Convergence of Discrete Conformal Maps of Surfaces

by

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Abstract

This dissertation proves a convergence result about discrete conformal structures on smooth manifolds.

A *discrete conformal structure* is a collection of triangulated piecewise constant-curvature manifolds where each constant-curvature piece is a simplex and each manifold in the collection has the same underlying triangulation. In this dissertation we define a discrete conformal structure on a triangulated topological manifold by assigning weights to each edge and each vertex in the triangulation.

A *discrete conformal map*, then, is a map between two piecewise constant-curvature manifolds in the same discrete conformal structure. These maps are not conformal in either the Riemannian geometry sense or the complex analysis sense, but they do share some properties with smooth conformal maps.

In the literature there are several specific kinds of discrete conformal structure that are usually studied separately. These include circle packing discrete conformal structures, circle patterns, and vertex scaling discrete conformal structures. This dissertation studies general discrete conformal structures, so the results herein hold for each of the particular examples of discrete conformal structures that are usually studied separately.

Furthermore, discrete conformal maps are usually defined only on piecewise constant-curvature manifolds. In this dissertation, however, we use a notion of Riemannian barycentric maps to allow us to define discrete conformal maps on smooth manifolds.

Our main result generalizes the Rodin-Sullivan Theorem from circle packing in two senses. First, our result is valid for Riemannian surfaces without boundary, while the Rodin-Sullivan Theorem only holds for the particular case of a Jordan domain in the complex plane. Second, we consider discrete conformal structures in general so we are not constrained to a particular discrete conformal structure such as circle packing or vertex scaling discrete conformal structures.

Chapter 1

Introduction

Since at least the 1960s mathematicians have been studying curved manifolds by approximating them using Euclidean simplices. One of the most influential such papers is [Reg61], and in the decades since its publication, many more questions about Riemannian manifolds and their structure (e.g. curvature) have been answered by using various techniques to discretize the question. This general technique is very powerful and has been used in many ways to study many kinds of smooth objects. Furthermore, in the realm of computation Euclidean simplices provide tractable ways to visualize and model surfaces and other manifolds.

In particular, conformal geometry is quite a tractable area to discretize. William Thurston in a 1985 talk at Purdue University conjectured a discrete analogue of the Riemann Mapping Theorem using circle packings to define a discrete analogue to a conformal map. This conjecture would go on to become the Rodin-Sullivan Theorem when Rodin and Sullivan proved it in [RS87]. This theorem has served as a blueprint for several later convergence theorems in this area. It is discussed in more detail in Appendix B.

Thurston's talk set off a flurry of activity in the area and now discrete conformal geometry is quite a mature field and yet still very active. It turns out that several different versions of discrete conformality can be defined. Two of the most well-studied of these are circle packing and vertex scaling discrete conformal structures. We discuss these briefly in §2.3.

The main purpose of this dissertation is to prove that discrete conformal mappings approximate smooth conformal mappings from the Riemannian geometry perspective. In particular, we use Riemannian barycentric maps to construct maps between piecewise Euclidean and Riemannian manifolds and use these maps to compare Riemannian metrics related by discrete conformal mappings. We consider general discrete conformal mappings and Riemannian manifolds that are not necessarily flat, which generalizes previous works for flat circle packing and vertex scaling discrete conformal structures.

In order to describe our techniques more concretely, we first set some notation. The following is meant to be intuitive, but we have referenced the sections where the rigorous definitions are given.

Let (M, g) and (N, h) be Riemannian manifolds of the same dimension and let $\Omega \subset M$ be a complete submanifold of M. Assume $\{\Omega_n\}$ is a sequence of submanifolds that limit to Ω. (We define this sequence more rigorously in Definition 36.) We will define a particular sequence of maps $\{\Phi_n : \Omega_n \to N\}$ and show that these maps converge to a smooth conformal map. More concretely, we show that the pullback metric $\Phi_n^* h$ is close to the metric g. This is the content of Proposition 47.

In order to define the sequence $\{\Phi_n\}$, we first find a sequence of triangulations T_n of Ω_n. These triangulations have to satisfy certain criteria, the most salient of which is that their edges must be geodesic segments and each simplex must have edge lengths such that if they were the edge lengths of a Euclidean simplex, that simplex would have positive volume. This allows us to assign a flat (Euclidean) simplex to each curved (Riemannian) simplex in T_n. We can build up a piecewise flat manifold $\Omega_{\Delta,n}$ by gluing the flat simplices together along their edges. This process gives a sequence of maps $\{\Psi_n : \Omega_{\Delta,n} \to \Omega_n\}$, which we call *Riemannian barycentric maps* and study in Chapter 3.

Once we have a sequence of piecewise flat manifolds $\Omega_{\Delta,n}$, we can define discrete conformal maps on them. Intuitively, a discrete conformal map is a map between piecewise flat manifolds with the same underlying triangulation. The only thing that changes between the two is the lengths of edges; the combinatorics of the triangulation are fixed. For each n, we define a discrete conformal map $\phi_n : \Omega_{\Delta,n} \to \widetilde{\Omega}_{\Delta,n}$. This sequence of maps is studied in Chapter 7.

Finally, we map the image piecewise flat manifolds $\widetilde{\Omega}_{\Delta,n}$ back into the smooth manifold (N, h) using another Riemannian barycentric map $\widetilde{\Psi}_n$. The map Φ_n is defined as a composition of Riemannian barycentric maps and a discrete conformal map, as in the following commutative diagram.

$$\begin{array}{ccc} (\Omega_n, g) & \xrightarrow{\Phi_n} & (\widetilde{\Omega}_n, h) \\ \uparrow{\scriptstyle \Psi_n} & & \uparrow{\scriptstyle \widetilde{\Psi}_n} \\ (\Omega_{\Delta,n}, g^{\Delta}) & \xrightarrow{\phi_n} & (\widetilde{\Omega}_{\Delta,n}, h^{\Delta}) \end{array}$$

FIGURE 1.1. Defining Φ_n

We estimate the pullback metric $\Phi_n^* h$ by first estimating pullback metrics under the component functions Ψ_n^{-1}, ϕ_n, and $\widetilde{\Psi}_n$ and then combining these estimates into our final statement. The maps Ψ_n^{-1} and $\widetilde{\Psi}_n$ are Riemannian barycentric maps for which we use estimates found in [DGW16]. To estimate the pullback metric across the discrete conformal map ϕ_n, we estimate edge lengths in the image in terms of edge lengths in the domain. This result does not depend on which particular discrete conformal structure we are working with since the proof uses a general framework set out in §2.2.

The advantages of the approach laid out in this dissertation are the following. First, we are not limited to working with a specific discrete conformal structure, but can work with several at once. Second, many of the results we prove do not rely on the dimension of the starting manifold. Finally, our result does not require Ω to be flat, as the Rodin-Sullivan Theorem does. We can define discrete conformal maps on curved manifolds by finding corresponding piecewise flat manifolds and defining the discrete conformal maps there.

