad \text{Electromagnetism} \tag{2.35}$$

where we remind the reader that the $q \sim Q$, so that the $\frac{Q^2}{Ebv^2}$ and $\frac{q^2}{Ebv^2}$ are also taken as small.

Just like in gravity, we can calculate the electromagnetic trajectories up into three parts, which was done in [8]. The full 2PM electromagnetic scattering trajectories for the particle can be written as

$$z^{\mu}_{\text{0PMem}} = z_0 \tag{2.36a}$$

$$z^{\mu}_{\text{1PMem}} = \frac{Qq}{mb} z_{g1} \tag{2.36b}$$

$$z^{\mu}_{\text{2PMem}} = \frac{Q^2q^2}{m^2b^2} z_{g2} + \frac{Qq^3}{m^2b^2} z_{\text{sf}} + \frac{Q^2q^2}{Mmb^2} z_{\text{mm}}, \tag{2.36c}$$

where z_0, z_{g1}, z_{g2}, z_{sf}, and z_{mm} are of order constant, have dimensions of length, and are multiplied by a term that reveals the PM expansion order. For the 2PM equation, the first term is analogous to the geodesic, the second term is analogous to the gravitational self force, and the third term is analogous to the matter mediated force. The coefficients of each term will each match their analogous gravitational scattering term when replacing Q with M and q with m.

The $z_{\rm g1}$ and $z_{\rm g2}$ contributions can be calculated by fixing the star in place and integrating the electromagnetic force perturbatively order by order. The $z_{\rm mm}$ term can be found by first calculating the perturbation of the star due to the particle's electromagnetic field, then calculating the star's perturbed electromagnetic field due to that perturbation, and finally calculating the effect of the star's perturbed electromagnetic field on the particle at a later time. Lastly, $z_{\rm sf}$ is calculated from electromagnetic self-force given by the Abraham-Lorentz-Dirac (ALD) equation [7].

To understand the coefficients in equation (2.36), note that any correction to the particle's motion is proportional to PM factor $\frac{Qq}{mb}$ while any correction to the star's motion is proportional to PM factor $\frac{Qq}{Mb}$. Finally the correction from self force from the ALD equation is proportional to PM factor $\frac{q^2}{mb}$. Multiplying these relevant factors will reveal the coefficients in front of each piece of the trajectory.

While there are a lot of similarities between gravity and electromagnetism, there are also some key differences. For the "geodesic piece", in gravity the metric is non-linear in M, while in electromagnetism the electromagnetic field is linear in Q (the third term in (2.17) has no analog in EM). Also, the electromagnetic self-force and matter mediated force have different coefficients, while in gravity, those two forces have the same coefficient.

The full 2PM trajectories are given in [8].[5] In chapter 5, we discuss the electromagnetic analog as it pertains to surprising 1PM effects, and in chapter 6, we will use the 2PM trajectories to calculate observables.

2.6 Order Counting Guide

The gravitational and electromagnetic PM expansions we consider are equivalent to expansions in the coupling constants G and k, respectively. These constants are set equal to one in geometrized

[5] Unlike our calculations, Saketh didn't start in the frame of one of the particles, and calculated the trajectories as a function of proper time rather than coordinate time.

units and are not visible in expressions, but we can count orders by using dimensional analysis. In particular, for any given term we can factor out a quantity with the correct physical dimensions[6] and then identify PM parameters in the remainder.

For example, consider the change in momentum for the geodesic motion. From (2.30), to leading order, the change in momentum is given by

$$\Delta p^x = -\frac{2\gamma Mm}{bv^2}(1+v^2). \tag{2.37}$$

Physically, momentum has dimensions of mass, so we factor out one of the masses,

$$\Delta p^x = m\left(\frac{M}{b}\frac{-2\gamma(1+v^2)}{v^2}\right) \sim 1\text{PM}, \tag{2.38}$$

showing that the term is 1PM on account of $M/b \ll 1$ being multiplied by a term of order constant (involving only velocity).

The gravitational and electromagnetic potentials are another good example. Both M/b and Q/b are 1PM as potentials, but the reasoning looks slightly different:

$$\frac{M}{b} \sim 1\text{PM}, \tag{2.39}$$

$$\frac{Q}{b} = \frac{M}{Q}\frac{Q^2}{Mb} \sim 1\text{PM}. \tag{2.40}$$

In (2.39), we note that the gravitational potential is dimensionless (there is nothing to factor out) and we identify the PM parameter $M/b \ll 1$. In (2.40), we note that the electromagnetic potential has dimensions of energy per unit charge, so we factor out M/Q and identify the EM PM parameter $Q^2/Mb \ll 1$.

We use a 2PM change in mass moment as a final example. Consider for example a term proportional to Q^3q/Eb. Since mass moment has physical dimensions of mass times length, we

[6]Since the particle velocities are order constant in the PM expansion, distance and time may be considered to have the same physical dimensions for these purposes, but mass, length, and charge must be distinct.

factor out Eb,

$$\Delta N \propto \frac{Q^3 q}{Eb} = Eb\left(\frac{Q^2}{Eb}\frac{qQ}{Eb}\right) \sim 2\text{PM}. \tag{2.41}$$

This mass moment shift is 2PM on account of the factors $Q^2/Eb \ll 1$ and $qQ/Eb \ll 1$.

3 Angular Momentum and Mass Moment

3.1 Conserved Quantities

In special relativity our conservation laws are derived from the conservation of stress energy given by

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.1}$$

The ten conserved quantities (energy, momentum, angular momentum, and mass moment) arise from the ten symmetries of the flat spacetime metric underlying the theory. The infinitesimal symmetry transformations are described by "Killing vector fields" satisfying

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0. \tag{3.2}$$

The Killing fields associated to each conserved quantity are

$$\begin{array}{cccc} & \xi^\mu_{p_x} = (0,1,0,0) & \xi^\mu_{L_x} = (0,0,-z,y) & \xi^\mu_{N_x} = (x,t,0,0) \\ \xi^\mu_E = (1,0,0,0) & \xi^\mu_{p_y} = (0,0,1,0) & \xi^\mu_{L_y} = (0,z,0,-x) & \xi^\mu_{N_y} = (y,0,t,0) \\ & \xi^\mu_{p_z} = (0,0,0,1) & \xi^\mu_{L_z} = (0,-y,x,0) & \xi^\mu_{N_z} = (z,0,0,t). \end{array} \tag{3.3}$$

To find the conservation law for a given Killing field we notice that, from (3.1) and (3.2), we have

$$\partial_\mu (T^{\mu\nu} \xi_\nu) = 0. \tag{3.4}$$

From here, we can integrate over a 4-volume and use the divergence theorem,

$$\int_{\mathcal{B}} \sqrt{|h|} d^3x T^{\mu\nu} \xi_\mu n_\nu = 0, \tag{3.5}$$

where $\mathcal{B}$ is the 3-dimensional boundary surface, $\sqrt{|h|}d^3x$ is the induced volume element on $\mathcal{B}$, and n_μ is the outward unit normal vector.[1] While through a closed surface the total integral is equal to zero, if we are just integrating the integrand through one piece on that surface, the integral can be non-zero. When the piece of the surface is timelike (normal vector is spacelike), we interpret the integral as the total conserved quantity at the "time" represented by the surface. When the piece is spacelike (normal vector is timelike), it represents the flux of radiation through that surface.

3.2 Choice of Surfaces

Different choices of the boundary $\mathcal{B}$ correspond to different choices in tracking the flow of a conserved quantity. We make two main choices, illustrated in figure 3.1 and described in detail in chapter 6. In one choice, the "box", we use constant-t slices for the initial and final time, whereas in the other choice, the "puzzle piece", we use constant-τ slices where

$$\tau = \pm\sqrt{t^2 - x^2 - y^2 - z^2}, \tag{3.6}$$

with $+$ for the final time and $-$ for the initial time. Radiative fluxes are always calculated on constant-r surfaces (vertical lines on the diagram); the main difference is whether initial and final values of conserved quantities are calculated at constant t or constant τ.

When looking at the observables in the scattering process, we often want to calculate the change of these observables from the beginning of the scattering process to the end. Using the box, t goes to negative and positive infinity at early and late time, respectively. Similarly, using the puzzle piece, we can take τ to go to negative and positive infinity at early and late times, respectively. Since the only length scale present at fixed PM order is the impact parameter b, physically these

[1] The normal vector satisfies $n^\mu n_\mu = +1$ when spacelike and $n^\mu n_\mu = -1$ when timelike. We assume that the surface is nowhere null.

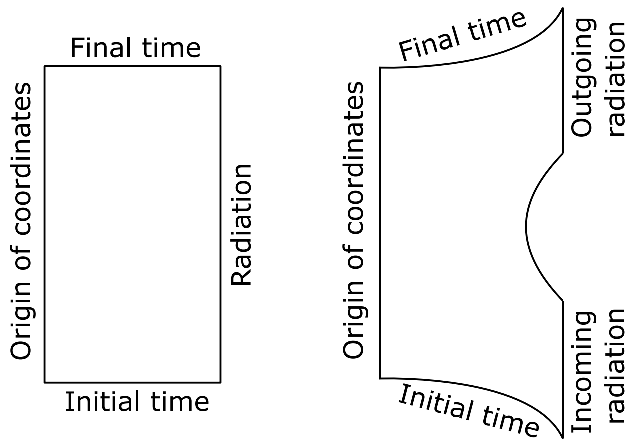

FIGURE 3.1: The “box” and the “puzzle piece” choices of integration region differ primarily in the choice of time slicing used to define the initial and final times. Full details are found in chapter 6.

limits correspond to

$$\frac{b}{|t|} \ll 1 \qquad \text{(box)} \tag{3.7}$$

$$\frac{b}{|\tau|} \ll 1. \qquad \text{(puzzle piece)} \tag{3.8}$$

When doing the early and late time expansions, we can say that taking the time parameter equal to positive or negative infinity is mathematically equivalent to taking the impact parameter to zero at constant PM order. The large-r expansion for calculating radiation is more subtle because the main contribution comes from t of order r. These limits are described in careful detail in chapter 6.

3.3 Flux in Electromagnetism

In electromagnetic scattering, we can write the stress energy tensor as a sum of the stress energy tensor of the particles and of the electromagnetic field separately,

$$T^{\mu\nu} = T^{\mu\nu}_{\text{mech}} + T^{\mu\nu}_{\text{EM}}. \tag{3.9}$$

The mechanical stress energy tensor is given by

$$T^{\mu\nu}_{\text{mech}} = \sum_I \gamma_I m_I v_I^\mu v_I^\nu \delta(\boldsymbol{x} - \boldsymbol{z}_I(t)), \tag{3.10}$$

where $v^\mu = (1, \mathbf{v})$, $\boldsymbol{x}$ is the spatial position and $\boldsymbol{z}$ is the trajectory of the particle. Dotting with a killing field (3.3) and integrating over a constant-t surface reproduces the formulas found in (1.9). Doing the same with a constant-τ surface gives the same formulas, except with the sum at fixed τ. At late times $\tau \to \infty$ when the particles move radially, this sum corresponds to adding particle conserved quantities together at the same proper time (instead of coordinate time).

The electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{3.11}$$

where F_{α_β} is the electromagnetic field tensor. Dotting with a Killing field and integrating over a constant-t surface reproduces the formulas found in (1.12). Doing the same with a constant-τ surface gives analogous formulas.

The detailed calculations of these integrals for electromagnetic scattering are done in chapter 5 with the box framework and chapter 6 with the puzzle piece framework. We comment here on one interesting aspect of the calculation.

Going back to equation (3.5), the Jacobian will be of order r^2, and the Killing Field order will depend on the conserved quantity. For energy and momentum, the Killing Vectors will be of order constant, and for angular momentum and mass moment, the Killing Vectors will be of order r. The interesting term to consider when performing the integral is the term of order constant, so for energy and momentum, we consider the components of the stress energy order $\frac{1}{r^2}$, and for angular momentum and mass moment, we consider the components order $\frac{1}{r^3}$.

From (3.11), the electromagnetic stress energy tensor is quadratic in the electromagnetic field tensor. The electromagnetic fields consist of a Coulombic term that is order $\frac{1}{r^2}$ and a radiative term that is order $\frac{1}{r}$. Thus, for energy and momentum, the interesting term to consider is the radiative term squared, while for angular momentum and mass moment, the interesting term is the cross product of the radiative term and the Coulomb term.

As an example, we can calculate the radiation of energy by,

$$E_{\text{rad}} = \int r^2 dt d\Omega T^{0i} n_i, \tag{3.12}$$

and the mass moment is given by

$$N^i_{\text{rad}} = \int r^3 dt d\Omega \left(\frac{t}{r} T^{ij} + n^i T^{0j} \right) n_j. \tag{3.13}$$

Here $i = 1, 2, 3$ represent spatial indices, $n^i = x^i/r$ is the radial unit vector and $d\Omega$ is the two-sphere area element. As can be seen, the relevant terms in the stress energy tensor will be order $(\frac{1}{r})^3$ for mass moment (and angular momentum) and order $(\frac{1}{r})^4$ for energy (and momentum).

As described in chapter 1, the above approach to conserved quantities does not work in gravitational theory. Instead, there are different definitions of conserved quantities that are useful in different contexts. In chapter 6 we consider these definitions and relate them to mechanical changes in the scattering problem.

3.4 Summary of Results

In this thesis, we did a variety of calculations all related to electromagnetic and gravitational scattering. In chapter 4, we calculated the gravitational self force of a particle moving at arbitrary velocity. At low velocities we find that it matches the Post-Newtonian result found in [6]. We also calculated the matter mediated force and checked our results against the low velocity limit found in [6]. Using the self force and the matter mediated force, along with the geodesic trajectories, we were able to calculate the full scattering angle in the center of mass frame for both the particle and the star (shown in equation (2.30)). This scattering angle matches the scattering angle first found by [9] in 1985. The scattering angle requires only the final momentum of the particle and the star.

We find that total energy and momentum are conserved throughout this process. Thus, for the interesting results, we must look at the position trajectories up to 2PM. We can split up the particle trajectory $\boldsymbol{z}(t)$ (where t is the coordinate time in the CEM frame) into the components perpendicular and parallel to the motion,

$$\boldsymbol{z}_{\parallel} = (\boldsymbol{z} \cdot \hat{\boldsymbol{p}})\hat{\boldsymbol{p}} \tag{3.14}$$

$$\boldsymbol{z}_{\perp} = \boldsymbol{z} - (\boldsymbol{z} \cdot \hat{\boldsymbol{p}})\hat{\boldsymbol{p}} \tag{3.15}$$

From our calculations, we notice $\Delta \boldsymbol{z}_{\perp p} = -\Delta \boldsymbol{z}_{\perp s}$, where p is for particle, s is for star, and Δ means final minus initial. We can then define the final impact parameter and change in impact parameter as

$$b_f = |\boldsymbol{z}^{+}_{\perp_p} - \boldsymbol{z}^{+}_{\perp_s}| = b + \Delta b \tag{3.16}$$

$$\Delta b = \frac{4\gamma M m}{b v^4}(1+v^2)\left(\frac{8}{3}v^3 - v + (1-3v^2)\operatorname{arctanh} v\right) \tag{3.17}$$

Since momentum is conserved and in the CEM frame the magnitude of the momentum of the particle and star are equal at early and late times, we can calculate the total angular momentum simply from

$$\Delta L = |\boldsymbol{p}^{+}|\Delta b = \frac{\gamma M m v}{E}\Delta b \tag{3.18}$$

where $\boldsymbol{p}^{+}$ and $\boldsymbol{p}^{-}$ refer to the particle's final and initial momentum respectively (the star's momentum will just be the opposite of the particle's). The magnitude is the same in the final state as it was initially $|\boldsymbol{p}^{+}| = \frac{\gamma M m b}{E}$.

Similarly, we can find the change in mass moment in both the parallel and perpendicular directions

$$\boldsymbol{N}_{\parallel} = \boldsymbol{N} \cdot \hat{\boldsymbol{p}} \tag{3.19}$$

$$\boldsymbol{N}_{\perp} = \boldsymbol{N} - (\boldsymbol{N} \cdot \hat{\boldsymbol{p}})\hat{\boldsymbol{p}}. \tag{3.20}$$

The results are

$$\Delta \boldsymbol{N}_{\parallel} = \frac{E(1-3v^2)}{v^3}(\boldsymbol{p}^+ + \boldsymbol{p}^-)\log\frac{\gamma_s}{\gamma_p} \tag{3.21}$$

$$|\Delta \boldsymbol{N}_{\perp}| = |E_p \Delta \boldsymbol{z}_{\perp p} + E_s \Delta \boldsymbol{z}_{\perp s}| = \frac{(E_p - E_s)\Delta b}{2} \tag{3.22}$$

where $\gamma_p = \frac{\gamma M + m}{E}$ $\gamma_p = \frac{M+\gamma m}{E}$, $E_p = \frac{m^2+\gamma M m}{E}$, and $E_s = \frac{M^2+\gamma M m}{E}$. Note that (3.18), (3.21), and (3.22) are respectively the same as equations 149, 159, and 160 of chapter 4 in a different form.

We refer to a change in mass moment as a "scoot", as in a person sitting on a chair and scooting forward: in this process the initial and final momentum are both zero but there is a shift in the position of the chair. The scoot in the direction of motion is distinct in that it begins at 1PM and it's entirely dependent on the choice of late-time expansion we do. If we expand instead in the proper time, large τ, the scoot in the direction parallel to motion will vanish.

To understand these issues better, we turn to the electromagnetic analog. We find that at 1PM, using t for the late time expansion, there is a similar mechanical scoot given by

$$\Delta \boldsymbol{N}_{\text{mech}} = -\frac{Qq}{\gamma^2 v^2}\log\frac{M+\gamma m}{m+\gamma M}(\hat{\boldsymbol{p}}^+ + \hat{\boldsymbol{p}}^-). \tag{3.23}$$

In electromagnetism, we are able to also calculate the stress energy tensor from the electromagnetic fields. Integrating over a constant t slice, we find that the mass moment in the fields scoots by the same exact value in the opposite direction as the mechanical scoot

$$\Delta \boldsymbol{N}_{\text{EM}} = \frac{Qq}{\gamma^2 v^2}\log\frac{M+\gamma m}{m+\gamma M}(\hat{\boldsymbol{p}}^+ + \hat{\boldsymbol{p}}^-). \tag{3.24}$$

Thus 1PM scoot is non-radiative in that it comes from the Coulomb field squared (a cross term between particle and star). Using constant-t slices, there is a non-radiative exchange of mass moment between particles and fields. On the other hand, if we use τ for the late-time expansion, we find that the mechanical and electromagnetic contributions individually vanish.

For this reason it is more convenient to present results using τ instead of t. Defined using τ, we find that the 2PM EM change in angular momentum and mass moment is

$$\Delta L = \frac{-4Qq\left(\gamma^2 v\left(-3MmQq + M^2q^2\gamma v^2 + m^2Q^2\gamma v^2\right) + 3MmQq\operatorname{arctanh} v\right)}{3EbMm\gamma^2 v^3} \tag{3.25}$$

$$|\Delta N_{\parallel}| = 0 \tag{3.26}$$

$$|\Delta N_{\perp}| = \frac{2qQ}{3b\gamma^3 EmMv^4}\Bigg(\gamma^2 v\Big(2\gamma^2 v^2\left(M^2q^2 - m^2Q^2\right) + 3qQ\left(M^2 - m^2\right) + 2\gamma mMv^2\left(q^2 - Q^2\right)\Big) + 3qQ\left(m^2 - M^2\right)\operatorname{arctanh} v\Bigg) \tag{3.27}$$

We also calculate the radiated angular momentum and mass moment, finding perfect agreement with these mechanical changes.

We can now circle back to the gravitational case and present those results using τ. For the trajectories, the effect is simply to remove the logarithmic parallel scoot, as expected from the electromagnetic analog. Comparing with radiated angular momentum and mass moment is more challenging due to ambiguities in these notions, but in chapter 6 we do find agreement for certain choices in the way these quantities are defined.

There are many similarities to notice between the change in mechanical angular momentum and mass moment between electromagnetism and gravity. In both cases, there is a 2PM scoot perpendicular to the motion, and a time-slicing dependent scoot in the direction of the motion. Both of these equations have a mysterious $\operatorname{arctanh} v$ that makes ΔL and $\Delta \boldsymbol{N}$ divergent in the ultra-relativistic limit. This signals the breakdown of our perturbation expansion at sufficiently large Lorentz factor.

The rest of the thesis consists of papers describing these calculations in detail. The notation is generally consistent with this section except that in the electromagnetic calculations we use m_1, m_2, q_1, and q_2 instead of m, M, q, and Q, respectively.

4 Self Force Effects in Post-Minkowskian Scattering

This is a paper I coauthored with Sam Gralla titled, ”Self-Force Effects in Post-Minkowskian Scattering”.

Self-force effects in post-Minkowskian scattering

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Abstract. We revisit the old problem of the self-force on a particle moving in a weak-field spacetime in the context of renewed interest in two-body gravitational scattering. We analytically calculate the scalar, electromagnetic, and gravitational self-force on a particle moving on a straight-line trajectory at a large distance from a Newtonian star, and use these results to find the associated correction to its motion. In the gravitational case we must also include the matter-mediated force, which acts at the same perturbative order as the gravitational self-force. We further augment the gravitational results with geodesic calculations at second order in the central body mass to determine the full, explicit solution to the two-body gravitational scattering problem at second post-Minkowskian order (2PM). We calculate the momentum transfer (which reproduces Westpfahl's old result), the change in mechanical angular momentum (which matches the radiative flux recently computed by Damour), and the change in mechanical mass moment (the time-space components of the angular momentum tensor), which has not previously appeared. Besides the new 2PM results of explicit trajectories and all conserved quantities, this work clarifies the role of gravitational self-force in PM scattering theory and provides a foundation for higher-order calculations.

1. Introduction

The study of self-force effects in curved spacetime began in 1960 with DeWitt and Brehme's foundational paper deriving the "tail integral" formula for electromagnetic self-force [1]. A few years later, DeWitt and DeWitt [2] evaluated this integral in the leading post-Newtonian (PN) approximation, i.e., for a charged particle moving slowly in the weak gravitational field of a point mass. This calculation was extended to the leading post-*Minkowskian* (PM) approximation (small-angle scattering with arbitrary initial velocity) by Westpfahl and Goller in 1980 [3] and the results were incorporated into Westpfahl's 1985 treatise on relativistic scattering [4]. These early PN and PM calculations illustrated the interesting physical effects of the self-force, but were (to quote the original) of "conceptual interest only, since the forces involved are far too small to be detected experimentally" [1].

In the 1990's new interest in the self-force emerged with the realization that *gravitational* self-force effects may in fact be of *practical* interest for gravitational-wave astronomy. The equations of gravitational self-force were formulated [5, 6] and it was realized that a scalar toy model [7] would provide a simpler starting point for computations. In light of these developments, Pfenning and Poisson [8] revisited the DeWitt-DeWitt PN calculation, clarifying the electromagnetic case and deriving analogous scalar and gravitational results. They illustrated the necessity of including additional "matter-mediated" forces in a consistent treatment of binary systems and connected the self-force method with the standard PN approach. Pfenning and Poisson's calculations clarified the role of the self-force formalism in binary dynamics and—in our view—were invaluable in establishing context for its more ambitious goal of providing accurate gravitational waveforms for relativistic binaries [9].

Motivated by the realization that gravitational *scattering* provides important information about binary dynamics and has deep connections to quantum processes and methods [10–33], in this paper we will analogously revisit the Westpfahl-Goller PM self-force calculations. We consider a particle moving on a nearly-straight trajectory at a great distance from a Newtonian star and calculate the leading effects of its (scalar, electromagnetic, or gravitational) self-field. We rederive the Westfpahl-Goller electromagnetic self-force and provide analogous scalar and gravitational results. For the gravitational case, we review the necessity of including the matter-mediated force and calculate it in closed form. We explicitly integrate the perturbed equations of motion in all three cases, providing parameterized trajectories in terms of elementary functions.

These results give the motion of the particle (mass m) in the initial rest frame of the star (mass M), meaning the frame where the star has asymptotically zero velocity in the infinite past. For gravitational scattering, we also determine the motion of the star by invoking the mass exchange symmetry of the problem. This requires augmenting our $O(Mm)$ self-force calculation with test-mass results at order $O(M^2)$ and changing to the center of energy-momentum (CEM) frame, where the symmetry is manifest as $M \leftrightarrow m$ together with rotation by π. This procedure yields the full parameterized trajectories of both bodies in 2PM scattering.

While the 2PM scattering problem has been studied by many different groups over the years, it appears that the full trajectories are a new result. From these trajectories we may compute any mechanical property of 2PM scattering. We rederive the momentum transfer (scattering angle) first computed by Westpfahl [4]. We compute the change in mechanical angular momentum, which matches the radiative angular momentum flux calculated by Damour [34]. We also compute the change in mechanical mass moment (the time-space component of the relativistic angular momentum tensor), a new result. We will be exploring further aspects of this change in mass moment in future publications [35, 36].

This paper is organized as follows. We begin with a prelude on the scattering angle that places our work in the context of the extensive recent interest in this quantity (Sec. 2). The remaining sections then derive the results, as follows. In Sec. 3 we review the construction of the scalar, electromagnetic, and gravitational retarded Green functions in a weakly curved spacetime at $O(M)$. In Sec. 4 we evaluate the tail integral to derive the self-forces at this order in the three cases. In Sec. 5 we derive the matter-mediated force that must be included in the gravitational case. In Sec. 6 we integrate the equations of motion and discuss physical quantities defined in the star frame. In Sec. 7 we add higher-order geodesic calculations and change to the CEM frame, providing the full 2PM trajectories and calculating associated physical quantities. Our metric signature is $-+++$ and we use Gaussian units with $G=c=1$. Covariant derivatives are denoted with a ∇ or a semicolon, while partial derivatives are denoted with a ∂ or a comma. Symmetrization is denoted with parentheses, e.g., $T_{(\alpha\beta)} = (1/2)(T_{\alpha\beta} + T_{\beta\alpha})$.

2. Prelude: scattering angle

The PM scattering angle has been the subject of intensive interest since Damour's 2016 analysis of its close relationship to the conservative dynamics of bound systems [10]. In order to place our work in the context of this active area of research, we now describe our approach through the lens of the scattering angle. Our calculation is organized in a joint perturbation series in the mass M of the star and the "charge" Q of the body, taken to be either the scalar charge q, the electric charge e, or the mass m. The relevant dimensionless parameters are

$$\frac{M}{bv^2} \ll 1, \qquad \frac{Q^2}{mb} \ll 1, \tag{1}$$

where b is the impact parameter and v is the initial relative velocity. We take the star to be at rest at zeroth order, and compute the leading self-force effects on the particle by evaluating the tail integral along straight line motion in the linearized Schwarzschild metric. In the gravitational case we must additionally solve for the $O(m/b)$ motion of the star and take into account the corrected gravitational forces on the particle—the so-called matter-mediated forces introduced by [8]. Integrating the equations gives the motion of the particle in the "star frame", meaning the frame where the star has asymptotically zero velocity in the distant past.

For the particle's scattering angle δ in the star frame, we find

$$\delta_{\rm scalar} = \frac{M}{bv^2}\left(2(1+v^2) - \frac{q^2}{mb}\frac{\pi}{4}(v^2+4\xi(1-v^2)) + \dots\right) + O\left(\frac{M^2}{b^2}\right) \tag{2}$$

$$\delta_{\rm em} = \frac{M}{bv^2}\left(2(1+v^2) - \frac{e^2}{mb}\frac{\pi}{4}(v^2+2) + \dots\right) + O\left(\frac{M^2}{b^2}\right) \tag{3}$$

$$\delta_{\rm grav} = \frac{M}{bv^2}\left(2(1+v^2) + \frac{m}{b}\frac{3\pi}{4}(v^2+4) + \dots\right) + O\left(\frac{M^2}{b^2}\right), \tag{4}$$

where ξ is the scalar coupling to curvature (see Eq. (7) below). To obtain the 2PM CEM-frame scattering angle χ, we first augment the $O(mM)$ result of (4) with an $O(M^2)$ test mass calculation. This geodesic contribution to the deflection $\delta_{\rm grav}$ turns out to have precisely the same numerical coefficient as the $O(Mm)$ term we computed:

$$\delta_{\rm grav} = \frac{M}{bv^2}\left(2(1+v^2) + \frac{m}{b}\frac{3\pi}{4}(v^2+4) + O\left(\frac{m^2}{b^2}\right)\right) + \frac{M^2}{b^2v^2}\frac{3\pi}{4}(v^2+4) + O\left(\frac{M^3}{b^3}\right). \tag{5}$$

Eq. (5) is the deflection angle of the particle as measured in the star frame. Denoting the CEM-frame deflection angle by χ, from a simple boost we find $\chi = (\tilde{E}/M)\delta$ at this order of approximation, where $\tilde{E} = \sqrt{M^2+m^2+2Mm\gamma}$ is the initial total energy in the initial CEM frame (with $\gamma = (1-v^2)^{-1/2}$ the initial Lorentz factor). We therefore derive

$$\chi = \frac{\tilde{E}}{bv^2}\left(2(1+v^2) + \left(\frac{m}{b}+\frac{M}{b}\right)\frac{3\pi}{4}(v^2+4)\right), \tag{6}$$

which is consistent to 2PM in the sense that it contains all terms that scale as λ^2 under $M \to \lambda M$ and $m \to \lambda m$. The parameter v is now interpreted as the relative velocity.

At this stage χ is the CEM-frame deflection angle *of the particle* (mass m). However, as the CEM frame is invariant under exchange of the two bodies, the CEM-frame deflection angle *of the star* (mass M) is determined by simply sending $m \leftrightarrow M$ in Eq. (6). But this formula is symmetric under the exchange, and we conclude that the star deflects by the same angle as the particle. We may therefore speak of *the* CEM-frame deflection angle at 2PM, given by Eq. (6). This reproduces Westpfahl's result [4].

It is worth emphasizing the logical role played by the exchange symmetry. The symmetry implies that $\chi_M = \chi_m(M \leftrightarrow m)$, where χ_M is the CEM-frame deflection angle of the mass M (the "star") and χ_m is the CEM-frame deflection angle of the mass m (the "particle"). In our approach, we *derive* that $\chi_M = \chi_m$ via direct calculation, resulting from the fact that the same coefficient $3\pi(v^2+4)/4$ appears in a self-force calculation and in a geodesic calculation. If one is willing to instead *assume* that $\chi_M = \chi_m$, then the agreement of these coefficients is guaranteed, and one can predict the self-force result from a geodesic calculation. Damour [17] has made this observation in the context of the "conservative dynamics" at all orders, where the two bodies by definition deflect by the same amount. While this is an enormously useful trick to obtain a portion of the dynamics from simple calculations, we emphasize that, without an independent argument that $\chi_M = \chi_m$, geodesic calculations alone cannot derive the scattering angle at 2PM or higher.

In this section we have given a preview of some results in a manner that illustrates key features of the approach. Our ensuing derivation of the complete 2PM trajectories follows a similar pattern: First, we first obtain the $O(Mm/b^2)$ trajectory of the particle in the star frame. Next we add in the $O(M^2/b^2)$ geodesic corrections. Finally, we calculate the particle trajectory in the the CEM frame and invoke the mass exchange symmetry to determine the trajectory of the star. We now describe these results.

3. Green functions in a weakly curved spacetime

In this section we review the construction of scalar, electromagnetic, and gravitational Green functions in a weakly curved spacetime. In any spacetime $g_{\mu\nu}$, these Green functions are defined by [8]

$$\Box G - \xi R G = -4\pi\delta_4(x, x'), \tag{7}$$

$$\Box G^{\alpha}{}_{\alpha'} - R^{\alpha}{}_{\beta} G^{\beta}{}_{\alpha'} = -4\pi\delta^{\alpha}{}_{\alpha'}\delta_4(x, x') \tag{8}$$

$$\Box \bar{G}^{\alpha\beta}{}_{\alpha'\beta'} + \mathcal{R}^{\alpha}{}_{\gamma}{}^{\beta}{}_{\delta}\bar{G}^{\gamma\delta}{}_{\alpha'\beta'} = -4\pi\delta^{(\alpha}{}_{\alpha'}\delta^{\beta)}{}_{\beta'}\delta_4(x, x') \tag{9}$$

where R is the Ricci scalar, $R_{\mu\nu}$ is the Ricci tensor, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, and

$$\mathcal{R}_{\mu\alpha\nu\beta} = 2R_{\mu\alpha\nu\beta} + 2R_{\mu(\alpha}g_{\beta)\nu} - R_{\mu\nu}g_{\alpha\beta} - 2R_{\alpha\beta}g_{\mu\nu} - Rg_{\mu(\alpha}g_{\beta)\nu} + Rg_{\mu\nu}g_{\alpha\beta}. \tag{10}$$

We have written $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ for the wave operator and $\delta_4(x, x') = \delta^{(4)}(x - x')/\sqrt{-g}$ for the invariant Dirac delta function. The scalar Green function G is a Green function for a massless scalar field with curvature coupling ξ. The electromagnetic Green function $G^{\alpha}{}_{\alpha'}$ is a Green function for the gauge field A^μ in Lorenz gauge, $\nabla_\mu A^\mu = 0$. The gravitational Green function $\bar{G}^{\alpha\beta}{}_{\alpha'\beta'}$ is a Green function for the trace-reversed metric perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - (1/2)h^{\mu}{}_{\mu}g_{\alpha\beta}$ in Lorenz gauge, $\nabla^\alpha \bar{h}_{\alpha\beta} = 0$. A Green function for the metric perturbation $h_{\mu\nu}$ is given by

$$G^{\alpha\beta}{}_{\alpha'\beta'} = \bar{G}^{\alpha\beta}{}_{\alpha'\beta'} - \frac{1}{2}g^{\alpha\beta}g_{\gamma\delta}\bar{G}^{\gamma\delta}{}_{\alpha'\beta'}. \tag{11}$$

Our normalization for the gravitational Green function follows Ref. [8], differing from the conventional one by a factor of four. (In particular, the integral $\int GT$ gives one-quarter the trace-reversed metric perturbation.) In all cases we consider the retarded Green function, i.e., the solution that vanishes when x' is not in the causal past of x.