But there are some disadvantages to our approach as well. The most important disadvantage is that we require some restrictive assumptions. In some cases, it is possible to prove these assumptions for that specific case. For example, in the case of circle packing discrete conformal structures on Jordan domains, the assumptions are proven as part of the proof of the Rodin-Sullivan Theorem. If this new approach is to have wide-ranging value, then some of these assumptions are going to need to be proven more generally. For more discussion of these assumptions, see Chapter 10.

1.1 Structure

This dissertation is organized as follows.

Chapter 2 introduces concepts and notation related to piecewise constant curvature manifolds and discrete conformal geometry. The last section in the chapter is a brief discussion of the two main examples of discrete conformal structures, circle packing and vertex scaling discrete conformal structures.

Chapter 3 lays out background for certain maps from piecewise flat manifolds into Riemannian manifolds. Theorem 16 is the main theorem from this section. We use it in Chapter 8 to estimate norms in certain pullback metrics.

In Chapters 4 and 5, we discuss two very important geometric lemmas that are integral to the proof of the Rodin-Sullivan Theorem, the Ring Lemma and the Hexagonal Packing Lemma. These two lemmas have analogues in the vertex scaling discrete conformal structure as well. At the end of Chapter 4, we conjecture a general version of the ring lemma that we then use later as an assumption. Similarly, the last section of Chapter 5 discusses an assumption that is a generalization of the Hexagonal Packing Lemma. We call this assumption *Local Discrete Conformal Rigidity* (LDCR) because the hexagonal combinatorics from the Rodin-Sullivan context are nowhere to be found.

Chapters 6 and 7 prove important theorems, especially Corollary 39, which is a convergence result about a sequence of functions that is defined in terms of a sequence of discrete conformal maps, and Theorem 42, which gives an estimate on the pullback metric $\phi_n^* h^\Delta$, where ϕ_n is a discrete conformal mapping. These two chapters lay the groundwork for the main result, Theorem 48 in Chapter 8.

In Chapter 9 we apply the results we have proven in order to give a new proof of the Rodin-Sullivan Theorem. It is not obvious which proof is better. In many ways, the original proof found in [RS87] is simpler and more clear than our proof. However, our proof has the advantage that it should generalize better.

In Chapter 10, we discuss the assumptions we make in more detail, in order to point towards areas where our result could be strengthened. This strengthening is almost certainly going to come from finding more general proofs for assumptions like the Hexagonal Packing Lemma or the Ring Lemma.

Finally, there are three appendices. The first is a review of well-known results in analysis and differential geometry that are helpful background for the work in this dissertation. Section A.4.3 may be new to some readers since manifolds with boundary are not always discussed at length in basic courses in differential geometry.

The second appendix is a discussion of the Rodin-Sullivan Theorem. This theorem serves as a very nice introduction to the area of discrete conformal geometry and is the main example to keep in mind while reading the rest of this dissertation.

The last appendix is a proof of a generalized Descartes Circle Theorem, using some clever linear algebra. This appendix has very little to do with the rest of the dissertation, but it could be the beginnings of a proof of a higher dimensional version of the Ring Lemma. This is because the circle packing version of the Ring Lemma can be proven using the standard Descartes Circle Theorem, so a more general Descartes Circle Theorem could eventually lead to a proof of a more general Ring Lemma.

Chapter 2

Discrete Conformal Geometry

2.1 Piecewise Constant Curvature Manifolds

2.1.1 Simplicial Complexes

There are many (broadly equivalent) conceptions of simplices and simplicial complexes. The definitions in this section are taken from [Hat02] and [Lee11], with the exception of the definitions of the standard and unit simplices Δ and D, which come from [DGW16].

In the definitions below, simplices are defined as certain subsets of Euclidean space. This is mostly for convenience. For an example of simplicial complexes defined entirely topologically, see [CMS84].

Definition 1. *Let* $\{v_0, v_1, \ldots, v_k\} \subset \mathbb{R}^n$ *be such that the set* $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ *is linearly independent. The* (linear) k-simplex spanned by $\{v_0, \ldots, v_k\}$ *is the convex hull of* $\{v_0, \ldots, v_k\}$ *and will be denoted by* $[v_0, \ldots, v_k]$. *The points* $\{v_0, \ldots, v_k\}$ *are called the* vertices *of the simplex.*

If σ *is a* k*-simplex, any simplex spanned by a nonempty subset of its vertices is called a* face *of* σ. *A face is* proper *if it is a proper subset of* σ.

We now introduce two special simplices that will feature prominently in what is to come.

Definition 2. *Let* $\{e_0, \ldots, e_k\}$ *be the standard basis of* $\mathbb{R}^k$.

The standard simplex $\Delta \subset \mathbb{R}^{k+1}$ *is the convex hull of the points* $\{e_0, \ldots, e_k\}$. *This simplex can also be written as*

$$\Delta = \left\{ \lambda \in \mathbb{R}^{k+1} \,:\, \sum_{i=0}^{k} \lambda^i = 1 \text{ and } \lambda^i \geq 0 \textit{ for every } i \right\}.$$

The unit simplex $D \subset \mathbb{R}^k$ *is the convex hull of* $\{0, e_1, \ldots, e_k\}$ *The unit simplex* D *can be more explicitly written as*

$$D = \left\{ u \in \mathbb{R} \,:\, \sum_{i=1}^{k} u^i \leq 1 \text{ and } u^i \geq 0 \textit{ for every } i \right\}.$$

Notice than any linear simplex $\sigma = [v_0, \ldots, v_k]$ has a canonical homeomorphism $x : \Delta \to \sigma$ given by $x(\lambda) = \sum \lambda^i v_i$. We will study this homeomorphism in more detail in §3.1.

Definition 3. *A* simplicial complex *is a collection K of simplices such that*

1. *If $\sigma \in K$, then every face of σ is also in K.*
2. *The intersection $\sigma \cap \tau$ of any two simplices is either empty or is a face of both σ and τ.*
3. *K is locally finite.*

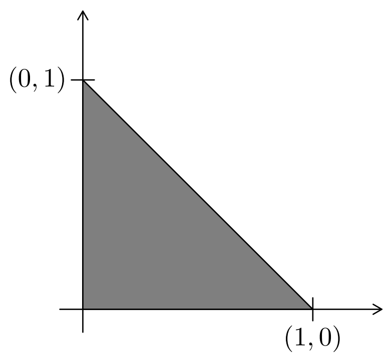

FIGURE 2.1. The 2-dimensional unit simplex D

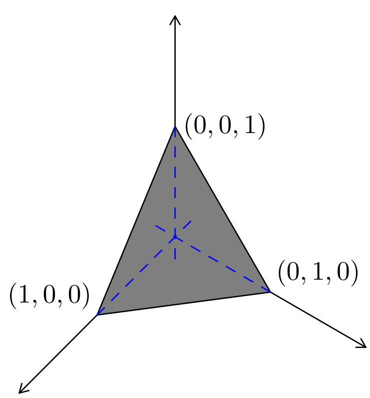

FIGURE 2.2. The 2-dimensional standard simplex Δ

2.1.2 Triangulations and piecewise constant curvature manifolds

Definition 4. *Let X be a topological space. A homeomorphism T between X and a simplicial complex K is called a* triangulation of X *and the pair (X, T) is called a* triangulated manifold.

Next we take a definition from [GT17]. In the following definition, the background geometry $\mathbb{G}$ is one of:

(a) Euclidean space $\mathbb{E}$,
(b) hyperbolic space $\mathbb{H}$, or
(c) spherical space $\mathbb{S}$.

Definition 5. *A* (triangulated) piecewise constant curvature manifold (M, T, ℓ) with background geometry $\mathbb{G}$ *is a triangulated manifold* (M, T) *together with a function* ℓ *on the edges of the triangulation such that each simplex can be embedded in* $\mathbb{G}$*, a space of constant curvature, as a nondegenerate simplex with edge lengths determined by* ℓ*.*

When the background geometry is Euclidean ($\mathbb{G} = \mathbb{E}$)*, hyperbolic* ($\mathbb{G} = \mathbb{H}$)*, or spherical* ($\mathbb{G} = \mathbb{S}$)*, such a manifold is called* piecewise flat*,* piecewise hyperbolic*, or* piecewise spherical *respectively.*

A piecewise flat manifold is often called a *piecewise linear* or *PL* manifold. An edge length function $\ell \in E(T)^*$ is sometimes called a *PL metric.* We do not use this terminology here, preferring to reserve the term "metric" to mean a Riemannian metric.

Piecewise constant curvature manifolds are the natural setting in which to define discrete conformal structures, which we will do in the next section.

The set of vertices (0-simplices) of T will be denoted $V(T)$, or just V when the triangulation T is clear from context. Similarly, the set of edges (1-simplices) will be denoted by either $E(T)$ or E and the set of triangles (2-simplices) by $F(T)$ or F. Occasionally (as in the next definition), we need to talk about directed edges. A *directed edge* is an element of $E(T)$ together with a choice of orientation. We label this set $E_+(T)$.

Let X be one of V, E, F, or E_+. Then the set of functions $\{f : X \to \mathbb{R}\}$ will be denoted by $X(T)^*$ or just X^* when there is no danger of confusion regarding which triangulation is meant. This is the standard notation for a dual space.

We next introduce a construction that ensures that the piecewise flat manifolds we work with have nice Poincaré dual structures.

Definition 6. *Suppose* (M, T) *is a triangulated manifold. An element* $d \in E_+(T)^*$ *is an assignment of* partial edge lengths *if* $\ell_{ij} = d_{ij} + d_{ji}$ *makes* (M, T, ℓ) *a piecewise* $\mathbb{G}$ *manifold and if for every triangle* $[v_i, v_j, v_k]$*,*

$$d_{ij}^2 + d_{jk}^2 + d_{ki}^2 = d_{ji}^2 + d_{ik}^2 + d_{kj}^2 \qquad \text{if } \mathbb{G} = \mathbb{E}, \tag{2.1}$$

$$\cosh(d_{ij})\cosh(d_{jk})\cosh(d_{ki}) = \cosh(d_{ji})\cosh(d_{kj})\cosh(d_{ik}) \qquad \text{if } \mathbb{G} = \mathbb{H}, \tag{2.2}$$

$$\cos(d_{ij})\cos(d_{jk})\cos(d_{ki}) = \cos(d_{ji})\cos(d_{kj})\cos(d_{ik}) \qquad \text{if } \mathbb{G} = \mathbb{S}, \tag{2.3}$$

These conditions on the squared partial edge lengths ensure that each triangle has a unique center, and hence that each simplex of any dimension has a unique center and a dual which intersects the simplex orthogonally at the center. This is the content of Proposition 4 of [Gli24].

We will not use the dual structure in this dissertation, but because of a classification theorem, Theorem 8 below, it is there in the background of each one of our discrete conformal structures. For examples of results that are proven using this dual structure, see [Gli11] and [Gli24].

2.2 Discrete Conformal Structures

This section mainly takes definitions from [GT17].

We begin by defining discrete conformal structure.

Definition 7. *A* discrete conformal structure $\mathcal{C}(M,T,U)$ *on a triangulated manifold* (M,T) *with background geometry* $\mathbb{G}$ *is a smooth map from an open set* $U \subset V(T)^*$ *to* $E_+(T)^*$ *that sends functions* f *on the vertices to a set of partial edge lengths* $\{d_{ij}\}$ *such that for each directed edge* $(v_i, v_j) \in E_+(T)$ *and vertex* $v_k \in V(T)$*,*

$$\frac{\partial \ell_{ij}}{\partial f_i} = d_{ij} \qquad \text{if } \mathbb{G} = \mathbb{E}, \tag{2.4}$$

$$\frac{\partial \ell_{ij}}{\partial f_i} = \tanh d_{ij} \qquad \text{if } \mathbb{G} = \mathbb{H}, \tag{2.5}$$

$$\frac{\partial \ell_{ij}}{\partial f_i} = \tan d_{ij} \qquad \text{if } \mathbb{G} = \mathbb{S}, \tag{2.6}$$

and

$$\frac{\partial d_{ij}}{\partial f_k} = 0$$

whenever k *is neither* i *nor* j*.*

Next we prove a classification theorem that will allow us to forget about partial edge lengths entirely. We will state the theorem in all three geometries, but we only prove the Euclidean case. The proof is similar in the other two geometries, but the calculations are harder. For details, see [GT17].

Theorem 8 (Theorem 4 of [GT17])**.** *Let* $\mathcal{C}(M,T,U)$ *be a discrete conformal structure with background geometry* $\mathbb{G}$ *on a surface* M*. Then there exists* $\alpha \in \mathbb{R}^{|V|}$ *and* $\eta \in \mathbb{R}^{|E|}$ *such that the conformal structure can be written as*

$$d_{ij} = \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i + f_j}}{\ell_{ij}}$$

with

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i + f_j}$$

if $\mathbb{G} = \mathbb{E}$*,*

$$\tanh d_{ij} = \frac{\alpha_i e^{2f_i}}{\sinh \ell_{ij}} \sqrt{\frac{1 + \alpha_j e^{2f_j}}{1 + \alpha_i e^{2f_i}}} + \frac{\eta_{ij} e^{f_i + f_j}}{\sinh \ell_{ij}}$$

with

$$\cosh \ell_{ij} = \sqrt{(1+\alpha_i e^{2f_i})(1+\alpha_j e^{2f_j})} + \eta_i e^{f_i+f_j}$$

if $\mathbb{G} = \mathbb{H}$*, or*

$$\tan d_{ij} = \frac{\alpha_i e^{2f_i}}{\sin \ell_{ij}} \sqrt{\frac{1-\alpha_j e^{2f_j}}{1-\alpha_i e^{2f_i}}} + \frac{\eta_{ij} e^{f_i+f_j}}{\sin \ell_{ij}}$$

with

$$\cos \ell_{ij} = \sqrt{(1-\alpha_i e^{2f_i})(1-\alpha_j e^{2f_j})} - \eta_{ij} e^{f_i+f_j}$$

if $\mathbb{G} = \mathbb{S}$*.*

Proof of the Euclidean case of Theorem 8. We assume just three things, that $\frac{\partial d_{ij}}{\partial f_k} = 0$, that $\frac{\partial \ell_{ij}}{\partial f_i} = d_{ij}$, and that $d_{01}^2 + d_{12}^2 + d_{20}^2 = d_{10}^2 + d_{21}^2 + d_{02}^2$ for every triangle $[v_0, v_1, v_2]$.