We will construct these Green functions in a weakly curved spacetime following the approach of Pfenning and Poisson [8]. The spacetime is described by a static Newtonian potential $\Phi(x,y,z)$, with metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) + O(\Phi^2). \tag{12}$$

The Newtonian mass density is given by Poisson's equation,

$$\nabla^2\Phi = 4\pi\rho. \tag{13}$$

Pfenning and Poisson [8] define three key biscalars from which all the Green functions follow by differentiation. These are the retarded Green function in flat spacetime

$$G_{\text{flat}}(x,x') = \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}, \tag{14}$$

together with additional biscalars A and B defined by

$$A(x,x') = \frac{1}{2\pi}\int G_{\text{flat}}(x,x'')\Phi(x'')G_{\text{flat}}(x'',x')d^4x'' \tag{15}$$

$$B(x,x') = \int G_{\text{flat}}(x,x'')\rho(x'')G_{\text{flat}}(x'',x')d^4x''. \tag{16}$$

The scalar Green function is given by Eqs. (3.11) and (3.14) of Ref. [8],

$$G = G_{\text{flat}} - 2A_{,tt'} - 2\xi B + O(\Phi^2), \tag{17}$$

while electromagnetic and gravitational Green functions are similarly expressed in terms of A and B in Eqs. (3.17), (3.21), (3.30), and (3.32).‡

These results hold for any weak-field spacetime. We now specialize to a star (mass M) that is compactly supported within some radius $\mathcal{R}$ and consider the Green functions only at distant spatial points,

$$|\mathbf{x}|, |\mathbf{x}'| \gg \mathcal{R}. \tag{18}$$

The Newtonian potential at these distances can be approximated by $\Phi = -M/r$, but the integrals (15) and (16) defining the biscalars (and ultimately providing the Green functions) involve the entire spacetime, including regions where this approximation is invalid. However, it turns out nevertheless to be consistent to make this replacement,

$$\Phi \to -\frac{M}{r}, \qquad \rho \to M\delta(\mathbf{x}), \tag{19}$$

after which the integrals yield [1, 8]

$$A = -\frac{M}{R}\Theta(T-R)\begin{cases}\log\dfrac{r+r'+R}{r+r'-R} & T < r+r' \\ \log\dfrac{T+R}{T-R} & T > r+r',\end{cases} \tag{20}$$

$$B = \frac{M}{rr'}\delta(T-r-r'), \tag{21}$$

‡ For Eq. (3.21), the reader should refer to the arXiv version of [8], since the journal version contains a typographical error.

where $r = |\boldsymbol{x}|$, $r' = |\boldsymbol{x}'|$, $R = |\boldsymbol{x} - \boldsymbol{x}'|$, and $T = t - t'$. Note that with the substitution $\Phi = -M/r$, the metric (12) agrees with the Schwarzschild metric in isotropic coordinates, expanded to first order in M.

We now justify the substitution (19) under the approximation (18). For the B integral, we will rely on the careful arguments of Sec. IVD of Ref. [8], which derive Eq. (21) directly. For the A integral, we note that the portion of the integration region where Φ differs significantly from $-M/r$ is negligible in the approximation (18). This may be visualized noting that the the integration region in (15) (and (16)) may be identified with the ellipsoid $(\boldsymbol{x}'' - \boldsymbol{x})^2 + (\boldsymbol{x}'' - \boldsymbol{x}')^2 = (t - t')^2$ in Euclidean space parameterized by $\boldsymbol{x}''$, which has foci at the two spatial points $\boldsymbol{x}'$ and $\boldsymbol{x}''$. As these two foci are by assumption located at large distances from the region of the star (Eq. (18)), that region occupies only a parametrically small portion of the ellipsoid, which can be neglected in the integral at leading order. Formally, one may approximate the integral using matched asymptotic expansions with small parameter $b/\mathcal{R}$, where b is the greater of r and r', defining a near-zone near-zone $r'' \ll b$ and a far-zone $r'' \gg \mathcal{R}$. One finds that the near-zone contribution vanishes at leading order in $b/\mathcal{R}$.

We now illustrate the properties of the resulting Green functions, using the scalar case as an example. Plugging Eqs. (14), (20) and (21) in to Eq. (17) and dropping the $O(\Phi^2)$ error, one finds

$$G = G^{\text{direct}} + G^{\text{tail}} \tag{22}$$

with

$$G^{\text{direct}} = \frac{\delta(T-R)}{R} - \frac{2M}{R} \log \frac{r + r' + R}{r + r' - R} \delta'(T - R) \tag{23}$$

$$G^{\text{tail}} = \left(\frac{4M}{T^2 - R^2} - \xi \frac{2M}{rr'} \right) \delta(T - r - r') - \frac{8MT}{(T^2 - R^2)^2} \Theta(T - r - r'). \tag{24}$$

We have grouped the terms into the "direct" and "tail" pieces in the Hadamard decomposition (e.g., [37]).

The direct piece of the Green function by definition has support only on the past light cone (i.e., when a future-directed null geodesic runs from x' to x). Here we see this property reflected in a series expansion in M; the first term involves the flat spacetime past light cone $T = R$, while the term proportional to M gives the first correction. Consistent to this perturbative order, the direct term can equivalently be written

$$G^{\text{direct}} = \frac{1}{R} \delta(\Sigma), \qquad \Sigma = T - R - 2M \log \frac{r + r' + R}{r + r' - R}, \tag{25}$$

where $\Sigma = 0$ describes the past light cone of x. On general grounds, it is also possible to express this term as $G^{\text{direct}} = \sqrt{\Delta}\Theta(T)\delta(\sigma)$, where Δ is the Van Vleck determinant and σ is Synge's world function (one-half the squared geodesic distance) [37]. This form of the Green function was explored in Ref. [38].

The tail portion of the Green function has support within the past light cone. Generically this support extends throughout the past light cone, but in the weakly

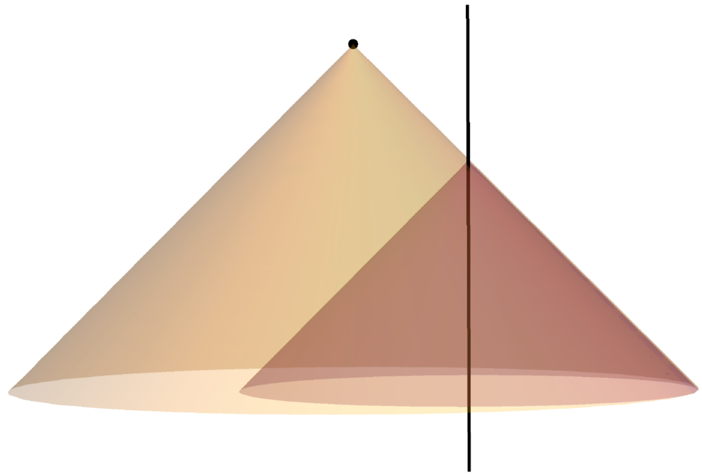

Figure 1. Support of the retarded Green function (scalar, electromagnetic, or gravitational) in the spacetime of a Newtonian star, when both points are at large spatial distances from the star. The star can be idealized as occupying a worldline at the spatial origin of coordinates, shown as a vertical line. The direct portion of the Green function has support on the past light cone, which can be represented as delta functions and derivatives on the flat-spacetime past light cone $T = R$ (gold). The tail portion of the Green function has support only on and within a secondary cone $T = r + r'$ emanating from the intersection of the past light cone with the star's worldline (beige).

curved spacetime when both points of Green function are at large distances from the star, the support is in fact only within a restricted region of the past light cone (Fig. 1). In particular, the tail portion vanishes when the spacetime points are sufficiently close together, becoming non-zero only when a signal has had time to "bounce" off the star ($T \geq r+r'$). In this approximation, therefore, the tail may be viewed as being caused by scattering of the direct part off of the singularity at $r = 0$. These properties extend to the electromagnetic and gravitational Green's functions, and in fact were first discussed in the electromagnetic case in the original work of DeWitt and DeWitt [1]. We are unaware of a deeper explanation for this surprising behavior of the retarded Green function.

4. Self-forces

Consider a point particle moving on a timelike geodesic $z^\mu(\tau)$ of a spacetime $g_{\mu\nu}$, parameterized by proper time τ with four-velocity u^μ. The scalar, electromagnetic, and gravitational self-forces are given by (in the conventions of Ref. [8])

$$f^{\alpha}_{(q)}(\tau) = q^2 \int_{-\infty}^{\tau^-} \left(\nabla^\alpha G + u^\alpha u^\beta \nabla_\beta G\right) d\tau', \tag{26}$$

$$f^{\alpha}_{(e)}(\tau) = e^2 \int_{-\infty}^{\tau^-} \left(\nabla^\alpha G_{\beta\alpha'} - \nabla_\beta G^{\alpha}{}_{\alpha'}\right) u^\beta u^{\alpha'} d\tau' \tag{27}$$

$$f^{\alpha}_{(m)}(\tau) = 2m^2 \int_{-\infty}^{\tau^-} \left(\nabla^\alpha G_{\beta\gamma\mu'\nu'} - 2\nabla_\gamma G^{\alpha}{}_{\beta\mu'\nu'} - u^\alpha u^\delta \nabla_\delta G_{\beta\gamma\mu'\nu'}\right) u^\beta u^\gamma u^{\mu'} u^{\nu'} d\tau'. \tag{28}$$

Here q is the scalar charge, e is the electric charge, and m is the mass; we denote the corresponding forces with a subscript featuring the associated symbol in parentheses. In these expressions, both spacetime points x and x' are evaluated on the worldline, i.e., $x = z(\tau)$ and $x' = z(\tau')$. For the bitenors ∇G and the tensor u, the prime (or lack thereof) on the index indicates which spacetime point is being considered. The upper limit τ^- indicates that the integral is to be stopped at $\tau' = \tau - \epsilon$ for some $\epsilon > 0$, with the limit $\epsilon \to 0$ taken after integration. This excludes the singular behavior of the Green function at coincidence, having the effect of picking out the tail contribution only.

We will consider the weakly curved spacetime (12) of a compact star and choose a geodesic that remains always distant from its typical radius $\mathcal{R}$. In computing the self-force, both points in the Green function are thus always in the large-distance regime (18), so we may use the definite expressions (20) and (21) for the biscalars underlying the Green functions. Letting b denote the impact parameter, we may summarize the key assumptions as

$$\Phi(x, y, z) \ll 1, \qquad b \gg \mathcal{R}. \tag{29}$$

That is, the spacetime is weakly curved everywhere, and the particle remains far from the star.

Under these approximations at first order beyond flat spacetime, the (scalar, electromagnetic, or gravitational) Green function is constructed from A and B given in Eqs. (20) and (21) and hence is sensitive only to the total mass M of the star. Thus, all results may be expressed in terms of a single small parameter,

$$\frac{M}{b} \ll 1. \tag{30}$$

We will organize our results in terms of this parameter; however, it should be borne in mind that the more restrictive conditions (29) must hold for the results to be valid.

To zeroth order in M/b, spacetime is flat and the particle geodesic is a straight line. We will choose the motion to be in the z direction, with separation from the star in the x direction,

$$z^\mu(\tau) = (t, b, 0, vt) + O(M/b), \tag{31}$$

where $v > 0$ is the (constant) velocity. The four-velocity is thus

$$u^\alpha = \gamma(1, 0, 0, v) + O(M/b), \tag{32}$$

where $\gamma = 1/\sqrt{1 - v^2}$.

4.1. Results

The tail portion of the Green function is non-zero first at $O(M/b)$, meaning that the $O(M/b)$ self-force depends only on the $O(1)$ straight-line motion of the charge. That is, for the purposes of computing the leading, $O(M/b)$ self-force, we may consistently take the particle to move on a straight line. Our task is therefore to compute the integrals (26), (27), and (28) for the worldline (31) and Green function constructed from (20) and (21) according to Eq. (17) and the description below.

The y component of the self-force is zero by symmetry (the motion is confined to the xz plane), while the t component is related to the z component by $f^t = vf^z$ according to the orthogonality condition $f^\mu u_\mu = 0$. We find that the x and z components can be expressed in terms of nine master integrals $\{\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i\}$ as

$$f^z_{(q)} = -2q^2\gamma^{-1}\left(\gamma^2 v\mathcal{A}_1 + \mathcal{A}_2 + \xi\gamma^2 v\mathcal{B}_1 + \xi\mathcal{B}_2\right) \tag{33}$$
$$f^x_{(q)} = -2q^2\gamma^{-1}\left(\mathcal{A}_3 + \xi\mathcal{B}_3\right) \tag{34}$$
$$f^z_{(e)} = -e^2\gamma\left(\mathcal{C}_3 - 2v\mathcal{A}_1 - 2\gamma^{-2}\mathcal{A}_2 - v\mathcal{B}_1 + (1+v^2)\mathcal{B}_2\right) \tag{35}$$
$$f^x_{(e)} = -e^2\gamma\left(\mathcal{C}_2 + v\mathcal{C}_1 - 2\gamma^{-2}\mathcal{A}_3 + (1+v^2)\mathcal{B}_3\right) \tag{36}$$
$$f^z_{(m)} = -2m^2\gamma\left(\gamma^{-2}\mathcal{A}_2 - 2\mathcal{C}_3 + v\mathcal{A}_1 + 2v(2+\gamma^2v^2)\mathcal{B}_1 - 2v^2\mathcal{B}_2\right) \tag{37}$$
$$f^x_{(m)} = -2m^2\gamma\left(\gamma^{-2}\mathcal{A}_3 - 2\mathcal{C}_2 - 2v\mathcal{C}_1 - 2v^2\mathcal{B}_3\right). \tag{38}$$

The definitions and results for the integrals are§

$$\mathcal{A}_1 = \int_{-\infty}^{t^-} \frac{d}{dt} A_{,tt'} dt' = \frac{Mv^2\left(r^2\left(1-3v^2\right) + 4rvz + z^2\left(v^2-3\right)\right)}{2r(r-vz)^4} \tag{39}$$
$$\mathcal{A}_2 = \int_{-\infty}^{t^-} A_{,tt'z} dt' = \frac{Mv\left(r-3vz\right)\left(r^2(1-3v^2) + 4rzv + z^2(v^2-3)\right)}{6r\left(r-vz\right)^5} \tag{40}$$
$$\mathcal{A}_3 = \int_{-\infty}^{t^-} A_{,tt'x} dt' = \frac{Mvb\left(2r^2v^3 - rz\left(1+3v^2\right) - \left(v^2-3\right)vz^2\right)}{2r(r-vz)^5}, \tag{41}$$

and

$$\mathcal{B}_1 = \int_{-\infty}^{t^-} \frac{d}{dt} B dt' = \frac{Mv\left(r^2v - 2rz + vz^2\right)}{r^3(r-vz)^2} \tag{42}$$
$$\mathcal{B}_2 = \int_{-\infty}^{t^-} B_{,z} dt' = -\frac{Mz\left(r^2\left(v^2+1\right) - 3rvz + v^2z^2\right)}{r^3(r-vz)^3} \tag{43}$$
$$\mathcal{B}_3 = \int_{-\infty}^{t^-} B_{,x} dt' = -\frac{Mb\left(r^2\left(v^2+1\right) - 3rvz + v^2z^2\right)}{r^3(r-vz)^3}, \tag{44}$$

§ Note that the form of the self-force in terms of the master integrals holds for any weakly curved spacetime, but the expressions we calculate for the integrals are valid only at large distances from a Newtonian star.

and

$$\mathcal{C}_1 = \int_{-\infty}^{t^-} \frac{d}{dt}(A_{,zx'} - A_{,xz'})dt' = \frac{Mv^2 b\,(2r^2 v - 3rz + vz^2)}{r^3(r - vz)^3} \tag{45}$$

$$\mathcal{C}_2 = \int_{-\infty}^{t^-} \frac{d}{dt}(A_{,xt'} - A_{,tx'})dt' = \frac{Mvb\,(2r^2 v - 3rz + vz^2)}{r^3(r - vz)^3} \tag{46}$$

$$\mathcal{C}_3 = \int_{-\infty}^{t^-} \frac{d}{dt}(A_{,zt'} - A_{,tz'})dt' = \frac{Mv\,(r^3\,(1 - 2v^2) + 3r^2 vz - 3rz^2 + vz^3)}{r^3(r - vz)^3}. \tag{47}$$

In formulating these integrals, we have changed variables from τ' to t', with the notation t^- indicating that the integration is to be stopped at $t^- + \epsilon$ for $\epsilon > 0$, after which the $\epsilon \to 0$ limit is to be taken. The unprimed total derivative is defined to be $d/dt = \partial_t + v\partial_z$, or equivalently ordinary differentiation after evaluation on the worldline. In these expressions, it is implicit that both the primed and unprimed points are to be evaluated on the worldline ($x = x' = b, y = y' = 0, z = vt, z' = vt'$) after derivatives are taken. The displayed results of these integrals are likewise to be evaluated at the present position of the particle, i.e., $z = vt$ and $r = \sqrt{b^2 + v^2 t^2}$. In this sense, the nine integrals are functions of t alone.

Our results for the electromagnetic self-force agree with Ref. [3]. Our results for the gravitational self-force are new, although analogous statements undoubtedly exist inside Westpfahl's 2PM calculation [4]. Our results for the scalar self-force are new.

4.2. Method of computation

We now describe our method of computation for the integrals (39)-(47). As explained in Sec. (3) above, the Green functions contain terms supported on $T = R$ (the direct portion, supported on the past light cone), terms supported on $T = r - r'$ (the part of the tail supported on the secondary light cone), and terms supported on $T < r - r'$ (the remainder of the tail). Given a spacetime point (t, x, y, z), the intersection of its past light cone with the worldline defines a retarded time t_1, while the intersection of its secondary light cone defines a "doubly retarded" time t_2. Since the direct portion of the Green function by construction does not contribute to the self-force, the important time is the doubly-retarded one,

$$t_2 = \gamma^2 \left(t - r - \sqrt{b^2 \gamma^{-2} + v^2 (t - r)^2} \right). \tag{48}$$

We also define doubly-retarded versions of some associated quantities,

$$r_2 = \sqrt{b^2 + v^2 t_2^2} \tag{49}$$

$$T_2 = t - t_2 \tag{50}$$

$$R_2 = \sqrt{(x - b)^2 + y^2 + (z - vt_2)^2}. \tag{51}$$

The delta functions $\delta(T - r - r')$ appearing in the Green function may be changed to the integration variable t' via

$$\delta(t - t' - r - \sqrt{b^2 + v^2 t'^2}) = \frac{\delta(t' - t_2)}{1 + v^2 \frac{t_2}{r_2}} \qquad (\text{if } t' < t), \tag{52}$$

where the restriction to $t' < t$ has allowed us to drop a second, unphysical root that does not contribute to the integral involving only $t' < t$.

Using Eqs. (48) and (52), the evaluation of the integrals is straightforward (especially for computer algebra software), but produces unduly complicated expressions. In order to obtain the simple forms given in Eqs. (39)-(47), we have made judicious choices (described below) of the order of differentiation, evaluation on the worldline, and integration, and also employed a number intermediate formulas holding after evaluation on the worldline. These formulas are

$$t_2 \overset{z}{=} \gamma^2 \left((1 + v^2)t - 2r\right), \tag{53}$$

$$r_2 \overset{z}{=} \gamma^2((1 + v^2)r - 2v^2 t), \tag{54}$$

$$r_2 + v^2 t_2 \overset{z}{=} r - v^2 t, \tag{55}$$

$$\partial_r t_2 \overset{z}{=} \frac{-r_2}{r - v^2 t}, \tag{56}$$

$$\partial_r r_2 \overset{z}{=} \frac{-v^2 t_2}{r - v^2 t}, \tag{57}$$

as well as

$$R \overset{z}{=} vT \tag{58}$$

$$T_2 \overset{z}{=} 2\gamma^2(r - v^2 t) \tag{59}$$

$$T_2^2 - R_2^2 \overset{z}{=} 4\gamma^2(r - v^2 t)^2 \tag{60}$$

$$\partial_x(T_2^2 - R_2^2) \overset{z}{=} \frac{-4\gamma^2 b}{r}((1 + v^2)r - 2v^2 t) \tag{61}$$

$$\partial_z(T_2^2 - R_2^2) \overset{z}{=} \frac{4\gamma^2 v}{r}(r^2 - rt(1 + 2v^2) + 2v^2 t^2), \tag{62}$$

where the symbol $\overset{z}{=}$ indicates that the equality holds when evaluated on the worldline $(x = b, y = 0, z = vt)$ after differentiation. The partial derivative with respect to r is used only for t_2 and r_2, in which case it means that t is held fixed.

We now illustrate the method of computation with a few examples, starting with $\mathcal{B}_1$. Plugging Eq. (21) into the integral in (42), we have

$$\mathcal{B}_1 = \left(\int_{-\infty}^{t} \frac{d}{dt} \left. \frac{M\delta(t - t' - r - r')}{rr'} \right|_{r' = \sqrt{b^2 + v^2 t'^2}} dt' \right) \Bigg|_{x=b,\ y=0,\ z=vt}, \tag{63}$$

where we remind the reader that $d/dt = \partial_t + v\partial_z$. Since the integrand vanishes in a neighborhood of $t' = t$ (having support only at $t' = t - r - r'$), we have replaced the original upper limit t^- with t. We can also pull the time derivative out of the integral for the same reason. Furthermore, for this total derivative we may evaluate on the worldline "early," i.e.

$$\mathcal{B}_1 = \frac{d}{dt}\int_{-\infty}^{t} \frac{M\delta(t-t'-r-r')}{rr'}\bigg|_{\substack{r'=\sqrt{b^2+v^2t'^2}\\ r=\sqrt{b^2+v^2t^2}}} dt'. \tag{64}$$

Using the delta function change of variables (52), the integral is calculated to be

$$\begin{aligned}
\mathcal{B}_1 &= \frac{d}{dt}\int_{-\infty}^{t} \frac{M}{rr'}\frac{\delta(t'-t_2)}{1+v^2\frac{t'}{r'}}\bigg|_{\substack{r'=\sqrt{b^2+v^2t'^2}\\ r=\sqrt{b^2+v^2t^2}}} dt' &(65)\\
&= \frac{d}{dt}\left(\frac{M}{r}\frac{1}{r_2+v^2t_2}\bigg|_{r=\sqrt{b^2+v^2t^2}}\right) &(66)\\
&= \frac{d}{dt}\left(\frac{M}{r(r-v^2t)}\bigg|_{r=\sqrt{b^2+v^2t^2}}\right) &(67)\\
&= -\frac{Mv^2(b^2+2t(tv^2+r))}{r^3(r-v^2t)^2}\bigg|_{r=\sqrt{b^2+v^2t^2}}, &(68)
\end{aligned}$$

where we use (55) in the third step. Eq. (68) reproduces the claimed result (42) after use of $z = vt$.

The integrals for $\mathcal{B}_2$ and $\mathcal{B}_3$ proceed similarly, except that the evaluation on the worldline must now be done after the unprimed derivative is taken. After moving the derivative outside the integral and performing the integral using Eq. (52), we are left with

$$\mathcal{B}_2 = \frac{\partial}{\partial z}\left(\frac{M}{r}\frac{1}{r_2+v^2t_2}\right)\bigg|_{x=b,\ y=0,\ z=vt}. \tag{69}$$

The expression for $\mathcal{B}_3$ identical except the derivative is $\partial/\partial x$. These derivatives may be evaluated and expressed back in terms of r and t using Eqs. (53)-(57), resulting in Eqs. (43) and (44) for $\mathcal{B}_2$ and $\mathcal{B}_3$.

We next turn to the $\mathcal{C}$ integrals. These are composed of derivatives of the biscalar A, which in general have (1) delta functions at $T = R$, (2) delta functions at $T = r + r'$, and (3) smooth functional dependence away from these special points. For the special combination of derivatives appearing in the $\mathcal{C}$ integrals, however, the smooth parts cancel out, leaving only the delta functions. The delta functions at $T = R$ are part of the direct portion of the Green function and do not contribute to the integration range $t' < t$. What remains is supported purely at $T = r - r'$,

$$\begin{aligned}
A_{,zx'} - A_{,xz'} &= \frac{2M(zx'-xz')}{rr'(r+r'-R)(r+r'+R)}\delta(T-r-r'), \qquad (\text{if } T \neq R) &(70)\\
A_{,xt'} - A_{,tx'} &= \frac{2M}{T^2-R^2}\left(\frac{x}{r}+\frac{x'}{r'}\right)\delta(T-r-r'), \qquad (\text{if } T \neq R) &(71)\\
A_{,zt'} - A_{,tz'} &= \frac{2M}{T^2-R^2}\left(\frac{z}{r}+\frac{z'}{r'}\right)\delta(T-r-r'), \qquad (\text{if } T \neq R). &(72)
\end{aligned}$$

The method of evaluation of the $\mathcal{C}$ integrals parallels that of $\mathcal{B}_1$, where one can first evaluate on the worldline, then perform the integral, and finally take the total derivative. The results may be simplified using Eqs. (53)-(62).

Finally, we turn to the $\mathcal{A}$ integrals, which are constructed from derivatives of $A_{tt'}$. Over the range of integration $t < t'$ we may again drop the "direct" terms supported on $T = R$, which leaves

$$A_{,tt'} = \frac{4MT}{(R^2 - T^2)^2}\Theta(T - r - r') + \frac{2M}{R^2 - T^2}\delta(T - r - r') \qquad (\text{if } T \neq R). \tag{73}$$

Evidently, the $\mathcal{A}$ integrals involve terms proportional to $\Theta(T - r - r')$ in addition to the delta functions we have already encountered. The Heaviside integrands turn out to be very simple once evaluated on the worldline—for $\mathcal{A}_2$ and $\mathcal{A}_3$ this means the relevant derivative must be taken *before* integration—and are easily expressed in terms of anti-derivatives evaluated at the doubly-retarded time t_2. The delta-function terms can be treated as before, and full expressions can be simplified using the formulas (53)-(62).

4.3. Limits

Eqs. (33)-(47) provide expressions for the scalar, electromagnetic, and gravitational self-forces as functions of time t, expressed as a ratio of polynomials in v, b, $r = \sqrt{b^2 + v^2t^2}$, and $z = vt$. Although relatively compact expressions can be obtained in the electromagnetic case, we find no additional insight from writing out these polynomials. However, it is helpful to examine the limits of low and high velocity.

For the low-velocity limit, we consider v to be small but allow vt to have any size, since the range of t is unbounded. Thus we expand Eqs. (33)-(38) in v at fixed r and z. Keeping through $O(v)$, we find that the results can be repackaged in vector notation as

$$\boldsymbol{f}_{(q)} = q^2\frac{2M}{r^3}\hat{\boldsymbol{r}} + \frac{1}{3}q^2\frac{d\boldsymbol{g}}{dt} + O(v^2), \tag{74}$$

$$\boldsymbol{f}_{(e)} = e^2\frac{M}{r^3}\hat{\boldsymbol{r}} + \frac{2}{3}q^2\frac{d\boldsymbol{g}}{dt} + O(v^2), \tag{75}$$

$$\boldsymbol{f}_{(m)} = -\frac{11}{3}m^2\frac{d\boldsymbol{g}}{dt} + O(v^2), \tag{76}$$

where $\boldsymbol{g} = -M/r^2\hat{\boldsymbol{r}}$ is the Newtonian gravitational acceleration and $\hat{\boldsymbol{r}}$ is the radial unit vector. This reproduces the low-velocity results derived in Refs. [1, 8], where we have made the additional assumption of straight-line motion. As observed in these references, the scalar and electromagnetic forces each consist of a dissipative piece equal to the self-force on a particle moving in flat spacetime subject to a Newtonian gravitational force (terms proportional to $d\boldsymbol{g}/dt$), together with a conservative force that must be attributed purely to the curvature of spacetime (terms proportional to M/r^3). The gravitational force contains a dissipative-type term with a "wrong sign" suggestive of radiation *anti*-damping instead of damping. While perhaps surprising, this should not be alarming since (1) the gravitational self-force is gauge-dependent, with the particle position not directly observable and (2) there are additional, matter-mediated forces that must be taken into account in this problem (Sec. 5 below). These issues were first discussed in Ref. [8] in the case of bound motion.

For the large-velocity limit, we begin by substituting $v = \sqrt{1-\gamma^{-2}}$ in Eqs. (33)-(38) and expanding for large γ. We find that the coefficients in this large-γ series blow up at $r = z$, a behavior that originates from the denominators in Eqs. (39)-(47). This divergence at $t \to -\infty$ signals the need for a separate expansion adapted to the distant past, which can be matched to the usual expansion to provide a uniformly valid approximation. The need for a second expansion is physically natural since the time to bounce a light signal off the star and return to the particle diverges in the ultrarelativistic limit.

5. Matter-mediated Force

The gravitational self-force provides an $O(mM/b^2)$ correction to the acceleration of the particle, which may be interpreted as the action of the particle's own gravitational field on its motion. A second physical effect acts at this same perturbative order: the particle's gravitational field accelerates the star at $O(m/b)$, and the new motion of the star changes the acceleration of the particle at $O(mM/b^2)$. Before tackling the calculation, it is helpful to review the formal origin of this additional force, which arises for any spacetime containing matter [8].

The derivation [5, 6, 39, 40] of the gravitational self-force assumes that the background spacetime is a vacuum solution of Einstein's equation. However, the key assumptions are local to the particle and will still hold if the particle is restricted to a *vacuum region* of an otherwise non-vacuum spacetime. The main difference is that the (far-zone) metric perturbation $h_{\mu\nu}$ will have a new source term $\delta T^{\rm star}_{\mu\nu}$ representing the perturbation to the background stress-energy that is induced by the presence of the particle, in addition to the point-particle stress-energy $T^{\rm particle}_{\mu\nu}$ of the particle itself. (For clarity we refer to the perturbed matter stress-energy as that of a star, although our comments apply more generally.) The metric perturbation similarly gains an extra term,

$$h_{\mu\nu} = h^{\rm particle}_{\mu\nu} + h^{\rm star}_{\mu\nu}, \tag{77}$$

with

$$h^{\rm particle}_{\mu\nu} = \int G_{\mu\nu}{}^{\mu'\nu'} T^{\rm particle}_{\mu'\nu'} \sqrt{-g} d^4x, \tag{78}$$

$$h^{\rm star}_{\mu\nu} = \int G_{\mu\nu}{}^{\mu'\nu'} \delta T^{\rm star}_{\mu'\nu'} \sqrt{-g} d^4x, \tag{79}$$

where $G^{\mu\nu}{}_{\mu'\nu'}$ is the gravitational Green function defined in Sec. 3 above. We may regard $h^{\rm particle}_{\mu\nu}$ as the metric perturbation generated directly by the particle (i.e., the self-field, which diverges at the particle) and $h^{\rm star}_{\mu\nu}$ as the metric perturbation due to the shift in the star's stress-energy induced by the presence of the particle (which is smooth at the particle). Both perturbations are proportional to the particle mass m; every use of $h_{\mu\nu}$ in this paper represents a term linear in m. The two terms propagate through the derivation, giving two corresponding terms in the final force on the particle,

$$f^\mu = f^\mu_{(m)} + f^\mu_{\rm mm}. \tag{80}$$

The first term is standard expression (28) for the self-force, while the second is an additional "matter-mediated" force [8],

$$f^\mu_{\rm mm} = -\frac{1}{2} m \left(g^{\mu\nu} + u^\mu u^\nu\right) \left(2\nabla_\beta h^{\rm star}_{\alpha\nu} - \nabla_\nu h^{\rm star}_{\alpha\beta}\right) u^\alpha u^\beta. \tag{81}$$

The matter-mediated force takes the form of a perturbed geodesic equation, i.e., ignoring the self-force, the particle would move on a geodesic of $g_{\mu\nu} + h^{\rm star}_{\mu\nu}$.