The structure of this proof is that we write and then solve a pair of partial differential equations that give us ℓ_{ij}^2. We then check that constants match.

First, we use (2.1) to say

$$\frac{\partial^2}{\partial f_i \partial f_j}\left(d_{ij}^2 - d_{ji}^2\right) = \frac{\partial^2}{\partial f_i \partial f_j}\left(d_{ik}^2 + d_{kj}^2 - d_{jk}^2 - d_{ki}^2\right) = 0.$$

Then we use the assumption that $\frac{\partial \ell_{ij}}{\partial f_i} = d_{ij}$ to find that

$$\frac{\partial^2 \ell_{ij}}{\partial f_i \partial f_j} = \frac{\partial d_{ij}}{\partial f_j} = \frac{\partial d_{ji}}{\partial f_i}$$

and from here, we do the following calculation:

$$\left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right)\left(d_{ij}^2 - d_{ji}^2\right) = 2d_{ij}\left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right)(d_{ij}) - 2d_{ji}\left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right)(d_{ji}) \quad (2.7)$$

$$= 2d_{ij}\left(\frac{\partial d_{ij}}{\partial f_i} + \frac{\partial d_{ij}}{\partial f_j}\right) - 2d_{ji}\left(\frac{\partial d_{ji}}{\partial f_i} + \frac{\partial d_{ji}}{\partial f_j}\right) \quad (2.8)$$

$$= 2d_{ij}\left(\frac{\partial}{\partial f_i}(\ell_{ij})\right) - 2d_{ji}\left(\frac{\partial}{\partial f_j}\ell_{ij}\right) \quad (2.9)$$

$$= 2(d_{ij}^2 - d_{ji}^2) \quad (2.10)$$

Next we take derivatives of both sides, once with respect to f_i, once with respect to f_j, remembering that the second mixed partial is 0, and we get the two equations below:

$$\frac{\partial^2}{\partial f_i^2}\left(d_{ij}^2 - d_{ji}^2\right) = 2\frac{\partial}{\partial f_i}(d_{ij}^2 - d_{ji}^2)$$

$$\frac{\partial^2}{\partial f_j^2}\left(d_{ij}^2 - d_{ji}^2\right) = 2\frac{\partial}{\partial f_j}(d_{ij}^2 - d_{ji}^2).$$

These are a pair of separable ODE's in $\frac{\partial}{\partial f_k}(d_{ij}^2 - d_{ji}^2), k \in \{i, j\}$, so we can solve them to get

$$\frac{\partial}{\partial f_i}(d_{ij}^2 - d_{ji}^2) = 2a_{ij}e^{2f_i}$$

and

$$\frac{\partial}{\partial f_j}(d_{ij}^2 - d_{ji}^2) = -2a_{ji}e^{2f_j}.$$

Note that a_{ji} has a negative coefficient by choice to make the calculation more convenient later.

Next we use (2.10) to say that

$$\begin{aligned} 2(d_{ij}^2 - d_{ji}^2) &= \left(\frac{\partial}{\partial f_i} + \frac{\partial}{\partial f_j}\right)\left(d_{ij}^2 - d_{ji}^2\right) \\ &= 2a_{ij}e^{2f_i} - 2a_{ji}e^{2f_j}, \end{aligned}$$

from which it follows immediately that $d_{ij}^2 - d_{ji}^2 = a_{ij}e^{2f_i} - a_{ji}e^{2f_j}$.

We have written $d_{ij}^2 - d_{ji}^2$ in a more helpful way, but the reason we want to do that is because we can write $\frac{\partial \ell_{ij}^2}{\partial f_i}$ in terms of $d_{ij}^2 - d_{ji}^2$, which is what we want to do here. We have the following:

$$\begin{aligned} \frac{\partial \ell_{ij}^2}{\partial f_i} &= 2\ell_{ij}\frac{\partial \ell_{ij}}{\partial f_i} \\ &= 2\ell_{ij}d_{ij} \\ &= \ell_{ij}(d_{ij} + d_{ji} + d_{ij} - d_{ji}) \\ &= \ell_{ij}^2 + \ell_{ij}(d_{ij} - d_{ji}) \\ &= \ell_{ij}^2 + (d_{ij}^2 - d_{ji}^2). \end{aligned}$$

We do a similar calculation with respect to f_j to find that

$$\frac{\ell_{ij}^2}{\partial f_j} = \ell_{ij}^2 - (d_{ij}^2 - d_{ji}^2).$$

Then finally we replace $d_{ij}^2 - d_{ji}^2$ with $a_{ij}e^{2f_i} - a_{ji}e^{2f_j}$ and we have our pair of differential equations:

$$\begin{aligned} \frac{\partial \ell_{ij}^2}{\partial f_i} &= \ell_{ij}^2 + a_{ij}e^{2f_i} - a_{ji}e^{2f_j} \\ \frac{\partial \ell_{ij}^2}{\partial f_j} &= \ell_{ij}^2 - a_{ij}e^{2f_i} + a_{ji}e^{2f_j}. \end{aligned}$$

We can treat each of these as an ODE and solve it using integrating factors to get the following pair of equations:

$$\begin{aligned}\ell_{ij}^2 &= a_{ij}e^{2f_i} + a_{ji}e^{2f_j} + c(f_j)e^{f_i}\\ \ell_{ij}^2 &= a_{ij}e^{2f_i} + a_{ji}e^{2f_j} + d(f_i)e^{f_j}.\end{aligned}$$

Note that $c(f_j)$ is a function of f_j but constant with respect to f_i and similarly, $d(f_i)$ is constant with respect to f_j. Hence we see immediately that

$$c(f_j)e^{f_i} = d(f_i)e^{f_j} = 2\eta_{ij}e^{f_i+f_j}$$

for some number η_{ij}.

Finally we need to check that the constants match. There are two parts to this. First we show that $a_{ij} = a_{ik}$, so we can replace a_{ij} with α_i, depending only on vertices. The second thing to show is that $\alpha_{i,jk} = \alpha_{i,jl}$ and $\eta_{ij,k} = \eta_{ij,l}$ for adjacent triangles $[v_i, v_j, v_k]$ and $[v_i, v_j, v_\ell]$ sharing the edge e_{ij}.

For the first part, we begin with the condition on squared edge lengths, slightly rewritten. We then differentiate both sides and simplify:

$$\begin{aligned}\frac{\partial}{\partial f_i}\left(d_{ij}^2 - d_{ji}^2 + d_{ki}^2 - d_{ik}^2\right) &= \frac{\partial}{\partial f_i}\left(d_{kj}^2 - d_{jk}^2\right)\\ \frac{\partial}{\partial f_i}\left(a_{ij}e^{2f_i} - a_{ji}e^{2f_j} + a_{ki}e^{2f_k} - a_{ik}e^{2f_i}\right) &= 0\\ 2e^{2f_i}\left(a_{ij} - a_{ik}\right) &= 0\\ a_{ij} - a_{ik} &= 0.\end{aligned}$$

Hence we have shown that $a_{ij} = a_{ik}$ and we can write it as α_i instead.

For the next part, we take ℓ_{ij}^2 written from the rule in two adjacent triangles, take a couple of derivatives and see that indeed, the constants must match:

$$\begin{aligned}\frac{\partial \ell_{ij}^2}{\partial f_i} &= 2\alpha_{i,jk}e^{2f_i} + 2\eta_{ij,k}e^{f_i+f_j}\\ &= 2\alpha_{i,jl}e^{2f_i} + 2\eta_{ij,l}e^{f_i+f_j}\\ \frac{\partial^2 \ell_{ij}^2}{\partial f_j \partial f_i} &= 2\eta_{ij,k}e^{f_i+f_j}\\ &= 2\eta_{ij,l}e^{f_i+f_j}.\end{aligned}$$

By the second derivative, the η's match and then once the η's match, the first derivative tells us the α's do as well. □

The above theorem allows us to reformulate our definition of discrete conformal structure into the following equivalent definition. The domain U of this map consists of all functions $f \in V(T)^*$ such that (M, T, ℓ) is a piecewise flat manifold. That is, every simplex is nondegenerate when the lengths of edges e_{ij} are given by $\ell_{ij}(f)$.

Definition 9. *Let (M, T) be a triangulated manifold and fix $\alpha \in \mathbb{R}^{|V|}, \eta \in \mathbb{R}^{|E|}$. The* discrete conformal structure with background geometry $\mathbb{G}$ *corresponding to α, η is the map $C_{\alpha,\eta} : U \subset V(T)^* \to E(T)^*$ given by $f \mapsto \ell(f)$, where for each edge $[v_i, v_j]$, the edge length ℓ_{ij} satisfies*

$$\ell_{ij}^2 = \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j} \qquad \text{if } \mathbb{G} = \mathbb{E},$$

$$\cosh \ell_{ij} = \sqrt{(1 + \alpha_i e^{2f_i})(1 + \alpha_j e^{2f_j})} + \eta_{ij} e^{f_i+f_j} \qquad \text{if } \mathbb{G} = \mathbb{H},$$

$$\cos \ell_{ij} = \sqrt{(1 - \alpha_i e^{2f_i})(1 - \alpha_j e^{2f_j})} - \eta_{ij} e^{f_i+f_j} \qquad \text{if } \mathbb{G} = \mathbb{S}.$$

The domain U of this map consists of all functions $f \in V(T)^*$ such that (M, T, ℓ) is a piecewise $\mathbb{G}$ manifold. That is, every simplex is nondegenerate when the lengths of edges e_{ij} are given by $\ell_{ij}(f)$. If $f \in U$, then f is called a discrete conformal factor*.*

Note that the domain U can very easily be empty. This happens, for example, in the Euclidean background when α and η are negative for every vertex and edge respectively. There may be interesting consequences of relaxing the requirement that the resultant edge lengths ℓ_{ij} must make (M, T, ℓ) a piecewise constant curvature manifold, but examining them would take us too far afield. Hence in this dissertation, we always assume that α and η are chosen such that there is at least one function $f \in V(T)^*$ whose resulting edge lengths give nondegenerate simplices.

Finally, we can now say what it means for two piecewise constant curvature manifolds to be discrete conformal.

Definition 10. *Two piecewise $\mathbb{G}$ manifolds with the same underlying triangulation T are said to be* discrete conformal *(or sometimes* discretely conformally equivalent*) when their edge lengths come from the same discrete conformal structure. That is, (M, T, ℓ) and $(M, T, \overline{\ell})$ are discrete conformal if there exist some $\alpha \in \mathbb{R}^{|V|}, \eta \in \mathbb{R}^{|E|}$ such that $\ell = C_{\alpha,\eta}(f)$ and $\overline{\ell} = C_{\alpha,\eta}(\overline{f})$ for some $f, \overline{f} \in V(T)^*$.*

For the rest of this dissertation we work only with piecewise flat manifolds, so whenever the background geometry of a discrete conformal structure is not specified, the assumption is that it is Euclidean.

2.3 Examples of (Flat) Discrete Conformal Structures

In this section we will briefly examine several specific discrete conformal structures, specifically circle packing, inversive distance, and vertex scaling discrete conformal structures.

These examples of discrete conformal structures all satisfy Definition 7 and hence as noted earlier, all have dual structures such that duals are orthogonal to their corresponding primals. Having nice dual structures like this is quite helpful in computer modeling and mesh generation, even without a discrete conformal structure. For an example of triangulations with nice dual structures that does not come from a discrete conformal structure as in Definition 7, see [MMdGD11].

2.3.1 Circle Packing

We begin with the circle packing discrete conformal structure.

A *circle packing* is a collection of closed discs residing on some Riemannian surface, where each closed disc is externally tangent to its neighbors. For our purposes, we restrict our attention to *univalent* packings, which are packings for which the collection of discs have disjoint interiors. Univalent packings are the most common, but packings that are not univalent are better suited to certain contexts, such as approximating complex functions with branch points. See [Ste05] for more discussion of multivalent circle packings.

Each circle packing has an associated triangulation T constructed by connecting the centers of every pair of adjacent circles with geodesic segments. The vertices of T are the centers of the circles, the edges are the geodesic segments connecting neighboring circle centers, and the faces are the triangles formed by these, one triangle for every triple of mutually tangent circles.

The circle packing discrete conformal structure has edge lengths defined as sums of radii of neighboring tangent circles. That is, the length ℓ_{ij} of the edge e_{ij} connecting vertices v_i and v_j is $\ell_{ij} = r_i + r_j$, where r_i, r_j are the radii of circles with centers v_i, v_j respectively.

If we define $f_i := \log r_i$, then

$$\begin{aligned}\ell_{ij}^2 &= (r_i + r_j)^2 \\ &= (e^{f_i} + e^{f_j})^2 \\ &= e^{2f_i} + e^{2f_j} + 2e^{f_i+f_j} \\ &= \alpha_i e^{2f_i} + \alpha_j e^{2f_j} + 2\eta_{ij} e^{f_i+f_j}\end{aligned}$$

where $\alpha_i = 1$ for every vertex v_i and $\eta_{ij} = 1$ for every edge e_{ij}. It is easily checked that

$$\begin{aligned}d_{ij} &= \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i+f_j}}{\ell_{ij}} \\ &= \frac{r_i^2 + r_i r_j}{r_i + r_j} \\ &= r_i,\end{aligned}$$

and hence the discrete conformal structure $C_{1,1}$ is the circle packing discrete conformal structure.