This elegant presentation of the matter-mediated force belies a severe practical difficulty: the perturbed star stress-energy $\delta T^{\mu'\nu'}_{\rm star}$ will in general be required to satisfy additional equations that involve the metric perturbation $h^{\rm star}_{\mu\nu}$ (and $h^{\rm particle}_{\mu\nu}$), making Eq. (79) of little use in actually computing $h^{\rm star}_{\mu\nu}$. Following Ref. [8], we will be able to circumvent this difficulty by taking advantage of the series expansion in M and modeling the star as a delta function to leading order. We are confident in the delta-function assumption because (1) no infinities arise in the subsequent calculation and (2) the more careful arguments given above for the computation of the Green function are equivalent to the assumption of a delta-function star. Still, we emphasize that this choice has not been justified with the same rigor as analogous claims made in Sec. 3 above.

In order to obtain the matter-mediated force (81) at $O(m^2 M)$ as desired, we will need the star's perturbation $h^{\rm star}_{\mu\nu}$ at order $O(mM)$. To determine this perturbation from Eq. (79), we will need the stress-energy of the star at $O(mM)$. In a given spacetime $\hat{g}_{\mu\nu}$ with timelike coordinate t, a point particle stress-energy tensor takes the form

$$T^{\mu\nu}_{\rm p.p.} = M U^\mu U^\nu \frac{\delta(\boldsymbol{x} - \boldsymbol{Z}(t))}{U^t \sqrt{-\hat{g}}}, \qquad U^\mu \hat{\nabla}_\mu U^\nu = 0, \tag{82}$$

where the second equation follows from conservation of stress-energy and indicates that $\boldsymbol{Z}(t)$ (four-velocity U^μ) is a geodesic of the spacetime. We will assume that this form holds for our star at first order in M. Consistent to this order, we may drop all M-dependent terms in $\boldsymbol{Z}$, U^μ, and $\hat{g}$. We will also drop terms of $O(m^2)$, since these are neglected everywhere in our calculation. Thus we use that

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + h^{\rm pf}_{\mu\nu} + O(M) + O(m^2), \tag{83}$$

where $h^{\rm pf}_{\mu\nu}$ (pf for "particle flat") is the metric perturbation due to the particle at $O(M^0 m^1)$, i.e., the leading piece of $h^{\rm particle}$. This is just the Lorenz-gauge linearized field of a point particle moving on a straight line in flat spacetime, which may be obtained by translating and boosting the linearized Schwarzschild metric in isotropic coordinates (or, alternatively, constructed using the $M = 0$ limit of the Green function discussed in Sec. 3 with a point particle source). For our choices (31) and (32) for the particle worldline, the non-zero components are

$$h^{\rm pf}_{00} = h^{\rm pf}_{zz} = \gamma^2(1+v^2)\frac{2m}{r_p} \tag{84}$$

$$h^{\rm pf}_{0z} = -2\gamma^2 v\frac{2m}{r_p} \tag{85}$$

$$h^{\rm pf}_{xx} = h^{\rm pf}_{yy} = \frac{2m}{r_p}, \tag{86}$$

where

$$r_p = \sqrt{(x-b)^2 + y^2 + \gamma^2(z-vt)^2}. \tag{87}$$

Assuming that the point particle form (82) holds for the star's stress-energy to $O(M)$ then gives

$$T^{\rm star}_{00} = M\delta(\boldsymbol{x}) - MZ^i(t)\partial_i\delta(\boldsymbol{x}) + \frac{mM\gamma^2(3v^2-1)}{\sqrt{b^2+\gamma^2v^2t^2}}\delta(\boldsymbol{x}) + O(m^2M) + O(M^2), \tag{88}$$

$$T^{\rm star}_{0i} = -M\frac{dZ^i}{dt}\delta(\boldsymbol{x}) + O(m^2M) + O(M^2), \tag{89}$$

$$T^{\rm star}_{ij} = O(m^2M) + O(M^2), \tag{90}$$

where we have used the fact that $Z^i = O(m)$ to expand the delta function.

The motion of the star $Z^i(t)$ is determined by the geodesic equation in the spacetime $\hat{g}_{\mu\nu}$ with $Z^\mu = (t,0,0,0)$ to leading order, which takes the form

$$\frac{d^2Z^i}{dt^2} = \frac{1}{2}\partial_i h^{\rm pf}_{00} - \partial_0 h^{\rm pf}_{0i} + O(Mm) + O(m^2). \tag{91}$$

Integrating Eq. (91) once, we find for the velocity $V^i = dZ^i/dt$ that

$$V^z(t) = \frac{m\gamma(3v^2-1)}{v\sqrt{b^2\gamma^{-2}+v^2t^2}} \tag{92}$$

$$V^x(t) = \frac{m\gamma(1+v^2)}{bv}\left(1 + \frac{vt}{\sqrt{b^2\gamma^{-2}+v^2t^2}}\right), \tag{93}$$

where we suppress the $O(Mm)$ and $O(m^2)$ error terms. We have fixed the integration constants by demanding that the star velocity vanish at early times. Integrating again, we find for the position $Z^i = (X, Y, Z)$ that

$$Z(t) = \frac{m\gamma(3v^2-1)}{v^2}\text{arctanh}\frac{vt}{\sqrt{b^2\gamma^{-2}+v^2t^2}} \tag{94}$$

$$X(t) = \frac{m\gamma(1+v^2)}{bv^2}\left(vt + \sqrt{b^2\gamma^{-2}+v^2t^2}\right). \tag{95}$$

For the X position, we have chosen the integration constant such that $X(t \to -\infty) = 0$. The analogous choice for the Z position is not possible since $Z(t)$ diverges logarithmically and early (and late) times. In the low-velocity limit of ordinary Newtonian dynamics, this divergence can be attributed to the $1/r^2$ force, which gives a log when twice integrated. Here we have chosen the integration constant so that $Z = 0$ at $t = 0$.

The metric perturbation $h^{\rm star}_{\mu\nu}$ of the star is given by Eq. (79). To determine the $O(mM)$ metric perturbation we will need only the $O(M^0)$ part of the Greeen function, i.e., the flat spacetime Green function. From the fundamental equation (9), the trace-reversed Green function is simply related to the scalar Green function $G_{\sf flat}$ (14) as

$$\bar{G}^{\alpha\beta}{}_{\alpha'\beta'} = \delta^{(\alpha}{}_{\alpha'}\delta^{\beta)}{}_{\beta'}\frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|)}{|\boldsymbol{x} - \boldsymbol{x}'|} + O(M), \tag{96}$$

meaning that we can determine $h^{\rm star}_{\mu\nu}$ to the relevant order $O(Mm)$ by

$$h^{\rm star}_{\mu\nu} = \bar{h}^{\rm star}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\bar{h}^{\rm star}_{\alpha\beta}, \qquad \bar{h}^{\rm star}_{\mu\nu} = 4\int \frac{\delta T^{\rm star}_{\mu\nu}(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3x'. \tag{97}$$

The star's perturbed stress-energy tensor $\delta T^{\rm star}_{\mu\nu}$ is identified as as the $O(m)$ piece of Eqs. (88)-(90) (recalling that $Z^i = O(m)$), and performing the integral gives

$$h^{\rm star}_{00} = \frac{2M}{r}\left[\left(\frac{Z_i|_{t-r}}{r} + V_i|_{t-r}\right)\frac{x^i}{r} + \frac{m\gamma(3v^2 - 1)}{\sqrt{b^2\gamma^{-2} + v^2(t-r)^2}}\right] \tag{98}$$

$$h^{\rm star}_{0i} = -\frac{4MV^i|_{t-r}}{r} \tag{99}$$

$$h^{\rm star}_{ij} = h^{\rm star}_{00}\delta_{ij}, \tag{100}$$

where the notation $Z_i|_{t-r}$ means to evaluate the function $Z_i(t)$ at $t - r$, i.e., $Z_i|_{t-r} = Z_i(t-r)$ (and likewise for V^i). These functions are given in Eq. (92)-(95) above.

The matter-mediated force is now given by Eq. (81). As $h^{\rm star}_{00}$ is already $O(M)$, the $O(M^0)$ metric and particle four-velocity may be used, giving

$$f^z_{\rm mm} = \frac{m\gamma^2}{2}\left((1+v^2)h^{\rm star}_{00,z} + 2vh^{\rm star}_{0z,z} - \gamma^2 v(3 - v^2)\frac{d}{dt}h^{\rm star}_{00} - 2\gamma^2\frac{d}{dt}h^{\rm star}_{0z}\right) \tag{101}$$

$$f^x_{\rm mm} = \frac{m\gamma^2}{2}\left((1+v^2)h^{\rm star}_{00,x} + 2vh^{\rm star}_{0z,x} - 2\frac{d}{dt}h^{\rm star}_{0x}\right), \tag{102}$$

where the terms are to be evaluated on the particle worldline ($x = b, y = 0, z = vt$). In writing this expression, we have used Eq. (100) to eliminate $h^{\rm star}_{ij}$.

To simplify the expressions (101) and (102) for the matter-mediated force, it is helpful to note that the square root in (98) is related to the doubly-retarded time t_2 defined in (48) by

$$\sqrt{b^2\gamma^{-2} + v^2(t-r)^2} = t - r - \gamma^{-2}t_2. \tag{103}$$

This square root also arises in the expressions (92)-(95) for V_i and Z_i after sending $t \to t - r$, as appears in $h^{\rm star}_{\mu\nu}$ via Eqs. (98)-(100). After eliminating all such square roots in favor of t_2, one may use Eqs. (53)-(57) to simplify the expressions when evaluated on the worldline. Doing so, we find

$$f^z_{\rm mm} = \frac{\gamma^5 m^2 M}{v^2 r^5 (r - vz)^3} \mathcal{F}_z(r, v, z) \tag{104}$$

$$f^x_{\rm mm} = \frac{\gamma^3 m^2 M}{v^2 r^5 (r - vz)^3} \frac{\mathcal{F}_x(r, v, z)}{b}, \tag{105}$$

where

$$\begin{aligned}
\mathcal{F}_z = {} & r^5 v \left(2v^8 - 8v^6 + v^5 + 9v^4 - 2v^3 + 2v^2 - 3v - 1\right) \\
& - r^4 z \left(4v^8 + 3v^7 - 5v^6 - 6v^5 + 37v^4 - 9v^3 - 15v^2 + 3\right) \\
& + 3r^3 z^2 \left(v^8 + 5v^7 - 2v^6 + 18v^5 - 15v^3 + 2v^2 + 4v - 1\right) \\
& - r^2 z^3 v \left(v^8 + 5v^7 - 2v^6 + 48v^5 + 24v^4 - 67v^3 + 18v^2 + 18v - 9\right) \\
& + 3rz^4 v^2 \left(9v^4 - 12v^3 + 6v^2 + 4v - 3\right) \\
& - 3z^5 v^3 \left(-3v^5 + 3v^4 - 2v^3 + 2v^2 + v - 1\right) \\
& - \left(r^2 - 3z^2\right) (vz - r)^3 \left(1 - 3v^2\right)^2 \operatorname{arctanh}[(rv - z)/(r - vz)]
\end{aligned} \tag{106}$$

and

$$\begin{aligned}
\mathcal{F}_x = {} & -2r^6(1 + v^2)(1 - 4v^4 + v^6) \\
& - r^5 z(2 - 9v + 8v^2 - 12v^3 + 6v^4 + 39v^5 - 6v^7) \\
& + r^4 z^2(3 + 6v - 15v^2 + 24v^3 - 23v^4 + 18v^5 + 11v^6) \\
& - r^3 z^3(-3 + 12v - 5v^3 + 21v^4 - 62v^5 + 18v^6 + 3v^7) \\
& - r^2 z^4 v(9 - 18v + 16v^2 - 3v^3 + v^4 + 32v^5 - 6v^6 + v^7) \\
& + 3rz^5 v^2(1 + v^2)(3 - 4v + 3v^2) \\
& + 3z^6(-1 + v)v^3(1 + v^2)^2 \\
& - 3b^2 z(-r + vz)^3(1 + v^2)(-1 + 3v^2)\operatorname{arctanh}[(rv - z)/(r - vz)].
\end{aligned} \tag{107}$$

This completes the calculation of the matter-mediated force.

Note that the matter-mediated force arises only because our background spacetime contains matter. If we had repeated our calculation using a black hole background instead of a star background, there would be no matter-mediated force at all, and the entire $O(mM)$ dynamics would be given by the gravitational self-force. Since we expect the binary dynamics to be independent of the body compositions at this order of approximation, the natural conclusion is that the Lorenz-gauge gravitational self-force must depend in detail on the central body composition, even in the limit where the particle is very distant compared to the typical size of the central body. This phenomenon has been seen previously for nonminimally coupled scalar fields, but not for minimally coupled scalar fields or electromagnetic fields [8,41–43]. We note, however, that the issue of the back-reaction of black hole motion on the particle is closely tied up with the question of the choice of gauge [44, 45].

6. Motion of the particle

In any spacetime, the equations of motion for a mass m moving under a four-force f^μ are

$$m\frac{d^2x^\mu}{d\tau^2} = -m\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + f^\mu. \tag{108}$$

We will integrate these equations in the weakly curved spacetime (12) with background particle trajectory (31), where the force is either the scalar self-force, the electromagnetic self-force, or the sum of the gravitational self-force and the matter-mediated force. In this section we work consistently to $O(M)$, dropping all terms of order $O(M^2)$ or higher. Expanding (108) gives

$$\frac{d^2t}{d\tau^2} = -\gamma^2\frac{2Mz}{r^3}v + \frac{vf^z}{m} \tag{109}$$
$$\frac{d^2z}{d\tau^2} = -\frac{Mz}{r^3} + \frac{f^z}{m} \tag{110}$$
$$\frac{d^2x}{d\tau^2} = -\gamma^2\frac{Mx}{r^3}(1+v^2) + \frac{f^x}{m}, \tag{111}$$

where the right-hand sides are to be evaluated on the background worldline ($x = b$, $z = vt$, and $r = \sqrt{b^2 + v^2t^2}$, with $t = \gamma\tau$). In writing these equations we have used the fact that $f^0 = vf^z$ to this order, as discussed above Eq. (33). In the scalar theory the rest mass m is not constant (see Appendix A), but the variation begins at $O(Mq^2)$ and hence contributes an $O(M^2)$ term to Eqs. (109)-(111), which is neglected. We therefore treat m as a constant.

Integrating once and choosing the perturbed four-velocity to vanish in the infinite past, we find consistent to $O(M)$ that

$$\frac{dt}{d\tau} = \gamma + \frac{2M\gamma}{r} + \frac{v}{\gamma m}\int_{-\infty}^{\gamma\tau} f^z dt', \tag{112}$$
$$\frac{dz}{d\tau} = \gamma v + \frac{M}{\gamma v r} + \frac{1}{\gamma m}\int_{-\infty}^{\gamma\tau} f^z dt', \tag{113}$$
$$\frac{dx}{d\tau} = -\frac{M\gamma}{bv}\left(1+v^2\right)\left(1+\frac{z}{r}\right) + \frac{1}{\gamma m}\int_{-\infty}^{\gamma\tau} f^x dt', \tag{114}$$

where the right-hand sides are evaluated on the background worldline. That is, we set $x = b$, $y = 0$ and $z = v\gamma\tau$ in the terms without integrals, and the integrals are taken along the background worldline parameterized by t', i.e., f^z and f^x are to be evaluated at $x = b$, $y = 0$, $z = vt'$. (The prime distinguishes this parameter t' from the t appearing on the left-hand side of (112), which represents the t coordinate of the particle's perturbed motion at proper time τ.) Integrating a second time, we find

$$t = \gamma\tau + \frac{2M}{v}\text{arctanh}\frac{z}{r} + vG^z(\gamma\tau), \tag{115}$$

$$z = v\gamma\tau + \frac{M}{\gamma^2 v^2}\text{arctanh}\frac{z}{r} + G^z(\gamma\tau), \tag{116}$$

$$x = b - \frac{M}{bv^2}(1+v^2)(r+z) + G^x(\gamma\tau), \tag{117}$$

where again the background motion ($x = b$, $y = 0$, $z = v\gamma\tau$) is to be inserted on the right-hand side, and we define

$$G^\mu(t) = \frac{1}{\gamma^2 m}\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt'' f^\mu(t''). \tag{118}$$

In these integrals, the force f^μ is evaluated on the background trajectory parameterized by t'', i.e., with $x = b$, $y = 0$, $z = vt''$. In writing the solution this way, we have chosen the integration constants so that the background quantities retain their physical meaning as initial values. That is, v is the initial velocity (with γ the initial Lorentz factor), and b is the initial x position, i.e., the impact parameter. Note that the integration constants associated with the position integrals for t and z do not affect the meaning of b and v and have been chosen for mathematical convenience. It is not possible to choose these so that the perturbation vanishes at early times, due to the logarithmic divergences associated with the long-range nature of the gravitational force (see discussion below Eq. (95)).

Sometimes it is more convenient to use t as the parameter. From Eqs. (115)-(117), consistent to $O(M)$, we have

$$z = vt + \frac{M(1-3v^2)}{v^2}\text{arctanh}\frac{z}{r} + \gamma^{-2}G^z(t), \tag{119}$$

$$x = b - \frac{M}{bv^2}(1+v^2)(r+z) + G^x(t), \tag{120}$$

where the right-hand-sides are now evaluated at $x = b, y = 0, z = vt$. The integrals for $G^x(t)$ and $G^z(t)$ are straightforward, if tedious, and in Appendix B we present the results for $x(t)$ and $z(t)$ in the scalar, electromagnetic, and gravitational cases. We also provide separate formulas for G^z and G^x in (B.18)–(B.23), so that the reader can efficiently reconstruct the proper-time parameterized trajectories given in Eqs. (115)-(117) above. This completes the derivation of the star-frame trajectories.

6.1. Physical Quantities

We now use the particle trajectories to compute physical quantities defined in the star frame. We will begin with the energy and momentum. The four-momentum of a point particle is given by $p^\mu = m dx^\mu/d\tau$. The initial energy-momentum p_0^μ of the particle is by definition

$$p_0^\mu = m\gamma(1,0,0,v), \tag{121}$$

which also follows from the $\tau \to -\infty$ limit of Eqs. (115)–(117), or equivalently from the $t \to -\infty$ limit of Eqs. (119) and (120). The final energy-momentum p_f^μ and is instead calculated from the late-time limit of these equations, using the different forces f^μ as appropriate. We will present the results as

$$p^\mu_{f,\text{scalar}} = p_0^\mu + \Delta p^\mu_{(M)} + \Delta p^\mu_{(q)} \tag{122}$$
$$p^\mu_{f,\text{EM}} = p_0^\mu + \Delta p^\mu_{(M)} + \Delta p^\mu_{(e)} \tag{123}$$
$$p^\mu_{f,\text{grav}} = p_0^\mu + \Delta p^\mu_{(M)} + \Delta p^\mu_{(m)} + \Delta p^\mu_{\text{mm}}, \tag{124}$$

where‖

$$\Delta p^\mu_{(M)} = \frac{Mm\gamma}{bv}\Big(0, -2(1+v^2), 0, 0\Big) \tag{125}$$
$$\Delta p^\mu_{(q)} = \frac{Mq^2\gamma}{b^2 v}\Big(0, \frac{\pi}{4}\left(v^2 + 4\xi\gamma^{-2}\right), 0, 0\Big) \tag{126}$$
$$\Delta p^\mu_{(e)} = \frac{Me^2\gamma}{b^2 v}\Big(0, \frac{\pi}{4}\left(2 + v^2\right), 0, 0\Big) \tag{127}$$
$$\Delta p^\mu_{(m)} = \frac{Mm^2\gamma v}{b^2}\Big(0, -\frac{7\pi}{4}, 0, 0\Big) \tag{128}$$
$$\Delta p^\mu_{\text{mm}} = \frac{Mm^2\gamma}{b^2 v}\Big(-\frac{2\gamma}{v}(1+v^2)^2, -\pi\left(3 - v^2\right), 0, -\frac{2\gamma}{v^2}(1+v^2)^2\Big). \tag{129}$$

Notice that the geodesic term (125) and the self-forces (126), (127), and (128) only modify the x component of the four-momentum, while the matter-mediated piece changes p^t and p^z as well.

The momentum change is directly related to the scattering angle δ, which for our initial conditions (121) is given by

$$\tan\delta = -\frac{p_f^x}{p_f^z}. \tag{130}$$

For our small-angle scattering problem, (i.e., consistent to $O(M)$) this may be approximated by

$$\delta = -\frac{p_f^x}{\gamma m v}. \tag{131}$$

The deflection angles then follow from Eqs. (122)–(129). The explicit expressions in the scalar, electromagnetic, and gravitational cases were displayed in Eqs. (2), (3), and

‖ In Eqs. (128), we have corrected a typo from the previous arXiv version. We thank Oliver Long for helping find this typo.

(4) above. In those equations, we have also restored the explicit $O(M/b)^2$ errors as well as inserted dots $(\dots)$ to indicate the presence of higher-order terms in the small parameters $q^2/(mb)$, $e^2/(mb)$, and m/b. Although the scalar and electromagnetic field equations are linear, these higher-order terms will still arise at least from iterating a self-force calculation using the corrected motion.

Finally, we discuss angular momentum. The angular momentum of a point particle about the origin of coordinates is $J^i = \epsilon^{ijk} x^j p^k$, where x^μ is the position and p^μ is the four-momentum. The initial angular momentum J_0^i of the particle is by definition

$$J_0 = -J_0^y = \gamma m v b, \tag{132}$$

which also follows from the $\tau \to -\infty$ limit of Eqs. (115)–(117) or the $t \to -\infty$ limit of Eqs. (119) and (120). The final angular momentum follows from the late-time limit of these equations. In the scalar and electromagnetic cases, we find

$$J_f^{\rm scalar} = J_0 \left(1 - \frac{2}{3}\frac{\gamma(1+v^2)}{v}\frac{Mq^2}{mb^2}\right) \tag{133}$$

$$J_f^{\rm EM} = J_0 \left(1 - \frac{4}{3}\frac{\gamma(1+v^2)}{v}\frac{Me^2}{mb^2}\right). \tag{134}$$

In the gravitational case we find that the corresponding "final angular momentum" of the particle diverges logarithmically in time, indicating that the star frame is not suitable for discussing the particle angular momentum. We will see in Sec. 7 that the difficulty disappears in the CEM frame.

7. 2PM gravitational scattering

In the previous section we determined the motion of the particle in the frame of the star, meaning the frame where the star has asymptotically zero velocity in the distant past. In the scalar and electromagnetic cases, it was consistent to keep the star at rest for the purposes of computing the particle trajectories at the given order of approximation. By contrast, in the gravitational case we required the $O(m)$ motion of the star (Eqs. (95) and (94)) in order to determine the $O(Mm)$ motion of the particle. However, for a complete 2PM treatment we are still missing the $O(mM)$ motion of the star as well as the second-order contributions, $O(M^2)$ and $O(m^2)$, to the motion.

Our strategy for determining these missing pieces is to determine the $O(M^2)$ motion of the particle from the geodesic equation in the Schwarzschild spacetime (Appendix B) and to invoke a body exchange symmetry in order to determine the motion of the star. It is intuitively clear that such a symmetry should exist, since the assignment of the words "particle" and "star" to the two bodies should not affect their motion in a framework where only the respective masses m and M appear in the final answer. However, our calculation has made coordinate choices which break this symmetry: our background configuration takes the star to be at rest, and our perturbation theory is correspondingly asymmetric, with gauge choices made in different ways for the star and particle. To make the exchange symmetry manifest we will have to change coordinates to treat the bodies symmetrically.

For clarity, let us first discuss the situation for two non-interacting point particles, which we name "particle" (mass m) and "star" (mass M) for consistency. In this case we can define the star frame, where the star has position $(X, Y, Z) = (0, 0, 0)$ and the particle has position $(x, y, z) = (b, 0, vt)$, and the center of energy-momentum (CEM) frame, where there is no net momentum and the center of energy is at the spatial coordinate origin. If the star frame has Minkowski coordinates (t, x, y, z) and the CEM frame has Minkowski coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$, then the frames are related by

$$\tilde{t} = \frac{M + \gamma m}{\tilde{E}} t - \frac{\gamma m v}{\tilde{E}} z \tag{135}$$

$$\tilde{z} = \frac{M + \gamma m}{\tilde{E}} z - \frac{\gamma m v}{\tilde{E}} t \tag{136}$$

$$\tilde{x} = x - b\frac{m(\gamma M + m)}{\tilde{E}^2}, \tag{137}$$

where $\gamma = (1 - v^2)^{-1/2}$ as before in this paper, and $\tilde{E}$ denotes the total energy in the CEM frame,

$$\tilde{E} = \sqrt{m^2 + M^2 + 2\gamma M m}. \tag{138}$$

This transformation involves a boost in the z direction to eliminate the net momentum, together with a translation in the x direction to eliminate the non-zero center of energy.

In the CEM frame of these non-interacting bodies, swapping the bodies is equivalent to rotation by 180°. That is, we have

$$\tilde{Z}(\tilde{t}, M, m) = -\tilde{z}(\tilde{t}, m, M), \qquad \tilde{X}(\tilde{t}, M, m) = -\tilde{x}(\tilde{t}, m, M), \tag{139}$$

where $\tilde{\mathbf{Z}} = (\tilde{X}, \tilde{Y}, \tilde{Z})$ is the CEM-frame spatial position of the star, while $\tilde{\mathbf{z}} = (\tilde{x}, \tilde{y}, \tilde{z})$ is the CEM-frame spatial position of the particle. For the gravitational scattering problem, we will *define* the "initial CEM frame" (CEM frame for short) by the same transformation (135)–(137) applied to what we have called the initial star frame, and then *impose* the swapping symmetry (139) in order to determine the motion of the star.¶ This is logically equivalent to repeating our entire calculation with the words "particle" and "star" interchanged and finding a coordinate transformation to make the combined solution respect the exchange symmetry.

The particle trajectory in the star frame, $\mathbf{z}(t)$, is given as Eq. (B.3) together with later supporting equations. Plugging these expressions into Eqs. (136) and (137) gives the CEM-frame trajectory $\tilde{\mathbf{z}}(t)$, but still parameterized by star-frame time t. To express in terms of the CEM-frame time $\tilde{t}$, we plug the trajectory $z(t)$ [Eq. (B.3)] into Eq. (135)

¶ The 1PM trajectory of the star used in the derivation of the matter-mediated force [Eqs.(94) and (95)] agrees with the 1PM trajectory determined from this method, showing that no further gauge transformations are required.

and solve for t, order by order in the PM expansion. Accurate to 2PM, we have

$$\begin{aligned} t = t_0 &+ \frac{mM}{m+\gamma M}\frac{\gamma^2\left(1-3v^2\right)}{v}\operatorname{arctanh}\frac{vt_0}{\sqrt{b^2+v^2t_0^2}} \\ &+ \frac{m^2M^2}{(m+\gamma M)^2}\frac{\gamma^4\left(1-3v^2\right)^2}{\sqrt{b^2+v^2t_0^2}}\operatorname{arctanh}\frac{vt_0}{\sqrt{b^2+v^2t_0^2}} \\ &+ \frac{\gamma^2 mv}{m+\gamma M}\left(z_{g2}(t_0)+z_m(t_0)+z_{\rm mm}(t_0)\right), \end{aligned} \tag{140}$$

where we have defined

$$t_0 = \frac{\gamma\tilde{E}}{m+\gamma M}\tilde{t}. \tag{141}$$

This completes the derivation of the 2PM trajectories of the particle and star, expressed in the initial CEM frame. The particle trajectory $\mathbf{z}(\tilde{t})$ is given by Eqs. (B.3), (B.6)–(B.9), (B.14)–(B.17), (136), (137), and (140), and the star trajectory $\mathbf{Z}(\tilde{t})$ is then given by Eq. (139).

7.1. Initial conditions

It is instructive to consider the "initial conditions" for the CEM-frame scattering problem, i.e., the $\tilde{t}\to-\infty$ behavior of the trajectories. Expanding at early times, we find

$$\tilde{\mathbf{z}} = \left(\frac{M(M+\gamma m)}{\tilde{E}^2}b,\ 0,\ \frac{M\gamma}{m+\gamma M}\left[v\tilde{t}-\frac{(1-3v^2)\tilde{E}}{v^2}\log\frac{2\tilde{E}\gamma v|\tilde{t}|}{b(m+\gamma M)}\right]\right)+O\left(\frac{\log|\tilde{t}|}{\tilde{t}}\right) \tag{142}$$

$$\tilde{\mathbf{Z}} = \left(-\frac{m(m+\gamma M)}{\tilde{E}^2}b, 0, \frac{-m\gamma}{M+\gamma m}\left[v\tilde{t}-\frac{(1-3v^2)\tilde{E}}{v^2}\log\frac{2\tilde{E}\gamma v|\tilde{t}|}{b(M+\gamma m)}\right]\right)+O\left(\frac{\log|\tilde{t}|}{\tilde{t}}\right). \tag{143}$$

The 1PM logarithmic correction is an unavoidable consequence of the long-range nature of the gravitational force, as already discussed below Eqs. (94) and (95). Notice, however, that there is no 2PM logarithmic correction.

The final trajectories of the bodies can be analogously calculated from the $\tilde{t}\to+\infty$ limit of the trajectories derived in the previous subsection. We will now report the initial and final values of the various conserved quantities.

7.2. Energy, momentum, and scattering angle

The initial four-momentum of the particle is

$$\tilde{p}_0^\mu = m\left(\frac{m+\gamma M}{\tilde{E}}, 0, 0, \frac{\gamma M v}{\tilde{E}}\right). \tag{144}$$

The final four-momentum of the particle is given as $\tilde{p}^\mu_f = \tilde{p}^\mu_0 + \Delta\tilde{p}^\mu$ with

$$\Delta\tilde{p}^\mu = \frac{\gamma m M}{bv}\left(0, -2\left(v^2+1\right) - \frac{3\pi}{4}\left(v^2+4\right)\frac{m+M}{b}, 0, -2(v^2+1)^2\frac{\tilde{E}}{bv^2}\right). \tag{145}$$

The four-momentum of the star is determined by sending $M \leftrightarrow m$ and $\mathbf{x} \to -\mathbf{x}$ (i.e., flipping the sign of the spatial components of the four-momentum). This shows that the mechanical energy is separately conserved for particle and star, and that mechanical momentum is exchanged, with the total mechanical momentum conserved. In particular, the particle and star deflect by the same amount χ satisfying $\tan\chi = -\tilde{p}^x_f/\tilde{p}^z_f$. Expanding this equation to 2PM order, we find

$$\chi = \frac{\tilde{E}}{bv^2}\left(2(v^2+1) + \frac{3\pi}{4}(v^2+4)\frac{M+m}{b}\right). \tag{146}$$

This result can also be derived by boosting the lab-frame deflection angle δ, as we did to arrive at (6) above.

7.3. Angular momentum

We define the initial and final angular momentum using the special-relativistic formula evaluated at early and late times, where the particles are widely separated. For the particle, the angular momentum formula is

$$\tilde{J} = -\tilde{J}^y = \tilde{p}^z\tilde{x} - \tilde{p}^x\tilde{z}. \tag{147}$$

Since $\tilde{x}(\tilde{t})$ and $\tilde{z}(\tilde{t})$ can diverge like $\tilde{t}$ at early and late times, in principle the $1/\tilde{t}$ corrections to $\tilde{p}^\mu$ may contribute to the early and late-time limits of Eq. (147). These corrections involve both the $1/\tilde{t}$ behavior of the trajectory as well as corrections due to the metric of the star. However, we find that the $1/\tilde{t}$ terms end up canceling in the expression (147) for the angular momentum at 2PM order. In practice, this means that one may use the initial and final momenta (144) and (145) in evaluating the angular momentum (147).

Taking the $\tilde{t} \to -\infty$ limit of Eq. (147), we find that the initial angular momentum of the particle is given by

$$\tilde{J}_0 = -\tilde{J}^y_0 = b\gamma v\frac{mM}{\tilde{E}}\frac{M(M+\gamma m)}{\tilde{E}^2}. \tag{148}$$

From the $\tilde{t} \to +\infty$ limit, we find that the final angular momentum is given by $\tilde{J}_f = \tilde{J}_0 + \Delta\tilde{J}$ with

$$\Delta\tilde{J} = \frac{2\gamma^2 M^2 m^2}{\tilde{E}bv^3}(1+v^2)\left(\frac{8}{3}v^3 - v + (1-3v^2)\operatorname{arctanh} v\right). \tag{149}$$

The angular momentum of the star is given by sending $m \leftrightarrow M$ in these equations. The total initial angular momentum is thus

$$\tilde{J}_0^{\rm tot} = \frac{b\gamma m M v}{\tilde{E}}. \tag{150}$$

The particle and star each lose the same amount of angular momentum, with the fractional change in the total mechanical angular momentum given by

$$\frac{\Delta \tilde{J}^{\rm tot}}{\tilde{J}_0^{\rm tot}} = \frac{4\gamma M m}{b^2 v^4}(1+v^2)\left(\frac{8}{3}v^3 - v + (1-3v^2)\operatorname{arctanh} v\right). \tag{151}$$

This agrees with Eq. (4.6) of Ref. [34], providing a direct check that the mechanical angular momentum lost matches the angular momentum radiated away in gravitational waves.