2.3.2 Inversive Distance Packings

One slight generalization of univalent circle packings is inversive distance packings. In these packings, neighboring circles are allowed to overlap or not intersect at all. For more on inversive distance packings see Appendix E of [Ste05].

Inversive distance packings are defined by assigning a weight σ_{ij} to every edge e_{ij}. This weight σ_{ij} is called the *inversive distance* between circles c_i and c_j. The squared edge lengths ℓ_{ij}^2 are then defined by

$$\ell_{ij}^2 = 2\sigma_{ij} r_i r_j + r_i^2 + r_j^2.$$

Elementary geometry shows that if two neighboring circles c_i and c_j overlap, then the inversive distance σ_{ij} is simply the cosine of their overlap angle (Φ_{ij} in Figure 2.3), which gives some intuition for what the edge weights σ_{ij} represent.

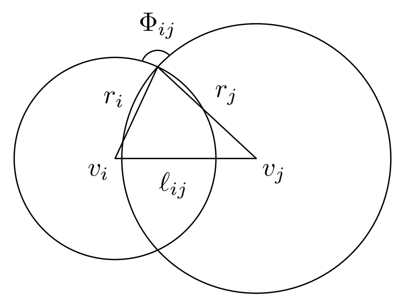

FIGURE 2.3. The parameter σ_{ij} is the cosine of Φ_{ij}

Inversive distance packings with small overlaps behave very similarly to univalent circle packings, but they belong to a different discrete conformal structure. To see this, rewrite ℓ_{ij}^2 as

$$\begin{aligned}\ell_{ij}^2 &= 2\sigma_{ij} r_i r_j + r_i^2 + r_j^2 \\ &= e^{2f_i} + e^{2f_j} + 2\sigma_{ij} e^{f_i+f_j},\end{aligned}$$

where $f_i := \log r_i$. This shows that inversive distance packings belong to $C_{1,\sigma}$ rather than $C_{1,1}$ like univalent circle packings.

Inversive distance circle packings with overlaps are sometimes called *circle patterns* and have been studied by, for example, Bücking in [Büc08] and [Büc18].

2.3.3 Vertex Scaling

The following definitions come from [LSW20], but they have been reworked to fit better with the notation used elsewhere.

The vertex scaling discrete conformal structure has edge lengths defined as multiples of some base edge length. That is, let $L \in E(T)^*$ be a set of edge lengths such that (M, T, L) is a piecewise flat manifold. A set of edge lengths $\ell \in E(T)^*$ is said to be *related (to L) by a vertex scaling* if there is some $w \in V(T)^*$ such that $\ell_{ij} = L_{ij} e^{w_i + w_j}$ for every edge e_{ij}. In this case, we write $\ell = w * L$.

It is easy to see that

$$\ell_{ij}^2 = L_{ij}^2 e^{2w_i + 2w_j},$$

so if we define $f := 2w$ and let $\alpha_i = 0$ for every vertex v_i and $\eta_{ij} = L_{ij}^2/2$ for every edge, then vertex scaling is the discrete conformal structure $C_{0,L^2/2}$.

Partial edge lengths do not get used much in the literature on vertex scaling conformal structures, but we can simply define

$$\begin{aligned} d_{ij} :&= \frac{\alpha_i e^{2f_i} + \eta_{ij} e^{f_i + f_j}}{\ell_{ij}} \\ &= \frac{L_{ij}^2/2 e^{f_i + f_j}}{L_{ij} e^{w_i + w_j}} \\ &= \frac{L_{ij}}{2} e^{w_i + w_j}, \end{aligned}$$

so $d_{ij} = \ell_{ij}/2$ in the vertex scaling case. In the literature, this discrete conformal structure has occasionally been called the *perpendicular bisector* discrete conformal structure since the duals to edges are exactly the perpendicular bisectors of those edges.

For more discussion of vertex scaling discrete conformal structures see [LSW20], [WZ20], [LWZ21], [GLW19], and [WGS15], among many others.

Chapter 3

Riemannian barycentric maps and coordinates

The discussion in this chapter closely follows [DGW16].

3.1 Parametrizing Euclidean Simplices

Recall the standard simplex Δ and the unit simplex D from Definition 2.

Given a (possibly degenerate) Euclidean n-simplex $\sigma = \operatorname{conv}(p_0, \ldots, p_n)$, we can parametrize it using either the standard simplex Δ or the unit simplex D. The two parametrizations are the following:

$$x : \Delta \to \sigma, \qquad x(\lambda) = \sum_{i=0}^{n} \lambda^i p_i \tag{3.1}$$

$$y : D \to \sigma, \qquad y(u) = Au + p_0, \tag{3.2}$$

where A is the matrix whose ith column is $p_i - p_0$.

Notice that x is simply the usual barycentric map.

Let g be the Euclidean metric on $\mathbb{R}^m$. Since the simplex σ is a subset of $\mathbb{R}^m$, the restriction $g|_\sigma$ is a metric on σ. We can pull this metric back to either Δ or D using x or y respectively.

We first consider the pullback metric x^*g on Δ. Let $v, w \in T_\lambda\Delta$ for some $\lambda \in \Delta$. Then x^*g is given by

$$\langle v, w \rangle_{x^*g} = \left\langle \sum_{i=0}^{n} v^i p_i, \sum_{j=0}^{n} w^j p_j \right\rangle_{\mathbb{R}^m}.$$

Now any tangent vector $v \in T_\lambda\Delta$ lies in the hyperplane $\sum_{i=0}^{n} x^i = 0$, so both v and w are such that $\sum v^i = \sum w^i = 0$. We can therefore use algebra to rewrite $\langle v, w \rangle_{x^*g}$ as

$$\langle v, w \rangle_{x^*g} = -\frac{1}{2} \sum_{i,j=0}^{n} |p_i - p_j|^2_{\mathbb{R}^m} v^i w^j, \tag{3.3}$$

which is particularly nice since $|p_i - p_j|$ is just ℓ_{ij}, the length of the edge e_{ij} connecting the vertices p_i and p_j.

Next we use y to pull g back to D. We have:

$$\langle v, w\rangle_{y^*g} = \langle Av, Aw\rangle_g \tag{3.4}$$

$$= \sum_{i,j=1}^{n} \langle p_i - p_0, p_j - p_0\rangle_{\mathbb{R}^m} v^i w^j. \tag{3.5}$$

These two metrics, x^*g and y^*g, are functionally identical since they are both pullbacks of g and, furthermore, Δ and D can be mapped isomorphically by a linear map with linear inverse. As such, we can parametrize a Euclidean simplex σ by either Δ or D, depending on which is most convenient to work with. Often we will choose which parametrization to use based on which pullback metric is more convenient between x^*g and y^*g.

Parametrizing by Δ gives us a symmetric formulation where none of the vertices are privileged, but it has the downside that the metric x^*g is not determined in the direction perpendicular to the tangent plane $T_\lambda\Delta$. On the other hand, parametrizing by D privileges the vertex p_0, but the metric y^*g is determined in all directions. Which parametrization we use is, again, a matter of convenience.

3.2 Parametrizing Riemannian Simplices

Next we parametrize Riemannian simplices using Δ. In order to see how to do this, let us examine Euclidean barycentric coordinates a little more carefully.

We have defined the Euclidean barycentric map $x : \Delta \to \sigma$ by $x(\lambda) = \sum_i \lambda^i p_i$. This definition does not generalize, so we need an alternative.

Consider the function

$$E_\lambda(a) := \sum_{i=0}^{n} \lambda^i |a - p_i|^2_{\mathbb{R}^m}.$$

We can calculate partial derivatives of this function as follows:

$$\begin{aligned}\frac{\partial E_\lambda}{\partial a^j} &= \sum_{i=0}^{n} \lambda^i \frac{\partial}{\partial a_j}\left(|a - p_i|^2_{\mathbb{R}^m}\right)\\ &= 2\sum_{i=0}^{n} \lambda^i (a^j - p_i^j)\\ &= 2\left(a^j - x^j(\lambda)\right).\end{aligned}$$

The last equality follows from the fact that $\sum_i \lambda^i = 1$ for any $\lambda \in \Delta$. Hence $a = x(\lambda)$ is a critical point of E_λ. Further, the Hessian H_{E_λ} is very simple, being twice the identity matrix. Thus the Hessian is positive definite and hence $x(\lambda)$ is the minimizer of the function $E_\lambda(a)$. We use a generalized version of $E_\lambda(a)$ to define Riemannian barycentric coordinates.

Let (M, g) be a complete m-dimensional Riemannian manifold with $m > 1$ and let Δ be the n-dimensional standard simplex

$$\Delta = \left\{ \lambda \in \mathbb{R}^{n+1} \,:\, \lambda^i \geq 0, \sum_i \lambda^i = 1 \right\}.$$

Let $p_0, \ldots, p_n$ be points in M and consider the function $E : M \times \Delta \to \mathbb{R}$ given by

$$E(a, \lambda) = \sum_{i=0}^{n} \lambda^i d_g^2(a, p_i), \tag{3.6}$$

where d_g is the Riemannian distance function on M. Once we are in the Riemannian setting, the function $E(\cdot, \lambda)$ may not have a minimizer or it may have multiple. However, the following proposition says that as long as we restrict our attention to a small enough subset of M, minimizers exist and are unique.

Proposition 11 (Proposition 13 of [DGW16])**.** *If the points p_i lie in a ball whose radius is less than half the convexity radius, then $E(\cdot, \lambda)$ has a unique minimizer.*

For our purposes, we are interested in a sequence of triangulations where the maximum edge length is going to zero, so we may as well assume that every simplex in every one of our triangulations lies inside a ball small enough to be a geodesic ball. Hence Karcher means always exist and are unique.

We can now define the (Riemannian) barycentric mapping, as follows:

Definition 12. *For a given $\lambda \in \Delta$, let $\Psi(\lambda)$ be the minimizer of $E(\cdot, \lambda)$. We call Ψ the* (Riemannian) barycentric mapping *with respect to vertices $p_0, \ldots, p_n$. Its image in M is called the corresponding* Karcher simplex.

It is shown in [Kar77] that local minimizers of $E(\cdot, \lambda)$ for fixed λ are zeros of the section $F : M \times \Delta \to TM$ given by

$$F(a, \lambda) = \sum_{i=0}^{n} \lambda^i X_i|_a, \qquad \text{where} \quad X_i = \frac{1}{2} \operatorname{grad} d^2(\cdot, p_i).$$

Notice that if $\lambda^i = 0$ for some i, then the value of $\Psi(\lambda)$ is independent of p_i. Hence each facet of Δ is mapped to a Karcher subsimplex and this subsimplex only depends on the vertices it contains. Furthermore, since Ψ is continuous, the Karcher subsimplices form the boundary of a Karcher simplex. That is, $\partial(\Psi(\Delta)) = \Psi(\partial\Delta)$.

Lemma 13. *Let Δ be the (closed) standard k-simplex. Then the Riemannian barycentric map $\Psi : \Delta \to M$ is a diffeomorphism onto its image.*

Proof. By the preceding discussion, Ψ is a diffeomorphism on the interior of Δ. All that remains is to show that Ψ is smooth on $\partial\Delta$.

Let $\lambda \in \partial\Delta$. We need to find a neighborhood U_λ of λ and a function $\Psi^{\text{ext}} : U_\lambda \to M$ such that Ψ^{ext} is smooth and agrees with Ψ on Δ.

The neighborhoods U_λ we choose will simply be small ϵ-balls about λ. That is,

$$U_\lambda = \left\{ x \in \mathbb{R}^{k+1} \,:\, \sum_{i=0}^{k} x^i = 1 \,\text{and}\, |x - \lambda|_{\mathbb{R}^{k+1}} < \epsilon \right\}.$$

Define the extended function $\Psi^{\text{ext}}(x)$ is the same way as we did $\Psi(\lambda)$, that is, as the local minimizer of $E(\cdot, a)$, where E is defined by (3.6). With this definition, it is immediate that Ψ^{ext} agrees with Ψ an $U_\lambda \cap \Delta$ since they are defined by the same expression.

For smoothness, we need that the minimizer of $E(\cdot, x)$ exists and is unique for each $x \in U_\lambda$. By the Remarks on page 100 of [DVW15], this is true as long as ϵ is chosen to be small enough. $\square$

The main result of [DGW16] is Theorem 15 below, which gives estimates of the pullback metric $\Psi^* g$ on Δ in terms of the Euclidean metric, g^Δ, defined by

$$\langle v, w \rangle_{g^\Delta} := -\frac{1}{2} \sum_{i,j=0}^{n} d_g^2(p_i - p_j)\, v^i w^j,$$

for all $v, w \in T_\lambda \Delta$.

Note that if we take a Euclidean n-simplex σ whose edge lengths ℓ_{ij} are determined by the corresponding geodesic edge length in M, $\ell_{ij} := d_g(p_i, p_j)$, then g^Δ is the pullback to Δ of the induced Euclidean metric on σ. We frequently refer to the induced Euclidean metric on σ as g^Δ. The advantage of doing so is that we can then study g^Δ by pulling it back to the unit simplex D instead of the standard simplex Δ. We do this, for example, in §6.2.

Now g^Δ is not a metric for every set of edge lengths we could choose. A necessary and sufficient condition for g^Δ to yield a metric is that there is a nondegenerate Euclidean simplex σ with edge lengths given by $\{\ell_{ij}\}$.

In fact, we will assume our triangulations satisfy a slighter stronger condition, namely, every Euclidean simplex (Δ, g^Δ) must be (ϑ, ϵ)-full, according to the following definition.

Definition 14. *A n-simplex σ with Riemannian metric g is* (ϑ, ϵ)-full *if all edges have length less than or equal to ϵ and*

$$n! vol_g(\sigma) \geq \vartheta \epsilon^n,$$

where $vol_g(\sigma)$ is the Riemannian volume.