7.4. Mass moment

Information about center of energy is encoded in the time-space cross-terms of the relativistic angular momentum tensor, which we will refer to as the *mass moment* $\boldsymbol{N}$. The special-relativistic formula for the mass moment of the particle is

$$\tilde{\boldsymbol{N}} = \tilde{p}^0 \tilde{\boldsymbol{z}} - \tilde{\boldsymbol{p}}\tilde{t}. \tag{152}$$

Expanding at $\tilde{t} \to -\infty$, we find

$$\tilde{N}^x = O\left(\frac{\log|\tilde{t}|}{\tilde{t}}\right) \tag{153}$$

$$\tilde{N}^z = -\gamma(1-3v^2)\frac{Mm}{v^2}\left(\log\frac{2v\tilde{E}|\tilde{t}|}{(m+\gamma M)b} - 1\right) + O\left(\frac{\log|\tilde{t}|}{\tilde{t}}\right) \tag{154}$$

The presence of the $\log \tilde{t}$ means that the the initial particle mass moment is not well-defined. Note also that the -1 in Eq. (154) arises from a $1/\tilde{t}$ correction to the late-time momentum. However, according to the symmetry (139), the star's early-time mass moment is given by sending $m \leftrightarrow M$ and multiplying by -1. Thus these features cancel out of the total system mass moment, which has the well-defined initial value of

$$\tilde{N}_0^{x,\rm tot} = 0 \tag{155}$$

$$\tilde{N}_0^{z,\rm tot} = -\gamma(1-3v^2)\frac{Mm}{v^2}\log\frac{M+\gamma m}{m+\gamma M}. \tag{156}$$

That is, provided we discuss the total mass moment, we may compute from (152) using the initial value (144) of the four-momentum, just as in the case of angular momentum.

The non-zero initial value of the system mass moment may suggest that we are not, after all, in the initial CEM frame. However, we have considered only the *mechanical* contribution to the mass moment, ignoring any effects of the gravitational field. In the electromagnetic analog problem [36], a contribution from the electromagnetic field precisely cancels this mechanical portion, such that the initial mass moment is indeed zero. The name "initial CEM frame" is thus fully justified in the electromagnetic case, and we will continue to use it in the gravitational case studied here.

The late-time behavior of the mass moment is precisely analogous to the early-time limit: the particle and star contributions are individually logarithmically divergent, but the total is well-defined and sensitive only to the late-time four-momentum. The final value of the system mass moment is given by[+]

$$\tilde{N}_f^{z,\mathrm{tot}} = \frac{\gamma M m}{v^2}(1-3v^2)\left(\log\frac{M+\gamma m}{m+\gamma M}\right). \tag{157}$$

$$\begin{aligned}\tilde{N}_f^{x,\mathrm{tot}} = 2(1+v^2)\frac{\gamma \tilde{E} M m}{b v^4}\Bigg[&-(1-3v^2)\log\frac{M+\gamma m}{m+\gamma M} \\ &+\left(v-\frac{8}{3}v^3-(1-3v^2)\mathrm{arctanh}\, v\right)\frac{M^2-m^2}{\tilde{E}^2}\Bigg].\end{aligned} \tag{158}$$

Consistent to 2PM, this may be written as

$$\tilde{N}_f^{z,\mathrm{tot}} = -\tilde{N}_0^{z,\mathrm{tot}}. \tag{159}$$

$$\tilde{N}_f^{x,\mathrm{tot}} = \chi \tilde{N}_0^{z,\mathrm{tot}} - \frac{\Delta J}{2\gamma v}\frac{M^2-m^2}{Mm}. \tag{160}$$

We thus find that there is a change in the system's mechanical mass moment as a result of the scattering, an effect we will refer to as a "scoot". Notice that there is a scoot at *first* post-Minkowksian order (the logarithmic term in $\tilde{N}_f^{z,\mathrm{tot}}$), together with 2PM corrections (the remaining terms in $\tilde{N}_f^{z,\mathrm{tot}}$ and $\tilde{N}_f^{x,\mathrm{tot}}$). We will discuss more details of this effect in a future publication [35].

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Appendix A. Mass evolution in the scalar theory

The equations of motion for a scalar charge do not preserve the rest mass m, but rather involve an evolution equation [7] analogous to the self-force (26),

$$\frac{dm}{d\tau} = -q^2\int_{-\infty}^{\tau^-} u^\mu \nabla_\mu G d\tau'. \tag{A.1}$$

[+] In Eqs. (157) and (159), we have corrected an error from the previous arXiv version. We thank Hongji Wei for helping find this error.

With the approximations of Sec. (4), this becomes

$$\frac{dm}{d\tau} = -2q^2(\mathcal{A}_1 + \xi\mathcal{B}_1), \tag{A.2}$$

where $\mathcal{A}_1$ and $\mathcal{B}_1$ are defined and given in Eqs. (39) and (42). Denoting the value of the mass in the asymptotic past by m_0, the time evolution of the mass is

$$m(\tau) = m_0 - \frac{2q^2}{\gamma}\int_{-\infty}^{\gamma\tau}(\mathcal{A}_1 + \xi\mathcal{B}_1)dt'. \tag{A.3}$$

where the integral is over the background worldline parameterized by t', i.e., $x = b$, $y = 0$, $z = vt'$. Using Eqs. (39) and (42), this integral is easily evaluated in closed form to give the full mass evolution $m(\tau)$. A notable property of this function is that it returns to m_0 in the future, i.e.,

$$m(\tau \to \pm\infty) = m_0. \tag{A.4}$$

That is, there is no change in rest mass over the scattering process.

Appendix B. Parameterized Trajectories

In this appendix we provide explicit formulae for the trajectory of the particle in the frame of the star, using the t coordinate as a parameter. For the gravitational case, the CEM-frame trajectories can be determined from these in the manner described in below Eq. (141) of the main text. We distinguish the contributions to the motion from the various forces as follows:

$$\mathbf{z}_{\text{scalar}} = (b + x_{g1} + x_q, 0, vt + z_{g1} + z_q) \tag{B.1}$$

$$\mathbf{z}_{\text{EM}} = (b + x_{g1} + x_e, 0, vt + z_{g1} + z_e) \tag{B.2}$$

$$\mathbf{z}_{\text{grav}} = (b + x_{g1} + x_{g2} + x_m + x_{\text{mm}}, 0, vt + z_{g1} + z_{g2} + z_m + z_{\text{mm}}) \tag{B.3}$$

The subscript q refers to the scalar self-force, the subscript e refers to the electromagnetic self-force, and the subscript m refers to the gravitational self-force. The subscript mm refers to the matter-mediated force, which acts at the same order as the gravitational self-force. The subscriptions $g1$ and $g2$ refer to the geodesic terms (gravitational forces) at order M and M^2, respectively. In presenting the results, we will evaluate all functions explicitly in terms of t, with the exceptions of

$$r = \sqrt{b^2 + v^2t^2} \tag{B.4}$$

$$s = b^2 + t^2v^2\gamma^{-2}. \tag{B.5}$$

Here r is just the radius of the particle as measured in flat spacetime with the background straight-line motion, while s is a positive quantity with no clear interpretation.

The first-order geodesic terms are given by

$$x_{g1} = -\frac{M}{bv^2}(1+v^2)(r+vt) \tag{B.6}$$

$$z_{g1} = \frac{M(1-3v^2)}{v^2}\text{arctanh}\frac{vt}{r}. \tag{B.7}$$

These were already provided in Eqs. (116) and (117) (and again in (119) and (120)), where the notation $z = vt$ was used. The second-order geodesic terms follow from a straightforward calculation using the Schwarzschild metric in isotropic coordinates. The results are*

$$\begin{aligned} x_{g2}(t) = \frac{M^2}{8bv^4}\Bigg(& \frac{8\left(v^2+1\right)^2 vt}{r} + 8\left(v^2+1\right)\left(3v^2-1\right)\log\frac{r+vt}{b} - \frac{3\pi v^3 t}{b}(4+v^2) \\ & + 2\left(v^2-2\right)^2 + \frac{8\left(3v^4+2v^2-1\right)vt}{r}\text{arctanh}\frac{vt}{r} - \frac{6\left(v^2+4\right)v^3 t}{b}\arctan\frac{vt}{b}\Bigg) \end{aligned} \tag{B.8}$$

$$\begin{aligned} z_{g2}(t) = \frac{M^2}{8bv^4}\Bigg(& -\frac{8\left(v^2+1\right)\left(v^2t^2\left(v^2+1\right)+2b^2\gamma^{-2}\right)}{br} - \frac{8\left(v^2+1\right)^2 vt}{b} \\ & + \frac{8b\left(1-3v^2\right)^2}{r}\text{arctanh}\frac{vt}{r} + 6v^2\left(4-9v^2\right)\left(\arctan\frac{vt}{b}+\frac{\pi}{2}\right)\Bigg). \end{aligned} \tag{B.9}$$

The self-force terms result from the integrals described in Sec. (6). The scalar results are

$$\begin{aligned} x_q(t) = \frac{\gamma M q^2}{24b^2 m}\Bigg(& \frac{2b^3\left(-12\xi+4(3\xi-1)v^2+3\right)}{s} - \frac{2brt}{\gamma^2 s^3}\Bigg(\frac{2b^2v^2t^2}{\gamma^2}\left(-12\xi+3(4\xi+1)v^2+v^4+4\right) \\ & + b^4\left(-12\xi+(12\xi+5)v^2-6v^4+4\right) + \frac{v^4t^4}{\gamma^4}\left(-12\xi+(12\xi+1)v^2+4\right)\Bigg) \\ & - \frac{6t\left((4\xi-1)v^2-4\xi\right)}{\gamma v}\left(\arctan\frac{r}{\gamma v b}+\arctan\frac{vt}{\gamma b}\right) - \frac{16b^7v^4}{s^3} + \frac{4b^5v^2\left(3v^2+4\right)}{s^2} \\ & + 2b\left(-12\xi+\frac{12\xi}{v^2}-4v-\frac{4}{v}+3\right)\Bigg) \end{aligned} \tag{B.10}$$

$$\begin{aligned} z_q(t) = \frac{\gamma M q^2}{24b^2 m}\Bigg(& \frac{2b^2\left(-12\xi+4(3\xi-2)v^2+3\right)vt}{\gamma^2 s} - \frac{2b^2 r}{\gamma^2 v s^3}\Big(\frac{2b^2v^2t^2}{\gamma^4}\left(12\xi+3v^2-4\right) \\ & + b^4\left(12\xi+(5-12\xi)v^2+2v^4-4\right) - \frac{v^4t^4}{\gamma^4}\left(-12\xi+3(4\xi-3)v^2+4\right)\Big) - \frac{16b^6v^5t}{\gamma^2 s^3} \\ & + \frac{6b\left((4\xi+1)v^2-4\xi\right)}{\gamma v^2}\left(\arctan\frac{r}{\gamma v b}+\arctan\frac{vt}{\gamma b}\right) + \frac{4b^4\left(v^2+4\right)v^3 t}{\gamma^2 s^2}\Bigg) \end{aligned} \tag{B.11}$$

* In Eq. (B.9), we have corrected an error from the previous arXiv version. We thank Hongji Wei for helping find this error.

$$x_e(t) = \frac{e^2\gamma M}{24b^2mv^2}\left(\frac{-4b^5v^4\left(3v^2+4\right)}{s^2} + \frac{2b^3v^2\left(10v^2+3\right)}{s} + \frac{16b^7v^6}{s^3}\right.$$
$$- \frac{2bv^2t}{\gamma^2s^3}r\left(2b^2v^2t^2\left(v^6-4v^4+v^2+2\right)+b^4\left(6v^4+v^2+2\right)+\frac{v^4t^4}{\gamma^4}\left(5v^2+2\right)\right)$$
$$\left.+\frac{6vt}{\gamma}\left(v^2+2\right)\left(\arctan\frac{r}{\gamma vb}+\arctan\frac{vt}{\gamma b}\right)-2b\left(8v^3-3v^2+8v-6\right)\right) \tag{B.12}$$

$$z_e(t) = \frac{e^2\gamma M}{24b^2mv^2}\left(\frac{16b^6v^7t}{\gamma^2s^3} - \frac{4b^4\left(v^2+4\right)v^5t}{\gamma^2s^2} - \frac{2b^2\left(10v^2-3\right)v^3t}{\gamma^2s}\right.$$
$$+\frac{2vb^2r\left(2b^2v^2t^2\left(3v^6+8v^4-13v^2+2\right)+b^4\left(2v^4-13v^2+2\right)-\frac{v^4t^4}{\gamma^4}\left(9v^2-2\right)\right)}{\gamma^2s^3}$$
$$\left.+\frac{6b\left(5v^2-2\right)}{\gamma}\left(\arctan\frac{r}{\gamma vb}+\arctan\frac{vt}{\gamma b}\right)\right) \tag{B.13}$$

$$x_m(t) = \frac{\gamma mM}{12b^2v}\left(\frac{2b^5v^3\left(3v^2+4\right)}{s^2} - \frac{b^3v\left(28v^2+45\right)}{s} + b\left(44v^2-21v+44\right) - \frac{8b^7v^5}{s^3}\right.$$
$$+\frac{vt}{\gamma^2}\frac{r}{s^3}\left(2b^3v^2t^2\left(v^6-22v^4-23v^2+44\right)+b^5\left(6v^4+19v^2+44\right)+\frac{bv^4t^4}{\gamma^4}\left(23v^2+44\right)\right)$$
$$\left.-\frac{21v^2t}{\gamma}\left(\arctan\frac{r}{\gamma vb}+\arctan\frac{tv}{\gamma b}\right)\right) \tag{B.14}$$

$$z_m(t) = \frac{\gamma mM}{12b^2v}\left(\frac{-8b^6v^6t}{\gamma^2s^3} + \frac{2b^4\left(v^2+4\right)v^4t}{\gamma^2s^2} - \frac{b^2\left(45-64v^2\right)v^2t}{\gamma^2s}\right.$$
$$-\frac{b^2r\left(2b^2v^2t^2\left(3v^6+62v^4-109v^2+44\right)+b^4\left(2v^4-67v^2+44\right)-\frac{v^4t^4}{\gamma^4}\left(63v^2-44\right)\right)}{2\gamma^2s^3}$$
$$\left.+\frac{69vb}{\gamma}\arctan\frac{\gamma v\left(-t\gamma^{-2}r+b^2\right)}{b\left(r+tv^2\right)}-\frac{69\pi vb}{2\gamma}\right) \tag{B.15}$$

$$
\begin{aligned}
x_{\rm mm}(t) = \frac{\gamma m M}{2b^2v^4}\Bigg(&\frac{2b^3\left(3-v^2\right)v^4}{s} - 2b\left(3v^4+2v^2-1\right)\left(\log\frac{b}{r+vt}+\operatorname{arctanh}(v)\right) \\
&- \frac{2b(v-1)\left(b^2\left(v^6+v^5-2v^4-3v^3+2v^2+1\right)+t^2v^2\left(v^5-4v^4-3v^3+v^2+1\right)\right)vt}{rs} \\
&- \frac{2b\left(3v^4+2v^2-1\right)vt}{r}\operatorname{arctanh}\frac{v\left(r-t\right)}{r-tv^2} - 2b\left(v^5-2v^4+2v^3+v^2+v-1\right) \\
&+ \frac{2\left(v^2-3\right)v^3t}{\gamma}\left(\arctan\frac{vt}{\gamma b}+\arctan\frac{r}{\gamma v b}\right)\Bigg)
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
z_{\rm mm}(t) = \frac{\gamma m M}{2b^2v^4}\Bigg(&\frac{2b^2\left(v^4-4v^2+3\right)v^5t}{s} + \frac{2b^2v^2t^2\left(-v^8+4v^7-2v^6-5v^5+4v^4+2v^2+v-3\right)}{rs} \\
&+ \frac{2b^4\left(v^7+v^6-7v^5+4v^4+v^3+v^2+v-2\right)-2v^4t^4\left(v^4-1\right)^2}{rs} \\
&+ \frac{2\left(v^2-3\right)v^2b}{\gamma}\arctan\frac{\gamma v\left(-t\gamma^{-2}r+b^2\right)}{b\left(r+tv^2\right)} + \frac{2\left(1-3v^2\right)^2b^2}{r}\operatorname{arctanh}\frac{v\left(t-r\right)}{r-tv^2} \\
&+ 2\left(v^2-1\right)\left(v^2+1\right)^2vt - \frac{\pi v^2\left(v^2-3\right)b}{\gamma}\Bigg)
\end{aligned}
\tag{B.17}
$$

The G^x and G^z integrals defined in Sec. 6 are given in terms of these definitions by

$$G^z_{\rm scalar}(t) = \gamma^2 z_q(t) \tag{B.18}$$
$$G^x_{\rm scalar}(t) = x_q(t) \tag{B.19}$$
$$G^z_{\rm EM}(t) = \gamma^2 z_e(t) \tag{B.20}$$
$$G^x_{\rm EM}(t) = x_e(t) \tag{B.21}$$
$$G^z_{\rm grav}(t) = \gamma^2(z_m + z_{\rm mm}) \tag{B.22}$$
$$G^x_{\rm grav}(t) = x_m + x_{\rm mm}. \tag{B.23}$$

References

[1] B. S. DeWitt and R. W. Brehme, "Radiation damping in a gravitational field," *Annals Phys.* **9** (1960) 220–259. 1, 3, 3, 4.3

[2] C. M. DeWitt and B. S. DeWitt, "Falling charges," *Physics Physique Fizika* **1** (1964) 3–20. [Erratum: Physics Physique Fizika 1, 145 (1964)]. 1

[3] K. Westpfahl and M. Goller, "Relativistic free-fall motion of a point charge in a weak gravitational field," *Lettere Al Nuovo Cimento (1971–1985)* **27** no. 6, (1980) 161–168. 1, 4.1

[4] K. Westpfahl, "High-Speed Scattering of Charged and Uncharged Particles in General Relativity," *Fortsch. Phys.* **33** no. 8, (1985) 417–493. 1, 2, 4.1

[5] Y. Mino, M. Sasaki, and T. Tanaka, "Gravitational radiation reaction to a particle motion," *Phys. Rev. D* **55** (1997) 3457–3476, arXiv:gr-qc/9606018. 1, 5

[6] T. C. Quinn and R. M. Wald, "An Axiomatic approach to electromagnetic and gravitational radiation reaction of particles in curved space-time," *Phys. Rev. D* **56** (1997) 3381–3394, arXiv:gr-qc/9610053. 1, 5

[7] T. C. Quinn, "Axiomatic approach to radiation reaction of scalar point particles in curved space-time," *Phys. Rev. D* **62** (2000) 064029, arXiv:gr-qc/0005030. 1, Appendix A

[8] M. J. Pfenning and E. Poisson, "Scalar, electromagnetic, and gravitational selfforces in weakly curved space-times," *Phys. Rev. D* **65** (2002) 084001, arXiv:gr-qc/0012057. 1, 2, 3, 3, 3, 3, 3, 2, 3, 4, 4.3, 5, 5, 5, 5

[9] L. Barack, "Gravitational self force in extreme mass-ratio inspirals," *Class. Quant. Grav.* **26** (2009) 213001, arXiv:0908.1664 [gr-qc]. 1

[10] T. Damour, "Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory," *Phys. Rev. D* **94** no. 10, (2016) 104015, arXiv:1609.00354 [gr-qc]. 1, 2

[11] T. Damour, "High-energy gravitational scattering and the general relativistic two-body problem," *Phys. Rev. D* **97** no. 4, (2018) 044038, arXiv:1710.10599 [gr-qc]. 1

[12] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, "General Relativity from Scattering Amplitudes," *Phys. Rev. Lett.* **121** no. 17, (2018) 171601, arXiv:1806.04920 [hep-th]. 1

[13] C. Cheung, I. Z. Rothstein, and M. P. Solon, "From Scattering Amplitudes to Classical Potentials in the Post-Minkowskian Expansion," *Phys. Rev. Lett.* **121** no. 25, (2018) 251101, arXiv:1808.02489 [hep-th]. 1

[14] D. A. Kosower, B. Maybee, and D. O'Connell, "Amplitudes, Observables, and Classical Scattering," *JHEP* **02** (2019) 137, arXiv:1811.10950 [hep-th]. 1

[15] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, "Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order," *Phys. Rev. Lett.* **122** no. 20, (2019) 201603, arXiv:1901.04424 [hep-th]. 1

[16] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, "Black Hole Binary Dynamics from the Double Copy and Effective Theory," *JHEP* **10** (2019) 206, arXiv:1908.01493 [hep-th]. 1

[17] T. Damour, "Classical and quantum scattering in post-Minkowskian gravity," *Phys. Rev. D* **102** no. 2, (2020) 024060, arXiv:1912.02139 [gr-qc]. 1, 2

[18] G. Kälin and R. A. Porto, "From Boundary Data to Bound States," *JHEP* **01** (2020) 072, arXiv:1910.03008 [hep-th]. 1

[19] G. Kälin and R. A. Porto, "From boundary data to bound states. Part II. Scattering angle to dynamical invariants (with twist)," *JHEP* **02** (2020) 120, arXiv:1911.09130 [hep-th]. 1

[20] G. Kälin, Z. Liu, and R. A. Porto, "Conservative Dynamics of Binary Systems to Third Post-Minkowskian Order from the Effective Field Theory Approach," *Phys. Rev. Lett.* **125** no. 26, (2020) 261103, arXiv:2007.04977 [hep-th]. 1

[21] G. Kälin and R. A. Porto, "Post-Minkowskian Effective Field Theory for Conservative Binary Dynamics," *JHEP* **11** (2020) 106, arXiv:2006.01184 [hep-th]. 1

[22] G. Mogull, J. Plefka, and J. Steinhoff, "Classical black hole scattering from a worldline quantum field theory," *JHEP* **02** (2021) 048, arXiv:2010.02865 [hep-th]. 1

[23] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "Universality of ultra-relativistic gravitational scattering," *Phys. Lett. B* **811** (2020) 135924, arXiv:2008.12743 [hep-th]. 1

[24] C. Cheung, N. Shah, and M. P. Solon, "Mining the Geodesic Equation for Scattering Data," *Phys. Rev. D* **103** no. 2, (2021) 024030, arXiv:2010.08568 [hep-th]. 1

[25] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Planté, and P. Vanhove, "Classical gravity from loop amplitudes," *Phys. Rev. D* **104** no. 2, (2021) 026009, arXiv:2104.04510 [hep-th]. 1

[26] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, "Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory," *Phys. Rev. Lett.* **126** no. 20, (2021) 201103, arXiv:2101.12688 [gr-qc]. 1

[27] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, "Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$," *Phys. Rev. Lett.* **126** no. 17, (2021) 171601, arXiv:2101.07254 [hep-th]. 1

[28] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, "Radiative Classical Gravitational

Observables at $\mathcal{O}(G^3)$ from Scattering Amplitudes," arXiv:2104.03957 [hep-th]. 1

[29] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, "Gravitational Bremsstrahlung from Reverse Unitarity," *Phys. Rev. Lett.* **126** no. 20, (2021) 201602, arXiv:2101.07255 [hep-th]. 1

[30] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "The eikonal approach to gravitational scattering and radiation at $\mathcal{O}$(G^3)," *JHEP* **07** (2021) 169, arXiv:2104.03256 [hep-th]. 1

[31] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, "Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach," arXiv:2106.08276 [hep-th]. 1

[32] D. Bini, T. Damour, and A. Geralico, "Radiative contributions to gravitational scattering," *Phys. Rev. D* **104** no. 8, (2021) 084031, arXiv:2107.08896 [gr-qc]. 1

[33] A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, "Classical gravitational scattering from a gauge-invariant double copy," *JHEP* **10** (2021) 118, arXiv:2108.04216 [hep-th]. 1

[34] T. Damour, "Radiative contribution to classical gravitational scattering at the third order in G," *Phys. Rev. D* **102** no. 12, (2020) 124008, arXiv:2010.01641 [gr-qc]. 1, 7.3

[35] S. E. Gralla and K. Lobo, "Gravitational Scoot," *in prep.* . 1, 7.4

[36] S. E. Gralla and K. Lobo, "Electromagnetic Scoot," arXiv:2112.01729 [gr-qc]. 1, 7.4

[37] E. Poisson, A. Pound, and I. Vega, "The Motion of point particles in curved spacetime," *Living Rev. Rel.* **14** (2011) 7, arXiv:1102.0529 [gr-qc]. 3, 3

[38] Y.-Z. Chu and G. D. Starkman, "Retarded Green's Functions In Perturbed Spacetimes For Cosmology and Gravitational Physics," *Phys. Rev. D* **84** (2011) 124020, arXiv:1108.1825 [astro-ph.CO]. 3

[39] S. E. Gralla and R. M. Wald, "A Rigorous Derivation of Gravitational Self-force," *Class. Quant. Grav.* **25** (2008) 205009, arXiv:0806.3293 [gr-qc]. [Erratum: Class.Quant.Grav. 28, 159501 (2011)]. 5

[40] A. Pound, "Self-consistent gravitational self-force," *Phys. Rev. D* **81** (2010) 024023, arXiv:0907.5197 [gr-qc]. 5

[41] T. D. Drivas and S. E. Gralla, "Dependence of Self-force on Central Object," *Class. Quant. Grav.* **28** (2011) 145025, arXiv:1009.0504 [gr-qc]. 5

[42] S. Isoyama and E. Poisson, "Self-force as probe of internal structure," *Class. Quant. Grav.* **29** (2012) 155012, arXiv:1205.1236 [gr-qc]. 5

[43] Y.-Z. Chu, K. Pasmatsiou, and G. D. Starkman, "Finite-size effects on the self-force," *Phys. Rev. D* **101** no. 10, (2020) 104020, arXiv:1910.02924 [gr-qc]. 5

[44] S. L. Detweiler and E. Poisson, "Low multipole contributions to the gravitational selfforce," *Phys. Rev. D* **69** (2004) 084019, arXiv:gr-qc/0312010. 5

[45] L. Barack, M. Colleoni, T. Damour, S. Isoyama, and N. Sago, "Self-force effects on the marginally bound zoom-whirl orbit in Schwarzschild spacetime," *Phys. Rev. D* **100** no. 12, (2019) 124015, arXiv:1909.06103 [gr-qc]. 5

5 Electromagnetic Scoot

This is a paper I coauthored with Sam Gralla titled, ”Electromagnetic Scoot”.

Electromagnetic Scoot

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Recent work on scattering of massive bodies in general relativity has revealed that the mechanical center of mass of the system (or, more precisely, its relativistic *mass moment*) undergoes a shift during the scattering process. We show that the same phenomenon occurs in classical scattering of charged particles in flat spacetime and study the effect in detail. Working to leading order in the interaction, we derive formulas for the initial and final values of the mechanical and electromagnetic energy, momentum, angular momentum, and mass moment. We demonstrate that the change in mechanical mass moment is balanced by an opposite change in the mass moment stored in the electromagnetic field. This is a non-radiative exchange between particles and field, analogous to exchange of kinetic and potential energy. A simple mechanical analogy is a person scooting forward on the floor, who exchanges mass moment with the floor. We therefore say that electromagnetic scattering results in an electromagnetic scoot.

I. INTRODUCTION

Scattering experiments, whether real or fictitious, offer a simple way to gain understanding of the physical implications of a theory. The small-deflection limit provides a further simplified testing ground where precise analytical results are usually possible. The study of small-angle scattering of comparable-mass particles in general relativity began in the 1980's with the derivation of the first-order [1] and second-order [2] deflection angle, and has recently seen a resurgence of interest due to connections with quantum scattering methods and the dynamics of bound systems. The new computational firepower thrown at this problem has resulted in spectacular progress, with the latest results now probing the *fourth* order in the small-angle approximation ([3–6] and references therein).

Inspired by this rich, interconnected set of results, we set about to understand gravitational scattering with a new approach using self-force methods [7]. We managed to reproduce the second-order results from the 1980s, but discovered, to our surprise, an overlooked feature of the problem that appears even at the *first* order beyond straight line motion. In addition to computing the energy, momentum, and angular momentum of the particles, we considered the last, overlooked conserved quantity: the mass moment. For a system of point particles, the mass moment is defined by

$$\boldsymbol{N}_{\text{mech}} = \sum_I E_I \boldsymbol{r}_I - t \sum_I \boldsymbol{p}_I, \tag{1}$$

where $\boldsymbol{r}_I$, E_I and $\boldsymbol{p}_I$ are the position, energy and momentum (respectively) of the particles labeled by I.[1] We include the subscript "mech" to emphasize that any field contributions have not been included in this formula.

[1] We set the speed of light to unity ($c = 1$) and regard relativistic mass and energy as equivalent. If we had not made this choice, we would divide the first term of (1) by c^2, ensuring that mass moment has units of mass times length.

The mass moment is the position-weighted energy of the system minus its total momentum times time, and its conservation reflects the uniform motion of the center of energy. It is numerically equal to the total energy times the center of energy at time $t = 0$, and it thereby tracks the position of the center of energy at a fiducial time. Although the total value can always be set to zero by a translation, the mass moment is additive (unlike the center of energy) and can therefore be budgeted like the energy, momentum, and angular momentum. That is, we can ask about *exchange* of mass moment between different degrees of freedom, or *radiation* of mass moment away to infinity. From a relativistic point of view, mass moment is inseparable from the angular momentum, since the two mix under boosts and only together form a relativistically invariant object (e.g., [8]).

In the scattering of two masses m_1 and m_2, the important mass scales are the initial total energy E_0 and the relativistic reduced mass μ,

$$E_0 = \sqrt{m_1^2 + m_2^2 + 2\gamma m_1 m_2}, \qquad \mu = \frac{m_1 m_2}{E_0}, \tag{2}$$

where γ is the initial relative Lorentz factor. In the small-deflection limit, the center of energy-momentum (CEM) frame scattering angle is proportional to [2]

$$\chi = \frac{GE_0}{bv^2} \ll 1, \tag{3}$$

where G is Newton's constant, b is the impact parameter, and v is the initial relative velocity. In our study of small-angle gravitational scattering through second order in χ [7], we found that the mechanical mass moment changes during the scattering process. At leading order in χ, the change is

$$\Delta \boldsymbol{N}_{\text{mech}} = 2\mu b \chi \gamma (1 - 3v^2) \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} \hat{\boldsymbol{z}}, \tag{4}$$

where $\hat{\boldsymbol{z}}$ is a unit vector pointing from particle 1 to particle 2 at early times. This form makes clear that the effect disappears in the Newtonian limit $v \to 0$, as it must.[2] However, plugging in for χ shows that the change in mass moment is in fact *independent of the impact parameter*,

$$\Delta \boldsymbol{N}_{\text{mech}} = 2\gamma \frac{1-3v^2}{v^2} G m_1 m_2 \log \frac{m_2+\gamma m_1}{m_1+\gamma m_2} \hat{\boldsymbol{z}}. \tag{5}$$

This suggests that the effect is not tied to the details of the small-angle scattering encounter and will exist in similar form for large-angle scattering with $\chi \gtrsim 1$.

These results surprised and puzzled us for a number of reasons, not least because we believed we were working in the CEM frame, where the total mass moment should be zero. We suspected that contributions from the gravitational field need to be included, but ran into obstacles related to the fact that field energy is fundamentally gauge-dependent in general relativity. It is also not entirely clear that spacetime can be treated as flat for the purposes of computing conserved quantities at early and late times, since these involve $1/t$ corrections that are formally the same order as gravitational field perturbations to the metric.

Work is underway to settle these issues in general relativity [9]. However, in the meantime, we can consider a simpler, electromagnetic analog where the problems do not arise. This problem is surely even older than the gravitational one, and many aspects of the calculation have undoubtedly been performed before (not least in the recent, mammoth exploration through third perturbative order [10]). However, we are unaware of any results on the mass moment, or even any mention of this quantity, in past work on electromagnetic scattering.

Consider, then, the small-angle scattering of classical charged particles. The leading deflection angle is proportional to [2]

$$\chi_{\text{EM}} = \frac{q_1 q_2}{\mu b v^2} \ll 1, \tag{6}$$

where q_1 and q_2 are the particle charges. (There is no explicit coupling constant analogous to G because we work in Gaussian units.) Computing at first order in χ_{EM}, we find a precisely analogous change in mass moment [Eq. (74) below],

$$\Delta \boldsymbol{N}_{\text{mech}} = -2\mu b \gamma^{-2} \chi_{\text{EM}} \log \frac{m_2+\gamma m_1}{m_1+\gamma m_2} \hat{\boldsymbol{z}} \tag{7}$$

$$= -\frac{2 q_1 q_2}{\gamma^2 v^2} \log \frac{m_2+\gamma m_1}{m_1+\gamma m_2} \hat{\boldsymbol{z}}. \tag{8}$$

Eq. (7) demonstrates the relationship to our perturbative calculation, while Eq. (8) shows that the change in mechanical mass moment is again independent of the impact parameter.

In the electromagnetic setting we can be confident, based on general theorems, that this change is in fact compensated by an equal and opposite change in the electromagnetic contribution to the mass moment. The EM field mass moment is given by

$$\boldsymbol{N}_{\text{EM}} = \int \mathcal{E} \boldsymbol{x} d^3x - t \int \boldsymbol{S} d^3x, \tag{9}$$

where $\mathcal{E}$ and $\boldsymbol{S}$ are the electromagnetic field energy and momentum densities, respectively. We argue that only the cross-term contributions (proportional to $q_1 q_2$) should be included in the point particle limit and explicitly evaluate these integrals at early and late times. We find that, indeed, the electromagnetic contribution exactly balances the mechanical one,

$$\Delta \boldsymbol{N}_{\text{EM}} = -\Delta \boldsymbol{N}_{\text{mech}}. \tag{10}$$

That is, there is no change in total mass moment, only an *exchange* between mechanical and electromagnetic degrees of freedom. The exchange is permanent: an electromagnetic scoot.

The electromagnetic problem thus provides a neat and tidy story that can be fully understood. The details and outcome of this calculation give insight into electromagnetic phenomena and lessons for seeking analogous understanding in the gravitational case. We discuss these points in detail at the conclusion of the manuscript.

While this paper was inspired by gravitational phenomena and is aimed substantially at researchers working in this area, we feel that the electromagnetic results are of interest in their own right. As such, we have endeavored to make the paper accessible to aficionados of electromagnetism who are not necessarily steeped in relativistic notation. We have therefore chosen to use vector notation throughout, eschewing the tensors that are standard in gravitational physics. However, we have retained the use of Gaussian units with the speed of light set equal to one, so that results can be easily compared with electromagnetic calculations in the high-energy and gravitational physics literature. Readers unfamiliar with these units can always restore constants like $4\pi\epsilon_0$ and c via dimensional analysis.

This paper is organized as follows. In Sec. II we set up the problem and derive the particle trajectories and electromagnetic fields through first order in the interaction. In Sec. III we calculate all conserved quantities at leading order at early and late times ($t \to \pm\infty$) and analyze the implications. We pay particular attention to the mass moment and also discuss the center of energy. We conclude with some discussion of the scoot phenomenon. An appendix describes the evaluation of certain integrals that arise in the analysis.

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[2] The appropriate Newtonian limit is $v \to 0$ at fixed b and χ, which preserves the condition $\chi \ll 1$ (small-angle deflection) assumed in our calculation. If one instead fixes b and E_0, the limit describes a bound system for which our scattering results are invalid.

II. SMALL-ANGLE SCATTERING OF RELATIVISTIC CHARGED PARTICLES

We will consider a scattering encounter between two classical charged particles 1 and 2 in the approximation of small deflection. To leading order, the particles move in straight lines, and the description is simplest in the frame where one particle is at rest. We will take particle 2 to be at rest at the origin and denote this frame with a prime. Choosing the motion to be in the z' direction and the transverse separation to be in the x' direction, the leading-order trajectories are simply

$$\boldsymbol{r}_1' = (b, 0, vt') \tag{11}$$
$$\boldsymbol{r}_2' = (0, 0, 0), \tag{12}$$

where b and v are constants interpreted as the impact parameter and relative velocity, respectively. Without loss of generality we assume that v is positive,

$$v > 0, \tag{13}$$

so that particle 1 moves in the $+z'$ direction.

We determine the corrected motion by integrating the Lorentz force law using the electric and magnetic fields produced by the background trajectories. There are no magnetic forces since $\dot{\boldsymbol{r}}_2' = 0$ and $\boldsymbol{B}_2' = 0$ to zeroth order, and the Lorentz force law becomes

$$m_1 \ddot{\boldsymbol{r}}_1' = \frac{q_1}{\gamma}\left(\boldsymbol{E}_2' - v^2(\hat{\boldsymbol{z}}' \cdot \boldsymbol{E}_2')\hat{\boldsymbol{z}}'\right) \tag{14}$$
$$m_2 \ddot{\boldsymbol{r}}_2' = q_2 \boldsymbol{E}_1', \tag{15}$$

where $\gamma = (1-v^2)^{-1/2}$ is the relative Lorentz factor. The electric fields produced at leading order are

$$\boldsymbol{E}_1' = q_1\gamma \frac{(x'-b)\hat{\boldsymbol{x}}' + y'\hat{\boldsymbol{y}}' + (z'-vt')\hat{\boldsymbol{z}}'}{[(x'-b)^2 + y'^2 + \gamma^2(z'-vt')^2]^{3/2}} \tag{16}$$
$$\boldsymbol{E}_2' = q_2 \frac{x'\hat{\boldsymbol{x}}' + y'\hat{\boldsymbol{y}}' + z'\hat{\boldsymbol{z}}'}{[x'^2 + y'^2 + z'^2]^{3/2}}. \tag{17}$$

The corrected motion is determined by integrating the right-hand-sides of Eqs. (14) and (15) using Eqs. (16) and (17). We choose the integration constants so that the perturbed velocity of each particle vanishes in the distant past (making v interpreted as the initial relative velocity), and so that the particles reach $z' = 0$ at $t' = 0$. We find

$$x_1' = b + \frac{q_1 q_2}{b m_1 \gamma v^2}\left(vt' + \sqrt{b^2 + v^2 t'^2}\right) \tag{18}$$
$$z_1' = vt' - \frac{q_1 q_2}{m_1 \gamma^3 v^2}\operatorname{arctanh}\frac{vt'}{\sqrt{v^2 t'^2 + b^2}} \tag{19}$$
$$x_2' = -\frac{q_1 q_2}{b m_2 v^2}\left(vt' + \sqrt{v^2 t'^2 + b^2\gamma^{-2}}\right) \tag{20}$$
$$z_2' = \frac{q_1 q_2}{m_2 \gamma^2 v^2}\operatorname{arctanh}\frac{vt'}{\sqrt{v^2 t'^2 + b^2\gamma^{-2}}}. \tag{21}$$

These equations provide the corrected particle trajectories. The second particle is no longer at rest, but its velocity is asymptotically zero at early times. We may thus interpret the primed frame as the "initial rest frame" of particle 2. Note, however, that the position of particle 2 in fact diverges logarithmically at early (and late) times. This is an unavoidable consequence of the long-range nature of the Coulomb force: an inverse-square force integrates up to a logarithmically divergent position.

A. Transformation to CEM frame

While the primed frame was convenient for finding the trajectories, it is conceptually less useful since it makes an explicit preference for one particle over the other, when no such preference exists in the problem. A more natural choice is the center of energy-momentum (CEM) frame, defined as the frame with no momentum $\boldsymbol{p}$ or mass moment $\boldsymbol{N}$,

$$\boldsymbol{p} = \boldsymbol{N} = 0. \tag{22}$$

In our relativistic scattering problem, these quantities receive contributions from both the particles and the electromagnetic field (see Sec. III). At leading order (neglecting the interaction), the particles move in straight lines [Eqs. (11) and (12)] and there is no contribution from the electromagnetic field. In this case the appropriate transformation consists of a Lorentz boost in the z direction and a translation in the x direction,

$$t = \frac{m_2 + \gamma m_1}{E_0} t' - \frac{\gamma m_1 v}{E_0} z' \tag{23}$$
$$z = \frac{m_2 + \gamma m_1}{E_0} z' - \frac{\gamma m_1 v}{E_0} t' \tag{24}$$
$$x = x' - b\frac{m_1(\gamma m_2 + m_1)}{E_0^2}, \tag{25}$$

where E_0 was introduced in Eq. (2) above.

Although this transformation was designed for the leading order motion, it turns out that no modification is necessary for the first perturbative correction. That is, we will see that Eqs. (23)-(25) still take us to the CEM frame as defined by (22). However, and quite surprisingly, we find that it is essential to take into account the contribution from the electromagnetic field, *even at early and late times, when the particles are infinitely separated.* This fact enables the electromagnetic scoot: a permanent exchange of mass moment between particle and field.

B. CEM-frame particle trajectories

The path of particle 1 in the CEM frame is given by plugging Eqs. (18) and (19) into Eqs. (23)–(25) and solving the resulting set of equations for $x_1(t)$ and $z_1(t)$, dropping terms nonlinear in $q_1 q_2$; the same procedure for particle 2 gives $x_2(t)$ and $z_2(t)$. The results are

$$x_1 = b_1 + \frac{q_1 q_2}{b m_1 \gamma v^2}\left(v t_1 + \sqrt{b^2 + v^2 t_1^2}\right) \tag{26}$$

$$z_1 = v_1\left(t - \frac{q_1 q_2 E_0}{m_1 m_2 \gamma^3 v^3}\operatorname{arctanh}\frac{v t_1}{\sqrt{b^2 + v^2 t_1^2}}\right) \tag{27}$$

$$x_2 = b_2 - \frac{q_1 q_2}{b m_2 \gamma v^2}\left(v t_2 + \sqrt{b^2 + v^2 t_2^2}\right) \tag{28}$$

$$z_2 = v_2\left(t - \frac{q_1 q_2 E_0}{m_1 m_2 \gamma^3 v^3}\operatorname{arctanh}\frac{v t_2}{\sqrt{b^2 + v^2 t_2^2}}\right), \tag{29}$$

where we define

$$t_1 = \frac{\gamma E_0}{m_1 + \gamma m_2} t, \qquad t_2 = \frac{\gamma E_0}{m_2 + \gamma m_1} t \tag{30}$$

$$b_1 = \frac{m_2(m_2 + \gamma m_1)}{E_0^2} b, \qquad b_2 = -\frac{m_1(m_1 + \gamma m_2)}{E_0^2} b \tag{31}$$

$$v_1 = \frac{\gamma m_2}{m_1 + \gamma m_2} v, \qquad v_2 = -\frac{\gamma m_1}{m_2 + \gamma m_1} v. \tag{32}$$

Notice that relabeling the particles ($m_1 \leftrightarrow m_2$ and $q_1 \leftrightarrow q_2$) is equivalent to reversing the sign of x and z. In other words, the configuration is invariant under swapping the particles and also rotating by 180° within the plane of their motion. This symmetry is a special property of the CEM frame. It was taken as the *definition* of the CEM frame in our recent work [7].

C. CEM-frame electromagnetic field

The leading electromagetic field is determined by the leading, straight-line motion of the charges. Since the sources have constant velocity, their fields are just the boosted Coulomb field, given for $I = 1, 2$ by

$$\mathbf{E}_I = \frac{q_I \gamma_I}{R_I^3}\left[(x - b_I)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + (z - v_I t)\hat{\boldsymbol{z}}\right] \tag{33}$$

$$\mathbf{B}_I = \frac{-q_I \gamma_I v_I}{R_I^3}\left[y\hat{\boldsymbol{x}} - (x - b_I)\hat{\boldsymbol{y}}\right], \tag{34}$$

where $\gamma_I = (1 - v_I^2)^{-1/2}$ and the boosted distance function is

$$R_I = \sqrt{(x - b_I)^2 + y^2 + \gamma_I^2 (z - v_I t)^2}. \tag{35}$$

Notice that the electric and magnetic fields are invariant under the operations $m_1 \leftrightarrow m_2$, $q_1 \leftrightarrow q_2$, $x \to -x$, $z \to -z$, a special property of the CEM frame [see discussion below Eq. (32)].

This completes the first-corrected description of the problem in the CEM frame: the particle trajectories are given in Eqs. (26)–(29), while the electric and magnetic fields are given in Eqs. (33) and (34).

III. ANALYSIS OF CONSERVED QUANTITIES

We will now discuss the behavior of the four conserved quantities: energy, momentum, angular momentum, and mass moment. For clarity, let us first imagine the situation where the particles are modeled by smooth, extended bodies. The budget for the system involves mechanical contributions from bodies 1 and 2 as well as the contribution from the electromagnetic field,

$$E = E_1 + E_2 + E_F \tag{36}$$

$$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_F \tag{37}$$

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{L}_F \tag{38}$$

$$\boldsymbol{N} = \boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{N}_F. \tag{39}$$

The form of the body contributions will depend on the particular model for the bodies, but the electromagnetic contribution is always given by

$$E_F = \frac{1}{8\pi}\int \left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) d^3x \tag{40}$$

$$\boldsymbol{p}_F = \frac{1}{4\pi}\int \left(\boldsymbol{E} \times \boldsymbol{B}\right) d^3x \tag{41}$$

$$\boldsymbol{L}_F = \frac{1}{4\pi}\int \boldsymbol{x} \times \left(\boldsymbol{E} \times \boldsymbol{B}\right) d^3x \tag{42}$$

$$\boldsymbol{N}_F = \frac{1}{8\pi}\int \left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \boldsymbol{x}\, d^3x - \boldsymbol{p}_F t. \tag{43}$$

Now let us consider the point particle limit. The particle conserved quantities take their standard relativistic forms,

$$E_I = \gamma_I m_I \tag{44}$$

$$\boldsymbol{p}_I = \gamma_I m_I \dot{\boldsymbol{r}}_I \tag{45}$$

$$\boldsymbol{L}_I = \boldsymbol{r}_I \times \boldsymbol{p}_I \tag{46}$$

$$\boldsymbol{N}_I = \gamma_I m_I \boldsymbol{r}_I - \boldsymbol{p}_I t, \tag{47}$$

where $I = 1, 2$ labels the particles. In these expressions the full Lorentz factors $\gamma_I = (1 - \dot{\boldsymbol{r}}_I^2)^{-1/2}$ must be used, as opposed to the background Lorentz factors appearing in Eqs. (33) and (34). The point particle limit is a significant simplification, since we now have definite expressions for the conserved quantities.

The point particle limit will also help with the field integrals, but a naive application brings trouble. Whereas Eqs. (40)-(43) are perfectly well-defined for extended bodies, they are divergent for point particles. The electric and magnetic fields grow like inverse distance squared as one approaches the particles, so that the densities of the conserved quantities grow like inverse distance to the fourth power. These singularities are not integrable, and all four conserved quantities are divergent. This issue is generally known as the "electron self-energy problem," although the problem is with the naive point particle limit, not with electrons.

To see how to proceed, let us consider the example of the energy,

$$E_F = \frac{1}{8\pi}\int \left[(\boldsymbol{E}_1+\boldsymbol{E}_2)^2 + (\boldsymbol{B}_1+\boldsymbol{B}_2)^2\right] d^3x. \tag{48}$$

This integral naturally splits into three contributions $E_f = E_{F1} + E_{F2} + E_{F\times}$ given by

$$E_{F1} = \frac{1}{8\pi}\int (\boldsymbol{E}_1^2 + \boldsymbol{B}_1^2) d^3x \tag{49}$$

$$E_{F2} = \frac{1}{8\pi}\int (\boldsymbol{E}_2^2 + \boldsymbol{B}_2^2) d^3x \tag{50}$$

$$E_{F\times} = \frac{1}{4\pi}\int (\boldsymbol{E}_1\cdot\boldsymbol{E}_2 + \boldsymbol{B}_1\cdot\boldsymbol{B}_2) d^3x. \tag{51}$$

The integrals for E_{F1} and E_{F2} are infinite on account of the inverse-square divergence of the electric and magnetic fields at the positions of the particles. However, there are several different ways to see that we can, and in fact *must*, drop these terms from the calculation.

The simplest reason is that E_{F1} and E_{F2} are proportional to q_1^2 and q_2^2, respectively, whereas the correction we consider is proportional to q_1q_2 [see Eqs. (26)–(29)]. We know that there is in fact no energy exchange between particle and field at order q_1^2 or q_2^2 (the self-force effects start at higher order [10]), so it is consistent to drop these terms from the energy budget. Including them properly would actually be quite subtle, since they contribute to the particle masses m_1 and m_2.[3] It is also useful to consider the slow motion limit, where one has the well known conservation of total (kinetic plus potential) energy. In this limit the cross-term integral evaluates precisely to the usual interaction energy $U = q_1q_2/(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$ [see Eq. (A9) below]. That is, reproducing the usual conservation of total energy requires dropping the self-energy terms.

For all these reasons, the correct budget for the conserved quantities in our problem is

$$E = E_1 + E_2 + E_{F\times} \tag{52}$$

$$\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_{F\times} \tag{53}$$

$$\boldsymbol{L} = \boldsymbol{L}_1 + \boldsymbol{L}_2 + \boldsymbol{L}_{F\times} \tag{54}$$

$$\boldsymbol{N} = \boldsymbol{N}_1 + \boldsymbol{N}_2 + \boldsymbol{N}_{F\times}, \tag{55}$$

where the particle contributions are given by Eqs. (44)-(47), while the cross-term field contributions are given by

$$E_{F\times} = \frac{1}{8\pi}\int \mathcal{E}_\times d^3x \tag{56}$$

$$\boldsymbol{p}_{F\times} = \frac{1}{4\pi}\int \boldsymbol{S}_\times d^3x \tag{57}$$

$$\boldsymbol{L}_{F\times} = \frac{1}{4\pi}\int \boldsymbol{x}\times\boldsymbol{S}_\times\, d^3x \tag{58}$$

$$\boldsymbol{N}_{F\times} = \frac{1}{8\pi}\int \mathcal{E}_\times \boldsymbol{x}\, d^3x - \boldsymbol{p}_{F\times} t. \tag{59}$$

Here we have introduced the cross-term energy and momentum densities as

$$\mathcal{E}_\times = \frac{1}{4\pi}\left(\boldsymbol{E}_1\cdot\boldsymbol{E}_2 + \boldsymbol{B}_1\cdot\boldsymbol{B}_2\right) \tag{60}$$

$$\boldsymbol{S}_\times = \frac{1}{4\pi}\left(\boldsymbol{E}_1\times\boldsymbol{B}_2 + \boldsymbol{E}_2\times\boldsymbol{B}_1\right). \tag{61}$$

All terms in the conserved quantity budgets (52)-(55) are now fully specified and mathematically well-defined.

A. Initial and final values: mechanical contribution

In order to understand the exchange of conserved quantities between particles and field, we will consider those quantities at early and late times in the scattering problem. Expanding the trajectories at $t\to\pm\infty$, we find

$$x_I = b_I + \Theta(t)\frac{2q_1q_2}{\mu b\gamma v^2} v_I t + O(1/t^2) \tag{62}$$

$$z_I = v_I t \mp v_I \frac{q_1q_2}{\mu\gamma^3v^3}\log\frac{2v|t_I|}{b} + O(1/t^2) \tag{63}$$

Here and below, the upper sign corresponds to late times, while the lower sign corresponds to early times. Notice that the z position of both particles is logarithmically divergent at early and late times. As discussed below Eq. (21), the divergence originates from the inverse-square nature of the electromagnetic force. The presence of the Heaviside function $\Theta(t)$ in the x position shows the scattering of the particles by a small angle δ:

$$\delta = \lim_{t\to\infty}\left(\frac{x_I - b_I}{z_I}\right) = \frac{2q_1q_2}{\mu b\gamma v^2}. \tag{64}$$

This result is well known (e.g., [10]).

Calculating the conserved quantities from Eqs. (44)-(47), we have

$$E_1 = \frac{m_1+\gamma m_2}{E_0}\left(m_1 - \frac{m_2}{E_0}\frac{q_1q_2}{\gamma v|t|}\right) + O(t^{-2}) \tag{65}$$

$$\boldsymbol{p}_1 = \left(\mu\gamma v - \frac{q_1q_2}{\gamma^2v^2}\frac{(m_1+\gamma m_2)^2}{E_0^2|t|}\right)\hat{\boldsymbol{z}} + \Theta(t)\frac{2q_1q_2}{bv}\hat{\boldsymbol{x}} + O(t^{-2}) \tag{66}$$

$$\boldsymbol{L}_1 = -\mu b\gamma v\frac{m_2(m_2+\gamma m_1)}{E_0^2}\hat{\boldsymbol{y}} + O(t^{-2}) \tag{67}$$

$$\boldsymbol{N}_1 = \mp\frac{q_1q_2}{\gamma^2v^2}\left(\log\frac{2\gamma v E_0|t|}{(m_1+\gamma m_2)b} - 1\right)\hat{\boldsymbol{z}} + O(t^{-2}) \tag{68}$$

[3] In the classic derivation of the self-force (e.g. [11]), the infinite self-energy is combined with a negatively infinite "bare mass", with the sum representing the finite particle mass m. In a rigorous derivation with extended bodies [12], one finds an analogous *finite* mass renormalization, with the observable mass *proven* to be a sum of material and field contributions, each of which is individually finite. This decomposition occurs even in the derivation of the Lorentz force law, irrespective of self-force corrections.

The values for particle 2 are given by exchanging $1 \leftrightarrow 2$ and sending $x \to -x$ and $z \to -z$ [see discussion below Eq. (32)]. The energy, momentum, and angular momentum have well-defined initial and final values (good limits as $t \to -\infty$ and $t \to +\infty$, respectively), but the mass moment inherits the logarithmic divergence of the position. If we instead consider the total mechanical contribution to the conserved quantities, we have

$$E_1 + E_2 = E_0 - \frac{q_1 q_2}{v|t|} \frac{m_1^2 + m_2^2 + 2m_1 m_2/\gamma}{E_0^2} \tag{69}$$

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = -\frac{q_1 q_2}{|t|} \frac{m_2^2 - m_1^2}{E_0^2} \hat{\boldsymbol{z}} + O(t^{-2}) \tag{70}$$

$$\boldsymbol{L}_1 + \boldsymbol{L}_2 = -\mu b \gamma v \hat{\boldsymbol{y}} + O(t^{-2}) \tag{71}$$

$$\boldsymbol{N}_1 + \boldsymbol{N}_2 = \mp \frac{q_1 q_2}{\gamma^2 v^2} \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} \hat{\boldsymbol{z}} + O(t^{-2}). \tag{72}$$

The total mechanical mass moment has well-defined initial and final values (limits as $t \to \pm\infty$), given by the lower sign and upper sign (respectively) in Eq. (72). These values are different, meaning there is a permanent change in mechanical mass moment. That is, if Δ represents final minus initial, and "mech" refers to the total contribution from the particles, we have

$$\Delta E_{\text{mech}} = 0, \quad \Delta \boldsymbol{p}_{\text{mech}} = 0, \quad \Delta \boldsymbol{L}_{\text{mech}} = 0, \tag{73}$$

but

$$\Delta \boldsymbol{N}_{\text{mech}} = -\frac{2 q_1 q_2}{\gamma^2 v^2} \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} \hat{\boldsymbol{z}}. \tag{74}$$

Since the total mass moment is conserved, there must be an opposing change in the electromagnetic field mass moment. We will now directly compute the field contributions to the conserved quantities in order to see the exchange explicitly.

B. Initial and final values: field contribution

We wish to compute the initial and final values of the field contributions to the conserved quantities. At early and late times, the particles are widely separated compared to their impact parameter, and one expects the problem to become effectively one-dimensional. This intuition is confirmed by dimensional analysis, as follows. At a given time t, the problem contains two length scales, b and vt. The cross-term field contributions can depend only on the ratio after a relevant dimensionful combination has been factored out,

$$E_{F\times} = \frac{q_1 q_2}{vt} f_E \left(\frac{b}{vt}, m_1, m_2, v \right) \tag{75}$$

$$\boldsymbol{p}_{F\times} = \frac{q_1 q_2}{vt} f_p \left(\frac{b}{vt}, m_1, m_2, v \right) \tag{76}$$

$$\boldsymbol{L}_{F\times} = q_1 q_2 f_L \left(\frac{b}{vt}, m_1, m_2, v \right) \tag{77}$$

$$\boldsymbol{N}_{F\times} = q_1 q_2 f_N \left(\frac{b}{vt}, m_1, m_2, v \right). \tag{78}$$

The functions f are just placeholders indicating functional dependence. These equations may also be derived mathematically by making the change of variables $\boldsymbol{x}' = \boldsymbol{x}/(vt)$ in the cross-term integrals (56)-(59) and noting that $b_I/(v_I t)$ is independent of I,

$$\frac{b_I}{v_I t} = \frac{b}{vt} \frac{(m_1 + \gamma m_2)(m_2 + \gamma m_1)}{E_0^2 \gamma}. \tag{79}$$

Eqs. (75)-(78) show that the leading behavior at large $|t|$ may be computed using the limit $b \to 0$ at fixed t. This confirms the intuition that the problem is one-dimensional at early (and late) times and allows us to use the $b = 0$ versions of the electric and magnetic fields to compute the cross-term field contributions. These integrals are evaluated in Appendix A. Noting our convention $v > 0$, the results are[4]

$$E_{F\times} = \frac{q_1 q_2}{v|t|} \frac{m_1^2 + m_2^2 + 2m_1 m_2/\gamma}{E_0^2} + O(t^{-2}) \tag{80}$$

$$\boldsymbol{p}_{F\times} = \frac{q_1 q_2}{|t|} \frac{m_2^2 - m_1^2}{E_0^2} \hat{\boldsymbol{z}} + O(t^{-2}) \tag{81}$$

$$\boldsymbol{L}_{F\times} = O(t^{-2}) \tag{82}$$

$$\boldsymbol{N}_{F\times} = \pm \frac{q_1 q_2}{\gamma^2 v^2} \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} \hat{\boldsymbol{z}} + O(t^{-2}). \tag{83}$$

Denoting these electromagnetic contributions with a subscript "EM", we see that the changes in electromagnetic conserved quantities (final minus initial) are

$$\Delta E_{\text{EM}} = 0, \quad \Delta \boldsymbol{p}_{\text{EM}} = 0, \quad \Delta \boldsymbol{L}_{\text{EM}} = 0, \tag{84}$$

and

$$\Delta \boldsymbol{N}_{\text{EM}} = \frac{2 q_1 q_2}{\gamma^2 v^2} \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} \hat{\boldsymbol{z}}. \tag{85}$$

The total values of conserved quantities remain constant, but there is an exchange of mass moment between mechanical and electromagnetic degrees of freedom [Eqs. (74) and (85)].

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[4] The analysis of the appendix did not establish the size of the error terms, and we have filled them in to match the mechanical values in Eqs. (69)–(72).

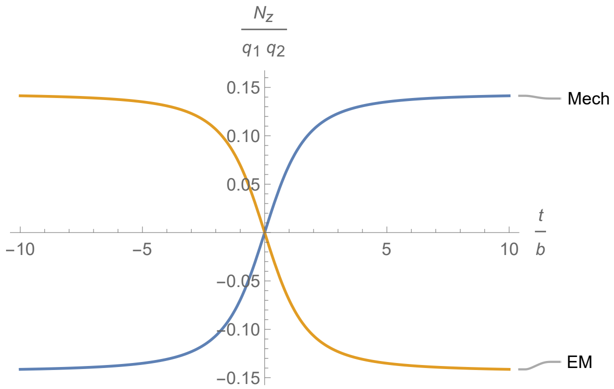

FIG. 1. Exchange of mass moment between mechanical and field degrees of freedom in small-angle electromagnetic scattering. We take speed $v = 1/2$ and mass ratio $m_2/m_1 = 2$.

C. Full time evolution

Eqs. (69)–(72) and (80)–(83) show that the initial and final values of the conserved quantities are

$$E = E_0, \quad \boldsymbol{p} = 0, \quad \boldsymbol{L} = -\mu b \gamma v \hat{\boldsymbol{y}}, \quad \boldsymbol{N} = 0. \tag{86}$$

Since there is no radiation in the problem (fields fall off like distance squared), it is clear that these values of the total (mechanical plus electromagnetic) conserved quantities must hold precisely at all times during the motion.[5] It was therefore sufficient to calculate the initial values of the conserved quantities in order to know their values for all time. Eq. (86) shows in particular that the conditions $\boldsymbol{p} = \boldsymbol{N} = 0$ defining the CEM frame [Eq. (22)] indeed hold for our solution to the scattering problem.

We were unable to explicitly evaluate the electromagnetic contributions at intermediate times, since the problem is no longer effectively one-dimensional. However, from the lack of radiation we know that the electromagnetic contributions are always equal and opposite to the mechanical ones,[5] which may be calculated from the trajectories (26)–(29). Fig. 1 shows the exchange in mass moment during the scattering process.

D. Mechanical center of energy

The mass moment is somewhat unfamiliar, and a reader might question why we have considered it at all. Given that the total momentum is zero, why not just consider the center of energy, which is an intuitive relativistic generalization of the center of mass? The simplest answer—already given in the introduction—is that the mass moment is additive, so it makes sense to talk about separate mechanical and electromagnetic contributions. However, one might still wonder about the behavior of the mechanical center of energy. We will find that this behavior is quite misleading!

Let us define the mechanical center of energy as

$$\boldsymbol{C} = \frac{\sum_I E_I \boldsymbol{r}_I}{\sum_I E_I}, \tag{87}$$

where as usual $I = 1, 2$ labels the particles. Computing this quantity from our results [starting either with the trajectories (26)–(29) or the conserved quantities (69)–(72)], one finds

$$\lim_{t\to\pm\infty} \boldsymbol{C} = \frac{\mp q_1 q_2}{\gamma^2 v^2 E_0}\left(\log\frac{m_2+\gamma m_1}{m_1+\gamma m_2} + \frac{m_2^2 - m_1^2}{E_0^2}\right)\hat{\boldsymbol{z}}. \tag{88}$$

The change in mechanical center of energy is thus

$$\Delta\boldsymbol{C} = -\frac{2 q_1 q_2}{\gamma^2 v^2 E_0}\left(\log\frac{m_2+\gamma m_1}{m_1+\gamma m_2} + \frac{m_2^2 - m_1^2}{E_0^2}\right)\hat{\boldsymbol{z}}. \tag{89}$$

This is a perfectly correct result given the definition (87), but it is hard to understand. Because the center of energy is not a conserved quantity, there is no way to discuss exchange between particles and field. And the specific form of (89) is quite puzzling because $\Delta\boldsymbol{C}$ does not vanish in the non-relativistic limit $v \ll 1$ (the second term survives).[6] It is well known that the mechanical center of mass is strictly conserved for non-relativistic two-body dynamics, where it is usually eliminated at the very start by passing to an effective single-particle description. How, then, can the mechanical center of energy fail to be conserved in the non-relativistic limit?

The answer is that the mechanical center of energy (87) does not actually reduce to the mechanical center of mass in the non-relativistic limit appropriate to the scattering problem. No matter how small the velocity, there will be a correction to the particles' net kinetic energy that balances the potential energy $q_1 q_2/|\boldsymbol{r}_1 - \boldsymbol{r}_2|$. This correction falls off only like the inverse distance between the particles and hence contributes to the mechanical center of energy (a distance-weighted average) even in the limit of infinite particle separation. The problematic second term in (89) is precisely (twice) this contribution.

We see that the mechanical center of energy does not have a very useful non-relativistic limit. By contrast, the mechanical mass moment properly reduces to the mechanical center of mass in the non-relativistic limit, in any frame with no net momentum.

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[5] Mathematically, one can easily show that there is no flux of energy, momentum, angular momentum, or mass moment through a large-radius sphere, because the relevant flux integrals fall off at least like inverse distance.

[6] To take the non-relativistic limit we must ensure that our small parameter χ_{EM} (6) remains small, which can be effected by expressing Eq. (89) in terms of χ_{EM} before letting $v \to 0$.

IV. DISCUSSION

We conclude with some discussion of the character and implications of these results. Let us begin with the size and direction of the electromagnetic scoot (8). The dimensional scale of the effect is set by the charges and initial velocity in the combination q_1q_2/v^2, which has units of mass moment (mass times distance). The size also depends on the dimensionless Lorentz factor γ and mass ratio m_2/m_1. In particular, the masses influence the scoot only through their ratio, with the absolute mass scale playing no role in setting the size. The direction of the scoot depends on the mass ratio as well as on the *signs* of the charges, i.e., whether the interaction is attractive or repulsive. If the interaction is attractive, the scoot is towards the final position of the heavier one, whereas if the interaction is repulsive, the scoot is towards the final position of the lighter one.

This association holds for the gravitational case (5) at sufficiently low velocities ($v < 1/\sqrt{3}$, to be precise), but breaks down above this threshold. The qualitative agreement between the gravitational and electromagnetic results at small velocity is expected given the well-known gravitomagnetic analogy. We have no intuition for the sign reversal in the gravitational scoot at $v = 1/\sqrt{3}$, where the whole effect vanishes. This reversal does not occur in electromagnetism.

In the introduction we commented that the gravitational and electromagnetic scoots have a universal character in that they are independent of impact parameter. Our study of the electromagnetic case sheds considerable light on this issue. The initial and final values of the mechanical mass moments are associated with logarithmic corrections to the particle position due to the long-range nature of the Coulomb force. These corrections will appear at initial and final times irrespective of the details of the scattering encounter. Similarly, our calculation of the electromagnetic field contributions relies only on the condition $|vt| \gg b$, which will occur at sufficiently early or late times in any scattering encounter.

In other words, our calculations show that *whenever charged particles interact only by electromagnetic forces, there will always be non-zero mechanical and electromagnetic contributions to the mass moment, even in the limit of wide separation.* These contributions are directed tangent to the particle separation, and hence will change in any scattering encounter that changes the orientation of the particles. In particular, if $\hat{\boldsymbol{r}}_{12}^{\text{initial}}$ is a unit vector pointing from particle 1 to particle 2 at early times and $\hat{\boldsymbol{r}}_{12}^{\text{final}}$ is a unit vector pointing from particle 1 to particle 2 at late times, then there will be a change in CEM-frame mechanical mass moment given by

$$\Delta \boldsymbol{N}_{\text{mech}} = \frac{q_1 q_2}{\gamma^2 v^2} \log \frac{m_2 + \gamma m_1}{m_1 + \gamma m_2} (\hat{\boldsymbol{r}}_{12}^{\text{final}} - \hat{\boldsymbol{r}}_{12}^{\text{initial}}). \quad (90)$$

In small-angle scattering we have $\hat{\boldsymbol{r}}_{12}^{\text{final}} \approx -\hat{\boldsymbol{r}}_{12}^{\text{initial}}$, reproducing the result of our explicit calculation [Eq. (8) or (74)]. In general scattering (at higher order in perturbation theory, or without any approximation), there may be additional terms due to radiative losses, but the "conservative" contribution (90) will always be present. In this sense the scoot is an unavoidable, and even trivial, consequence of the displacement between mechanical and electromagnetic mass moment that persists at large separation.

Why is this non-zero displacement present? Here we can offer only mathematical reasoning. The Coulomb force falls off as $1/d^2$ with increasing particle separation d. This means that energies and momenta will receive corrections from the interaction at order q_1q_2/d, and the terms $E_I \boldsymbol{r}_I$ and $\boldsymbol{p}_I t$ present in the mass moment (1) will have a finite limits at early and late times, where $d \sim \boldsymbol{r} \sim vt$. Similar comments apply to the electromagnetic cross-term energy and momentum. These various contributions to the total mass moment depend on a variety of parameters (q_1, q_2, m_1, m_2, v), and it would be surprising if they were all individually zero at all parameter values. We may set one linear combination to zero by choice of frame (the center of energy frame), but there will still be non-zero contributions from different degrees of freedom of the system.

These electromagnetic results provide context for the gravitational problem. They provide encouragement that the gravitational result (5) is not an artifact of some peculiar choice of gauge but rather a bona-fide physical effect worthy of further exploration. They also reveal that a proper accounting of the conserved quantities will undoubtedly require consideration of log corrections to particle position as well as gravitational field contributions. These effects are surely an integral part of the initial and final configurations for the gravitational scattering problem, and will be relevant for any rigorous formulation of scattering as a map from past timelike infinity to future timelike infinity. While the study of spacelike and null infinity in general relativity is rather mature, comparably little is known about timelike infinity, especially when matter is present. We hope that our electromagnetic results will be helpful in establishing a rigorous framework for the general relativistic scattering of massive bodies.

ACKNOWLEDGEMENTS

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Appendix A: Field cross-term integrals at zero impact parameter

In Sec. III B we showed that the leading behavior of the field cross-term integrals at early and late times may be determined from the integrals evaluated at zero impact parameter. In this appendix we evaluate the relevant integrals. We do not assume $v > 0$ in this appendix.

Setting $b = 0$ in Eqs. (33) and (34) and changing to cylindrical coordinates $\rho^2 = x^2 + y^2$ and $\tan\phi = y/x$, we have

$$\tilde{\mathbf{E}}_I = \frac{q_I \gamma_I}{\tilde{R}_I^3} \left[\rho \hat{\boldsymbol{\rho}} + (z - v_I t)\hat{\boldsymbol{z}}\right] \tag{A1}$$

$$\tilde{\mathbf{B}}_I = \frac{q_I \gamma_I v_I}{\tilde{R}_I^3} \rho \hat{\boldsymbol{\phi}} \tag{A2}$$

$$\tilde{R}_I = \sqrt{\rho^2 + \gamma_I^2 (z - v_I t)^2}, \tag{A3}$$

where the tilde stands for evaluation at $b = 0$. The cross-term energy density is

$$\tilde{\mathcal{E}}_\times = \frac{1}{4\pi}\left(\tilde{\boldsymbol{E}}_1 \cdot \tilde{\boldsymbol{E}}_2 + \tilde{\boldsymbol{B}}_1 \cdot \tilde{\boldsymbol{B}}_2\right) \tag{A4}$$

$$= \frac{1}{4\pi} \frac{q_1 q_2 \gamma_1 \gamma_2}{\tilde{R}_1^3 \tilde{R}_2^3}\left(\rho^2 (1 + v_1 v_2) + k_1 k_2\right), \tag{A5}$$

where we define

$$k_I = z - v_I t. \tag{A6}$$

To compute the total energy we first perform the integration over ρ and ϕ, yielding

$$\tilde{E}_{F\times} = \int \tilde{\mathcal{E}}_\times \rho d\rho d\phi dz \tag{A7}$$

$$= \frac{q_1 q_2}{2} \int_{-\infty}^{\infty} \frac{s_1 s_2 + \gamma_1 \gamma_2 (1 + v_1 v_2)}{(s_1 \gamma_1 k_1 + s_2 \gamma_2 k_2)^2} dz, \tag{A8}$$

where $s_I = k_I/|k_I|$ denotes the sign of k_I. The remaining integrand is discontinuous at $k_1 = 0$ and $k_2 = 0$, and the integral must be split up at the corresponding values $z = v_1 t$ and $z = v_2 t$. Performing these integrals, we find

$$\tilde{E}_{F\times} = q_1 q_2 \frac{1 + v_1 v_2}{|v_1 t - v_2 t|}$$
$$= \frac{q_1 q_2}{E_0^2 |vt|}\left(m_1^2 + m_2^2 + 2\frac{m_1 m_2}{\gamma}\right). \tag{A9}$$

In the non-relativistic limit $v \ll 1$, this reproduces the usual interaction energy $q_1 q_2/|z_1 - z_2|$ of the one-dimensional problem.

The cross-term momentum density is given by

$$\tilde{\boldsymbol{S}}_\times = \frac{1}{4\pi}\left(\tilde{\boldsymbol{E}}_1 \times \tilde{\boldsymbol{B}}_2 + \tilde{\boldsymbol{E}}_2 \times \tilde{\boldsymbol{B}}_1\right) \tag{A10}$$

$$= \frac{-1}{4\pi} \frac{q_1 q_2 \gamma_1 \gamma_2}{\tilde{R}_1^3 \tilde{R}_2^3}\left((v_1 + v_2)\rho^2 \hat{\boldsymbol{z}} - \rho(k_1 v_2 + k_2 v_1)\hat{\boldsymbol{\rho}}\right). \tag{A11}$$

Following similar steps as before, the total momentum is

$$\tilde{\boldsymbol{p}}_{F\times} = \int \tilde{\boldsymbol{S}}_\times \rho d\rho d\phi dz \tag{A12}$$

$$= -\frac{\gamma_1 \gamma_2 q_1 q_2}{2} \hat{\boldsymbol{z}} \int_{-\infty}^{\infty} \left(\frac{s_1 k_1 \gamma_1 - s_2 k_2 \gamma_2}{\gamma_1^2 k_1^2 - \gamma_2 k_2^2}\right)^2 dz \tag{A13}$$

$$= q_1 q_2 \frac{v_1 + v_2}{|v_1 t - v_2 t|}\hat{\boldsymbol{z}} \tag{A14}$$

$$= \mathrm{sign}(v)\frac{q_1 q_2}{|t|}\frac{m_2^2 - m_1^2}{E_0^2}\hat{\boldsymbol{z}}, \tag{A15}$$

where we note that $\mathrm{sign}(v_1 - v_2) = \mathrm{sign}(v)$.

The angular momentum density $\boldsymbol{x} \times \tilde{\boldsymbol{S}}_\times$ is proportional to $\hat{\boldsymbol{\phi}}$ and has magnitude independent of ϕ, so the total cross-term angular momentum vanishes,

$$\tilde{\boldsymbol{L}}_{F\times} = 0. \tag{A16}$$

The cross-term mass moment is built from the cross-term momentum, which we have already computed, together with the position-weighted average of the energy density. The relevant integral for the latter is

$$\int \tilde{\mathcal{E}}_\times \boldsymbol{x} d^3x = \frac{q_1 q_2}{2}\hat{\boldsymbol{z}} \int_{-\infty}^{\infty} \frac{s_1 s_2 + \gamma_1 \gamma_2 (1 + v_1 v_2)}{(s_1 \gamma_1 k_1 + s_2 \gamma_2 k_2)^2} z dz$$
$$= q_1 q_2 \mathrm{sign}(v_1 t - v_2 t)\hat{\boldsymbol{z}}\left(\frac{v_1 + v_2}{v_1 - v_2} + \frac{\log(\gamma_2/\gamma_1)}{\gamma_1^2 \gamma_2^2 (v_1 - v_2)^2}\right)$$
$$= -q_1 q_2 \mathrm{sign}(vt)\hat{\boldsymbol{z}}\left(\frac{m_1^2 - m_2^2}{E_0^2} + \frac{1}{\gamma^2 v^2}\log\frac{m_1 + \gamma m_2}{m_2 + \gamma m_1}\right). \tag{A17}$$

One subtlety of this calculation is worth noting. In computing the integral over z, we find that the antiderivative is logarithmically divergent at $z \to \pm\infty$, with the divergence canceling out of the final answer. This indicates that the integral over "all space" is not absolutely convergent and hence can depend on the manner in which the limit is taken. By using cylindrical coordinates we have taken the limit using increasingly large cylindrical regions. We have checked numerically that the answer is the same if we instead use increasingly large spherical regions. We expect that any sufficiently symmetric choice of region will result in the same answer, and conclude that the cylindrical (or spherical) approach is a reasonable definition for the total electromagnetic mass moment in all of space.

The mass moment is given by adding $\tilde{\boldsymbol{p}}_{F\times} t$ to the integral computed in (A17). This cancels the first term, leaving

$$\tilde{\boldsymbol{N}}_{F\times} = -q_1 q_2 \mathrm{sign}(vt)\frac{1}{\gamma^2 v^2}\log\frac{m_1 + \gamma m_2}{m_2 + \gamma m_1}\hat{\boldsymbol{z}}. \tag{A18}$$

This completes the calculation of the cross-term field integrals at zero impact parameter.

[1] M. Portilla, "SCATTERING OF TWO GRAVITATING PARTICLES: CLASSICAL APPROACH," *J. Phys. A* **13** (1980) 3677–3683. I

[2] K. Westpfahl, "High-Speed Scattering of Charged and Uncharged Particles in General Relativity," *Fortsch. Phys.* **33** no. 8, (1985) 417–493. I, I, I

[3] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, "Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$," *Phys. Rev. Lett.* **126** no. 17, (2021) 171601, arXiv:2101.07254 [hep-th]. I

[4] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, "Dynamics of Binary Systems to Fourth Post-Minkowskian Order from the Effective Field Theory Approach," arXiv:2106.08276 [hep-th].

[5] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, "Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at $O(G^4)$," arXiv:2112.10750 [hep-th].

[6] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, "Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-eccentricity Expansion," arXiv:2112.11296 [hep-th]. I

[7] S. E. Gralla and K. Lobo, "Self-force effects in post-Minkowskian scattering," arXiv:2110.08681 [gr-qc]. I, I, II B

[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. W. H. Freeman, San Francisco, 1973. I

[9] S. E. Gralla and K. Lobo, "Gravitational Scoot," *in prep.* . I

[10] M. V. S. Saketh, J. Vines, J. Steinhoff, and A. Buonanno, "Conservative and radiative dynamics in classical relativistic scattering and bound systems," *Phys. Rev. Res.* **4** no. 1, (2022) 013127, arXiv:2109.05994 [gr-qc]. I, III, III A

[11] E. Poisson, "An Introduction to the Lorentz-Dirac equation," arXiv:gr-qc/9912045. 3

[12] S. E. Gralla, A. I. Harte, and R. M. Wald, "A Rigorous Derivation of Electromagnetic Self-force," *Phys. Rev. D* **80** (2009) 024031, arXiv:0905.2391 [gr-qc]. 3

6 Frames and Slicings for Angular Momentum in Post-Minkowski Scattering

This is a paper I coauthored with Sam Gralla and Hongji Wei titled, "Frames and Slicings for Angular Momentum in Post-Minkowski Scattering".

Frames and Slicings for Angular Momentum in Post-Minkowski Scattering

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In relativistic physics, angular momentum is paired with a lesser known conserved quantity, the "mass moment", which appears as the time-space components of the angular momentum tensor. Calculations of mass moment in electromagnetic and gravitational scattering of point particles have led to some puzzling behavior in which the radiated mass moment does not appear to match the corresponding mechanical change. We review the issues and show how the freedoms of time slicing and asymptotic frame may be used to bring all known results into agreement. The key points are to use hyperboloidal time slices and to allow the perturbative and asymptotic frames to differ by an independent Bondi-Metzner-Sachs (BMS) transformation at early and late times. The relevant BMS transformation involves a translation found recently by Riva, Vernizzi, and Wong. Building on this work, we conjecture a flux balance law for all orders in the post-Minkowski expansion.

I. INTRODUCTION

The perturbative description of small-angle gravitational scattering is known as the post-Minkowksian (PM) expansion. Recently there has been much interest in PM scattering due to connections with fundamental and astrophysical questions—see Ref. [1] and many subsequent references. As part of this program, some subtleties have arisen regarding the notion of angular momentum [2–13]. Some of the confusion surrounds the time-space components of the angular momentum tensor, or "mass moment". In particular, the change in mechanical mass moment of the particles [5] (or, *scoot*[1]) has been found to disagree with the mass moment radiated during the scattering encounter [8]. The disagreement occurs at 1PM order, where the particles scoot but there is no balancing radiation at all, and also at 2PM (and presumably higher orders), where the mechanical scoot disagrees with the radiative flux. However, it is not totally clear that the mechanical and radiative calculations use compatible definitions and gauge conditions, so it is not obvious even whether agreement should be expected.

The electromagnetic (EM) analog problem has provided some intuition [6, 7]. The 1PM disagreement may be resolved by realizing that the EM field makes *non-radiative* contributions to the initial and final mass moment [7], balancing the mechanical scoot. However, it is difficult to pursue a similar resolution in the gravitational case, since the gravitational field does not have a well-defined stress-energy tensor. In this paper we will provide an alternative treatment of angular momentum in EM scattering that generalizes better to the gravitational case, and discuss an approach to the gravitational problem that restores agreement at least through 2PM.

The main important point is to move away from constant-t slices and instead formulate conservation laws with hyperboloidal slices, as in the definition of timelike infinity in general relativity [14–18]. This means that when calculating initial/final particle contributions, we add together the particle mass moments at equal proper time (instead of equal coordinate time). In the EM case, we check non-perturbatively that this eliminates non-radiative scoots and restores agreement between the mechanical change and the radiated mass moment, and we demonstrate the agreement explicitly through 2PM. In the gravitational case, we find that analogous 2PM agreement is obtained between mechanical calculations [5][2] and pseudotensor [8] or amplitudes-inspired [9] definitions of angular momentum flux.

The pseudotensor and amplitdes-inspired definitions of angular momentum rely on a choice of a preferred background flat metric. It is also interesting to consider the Bondi-Metzner-Sachs (BMS) framework [19–21], which only requires asymptotic flatness. However, in this case there is an ambiguity in relating coordinate choices in the PM calculation ("PM frame") with coordinate choices in the asymptotic analysis ("BMS frame"). One can make a natural identification in any region where the particles are widely separated (i.e., at initial or final times), but this identification is *not* preserved by time-evolution. In other words, if the PM and BMS frames are identified at early times, they can still differ by a BMS transformation at late times. Ref. [13] has found a formula that we interpret as the translation and supertranslation required for the BMS and PM frames to remain the "same" at late times. The transformation accounts for the discrepancy between BMS and PM fluxes, bringing all approaches into agreement.

This paper is organized as follows. In Sec. II we review conserved quantities in special relativity and discuss the freedom of integration surface. For the EM problem, we discuss the early/late field contributions in Sec. III, the corresponding mechanical contributions in

[1] The mass moment is equal to the energy times the center of energy evaluated at some canonical time. A change in mechanical mass moment occurs when a person scoots forward on the floor, moving her center of mass forward.

[2] We also discovered a computational error in Ref. [5], which has now been corrected in the arXiv version. Agreement requires both correcting the error and changing to hyperboloidal slicing.

Sec. IV, and compute the 2PM scoot (change in mass moment) in Sec. V. For the gravitational problem, we provide an overview in Sec. VI and consider the BMS framework in PM scattering in Sec. VII. We use units with $c = G = 4\pi\epsilon_0 = 1$.

II. CONSERVATION LAWS IN SPECIAL RELATIVITY

In this section we review angular momentum in special relativity and discuss two natural choices for formulating its global conservation. Given a conserved stress-energy tensor $T^{\mu\nu}$ (satisfying $\nabla_\mu T^{\mu\nu} = 0$) and a Killing field ξ^μ (satisfying $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$), we may construct a conserved current $T^{\mu\nu}\xi_\mu$,

$$\nabla_\nu (T^{\mu\nu}\xi_\mu) = 0. \tag{1}$$

Integrating this equation over some spacetime four-volume and using Stokes theorem, we find

$$\int_{\mathcal{B}} T_{\mu\nu}\xi^\mu n^\nu \sqrt{|h|} d^3x = 0, \tag{2}$$

where n^μ is the unit normal to the boundary $\mathcal{B}$ of the volume, which has induced metric h and area element $\sqrt{|h|}d^3x$.[3] We will consider two choices for the boundary $\mathcal{B}$.

A. The box

The simplest choice for $\mathcal{B}$ consists of initial and final $t =$ const surfaces bounded by $r =$ const spheres at large radius, together with a timelike surface consisting of the sphere cross time. In a one-dimensional projection this has the appearance of a box (Fig. 1 left).

We will focus on the time-space components of the angular momentum tensor, otherwise known as the "mass moment". These components are accessed by choosing boost Killing fields for ξ^μ,

$$\xi_{(i)} = x^i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x^i}, \tag{3}$$

where $i = 1, 2, 3$. When evaluated on a spacelike surface Σ, these give the components of the mass moment vector N^i at the "time" represented by Σ,

$$N^i_{(\Sigma)} = \int_\Sigma T_{\mu\nu}\xi^\mu_{(i)} \hat{n}^\nu \sqrt{|h|} d^3x, \tag{4}$$

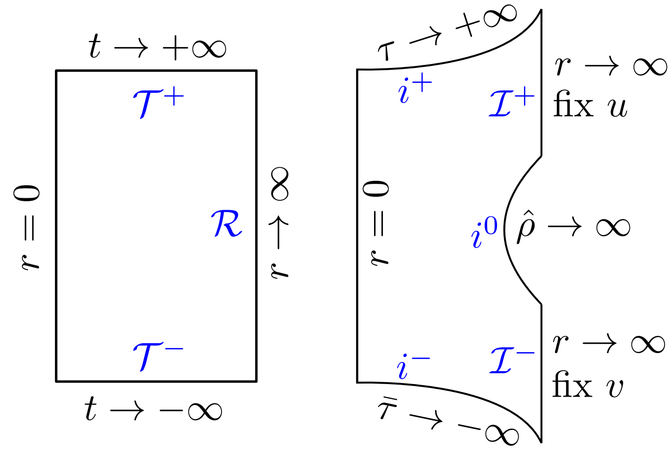

FIG. 1. The box (left) and the puzzle piece (right) choices for the integration boundary $\mathcal{B}$ in expressing conservation laws. The box is drawn taller than it is wide, since one takes $t \to \pm\infty$ at a (slightly) faster rate than $r \to \infty$. The puzzle piece involves five separate limits, reviewed in the text.

where $\hat{n}^\nu$ is the *future-directed* unit normal. For a $t =$ const surface this becomes

$$N^i_{(t)} = \int (T_{00}x^i + T_{i0}t) d^3x \tag{5}$$

$$= \int \mathcal{E} x^i d^3x - p^i t, \tag{6}$$

where $\mathcal{E} = T^{00}$ is the energy density and $p^i = \int T^{i0} d^3x$ is the total momentum. We therefore recognize the mass moment as equal to the value of the total energy times the center of energy at time $t = 0$.

When evaluated on a timelike surface, the integral of (2) has the interpretation of (minus) the total flux of mass moment leaving the system, which we denote by ΔN^i. For an $r =$ const surface we we have

$$\Delta N^i = \int T_{\mu j}\xi^\mu_{(i)} n^j r^2 d\Omega dt. \tag{7}$$

where $d\Omega = \sin\theta d\theta d\phi$ is the area element on the unit two-sphere and $n^i = x^i/r$ is the outward normal.

To make the box precise we must specify the order of limits for $t \to \infty$ and $r \to \infty$. Different choices correspond to different accounting systems for the conserved quantities. The conceptually familiar choice is to capture particles on the bottom and top (representing initial and final state) and radiation on the sides (representing flux leaving the system).[4] This requires a careful simultaneous limit, as now describe.

We describe the top/bottom of the box by $t = \pm t_0$ and the edge by $r = r_0(t_0)$, where $r_0(t_0)$ is some function.

[3] We assume that all parts of the boundary are either timelike or spacelike; n^μ is outward with $n^\mu n_\mu = +1$ for timelike portions, and inward with $n^\mu n_\mu = -1$ for spacelike portions.

[4] Alternatively, one could capture both particles and radiation on the bottom/top, thinking of the radiation as part of the initial/final state.

As $t_0 \to \infty$, the box must get wider fast enough that it outruns all massive particles, ensuring that they are captured on the top (and similarly for particles entering from the bottom). However, it must also get taller fast enough to capture radiation at arbitrarily late times (and analogously for early times). These requirements may be simultaneously met by taking the "corner velocity" $r_0'(t_0)$ to asymptotically approach 1 from below. The light moves faster than the corners and ends up on the side; the particles move slower and end on on the top/bottom. For definiteness we can choose $r_0 = t_0 - \sqrt{t_0}$. Thus we define

- *Final surface* $\mathcal{T}^+$: The constant-t surface with $r < t_0 - \sqrt{t_0}$ as $t_0 \to \infty$.
- *Radiation surface* $\mathcal{R}$: The constant-r surface with $r = t_0 - \sqrt{t_0}$ as $t_0 \to -\infty$.
- *Initial surface* $\mathcal{T}^-$: The constant-t surface with $r < t_0 - \sqrt{t_0}$ as $t_0 \to -\infty$.

Evaluating Eq. (2) on the box yields three contributions,

$$N^i_{\mathcal{T}^+} - N^i_{\mathcal{T}^-} = \Delta N^i_{\mathcal{R}}, \tag{8}$$

where the subscript $\mathcal{T}_+/\mathcal{T}_-$ refers to Eq. (5) using the final/initial surface, while the subscript $\mathcal{R}$ refers to Eq. (7) on the radiation surface. We thus have a conservation law stating that the change in mass moment is balanced by radiated mass moment. This is a formalization of the traditional procedure for understanding the flow of conserved quantities in scattering problems.

B. The puzzle piece

In the relativity literature, asymptotic conservation laws are usually formulated in a different way, involving five asymptotic regions, each with a separate adapted coordinate system. These may be defined relative to spherical coordinates (t, r, θ, ϕ) as

- *Future timelike infinity*, denoted i^+, is the limit $\tau \to \infty$ fixing (ρ, θ, ϕ), where

  $$t = \tau \cosh\rho, \qquad r = \tau \sinh\rho. \tag{9}$$

  This limit tracks outgoing particles with proper time τ, constant rapidity $\rho > 0$, and constant spatial direction (θ, ϕ). The points of i^+ therefore label outgoing subluminal velocities. Since $\tau = \text{const}$ surfaces are spacelike hyperboloids, we can think of i^+ as a spatial hyperboloid.

- *Future null infinity*, denoted $\mathcal{I}^+$, is the limit $r \to \infty$ fixing (u, θ, ϕ), where

  $$u = t - r. \tag{10}$$

  This limit tracks outgoing massless particles with affine parameter r, constant retarded time u, and constant spatial direction (θ, ϕ). The points of $\mathcal{I}^+$ therefore label parcels of outgoing radiation. Since $r = \text{const}$ surfaces are timelike cylinders, we can think of $\mathcal{I}^+$ as a timelike cylinder.[5]

- *Spatial infinity*, denoted i^0, is the limit $\hat{\rho} \to \infty$ fixing $(\hat{\tau}, \theta, \phi)$, where

  $$t = \hat{\rho} \sinh\hat{\tau}, \qquad r = \hat{\rho}\cosh\hat{\tau}. \tag{11}$$

  This limit follows trajectories with velocity larger than light, so that the points of i^0 label outgoing superluminal velocities. Since $\hat{\rho} = \text{const}$ surfaces are timelike hyperboloids, we can think of i^0 and a timelike hyperboloid.

- *Past null infinity*, denoted $\mathcal{I}^-$, is the limit $r \to \infty$ fixing (v, θ, ϕ), where

  $$v = t + r. \tag{12}$$

  This limit follows incoming massless particles back in time and is the time-reverse of the limit defining $\mathcal{I}^+$, producing a timelike cylinder whose points label parcels of incoming radiation.

- *Past timelike infinity*, denoted i^-, is the limit $\bar{\tau} \to -\infty$ fixing $(\bar{\rho}, \theta, \phi)$, where

  $$t = \bar{\tau}\cosh\bar{\rho}, \qquad r = -\bar{\tau}\sinh\bar{\rho}. \tag{13}$$

  This limit follows incoming massive particles back in time (these have rapidity $\bar{\rho} > 0$) and is the time-reverse of the limit defining i^+, producing a spatial hyperboloid whose points label incoming subluminal velocities.

Drawing together the five surfaces associated with the five limits produces a "puzzle piece" shape (Fig. 1 right; see [18] for further discussion). As with the box, we must specify the order of limits. Consider first the top corner, linking i^+ with $\mathcal{I}^+$. As $\tau_0 \to \infty$ we integrate over some range $\rho < \rho_0(\tau_0)$ for some function $\rho_0(\tau_0)$. Because $r_0/t_0 = \tanh\rho_0$ for the top corner, any function approaching infinity as $\tau \to \infty$ will satisfy the requirement that the corner's velocity asymptotically approaches 1 from below (see discussion in Sec. II A above.) At future null infinity $\mathcal{I}^+$ one integrates over u at fixed r. The upper cutoff $u_0 = \tau_0 e^{-\rho_0}$ approaches infinity as $\tau_0 \to \infty$ as long as ρ_0 approaches infinity slower than logarithmically. For definiteness, we take $\rho_0(t) = \log\sqrt{\tau_0}$ so that $u_0 = \sqrt{\tau_0}$ (much as we took $u_0 = \sqrt{t_0}$ in the case of the

[5] This contrasts with the conformal approach to asymptotics [22], where instead $\mathcal{I}^+$ is a null surface. However, the two concepts are equivalent—see Ref. [18] for further discussion.

box). Similar statements hold for the other corners of the puzzle piece.

Choosing the puzzle piece for our surface $\mathcal{B}$ in Eq. (2) gives an alternative formulation of conservation laws. Since $\tau = \sqrt{t^2 - r^2}$, the future-directed unit normal to a $\tau = \text{const}$ surface (with $\tau > 0$) is $\hat{n}^\mu = \tau^{-1}(t, x, y, z)$ when expressed in Cartesian coordinates $x^\mu = (t, x, y, z)$. Using this formula in Eq. (4), the total mass moment on a constant-τ surface is

$$N^i_{(\tau)} = \tau^2 \int d\rho d\theta d\phi \sinh^2\!\rho \sin\theta (T_{0\nu} x^\nu x^i + T_{i\nu} x^\nu t), \tag{14}$$

which may be compared to Eq. (5) for a $t = \text{const}$ slice. Performing the same exercise for the $\bar{\tau} = \text{const}$ surface used at early times, we find the same expression

$$N^i_{(\bar{\tau})} = \bar{\tau}^2 \int d\bar{\rho} d\theta d\phi \sinh^2\!\bar{\rho} \sin\theta (T_{0\nu} x^\nu x^i + T_{i\nu} x^\nu t). \tag{15}$$

Now we may define the contributions from the timelike infinities $i^\pm$ to be

$$N^i_{i^+} = \lim_{\tau \to \infty} N^i_{(\tau)} \tag{16}$$

$$N^i_{i^-} = \lim_{\bar{\tau} \to -\infty} N^i_{(\bar{\tau})}. \tag{17}$$

The contributions from $\mathcal{I}^\pm$ involve $r = \text{const}$ surfaces and hence take the same form as (7), only evaluated in different limits. In particular, we can define the contributions from the null infinities $\mathcal{I}^\pm$ to be

$$\Delta N^i_{\mathcal{I}^+} = \lim_{\substack{r \to \infty \\ \text{fix} u}} \int T_{\mu j} \xi^\mu_{(i)} n^j r^2 d\Omega du. \tag{18}$$

$$\Delta N^i_{\mathcal{I}^-} = \lim_{\substack{r \to \infty \\ \text{fix} v}} \int T_{\mu j} \xi^\mu_{(i)} n^j r^2 d\Omega dv, \tag{19}$$

where we remind the reader that $n^i = x^i/r$ is the outward normal to the unit sphere.

The remaining portion of the puzzle piece is spatial infinity i^0. While important in gravitational theory as a mathematical link between $\mathcal{I}^-$ and $\mathcal{I}^+$, it is entirely trivial in electromagnetic scattering. Neither the particles nor their radiation reach i^0, and the Coulomb fields fall off too rapidly to contribute flux (see discussion at the end of Sec. III below). The conservation law (2) evaluated on the puzzle piece for EM scattering therefore has four contributions,

$$N^i_{i^+} - N^i_{i^-} = \Delta N^i_{\mathcal{I}^-} + \Delta N^i_{\mathcal{I}^+}. \tag{20}$$

The left-hand side is interpreted as the change in mass moment (final i^+ minus initial i^-), while the right-hand side is the total radiative flux of mass moment, including incoming $\mathcal{I}^-$ and outgoing $\mathcal{I}^+$ radiation. Eq. (20) for the puzzle piece may be compared with Eq. (8) for the box.

III. EM FIELD CONTRIBUTIONS

We now consider the electromagnetic field stress-energy due to point particles moving on some specified worldlines $\boldsymbol{r}_I(t)$ with velocity $\boldsymbol{v}_I$ and acceleration $\boldsymbol{a}_I$. For each particle we define

$$\boldsymbol{R}_I(t) = \boldsymbol{r} - \boldsymbol{r}_I(t), \quad \hat{\boldsymbol{R}}_I = \boldsymbol{R}_I / R_I. \tag{21}$$

The field is a sum of Coulomb and radiation contributions from each,

$$F^{\mu\nu} = \sum_{a=1}^{n} \left(F^{\mu\nu}_{(C),a} + F^{\mu\nu}_{(R),a} \right), \tag{22}$$

whose electric and magnetic fields are

$$\boldsymbol{E}_{C,I} = \frac{q_I}{R_I^2} \frac{\hat{\boldsymbol{R}}_I - \boldsymbol{v}_I}{(1 - v_I^2)^{-1}(1 - \hat{\boldsymbol{R}}_I \cdot \boldsymbol{v}_I)^3} \bigg|_{\text{ret}} \tag{23}$$

$$\boldsymbol{E}_{R,I} = \frac{q_I}{R_I} \frac{\hat{\boldsymbol{R}}_I \times ((\hat{\boldsymbol{R}}_I - \boldsymbol{v}_I) \times \boldsymbol{a}_I)}{(1 - \hat{\boldsymbol{R}}_I \cdot \boldsymbol{v}_I)^3} \bigg|_{\text{ret}} \tag{24}$$

$$\boldsymbol{B}_I = \hat{\boldsymbol{R}}_I|_{\text{ret}} \times \boldsymbol{E}_I, \tag{25}$$

The subscript "ret" indicates evaluation at the retarded time t_r satisfying $R_I(t_r) = t - t_r$. The Coulomb field falls off like $1/r^2$, while the radiation field falls off like $1/r$.

Now suppose that the particles are widely separated at late times, so that their velocities are asymptotically constant,

$$\boldsymbol{v}_I = \boldsymbol{V}_I + O(1/t). \tag{26}$$

Here $\boldsymbol{V}_I$ is a constant independent of time. The position and acceleration then obey

$$\boldsymbol{r}_I = \boldsymbol{V}_I t + O(\log t) \tag{27}$$

$$\boldsymbol{a}_I = O(1/t^2). \tag{28}$$

The falloff of the acceleration means that the Coulomb field dominates at late times. The leading contribution is that of a constant-velocity charge.

The integrand (5) for the mass moment on a constant-t surface involves the stress-energy multiplied by distance or time, and integrated over space. For large r at constant t, the stress-energy falls off like $1/r^4$ (from the Coulomb field squared), so the integrand can fall off as slowly as $1/r$. Thus there can be a contribution from the large-r region as the cutoff r_0 is taken to infinity, from a product of two Coulomb fields. This contribution was computed for two-particle scattering at early and late times in Ref. [7].[6] At early or late times, in a frame where the particles are asymptotically colinear, the result is

$$\boldsymbol{N}_{\mathcal{T},\mathrm{EM}\times} = -\frac{q_1 q_2}{\gamma^2 v^2}\log\frac{\gamma_2}{\gamma_1}\hat{\boldsymbol{r}}_{12}, \tag{29}$$

where $\hat{\boldsymbol{r}}_{12}$ is the unit vector pointing from particle 1 to particle 2. Here $\mathcal{T}$ represents either $\mathcal{T}^+$ or $\mathcal{T}^-$, γ_I are the asymptotic Lorentz factors of the particles $a = 1, 2$, while γ and v are the relative Lorentz factor and velocity. The notation "EM×" indicates that this is the contribution from the cross-term (particle 1 times particle 2) in the electromagnetic stress-energy. The formally-infinite self-field terms would require a separate treatment; these contribute to the renormalized center of mass [23] and are not expected to feature in the electromagnetic sector of the scattering problem.

The purely kinematical contribution (29) to the system mass moment must be included at both early and late times in order for mass moment to be globally conserved. However, it is always canceled by a corresponding mechanical contribution arising from the $\log t$ corrections to the position (27) (see Sec. IV and Ref. [7]). This bookkeeping annoyance is unavoidable if $t =$ const slices are used.

However, if we instead evaluate the electromagnetic mass moment on a hyperboloidal slice $\tau =$ const, we find that the late-time integral vanishes—in fact, the *integrand* itself vanishes. This may be seen directly by constructing the electromagnetic stress-energy from Eqs. (22)-(25) and using Eq. (14) at large $\tau \to \infty$. We work in Cartesian coordinates but consider $\tau \to \infty$ fixing (ρ, θ, ϕ). In this case we have $\boldsymbol{r}_i \sim \tau$, $\boldsymbol{v}_i \sim 1$ and $\boldsymbol{a}_i \sim \tau^{-2}$ [similar to Eqs. (26)–(28)] as well as $x^i \sim \tau$. These orderings imply that the Coulomb fields $F^{\mu\nu}_{(C),i}$ scale as τ^{-2}, while the radiation fields scale as $F^{\mu\nu}_{(R),i}$ scale as τ^{-3}. Given the form of (14), only products of Coulomb terms can survive the large-τ limit. Plugging in any such product of Coulomb terms, one finds by direct calculation that the integrand of (14) vanishes. This calculation can be done most easily by using (26) to replace each Coulomb field by the Coulomb field of a constant-velocity charge.

We therefore learn that the electromagnetic field never makes any contribution to the late-time mass moment on the puzzle piece. An identical argument establishes the same for the early-time mass moment,

$$N_{i^\pm,\mathrm{EM}} = 0. \tag{30}$$

Here "$i^\pm$, EM" refers to the electromagnetic contribution on i^+ or i^-.[7] Working on the puzzle piece, we see that the initial (i^-) and final (i^+) contributions are purely mechanical. This conclusion holds for arbitrary n-body scattering.

By a similar argument one can check that the ordinary spatial angular momentum integrand also vanishes on $i^\pm$. The energy and momentum integrands are non-zero, but the (cross-term) integrals vanish. We therefore conclude that all electromagnetic contributions to conserved quantities vanish on $i^\pm$.

There will, of course, be contributions from the EM stress-energy on $\mathcal{I}^+$: this is the radiative flux that we will consider explicitly in Sec. V below. It is easy to check directly from (23)–(25) that there are no contributions from i^0 or $\mathcal{I}^-$. The lack of contribution from $\mathcal{I}^-$ is due to the lack of incoming radiation in the retarded fields produced by the particles.

IV. MECHANICAL CONTRIBUTIONS AT 1PM

It is instructive to examine the form of the mechanical contributions to mass moment, for both the box and the puzzle piece. As a definite example we will consider 1PM scattering of point charges with masses m_1, m_2 and charges q_1, q_2. From Eqs. (26)-(29) of Ref. [7], the 1PM trajectories (first correction to straight line motion) may be written

$$x_1 = \frac{m_2\gamma_2}{E_0}b + \frac{q_1 q_2}{b m_1 \gamma_1 v^2}\left(vt + X_1\right) \tag{31}$$

$$z_1 = \frac{\gamma m_2}{\gamma_1 E_0}v\left(t - \frac{q_1 q_2 E_0}{m_1 m_2 \gamma^3 v^3}\operatorname{arctanh}\frac{vt}{X_1}\right) \tag{32}$$

$$x_2 = -\frac{m_1\gamma_1}{E_0}b - \frac{q_1 q_2}{b m_2 \gamma_2 v^2}\left(vt + X_2\right) \tag{33}$$

$$z_2 = -\frac{\gamma m_1}{\gamma_2 E_0}v\left(t - \frac{q_1 q_2 E_0}{m_1 m_2 \gamma^3 v^3}\operatorname{arctanh}\frac{vt}{X_2}\right) \tag{34}$$

where we define

$$E_0 = \sqrt{m_1^2 + m_2^2 + 2\gamma m_1 m_2} \tag{35}$$

$$\gamma_1 = \frac{\gamma m_2 + m_1}{E_0}, \quad \gamma_2 = \frac{\gamma m_1 + m_2}{E_0} \tag{36}$$

$$X_i = \sqrt{(\gamma_i^2/\gamma^2)b^2 + v^2 t^2}. \tag{37}$$

[6] Ref. [7] considered 1PM scattering and computed the cross term (field of particle 1 times field of particle 2) EM contribution to mass moment at early and late times. As noted in that reference, the effect is not limited to 1PM scattering—it is a general feature of two-particle scattering. The radiative fields do not contribute on $\mathcal{T}^+$ and $\mathcal{T}^-$ by construction, and for two-particle scattering, we may always boost and rotate so that the particles move in the same direction at asymptotically early or late times. The integrals for the cross-term contribution to the electromagnetic mass moment become identical to those of Appendix.A of Ref. [7]. The magnitude (29) of the mass moment follows from (A15) and (A17) of that reference, noting that $\gamma_1 = (m_1 + \gamma m_2)/E_0$ and $\gamma_2 = (\gamma m_1 + m_2)/E_0$ with $E_0^2 = m_1^2 + m_2^2 + 2\gamma m_1 m_2$.

[7] In the analogous box equation (29), we were forced to restrict to the cross-term contributions, ignoring the infinite self-field contributions. Such a restriction is not necessary here; the hyperboloidal slices in effect regularize the self-field infinities for mass moment (and also angular momentum).

Here the particles move in the xz plane with initial separation (impact parameter) b in the x direction and initial relative velocity v (relative Lorentz factor γ) in the z direction. The initial individual Lorentz factors are denoted γ_1 and γ_2, and the initial total energy is denoted E_0. These are also the final Lorentz factors and total energy; there is no radiation or energy exchange at this order. The small parameters in this expansion are $q_1q_2/(bm_I)$, where $I = 1, 2$ labels the particles. In this section we keep consistently to linear order in these parameters, corresponding to order 1PM in the formal expansion.

We may compute the particle mass moments from these trajectories as

$$\boldsymbol{N}_I = E_I \boldsymbol{r}_I - \boldsymbol{p}_I t, \tag{38}$$

where E_I and $\boldsymbol{p}_I$ are the special-relativistic energy and momenta of the point particles $a = 1, 2$. We will consider the mass moments of each particle at early and late times. The transverse (x) components reflect the symmetric displacement from the origin in this frame at 1PM order,

$$N_1^x = -N_2^x = \frac{\gamma_1\gamma_2 m_1 m_2}{E_0}, \quad t \to \pm\infty. \tag{39}$$

The longitudinal (z) components are more interesting. Expanding at early and late times, we have

$$N_1^z \sim \mp \frac{q_1 q_2}{\gamma^2 v^2}\left(\log\frac{2\gamma v|t|}{\gamma_1 b} - 1\right), \quad t \to \pm\infty \tag{40}$$

$$N_2^z \sim \pm \frac{q_1 q_2}{\gamma^2 v^2}\left(\log\frac{2\gamma v|t|}{\gamma_2 b} - 1\right), \quad t \to \pm\infty. \tag{41}$$

Here $\sim$ indicates asymptotic equality, and we have kept all finite and divergent terms. (The error is $O(|t|^{-1})$.) Because $\gamma_1 \neq \gamma_2$ in general, the sum $\boldsymbol{N}_1 + \boldsymbol{N}_2$ does not in general vanish at early or late times,

$$N_1^z + N_2^z = \mp \frac{q_1 q_2}{\gamma^2 v^2}\log\frac{\gamma_2}{\gamma_1}, \qquad t \to \pm\infty \tag{42}$$

Noting that $\hat{\boldsymbol{r}}_{12} = \mp\hat{\boldsymbol{z}}$ as $t \to \pm\infty$, we see that this residual non-zero contribution precisely balances the electromagnetic mass-moment (29), making the total mass moment zero at both early and late times. However, since the signs of the EM and mechanical contributions both reverse from early to late times, there is net exchange of mass moment between the particles and the field. Note that this contribution proportional to q_1q_2 is 1PM order: factoring out the dimensions $m_I b$ of mass moment (where $i = 1, 2$ corresponds to either mass), we see the small parameter $q_1q_2/(m_I b)$.

Notice that we can make the expressions for $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ look more similar by using proper time,

$$N_1^z \sim \mp \frac{q_1 q_2}{\gamma^2 v^2}\left(\log\frac{2\gamma v|\tau_1|}{b} - 1\right), \quad \tau_1 \to \pm\infty \tag{43}$$

$$N_2^z \sim \pm \frac{q_1 q_2}{\gamma^2 v^2}\left(\log\frac{2\gamma v|\tau_2|}{b} - 1\right), \quad \tau_2 \to \pm\infty, \tag{44}$$

where $\tau_1 = t/\gamma_1$ and $\tau_2 = t/\gamma_2$ are proper time parameters for the two particles. Thus, if we instead add the mass moments together at the same proper time $\tau = \tau_1 = \tau_2$, we do find cancellation. This constant-τ addition is exactly what we do when evaluating the particles' mechanical stress-energy on the hyperboloid via (14) and (9) (or (15) and (13) at early times). That is, the hyperboloid provides exactly the needed surface to get the log contributions to mass moment to cancel, and thereby to remove the non-radiative scoot.

V. 2PM EM SCOOT

We now fill a gap in the literature by computing the scoot at 2PM in EM scattering. We work on the puzzle piece so that the scoot is purely radiative. We first compute the radiated mass moment at 2PM using the 1PM trajectories presented in the previous section, and then confirm that it matches the change in mechanical mass moment using the 2PM trajectories of Ref. [6].

According to Eq. (18), the flux of mass moment through $\mathcal{I}^+$ is given integrating $T_{\mu j}\xi^\mu_{(i)} n^j r^2$ over a large sphere at each retarded time u. We again work with Cartesian components but take the limit at fixed (u, θ, ϕ). The boost Killing field components $\xi^\mu_{(i)}$ grow linearly with r at fixed u, so a finite, non-zero contribution in the limit comes from the $1/r^3$ part of the stress-energy tensor. This means that radiated mass moment comes entirely from cross-terms between $O(1/r^2)$ Coulomb fields and $O(1/r)$ radiation fields, just as occurs with ordinary angular momentum (e.g., [24]). This contrasts with the radiated energy and momentum, which involve the radiation field squared. Since the Coulomb field begins at lower PM order than the radiation field, radiation of mass moment and angular momentum occurs at a lower PM order than radiation of energy and momentum [6, 25].

Following standard terminology, the PM order of an expression is equal to the number of powers of the coupling constant (G for gravity, $k = (4\pi\epsilon_0)^{-1}$ for EM) appearing in the expression. (However, we set these constants to one, so that in practice the PM order is determined by dimensional analysis.) We will denote terms that scale like k^n and faster as $O(n\text{PM})$. Since we compute radiation, we will also expand at large r fixing u. In this section, error terms $O(1/r^n)$ refer to r at fixed u.

We denote the constant 0PM velocity by $\boldsymbol{V}_I$,

$$\boldsymbol{r}_I = \boldsymbol{V}_I t + O(1\text{PM}). \tag{45}$$

The 0PM retarded time expanded at large r (fixing u) takes a simple form,

$$t_r = u\frac{1}{1 - \boldsymbol{V}_I\cdot\hat{\boldsymbol{r}}} + O\left(1\text{PM}, \frac{1}{r}\right), \tag{46}$$

with the notation $O(x, y) = O(x) + O(y)$. (We drop the label I on the retarded time t_r.) The retarded position is effectively the origin in this limit,

$$\boldsymbol{R}_I|_{t_r} = \boldsymbol{r} + O(1\text{PM}, r^0). \tag{47}$$

The leading Coulomb field (23) is thus

$$\boldsymbol{E}_{C,I} = \frac{q_I(\hat{\boldsymbol{r}} - \boldsymbol{V}_I)}{r^2\Gamma_I^2(1-\boldsymbol{V}_I\cdot\hat{\boldsymbol{r}})^3} + O\left(2\text{PM}, \frac{1}{r^3}\right) \tag{48}$$

$$\boldsymbol{B}_{C,I} = \hat{\boldsymbol{r}} \times \boldsymbol{E}_{C,I} + O\left(2\text{PM}, \frac{1}{r^3}\right), \tag{49}$$

where $\gamma_I = (1 - \boldsymbol{V}_I)^{-1/2}$ is the (constant) 0PM Lorentz factor already given in Eq. (36). The leading radiation field is given by

$$\boldsymbol{E}_{R,I} = \frac{q_I\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} - \boldsymbol{V}_I) \times \boldsymbol{a}_I|_{t_r})}{r(1-\boldsymbol{V}_I\cdot\hat{\boldsymbol{r}})^3} + O\left(3\text{PM}, \frac{1}{r^2}\right) \tag{50}$$

$$\boldsymbol{B}_{R,I} = \hat{\boldsymbol{r}} \times \boldsymbol{E}_{R,I} + O\left(3\text{PM}, \frac{1}{r^2}\right), \tag{51}$$

Noting that the flux integral (7) has a hidden power of $1/k$ in front of it, the 2PM mass moment flux involves only the terms displayed in Eqs. (46)–(51). It will happen that the u-dependence of the integrand comes entirely from the acceleration, so it is helpful to note the u-integral of $\boldsymbol{a}(t_r)$. Using (46) to change variables, we find

$$\int_{-\infty}^{\infty} \boldsymbol{a}_I|_{t_r} du = \Delta\boldsymbol{v}_I(1-\boldsymbol{V}_I\cdot\hat{\boldsymbol{r}}) + O\left(2\text{PM}, \frac{1}{r}\right). \tag{52}$$

where $\Delta\boldsymbol{v}_I = \int_{-\infty}^{\infty}\boldsymbol{a}_I(t)dt$ is the change in velocity of particle I. Using the trajectories (31)–(34), we find

$$\Delta\boldsymbol{v}_1 = \frac{2q_1q_2}{b\gamma_1 m_1 v}\hat{\boldsymbol{x}} \tag{53}$$

$$\Delta\boldsymbol{v}_2 = -\frac{2q_1q_2}{b\gamma_2 m_2 v}\hat{\boldsymbol{x}}. \tag{54}$$

We have now assembled all the ingredients needed to compute the radiative flux (18). The Cartesian components of the stress-energy tensor may be constructed using Eqs.(48)–(51) for $\boldsymbol{E}$ and $\boldsymbol{B}$. The time integral in (18) can be performed with the aid of (52)–(54), and the angular integrals are also straightforward. This gives the flux of mass moment $\Delta N^i_{\mathcal{I}^+}$. We also calculate the flux of angular momentum $\Delta L^y_{\mathcal{I}^+}$ [Eq. (18) using $\xi = -z\partial_x + x\partial_z$] in the same way. The results are

$$\Delta N^x_{\mathcal{I}^+} = \frac{2q_1q_2}{3bE_0m_1m_2v}\left(2\gamma\left(m_2^2q_1^2 - m_1^2q_2^2\right) + 2m_1m_2\left(q_1^2-q_2^2\right) + \frac{3q_1q_2}{\gamma^3v^3}\left(m_1^2-m_2^2\right)\left(\text{arctanh}\, v - \gamma^2 v\right)\right) \tag{55}$$

$$\Delta N^z_{\mathcal{I}^+} = 0 \tag{56}$$

$$\Delta L^y_{\mathcal{I}^+} = \frac{-4q_1q_2}{3bE_0m_1m_2v}\left(\gamma v\left(m_2^2q_1^2 + m_1^2q_2^2\right) + \frac{3q_1q_2}{\gamma^2v^2}m_1m_2\left(\text{arctanh}\, v - \gamma^2 v\right)\right). \tag{57}$$

These should each equal the mechanical change in the corresponding conserved quantity, as long as the mechanical contributions are calculated at early/late *proper* times (corresponding to the hyperboloidal slices of the puzzle piece). The 2PM change in mechanical angular momentum was calculated by Ref. [6] using proper time, and indeed our (57) agrees with Eq. (3.16) of that reference. The change in mechanical mass moment can also be calculated from the trajectories given in Ref. [6].[8] We use Eq. (38) to calculate the individual mass moments of each particle, add them together at the same proper time τ to find the total mechanical mass moment $N^{\text{mech}}(\tau)$, and calculate the difference between the final ($\tau \to \infty$) and initial ($\tau \to -\infty$) values. We find that the change in mass moment matches Eqs. (55) and (56). Thus there is complete agreement between radiated and mechanical contributions through 2PM in EM scattering.

VI. CONSERVATION LAWS IN GENERAL RELATIVITY

Conservation laws are more difficult to formulate in general relativity because of the large coordinate freedom and the lack of a local stress-energy tensor for the gravitational field. To recover familiar special-relativistic notions, one must consider some kind of expansion about flat spacetime. One can take the expansion to hold globally, as in the PM expansion, or only asymptotically, as needed for non-perturbative scattering. The latter approach is more general but also comes with a more general freedom of asymptotic frame, including so-called supertranslations in addition to the usual Poincaré freedom [19–21]. (The collection of all transformations forms the BMS group.) The supertranslation freedom presents subtleties when comparing the two frameworks, and we

[8] The trajectories given in Eq. (A25) of Ref. [6] contain typographical errors; the correct expressions were provided to us by M.V.S. Saketh in a private communication.

will see that even the Poincaré degrees of freedom in the frameworks do not necessarily map in the obvious way. In this section we consider both frameworks in the context of PM scattering and discuss how various choices may be made to bring 2PM results into agreement.

A. PM expansion

In the PM expansion the bodies are represented as point particles moving on worldlines $z_I^\mu(\tau)$ for $I = 1, 2$ in a nearly flat spacetime. The motion of the particles is determined by the geodesic equation (or more general equations, if spin or higher moments are included) in the self-consistent spacetime, with the singular metric perturbations suitably regularized. We will not review the literature here but merely quote results; we refer the reader to Ref. [26] for a foundational reference.

The PM expansion has a Poincaré freedom associated with the flat background metric as well as the gauge freedom of infinitesimal diffeomorphisms (i.e., diffeomorphisms expanded in G that reduce to the identity as $G \to 0$). The perturbed particle positions are gauge-dependent, but one expects the numerical values to become meaningful in some preferred class of gauges, at least at early and late times when the particles are widely separated. In particular, one would expect that in such gauges, it is sensible to discuss the initial and final values of the 10 Poincaré charges just by using special-relativistic formulae with the particle trajectories z_I^μ.

Using a gauge that arises naturally in self-force calculations ("Lorenz gauge")[9], two of us calculated the full 2PM trajectories and reported on the change in angular momentum and mass moment [7] computed on constant-t slices. The result contains logarithmic terms precisely analogous to those considered in the EM case above. However, as in the EM case, these may be eliminated by using constant-τ slices instead. Using the trajectories in the appendix of Ref. [7],[10] we find

$$N^x_{i+} - N^x_{i-} = 2(1+v^2)\frac{\gamma m_1 m_2}{b v^4 E_0}(m_1^2 - m_2^2) f(v) \quad (58)$$

$$N^z_{i+} - N^z_{i-} = 0 \quad (59)$$

$$L^y_{i+} - L^y_{i-} = 4(1+v^2)\frac{\gamma^2 m_1^2 m_2^2}{b v^3 E_0} f(v) \quad (60)$$

with

$$f(v) \equiv \left(\frac{8}{3}v^3 - v + (1-3v^2)\operatorname{arctanh} v\right). \quad (61)$$

[9] The Lorenz gauge was used for portions of the calculation involving the linearized Einstein equation; certain non-linear terms were handled using isotropic coordinates for the Schwarzschild metric.

[10] We have also fixed a computational error in the trajectories of Ref. [7], which has the effect of removing the last terms in Eqs. (157) and (159) therein. We have corrected the arXiv version and submitted an erratum.

One would expect that these mechanical changes are balanced by a radiative flux of angular momentum and mass moment, such that the totals are conserved. However, the definitions of radiated angular momentum and mass moment are more subtle in the gravitational case, since there is no local, gauge-invariant stress-energy tensor for the gravitational field. We refer the reader to Refs. [3, 8, 9, 11–13, 27, 28] for discussions of the various subtleties involved in defining angular momentum in the PM expansion. Here we will simply note that the 2PM mechanical changes (58)–(60) do match the radiated fluxes computed by these authors; in particular see Refs. [8, 9]. The use of constant-τ slices to compute mechanical contributions is essential for this agreement; otherwise one has logarithmic terms in the mechanical results, which do not match any corresponding terms in the fluxes of Refs. [8, 9].

Another approach to angular momentum in PM scattering has been taken by Bini and Damour [11]. The idea is to integrate out the electromagnetic field, replacing the local field theory description with a multi-particle action whose interaction terms are non-local in time. This action then has a full Poincaré group of Noether charges, which are conserved for the time-symmetric dynamics. The charges associated with boosts differ from the mechanical mass moment by interaction terms, which (at early and late times) become precisely logarithmic terms we have emphasized. The use of this interaction-corrected mass moment is an alternative way to remove the log terms from the mechanical scoot and restore agreement with radiative calculations.

VII. BMS ANGULAR MOMENTUM IN THE PM EXPANSION

The most generally applicable definition of angular momentum in the relativity literature is based on the Bondi framework [19–21]. One introduces an asymptotic structure at null infinity and considers the BMS group of transformations that preserve it. To each BMS element one may assign a charge $Q(u)$ together with a flux integral encoding its change with time [29–31]. The charges associated with BMS elements that are asymptotically rotations may be considered angular momenta. However, the BMS group is infinite-dimensional (having supertranslations in addition to translations), and there is an infinite number of such "angular momenta". In essence, the freedom of origin in the classical angular momentum is promoted to a whole freedom of a function on the sphere. Much effort has been devoted to studying angular momentum in this framework; we refer to Ref. [32] for a particularly clear presentation.

It is natural to ask how the Bondi framework can account for angular momentum flow in PM scattering, and important steps were taken in Refs. [10, 13]. In this section we will review these results with a somewhat different viewpoint, and conclude with a conjecture on how they may generalize to higher PM order.

We use the notation of Ref. [18] except that we reverse the roles of M and m: here the Bondi mass aspect is denoted M and the particle masses are denoted m_n. In this coordinate-based approach to the Bondi framework, one has a notion of Bondi coordinates (u, r, θ, ϕ) in which the asymptotic $r \to \infty$ expansion takes a certain form. This form involves tensors on the two-sphere (indices $A, B, C, \dots$) that also depend on time u. The physical information is contained in the triple $\{M, C_{AB}, N_A\}$, composed of the Bondi mass aspect, shear, and angular momentum aspect, respectively. To each coordinate transformation preserving the asymptotic form one associates a charge, and after modding out the zero-charge transformations one is left with the BMS group. For a given spacetime, a definite choice of $\{M, C_{AB}, N_A\}$ fixes the BMS freedom; we will say that such a choice constitutes a "BMS frame".

The BMS transformations and their charges may be represented by a scalar $T(\theta, \phi)$ and conformal Killing field $Y^A(\theta, \phi)$ on the sphere. The $\ell = 0, 1$ harmonics of T correspond to the four spacetime translations and associated energy and momenta, while the higher harmonics correspond to supertranslations and associated supermomenta. The six different choices of Y^A map to the six rotations/boosts and six different components of the angular momentum tensor. The map to conserved quantities in our conventions is shown in table I. In the simplifying case of a purely electric shear (expected to occur for early and late times), the conserved charges take simple forms (e.g., Eqs. 108-109 of [18]),

$$Q_T = \frac{1}{4\pi} \int_{S^2} M T d\Omega \,, \tag{62}$$

$$Q_Y = \frac{1}{8\pi} \int_{S^2} N_A Y^A d\Omega \,. \tag{63}$$

We are particularly interested in the dependence of the Lorentz charges Q_Y on the choice of Bondi frame. Assuming $\dot{C}_{AB} = 0$ in addition to purely electric shear (as expected to hold for early and late times), the change in a Lorentz charge Q_Y under a supertranslation T is (see, e.g., Eq. (A20) below)

$$\delta_T Q_Y = \frac{1}{8\pi} \int d\Omega Y^A (3M \partial_A T + T \partial_A M). \tag{64}$$

Although this formula is derived for an infinitesimal supertranslation, the supertranslation-invariance of M implies that it holds for a finite supertranslation as well.

A. Single particle

Using the Bondi framework to understand PM scattering requires finding an asymptotic coordinate transformation from the PM metric to Bondi coordinates. This is rather non-trivial at intermediate times, but at early and late times one would expect the metric to behave as some kind of weakly nonlinear superposition of stationary metrics that have been boosted and translated. To make progress, it is natural to start with Bondi coordinates for a boosted, translated Schwarzschild metric.

charge	name	generator
E	Energy	$T = 1$
P^i	Momentum	$T = n^i(\theta, \phi)$
$P_{\ell m}$	Supermomentum	$T = Y_{\ell m}(\theta, \phi)$
L^i	Angular Momentum	$Y^A = -\epsilon^{AB} \partial_B n^i(\theta, \phi)$
N^i	Mass Moment	$Y^A = \partial^A n^i(\theta, \phi)$

TABLE I. Conventions for BMS Charges and generators. We use the standard orientation $\epsilon_{\theta\phi} = + \sin\theta$.

There is, of course, a whole BMS freedom in attaching a BMS frame to the Schwarzschild metric. To reflect the idea of a boosted, translated Schwarzshild metric, we can insist that the 10 Poincaré charges match those of a particle in special relativity on a straight-line trajectory,

$$x^i(t) = v^i(t - T) + b^i, \tag{65}$$

where v^i, T and b^i are constants. We use the notation $Q^{\rm mech}$ to denote this collection of charges,

$$\begin{gathered} Q^{\rm mech} = \{E, p^i, L^i, N^i\} \text{ with:} \\ E = \gamma m, \qquad\qquad p^i = \gamma m v^i \\ L^i = \gamma m \epsilon^i{}_{jk} b^j v^k, \quad N^i = \gamma m (v^i - b^i T). \end{gathered} \tag{66}$$

(We only need to match seven charges in order to fix the seven free constants v^i, b^i, T. The other three rotations are absorbed in the vector notation.)

To find a BMS frame for the Schwarzschild metric which has these charges, first note the outgoing Eddington-Finklestein coordinates for a black hole of mass m provides a Bondi frame with

$$M = m \tag{67}$$

$$C_{AB} = 0 \tag{68}$$

$$N_A = 0. \tag{69}$$

To make the Poincaré charges match, we can perform an asymptotic boost and translation. We first perform the boost and then the translation. The relevant formulas were derived in App. A. Under the boost, the mass aspect changes to Eq. (A16), while the shear and angular momentum aspect remain vanishing. Under the translation, the mass aspect and shear remain invariant, while angular momentum aspect changes according to Eq. (A20). Expressing these results covariantly for a boost in the v^i direction, the result is

$$M^{\sf nat} = \frac{m}{\gamma^3 (1 - v_i n^i)^3} \tag{70}$$

$$C^{\sf nat}_{AB} = 0 \tag{71}$$

$$N^{\sf nat}_A = 3 M D_A B + B D_A M \tag{72}$$

with

$$B = b_i n^i - T. \tag{73}$$

One can also perform a supertranslation without affecting the Poincaré charges, but this just introduces additional complication (non-zero shear) with no clear benefit. This frame is a "good cut" since it has vanishing shear [33].

The frame (70)-(72) is in some sense the most natural Bondi frame for a boosted, translated, spinless particle. We therefore call it the "natural" frame, denoted with superscript "nat". The charges of this frame match the mechanical charges,

$$Q^{\text{mech}} = Q^{\mathsf{nat}}. \tag{74}$$

Notice the font distinction: we use a sans-serif font "nat" to denote a Bondi frame, whereas the original font "mech" is used to denote the collection of charges (81) associated with the point particle (104). Of course, the Bondi frame nat has (infinitely many) more charges than the ten in the collection Q^{mech}; here we mean that the Poincaré charges match. The remaining charges (the supermomenta) are just the $\ell \geq 2$ parts of the mass aspect M^{nat} given in (70).

Alternatively, we could begin with a Schwarzschild metric that has been translated and boosted using special-relativistic formulas, and attempt to find a natural asymptotic coordinate transformation to Bondi coordinates. This was done to $O(1/r)$ in Ref. [10], giving the mass aspect and shear, and to $O(1/r^2)$ in Ref. [13], providing the angular momentum aspect as well. The resulting Bondi frame is instead

$$M^{\mathsf{harm}} = \frac{m}{\gamma^3(1 - v_i n^i)^3} \tag{75}$$

$$C_{AB}^{\mathsf{harm}} = -(2D_A D_B - \gamma_{AB} D^2) S \tag{76}$$

$$N_A^{\mathsf{harm}} = 3 M D_A (B + S) + (B + S) D_A M, \tag{77}$$

where now we introduce

$$S = -2\gamma m (1 - v_i n^i) \log\left(\gamma(1 - v_i n^i)\right). \tag{78}$$

Since the calculations of Ref. [13] begin with harmonic gauge, we call this the "harmonic frame" (labeled with superscript "harm"). The harmonic frame satisfies the "nice cut" condition [34]:

$$\left(-\frac{1}{4} D^A D^B C_{AB}^{\mathsf{harm}} + M_{\ell\geq 2}^{\mathsf{harm}}\right) = 0. \tag{79}$$

Since the natural and harmonic frames refer to the same (Schwarzschild) spacetime, they must be related by a BMS transformation. Referring to Eq. (A20), we see that the difference is a translation and supertranslation encoded in the function S. The BMS charges are also in general different, which means that BMS Poincaré charges do *not* match the mechanical Poincaré charges of the point particle. Instead, we have

$$Q^{\mathsf{harm}} = Q^{\mathsf{nat}} + \delta_S Q = Q^{\text{mech}} + \delta_S Q, \tag{80}$$

where $\delta_S Q$ is given for Lorentz charges Q_Y in Eq. (64).[11] That is, if Bondi charges are computed in the harmonic frame, then an additional translation and supertranslation are required to to recover the corresponding mechanical charges,

$$Q^{\text{mech}} = Q^{\mathsf{harm}} - \delta_S Q. \tag{81}$$

This may be compared with Eq. (3.6) of Ref. [13], noting that $\delta_T Q_Y$ is equal to $j(M, T)$ in their notation. Ref. [13] proposed this formula as part of a definition of mechanical charges; here we emphasize the interpretation that the harmonic frame is translated and supertranslated relative to the natural frame that inherits the mechanical values.

B. Multiple particles

We now promote single-particle formulas of the previous section to the multi-particle context in a natural way. For the mass aspect and shear, which appear at $O(1/r)$ in the Bondi expansion and hence obey linear equations, a simple superposition will suffice. It is thus natural to expect that "natural" Bondi frames $\mathsf{nat}^\pm$ can be defined at early/late times in n-particle scattering by[12]

$$\lim_{u\to\pm\infty} M^{\mathsf{nat}^\pm} = \lim_{t\to\pm\infty} \sum_{n=1}^{N} \frac{m_n}{\gamma_n^3 (1 - \boldsymbol{v}_n \cdot \boldsymbol{n})^3} \tag{82}$$

$$\lim_{u\to\pm\infty} C_{AB}^{\mathsf{nat}^\pm} = 0. \tag{83}$$

Here m_n, γ_n, v_n refer to the point particle positions in harmonic gauge. Eqs. (82) and (83), promote Eqs. (70) and (71), to the multi-particle context by superposition.

These conditions only partially fix the BMS frame $\mathsf{nat}^\pm$. The mass aspect formula (82) fixes the boost and rotation degrees of freedom (given the known values of m_n and $\boldsymbol{v}_n$), while the no-shear condition (83) fixes the $\ell \geq 2$ supertranslations. (See Eq. (A18) and (A5) for the transformation law of the shear.) However, the differential operator in (89) annihilates $\ell = 0, 1$ modes of $S^\pm$, so this condition has no effect on the translation degrees of freedom.

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[11] Since the mass moment M is invariant under supertranslations, it is the same in the nat and harm frames.

[12] Since the particle velocities are finite at late times, the limit in Eq. (82) is independent of whether we use coordinate time or proper time. We have expressed the limits as $t \to \pm\infty$ for notational compactness; if we instead used proper time, these would have to be given separately as $\tau \to \infty$ and $\bar{\tau} \to -\infty$.

To fix the translations, one option would be to provide a similar formula for the angular momentum aspect, promoting Eq. (72) to the multi-particle context. However, since the angular momentum aspect appears at $O(1/r^2)$ where the equations are no longer linear, it is not clear what form a suitable promotion would take. Resolving this question would require finding a coordinate transformation from a multi-particle harmonic-gauge spacetime to a natural Bondi frame, which is a task of considerable complexity. However, it should still be possible to find a Bondi frame whose charges agree with the mechanical charges, so we can fix the translation degree of freedom of our natural frame by imposing

$$Q^{\text{mech}^\pm} = \lim_{u\to\pm\infty} Q^{\mathsf{nat}^\pm}, \tag{84}$$

promoting Eq. (74). Here by mechanical charges we mean

$$Q^{\text{mech}^+} = \lim_{\tau\to\infty} Q^{\text{mech}} \tag{85}$$

$$Q^{\text{mech}-} = \lim_{\bar\tau\to-\infty} Q^{\text{mech}}, \tag{86}$$

where we are careful to use the hyperboloidal slicing so of the puzzle piece to define early ($\bar\tau \to -\infty$) and late ($\tau \to \infty$) time limits. This distinction matters for the Lorentz charges (angular momentum and mass moment).

Eqs. (82), (83), and (84) define Bondi frames nat^+ and nat^- in terms of the PM trajectories at late and early times, respectively. Both frames satisfy the good cut condition (vanishing shear). It is well known that this condition is not preserved under evolution, an effect associated with gravitational memory (e.g., [35]). Thus these two frames are indeed distinct,

$$(\mathsf{nat}^+) \neq (\mathsf{nat}^-). \tag{87}$$

Instead, the two frames will be related by a BMS transformation. The gravitational memory induces to a supertranslation, but in principle there could also be translations, rotations, and boosts.

We can analogously define early/late harmonic Bondi frames by promoting Eqs. (75), (76), and (81),

$$\lim_{u\to\pm\infty} M^{\mathsf{harm}^\pm} = \lim_{t\to\pm\infty} \sum_{n=1}^{N} \frac{m_n}{\gamma_n^3(1-\boldsymbol{v}_n\cdot\boldsymbol{n})^3} \tag{88}$$

$$\lim_{u\to\pm\infty} C_{AB}^{\mathsf{harm}^\pm} = -(2D_AD_B - \gamma_{AB}D^2)S^\pm, \tag{89}$$

$$\lim_{u\to\pm\infty} Q^{\mathsf{harm}^\pm} = Q^{\text{mech}^\pm} + \lim_{u\to\pm\infty} \delta_{S^\pm} Q, \tag{90}$$

where we now introduce a multi-particle version of S,

$$S^\pm = \lim_{t\to\pm\infty} \Big(-2G \sum_{n=1}^{N} m_n\gamma_n(1-\boldsymbol{v}_n\cdot\boldsymbol{n}) \times \log\left(\gamma_n(1-\boldsymbol{v}_n\cdot\boldsymbol{n})\right)\Big). \tag{91}$$

Eqs. (88), (89) and (90) define harmonic Bondi frames harm^+ and harm^- in terms of the PM worldlines at late and early times. The natural and harmonic frames are related at all times u by a translation and supertranslation,

$$Q_Y^{\mathsf{nat}^\pm} = Q_Y^{\mathsf{harm}^\pm} - \delta_{S^\pm} Q. \tag{92}$$

The change $\delta_S Q(u)$ is given in terms of the mass aspect $M(u,\theta,\phi)$ in Eq. (64). This mass aspect is the same in the harmonic and natural frames (at all times) and is given at early/late times by Eq. (82) or (88).

These definitions allow us to restate a key idea of Ref. [13] as the conjecture that, in contrast to the natural frames, the two harmonic Bondi frames harm^+ and harm^- are the same at 2PM order,

$$(\mathsf{harm}^+) = (\mathsf{harm}^-) \qquad \text{at 2PM}. \tag{93}$$

More explicitly, the conjecture is that Bondi data satisfying the early-time conditions [the minus branch of Eqs. (88), (89) and (90)] will evolve to Bondi data satisfying the late-time conditions [the plus branch of the same equations] when the 2PM harmonic-gauge evolution equations are used. The conjecture has not been fully checked, but it predicts the flux balance laws proposed in Ref. [13].

We will express the mechanical change in terms of Bondi fluxes in the initial natural frame nat^-,

$$\lim_{u\to+\infty} Q_Y^{\mathsf{nat}^-} - \lim_{u\to-\infty} Q_Y^{\mathsf{nat}^-} = \mathcal{F}_Y^{\mathsf{nat}^-}, \tag{94}$$

where the Bondi flux is[13]

$$\mathcal{F}_Y = \frac{1}{8\pi}\int_{-\infty}^{\infty} du \int d\Omega Y^A \Big[\frac{1}{4}D_B\left(\dot C^{BC}C_{CA}\right) + \frac{1}{2}C_{AB}D_C\dot C^{BC} + \frac{u}{8}D_A\left(\frac{1}{8}\dot C_{BC}\dot C^{BC}\right)\Big]. \tag{95}$$

The superscript nat^- in (94) indicates to use the shear in the nat^- frame. This frame is convenient because the Bondi angular momentum flux *vanishes* at 2PM order [10],

$$\mathcal{F}_Y^{\mathsf{nat}^-} = O(\text{3PM}). \tag{96}$$

To see this, first note that there is a hidden prefactor of $1/G$ in (95), such that the flux is one order lower than the integrand. The last term contributes to the flux first at 3PM in any gauge since $\dot C_{BC}$ is of order 2PM. The special property of nat^- is that C_{AB} vanishes in the infinite past

[13] The Bondi flux for Lorentz charges Q_Y is expressed in Eq. (123) in Ref. [18]. By assuming that we work in vacuum and that there is no magnetic part of shear at early and late times, we use Eqs. (110), (121), together with the evolution equation for m [Eq. (111) in Ref. [18]] to arrive at Eq. (95). In this process we drop a term proportional to $-\frac{u}{4}Y^AD_AD_BD_C\dot C^{BC}$. This term vanishes under integration on account of the orthogonality of angular harmonics: C_{AB} is $\ell \geq 2$, while Y^A is $\ell = 1$.

$u \to -\infty$, so that C_{AB} itself (and not just $\dot{C}_{AB}$) is of order 2PM. In this case the first two terms also contribute to the flux starting at 3PM.

From Eq. (94) using Eqs. (84) and (92), we find

$$\lim_{u\to+\infty}\left(Q_Y^{\mathsf{harm}^-} - \delta_{S^-}Q_Y\right) - Q_Y^{\mathrm{mech}^-} = \mathcal{F}_Y^{\mathsf{nat}^-}. \tag{97}$$

The 2PM equivalence of harm^+ and harm^-, together with the vanishing of the nat^- flux at this order (96), now implies that

$$\lim_{u\to\infty}\left(Q_Y^{\mathsf{harm}^+} - \delta_{S^-}Q_Y\right) - Q_Y^{\mathrm{mech}^-} = O(3\mathrm{PM}), \tag{98}$$

and we can use (92) again to write

$$\lim_{u\to\infty}\left(Q_Y^{\mathsf{nat}^+} + \delta_{S^+}Q_Y - \delta_{S^-}Q_Y\right) - Q_Y^{\mathrm{mech}^-} = O(3\mathrm{PM}). \tag{99}$$

A second application of Eq. (84) then implies

$$Q_Y^{\mathrm{mech}^+} + \lim_{u\to\infty}\left(\delta_{S^+}Q_Y - \delta_{S^-}Q_Y\right) - Q_Y^{\mathrm{mech}^-} = O(3\mathrm{PM}). \tag{100}$$

Or, noting the linearity of $\delta_S Q_Y$ in S, we have

$$Q_Y^{\mathrm{mech}^+} - Q_Y^{\mathrm{mech}^-} = -\lim_{u\to\infty}\delta_{\Delta S}Q_Y + O(3\mathrm{PM}), \tag{101}$$

where $\Delta S \equiv S^+ - S^-$ so that

$$\lim_{u\to\infty}\delta_{\Delta S}Q_Y = \frac{1}{8\pi}\int d\Omega Y^A\big[3M^+\partial_A(S^+ - S^-) + (S^+ - S^-)\partial_A M^+\big], \tag{102}$$

where $S^\pm$ are defined in (91), and for convenience we denote the late-time mass aspect by M^+,

$$M^+ = \lim_{\tau\to\infty}\sum_{n=1}^{N}\frac{m_n}{\gamma_n^3(1-\boldsymbol{v}_n\cdot\boldsymbol{n})^3}. \tag{103}$$

The integral in Eq. (102) can be evaluated for each choice of Y^A, corresponding to the components of angular momentum and mass moment. This calculation was done in Ref. [13] in a different notation [See Eq. (3.25) therein], but we reproduce it here for clarity and completeness. The integrand is built from M^+, S^+, and S^-, which depend on the harmonic-gauge trajectories via the initial and final velocities of the particles. Since $S^\pm$ has a hidden factor of G in front, we will need only the 1PM trajectories to determine the 2PM $S^\pm$. In the canonical scattering setup we consider (center of energy frame with initial motion along z and transverse separation along x), the particle four-momenta $p^\mu_{a,\pm}$ at late $(+)$ and early $(-)$ times are given by Eqs. (144) and (145) of Ref. [5] as

$$p^\mu_{i,+} = p^\mu_{i,-} + \Delta p^\mu_i, \tag{104}$$

with

$$p^\mu_{1,-} = \left(\frac{m_1^2+\gamma m_1 m_2}{E_0}, 0, 0, \frac{\gamma v m_1 m_2}{E_0}\right) \tag{105}$$

$$p^\mu_{2,-} = \left(\frac{m_2^2+\gamma m_1 m_2}{E_0}, 0, 0, -\frac{\gamma v m_1 m_2}{E_0}\right) \tag{106}$$

$$\Delta p^\mu_1 = -\Delta p^\mu_2 \tag{107}$$

$$= \left(0, -\frac{2m_1 m_2\gamma}{bv}(1+v^2), 0, 0\right) + O(2\mathrm{PM}). \tag{108}$$

Plugging Eqs. (104)–(108) into Eqs. (103) and (91), we obtain

$$M^+ = \sum_{i=1,2}\frac{m^4}{p^0_{i,+} - p^z_{i,+}\cos\theta} + O(2\mathrm{PM}) \tag{109}$$

$$S^+ - S^- = -\frac{4m_1 m_2\gamma(1+v^2)}{bv}\sin\theta\cos\phi \times \log\left(\frac{m_1 + m_2\gamma(1-v\cos\theta)}{m_2 + m_1\gamma(1-v\cos\theta)}\right) + O(3\mathrm{PM}). \tag{110}$$

Note that $p^\mu_{i,+} = p^\mu_{i,-}$ at the 1PM order displayed in (109); we write $p^\mu_{i,+}$ for aesthetic reasons.

According to the conjecture (93), the integral (102) should match the mechanical change (with a minus sign–see Eq. (101)). As far as we are aware, the mechanical change in angular momentum has not been computed in harmonic gauge, but results in a related gauge[14] [5] were presented as Eqs. (58)–(60) above. Plugging Eqs. (109) and (110) into Eq. (102) and evaluating the integral for the relevant choices of Y^A (see table I), we find an exact match. This provides support for the conjecture (93).

C. Discussion

In this section we have reproduced and repackaged the asymptotic calculations of Ref. [13] and compared with the mechanical calculations of Ref. [5]. The main difference of interpretation is that Ref. [13] regard the term $\delta_{\Delta S}Q_Y$ as due to a "static" contribution to angular momentum, where as we view it the effect of a translation and supertranslation needed to reconcile PM and Bondi results. (The (super)translation is applied at late times $u \to \infty$, but its form is determined non-locally, using early-time information as well.) This interpretation has content: the Bondi and PM definitions of angular momentum are sufficiently distinct that, in principle, there could have been no reconciliation between them, and one would have been forced to "choose" between two compelling definitions of angular momentum. Instead, we find that the two different results are related by a BMS

[14] Both the harmonic gauge and the Lorenz gauge of [5] involve hyperbolic wave equations, where causality is manifest.

transformation, indicating that the difference is just a subtlety about frames in which the conserved quantities are defined.

It is natural to consider the physical interpretation of the (super)translation $\Delta S = S^+ - S^+$ that relates the natural-frame BMS results and the Lorenz-gauge PM results. The $\ell \geq 2$ modes (i.e., the pure supertranslations) are precisely those that eliminate the change in Bondi shear due to the linear memory effect, which is the total memory effect at this PM order. In this context one sometimes says that the BMS frame is *supertranslated* as a result of the scattering process [35]. We may describe the $\ell = 0, 1$ modes of ΔS as an additional *translation* of the frame as well, but there is a key difference to bear in mind: whereas the supertranslation can be defined intrinsically on $\mathcal{I}^+$, the translation is only *relative* to the harmonic-gauge PM frame.

In particular, the "supertranslation of the BMS frame" due to the passage of radiation can be defined as the supertranslation needed to eliminate the change in Bondi shear. Physically, nearby freely falling observers can measure that their final relative distances differ from those recorded prior to the passage of the radiation—the gravitational memory. By contrast, we are unaware of any corresponding intrinsic definition of the "translation of the BMS frame" from the Bondi data $\{M, C_{AB}, N_A\}$. Physically, we expect that no such definition will exist, since there is no way for freely falling observes to locally measure whether they have "moved" as a result of the passage of the radiation (absolute distances cannot be measured). Such a translation must be relative to something in the bulk; at present, the best we can say is that the BMS frame is translated relative to the PM frame.

Ascribing physical meaning to the translation requires ascribing physical meaning to the BMS and PM frames. For the natural BMS frame, we can note that the angular coordinates (θ, ϕ) label freely falling observers in the asymptotically flat region $r \to \infty$ with proper time u, and the good cut condition corresponds to initially synchronized clocks. For the PM frame, we can note that the harmonic (or Lorenz) gauge gives rise to causal propagation through the bulk. Since both frames have physically appealing properties, the relative translation between the frames may very well have physical significance. It would be interesting to understand this better.

D. Conjecture

A related question is whether the conjecture (93) can be promoted to higher PM order. Could it be that the harm^- frame evolves to the harm^+ at all orders in the PM expansion? The answer is no: The non-linear memory effect [36] guarantees that there will be a change in Bondi shear in any process involving radiation of energy. The harm^+ and harm^- frames must therefore differ at least by the supertranslation generated by (see, e.g., Eq. (117) of Ref. [18])[15]

$$S_{\mathrm{NLM}} = \frac{1}{2}[D^2(D^2+2)]^{-1} \int \dot{C}_{AB}\dot{C}^{AB} du, \tag{111}$$

with "NLM" standing for "non-linear memory". However, one may conjecture that this is the only difference,

$$\mathsf{harm}^+ = \mathcal{S}_{\mathrm{NLM}}(\mathsf{harm}^-), \tag{112}$$

where $\mathcal{S}_{\mathrm{NLM}}$ represents the action of the supertranslation generated by (111).

This conjecture predicts flux-balance laws for the Lorentz charges. Repeating the steps leading to (101), we see that the updated law is

$$Q_Y^{\mathrm{mech}^+} - Q_Y^{\mathrm{mech}^-} = \mathcal{F}_Y^{\mathsf{nat}^-} - \lim_{u\to\infty} \delta_\chi Q_Y, \tag{113}$$

where now

$$\chi = S^+ - S^- + S_{\mathrm{NLM}}. \tag{114}$$

In effect, the change in mechanical charges is given by three terms: the flux of angular momentum in the natural frame (the initial good cut whose charges agree with the initial Poincaré charges), the correction due to gravitational memory (the $\ell \geq 2$ parts of $S^+ - S^-$ together with S_{NLM}), and the further correction found by Ref. [13] (the $\ell = 0, 1$ parts of $S^+ - S^-$). It would be interesting to check this conjecture at higher PM order.

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Appendix A: BMS transformation laws

In this appendix, we derive simplified formulas for the change in the Bondi data $\{m, C_{AB}, N_A\}$ under BMS transformations in some special cases of relevance for deriving Eqs. (70)–(72) of the main text. We rely heavily on the formulas in Ref. [32]. However, note that the angular momentum N_A defined in Ref. [32] differs from our N_A by a term $uD_A m$. We denote the N_A of [32] by $\tilde{N}_A$, and the relationship to our N_A is

$$N_A = \tilde{N}_A - uD_A m. \tag{A1}$$

[15] In Eq. (111), it is understood that the inverse operator $[D^2(D^2+2)]^{-1}$ acts on the space of functions with no $\ell = 0, 1$ modes, and that these modes are removed from $\dot{C}_{AB}\dot{C}^{AB}$. Thus S_{NLM} contains only $\ell \geq 2$ modes.

We adopt the convention where $x^A = \{\theta, \phi\}$ are spherical coordinates, with the two-sphere metric γ_{AB}. We write the covariant derivative compatible with γ_{AB} as D_A. Bondi coordinates $\{u, r, x^A\}$, along with the definition of metric components (Bondi data $\{m, C_{AB}, N_A\}$) and conserved charges are along with [32]. Symmetrization and antisymmetrization are denoted as () and [] over indices respectively[16] We use dots to indicate a u-derivative (such as the Bondi news is denoted as $\dot{C}_{AB}$). We consider the vacuum case $T_{\mu\nu} = 0$.

Any symmetric rank-2 tensor C_{AB} on the sphere can be decomposed into an "electric part" C and "magnetic part" Ψ:

$$C_{AB} = (-2D_A D_B + \gamma_{AB} D^2)C + \epsilon_{C(A} D_{B)} D^C \Psi, \quad \text{(A2)}$$

where ϵ_{AB} is the Levi-Civita tensor on the sphere with the convention $\epsilon_{\theta\phi} = \sin\theta$. (See Eq. (105) in Ref. [18] or Eq. (2.24) in Ref. [32].) Note that the $\ell = 0, 1$ parts of C and Ψ will not contribute to the expression Eq. (A2).

Using the commutators of the spherical derivatives together with Riemann curvature of the unit sphere $R_{ABCD} = \gamma_{AC}\gamma_{BD} - \gamma_{AD}\gamma_{BC}$, we can select out the electric and magnetic parts of C_{AB} by applying derivative operators

$$D^A D^B C_{AB} = -D^2(D^2 + 2)C \quad \text{(A3)}$$

$$D_{[B} D^C C_{A]C} = \frac{1}{2}\epsilon_{C[A} D_{B]} D^C (D^2 + 2)\Psi. \quad \text{(A4)}$$

We will only consider cases where the magnetic part vanishes and the shear is constant $\dot{C}_{AB} = 0$,

$$C_{AB} = (-2D_A D_B + \gamma_{AB} D^2)C, \quad \text{(A5)}$$

where C is constant in time, $\dot{C} = 0$. note that $\ell = 0, 1$ parts of C do not contribute to C_{AB}.

From the vanishing of the Bondi news $\dot{C}_{AB} = 0$ and the stress energy $T_{\mu\nu} = 0$, it follows from the evolution equation Eq. (2.11a) in [32] that $\dot{m} = 0$. For the remainder of the appendix we thus have

$$\dot{m} = \dot{C}_{AB} = 0. \quad \text{(A6)}$$

However, the angular momentum aspect N_A will in general depend on time u.

[16] Denoting $P(A_1 \dots A_n)$ as permutations, symmetrized or antisymmetrized tensors are given by

$$T_{(A_1 \dots A_n)} = \frac{1}{n!} \sum_{P(A_1 \dots A_n)} T_{P(A_1 \dots A_n)}$$

$$T_{[A_1 \dots A_n]} = \frac{1}{n!} \sum_{P(A_1 \dots A_n)} (-1)^P T_{P(A_1 \dots A_n)}.$$

1. Finite boost from a good cut

We now consider the special case where the shear vanishes, known as a "good cut",

$$C_{AB} = 0. \quad \text{(A7)}$$

Noting also that $\dot{m} = 0$ (A6), we may obtain the action of an infinitesimal Lorentz transformation from Eqs. (2.18a)-(2.18c) in [32],

$$\delta_Y m = \frac{3}{2} m\psi + Y^A D_A m \quad \text{(A8)}$$

$$\delta_Y C_{AB} = 0 \quad \text{(A9)}$$

$$\delta_Y \tilde{N}_A = (1 + \frac{1}{2}u)\psi \tilde{N}_A + \mathcal{L}_Y \tilde{N}_A + \frac{3}{2} u m D_A \psi, \quad \text{(A10)}$$

where $\mathcal{L}_Y$ is the Lie derivative on the sphere with respect to Y^A, and $\psi = D_A Y^A$. The notation $\delta_Y X$ indicates the change in X after an infinitesimal coordinate transformation generated by the associated vector field ξ^μ given as Eq. (2.16) of Ref. [32]. We restricted to a Lorentz transformation by setting $\alpha = 0$ in that equation. We also used the equations of motion for $\tilde{N}_A$ [Eq. (2.11b) in Ref. [32]].

We can now express the change in N_A using Eqs. (A8), (A9), and (A1):

$$\delta_Y N_A = \delta_Y \tilde{N}_A - u D_A \delta m = (\mathcal{L}_Y + \psi) N_A. \quad \text{(A11)}$$

We now consider the action of a finite boost acting on an "initial" configuration

$$m = M, \quad C_{AB} = 0, \quad N_A = 0, \quad \text{(A12)}$$

where M is a constant independent of (θ, ϕ) (as well as u). This requires integrating up the equations after choosing Y^A to effect a boost in the z direction,

$$Y^A = D^A \cos\theta. \quad \text{(A13)}$$

First consider the mass aspect (A9). If w parameterizes the finite boost, then m becomes a function $m(w; \theta, \phi)$ satisfying

$$\delta_Y m = -\frac{\partial m}{\partial w} \quad \text{(A14)}$$

with the sign chosen so that positive w corresponds to a boost in the $+z$ direction. The normalization of Y^A in (A13) ensures that w is equal to the rapidity of the finite boost. We thereby obtain a differential equation for m,

$$-\frac{\partial m}{\partial w} = -3mx + (1 - x^2)\frac{\partial m}{\partial x}, \quad \text{(A15)}$$

where $x = \cos\theta$. One can check by direct calculation that a solution is

$$m = \frac{M}{(\cosh w - x \sinh w)^3}, \quad \text{(A16)}$$

as first shown in Ref. [19]. This is the unique solution by the Cauchy–Kovalevskaya theorem. The covariant version of this expression is Eq. (70).

Eq. (A9) shows that the shear is invariant, while Eq. (A11) shows that the change in mass aspect N_A is a linear operator acting on N_A. In both cases the unique solution with zero initial data is zero, so the boosted configuration also has $C_{AB} = N_A = 0$.

2. (Super)translation

We now consider a supertranslation acting on a Bondi frame satisfying (A5) and (A6) (but not assuming the good cut condition). Choosing $Y = 0$ and $\alpha = T$ in Eq. (2.13) of Ref. [32], the change of Bondi data can be computed from (2.18a-2.18c) in Ref. [32]:

$$\delta_T m = 0 \tag{A17}$$
$$\delta_T C = T \tag{A18}$$
$$\delta_T \tilde{N}_A = 3m D_A T + T D_A m, \tag{A19}$$

where we used the equations of motion for $\tilde{N}_A$ [Eq. (2.11b) in Ref. [32]] and also employed (A4) with $\Psi = 0$. In terms of N_A (A1) we have

$$\begin{aligned} \delta_T N_A &= \delta_T \tilde{N}_A - u D_A \delta_T m \\ &= 3m D_A T + T D_A m, \end{aligned} \tag{A20}$$

i.e., $\delta_T N_A = \delta_T \tilde{N}_A$ since $\delta_T m = 0$.

These infinitesimal results integrate up trivially to finite supertranslations. First, Eq. (A17) shows that m is invariant under the supertranslation. Next, Eq. (A18) shows that the infinitesimal change in C does not depend on C, so the equation is trivially solved as $C = wT$, if w is the group parameter and T is the representative of the infinitesimal supertranslation. In this case we simply send $wT \to T$ and express the finite supertranslation by the new T. Similar comments apply to (A20): since m is invariant (independent of w), the right-hand-side is similarly independent of γ. Thus the formulas for finite supertranslations are identical to the infinitesimal ones.

[1] T. Damour, "Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory," *Phys. Rev. D* **94** no. 10, (2016) 104015, arXiv:1609.00354 [gr-qc]. I

[2] T. Damour and N. Deruelle, "Radiation Reaction and Angular Momentum Loss in Small Angle Gravitational Scattering," *Phys. Lett. A* **87** (1981) 81. I

[3] T. Damour, "Classical and quantum scattering in post-Minkowskian gravity," *Phys. Rev. D* **102** no. 2, (2020) 024060, arXiv:1912.02139 [gr-qc]. VI A

[4] T. Damour, "Radiative contribution to classical gravitational scattering at the third order in G," *Phys. Rev. D* **102** no. 12, (2020) 124008, arXiv:2010.01641 [gr-qc].

[5] S. E. Gralla and K. Lobo, "Self-force effects in post-Minkowskian scattering," *Class. Quant. Grav.* **39** no. 9, (2022) 095001, arXiv:2110.08681 [gr-qc]. I, 2, VII B, VII B, VII C, 14

[6] M. V. S. Saketh, J. Vines, J. Steinhoff, and A. Buonanno, "Conservative and radiative dynamics in classical relativistic scattering and bound systems," *Phys. Rev. Res.* **4** no. 1, (2022) 013127, arXiv:2109.05994 [gr-qc]. I, V, V, 8

[7] S. E. Gralla and K. Lobo, "Electromagnetic scoot," *Phys. Rev. D* **105** no. 8, (2022) 084053, arXiv:2112.01729 [gr-qc]. I, III, III, 6, IV, VI A, VI A, 10

[8] A. V. Manohar, A. K. Ridgway, and C.-H. Shen, "Radiated Angular Momentum and Dissipative Effects in Classical Scattering," *Phys. Rev. Lett.* **129** no. 12, (2022) 121601, arXiv:2203.04283 [hep-th]. I, VI A, VI A, VI A

[9] P. Di Vecchia, C. Heissenberg, and R. Russo, "Angular momentum of zero-frequency gravitons," *JHEP* **08** (2022) 172, arXiv:2203.11915 [hep-th]. I, VI A, VI A, VI A

[10] G. Veneziano and G. A. Vilkovisky, "Angular momentum loss in gravitational scattering, radiation reaction, and the Bondi gauge ambiguity," *Phys. Lett. B* **834** (2022) 137419, arXiv:2201.11607 [gr-qc]. VII, VII A, VII B

[11] D. Bini and T. Damour, "Radiation-reaction and angular momentum loss at the second post-Minkowskian order," *Phys. Rev. D* **106** no. 12, (2022) 124049, arXiv:2211.06340 [gr-qc]. VI A, VI A

[12] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "Classical gravitational observables from the Eikonal operator," *Phys. Lett. B* **843** (2023) 138049, arXiv:2210.12118 [hep-th].

[13] M. M. Riva, F. Vernizzi, and L. K. Wong, "Angular momentum balance in gravitational two-body scattering: Flux, memory, and supertranslation invariance," *Phys. Rev. D* **108** no. 10, (2023) 104052, arXiv:2302.09065 [gr-qc]. I, VI A, VII, VII A, VII A, VII A, VII A, VII B, VII B, VII B, VII C, VII C, VII D

[14] U. Gen and T. Shiromizu, "Timelike infinity and asymptotic symmetry," *Journal of Mathematical Physics* **39** no. 12, (Dec., 1998) 6573–6592, arXiv:gr-qc/9709009 [gr-qc]. I

[15] C. Cutler, "Properties of spacetimes that are asymptotically flat at timelike infinity," *Classical and Quantum Gravity* **6** no. 8, (Aug., 1989) 1075–1097.

[16] J. Porrill, "The Structure of Timelike Infinity for Isolated Systems," *Proceedings of the Royal Society of London Series A* **381** no. 1781, (June, 1982) 323–344.

[17] S. Chakraborty, D. Ghosh, S. J. Hoque, A. Khairnar, and A. Virmani, "Supertranslations at timelike infinity," *Journal of High Energy Physics* **2022** no. 2, (Feb., 2022) 22. https://doi.org/10.1007/JHEP02(2022)022.

[18] G. Compère, S. E. Gralla, and H. Wei, "An asymptotic framework for gravitational scattering," *Class. Quant. Grav.* **40** no. 20, (2023) 205018, arXiv:2303.17124 [gr-qc]. I, II B, 5, VII, 13, VII D, A

[19] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems," *Proceedings of the Royal Society of London Series A* **269** no. 1336, (Aug., 1962) 21–52. I, VI, VII, A 1

[20] R. K. Sachs, "Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time," *Proceedings of the Royal Society of London Series A* **270** no. 1340, (Oct., 1962) 103–126.

[21] R. Sachs, "Asymptotic Symmetries in Gravitational Theory," *Phys. Rev.* **128** no. 6, (Dec., 1962) 2851–2864. I, VI, VII

[22] R. Penrose, "Asymptotic properties of fields and space-times," *Phys. Rev. Lett.* **10** (1963) 66–68. 5

[23] S. E. Gralla, A. I. Harte, and R. M. Wald, "A Rigorous Derivation of Electromagnetic Self-force," *Phys. Rev. D* **80** (2009) 024031, arXiv:0905.2391 [gr-qc]. III

[24] B. Bonga and E. Poisson, "Coulombic contribution to angular momentum flux in general relativity," *Phys. Rev. D* **99** no. 6, (2019) 064024, arXiv:1808.01288 [gr-qc]. V

[25] R. Lapiedra, F. Marques, and A. Molina, "CLASSICAL PREDICTIVE ELECTRODYNAMICS OF TWO CHARGES WITH RADIATION: ENERGY AND THREE MOMENTUM BALANCE AND SCATTERING CROSS-SECTIONS. 2," *J. Math. Phys.* **20** (1979) 1316–1320. V

[26] L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, "Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity," *Gen. Rel. Grav.* **13** (1981) 963–1004. VI A

[27] S. Mougiakakos, M. M. Riva, and F. Vernizzi, "Gravitational Bremsstrahlung in the post-Minkowskian effective field theory," *Phys. Rev. D* **104** no. 2, (2021) 024041, arXiv:2102.08339 [gr-qc]. VI A

[28] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, "Classical Gravitational Bremsstrahlung from a Worldline Quantum Field Theory," *Phys. Rev. Lett.* **126** no. 20, (2021) 201103, arXiv:2101.12688 [gr-qc]. VI A

[29] T. Dray and M. Streubel, "Angular momentum at null infinity," *Class. Quant. Grav.* **1** no. 1, (1984) 15–26. VII

[30] R. M. Wald and A. Zoupas, "A General definition of 'conserved quantities' in general relativity and other theories of gravity," *Phys. Rev. D* **61** (2000) 084027, arXiv:gr-qc/9911095.

[31] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **12** (2011) 105, arXiv:1106.0213 [hep-th]. VII

[32] E. E. Flanagan and D. A. Nichols, "Conserved charges of the extended Bondi-Metzner-Sachs algebra," *Phys. Rev. D* **95** no. 4, (2017) 044002, arXiv:1510.03386 [hep-th]. [Erratum: Phys.Rev.D 108, 069902 (2023)]. VII, A, A, A, A, A 1, A 1, A 1, A 2, A 2, A 2

[33] E. T. Newman and R. Penrose, "Note on the Bondi-Metzner-Sachs group," *J. Math. Phys.* **7** (1966) 863–870. VII A

[34] O. M. Moreschi, "Supercentre of mass system at future null infinity," *Classical and Quantum Gravity* **5** no. 3, (Mar, 1988) 423. https://dx.doi.org/10.1088/0264-9381/5/3/004. VII A

[35] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," *JHEP* **01** (2016) 086, arXiv:1411.5745 [hep-th]. VII B, VII C

[36] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," *Phys. Rev. Lett.* **67** (1991) 1486–1489. VII D

Bibliography

[1] R. M. Wald, *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984. 2

[2] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems," *Proceedings of the Royal Society of London Series A* **269** no. 1336, (Aug., 1962) 21–52. 1

[3] R. K. Sachs, "Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time," *Proceedings of the Royal Society of London Series A* **270** no. 1340, (Oct., 1962) 103–126. 1

[4] R. Sachs, "Asymptotic Symmetries in Gravitational Theory," *Phys. Rev.* **128** no. 6, (Dec., 1962) 2851–2864. 1

[5] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*. Cambridge University Press, 7, 2019. 2.1

[6] M. J. Pfenning and E. Poisson, "Scalar, electromagnetic, and gravitational selfforces in weakly curved space-times," *Phys. Rev. D* **65** (2002) 084001, `arXiv:gr-qc/0012057`. 2.2, 2.3, 3.4

[7] E. Poisson, A. Pound, and I. Vega, "The Motion of point particles in curved spacetime," *Living Rev. Rel.* **14** (2011) 7, `arXiv:1102.0529 [gr-qc]`. 2.3, 2.5

[8] M. V. S. Saketh, J. Vines, J. Steinhoff, and A. Buonanno, "Conservative and radiative dynamics in classical relativistic scattering and bound systems," *Phys. Rev. Res.* **4** no. 1, (2022) 013127, `arXiv:2109.05994 [gr-qc]`. 2.5, 2.5

[9] K. Westpfahl, "High-Speed Scattering of Charged and Uncharged Particles in General Relativity," *Fortsch. Phys.* **33** no. 8, (1985) 417–493. 3.4