*If T is a triangulation such that every simplex in T is (ϑ, ϵ)-full, then we say that T is a (ϑ, ϵ)-*full triangulation*.*

In the following theorem, n is the dimension of the simplex while C_0, C_1 are constants depending on the curvature of the manifold M. More explicitly, $C_0 = \|R\|_\infty$ is the supremum over the manifold M of the usual pointwise 2-norm of the Riemannian curvature tensor and $C_1 = \|\nabla R\|_\infty$ is the supremum over M of the pointwise 2-norm of the covariant derivative of the Riemannian curvature tensor. See [DGW16] for more discussion and a proof.

Theorem 15 (Theorem 2 of [DGW16])**.** *There exist constants $\alpha = \alpha(n, \vartheta, C_0, C_1)$, $\beta = \beta(n, \vartheta, C_0)$, and $\gamma = \gamma(n, \vartheta, C_0, C_1)$ such that if $\epsilon < \alpha$ and (Δ, g^Δ) is a (ϑ, ϵ)-full simplex then*

$$|(\Psi^* g - g^\Delta)(v, w)| \leq \beta \epsilon^2 |v|_{g^\Delta} |w|_{g^\Delta},$$

and

$$|\nabla^e \Psi^* g(u, v, w)| \leq \gamma \epsilon |u|_{g^\Delta} |v|_{g^\Delta} |w|_{g^\Delta}$$

for tangent vectors $u, v, w \in T_\lambda \Delta$ at any $\lambda \in \Delta$.

The following is a direct consequence of Theorem 15.

Theorem 16. *Let (M, g) be a Riemannian manifold and let (Δ, g^Δ) be a (ϑ, ϵ)-full simplex, where ϵ is small enough so that the Riemannian barycentric map $\Psi : \Delta \to M$ is well-defined. Then there exists a constant β, depending on the dimension and Riemannian curvature of M, such that*

$$(1 - \beta\epsilon)\,|X|^2_{g^\Delta} \leq |X|^2_{\Psi^* g} \leq (1 + \beta\epsilon)\,|X|^2_{g^\Delta} \tag{3.7}$$

for $X \in T_\lambda \Delta$ and

$$(1 - \beta\epsilon)|X|^2_g \leq |X|^2_{(\Psi^{-1})^* g^\Delta} \leq (1 + \beta\epsilon)|X|^2_g \tag{3.8}$$

for $X \in T_p M$ for every λ in Δ and every p in $\Psi(\Delta)$.

Proof. Let λ be a point in Δ and let $X \in T_\lambda \Delta$. Then by Theorem 15,

$$\left|\left(\Psi^* g - g^\Delta\right)(X, X)\right| \leq \beta \epsilon^2 |X|^2_{g^\Delta}.$$

We can rewrite this inequality as

$$(1 - \beta\epsilon^2)|X|^2_{g^\Delta} \leq |X|^2_{\Psi^* g} \leq (1 + \beta\epsilon^2)|X|^2_{g^\Delta}. \tag{3.9}$$

which immediately gives us (3.7) since $1 - \beta\epsilon \leq 1 - \beta\epsilon^2$ and $1 + \beta\epsilon^2 \leq 1 + \beta\epsilon$ for small ϵ.

Next, let $X \in T_p M$ where $p \in \Psi(\Delta)$. Since Ψ and Ψ^{-1} are diffeomorphisms on Δ and $\Psi(\Delta)$ respectively, there is some $Y \in T_{\Psi^{-1}(p)} \Delta$ such that $X = \Psi_* Y$.

By the left hand inequality of (3.9),

$$\begin{aligned} |Y|^2_{g^\Delta} &\le \frac{|Y|^2_{\Psi^* g}}{1-\beta\epsilon^2} \\ &\le |Y|^2_{\Psi^* g}(1+\beta\epsilon), \end{aligned}$$

since $1/(1-\beta\epsilon^2) \le 1+\beta\epsilon$ for small ϵ.

Similarly, the right hand inequality of (3.9) gives

$$\begin{aligned} |Y|^2_{g^\Delta} &\ge \frac{|Y|^2_{\Psi^* g}}{1+\beta\epsilon^2} \\ &\ge |Y|^2_{\Psi^* g}(1-\beta\epsilon). \end{aligned}$$

Now since $Y = (\Psi^{-1})_* X$, we can replace Y in the previous inequalities, combining them both to get

$$\left|(\Psi^{-1})_* X\right|^2_{\Psi^* g}(1-\beta\epsilon) \le \left|(\Psi^{-1})_* X\right|^2_{g^\Delta} \le \left|(\Psi^{-1})_* X\right|^2_{\Psi^* g}(1+\beta\epsilon),$$

which simplifies to

$$|X|^2_g\,(1-\beta\epsilon) \le |X|^2_{(\Psi^{-1})^* g^\Delta} \le |X|^2_g(1+\beta\epsilon),$$

as required.

□

Chapter 4

The Ring Lemma

A key step in the proof of the Rodin-Sullivan Theorem (Theorem 82 in the Appendix) is an invocation of the Ring Lemma, which is a result that gives a bound on the ratio of radii of neighboring circles in a univalent circle packing. In the Rodin-Sullivan Theorem this result establishes quasiconformality of the circle packing maps. There is an analogous result in the context of vertex scaling, and it is used in much the same way as the circle packing ring lemma.

In our main result, we assume our sequences of triangulations satisfy a version of the ring lemma. This allows us to maintain control on ratios of exponentials of discrete conformal factors at neighboring vertices, which comes into play at a couple of key points.

In this chapter we discuss and prove the circle packing and vertex scaling ring lemmas, as well as stating our (conjectured) generalized ring lemma. For this chapter, we work in two dimensions.

4.1 Circle Packing Ring Lemma

Let c be a circle in a circle packing. The *flower* centered at c is the closed set consisting of c and its interior, all circles tangent to c and their interiors, and the interiors of all the triangular interstices formed by these circles.

The statement of the Ring Lemma is as follows:

Lemma 17 (Lemma 8.2 of [Ste05])**.** *For each integer $k \geq 3$ there exists a constant $\mathfrak{c}(k) > 0$ such that if F is any univalent k-flower of circles in the Euclidean or hyperbolic plane having a central circle of radius r_0, then the radius r of each petal satisfies $r \geq \mathfrak{c}(k) r_0$.*

Proof. Let F be a univalent k-flower in the (Euclidean or hyperbolic) plane, scaled so that the central circle has radius $r_0 = 1$.

First note that if all k petals have the same radius r, then there is a unique r for which the tangency pattern of the flower is preserved. This r is easy to calculate, but we do not need a specific value.

If one petal in the flower shrinks, another has to grow to compensate, because otherwise the petals will not close up around the central circle. Thus it is impossible for every petal to have a radius smaller than r, the radius they would all have if their radii were equal. Hence at least one petal must have radius larger than r. Choose

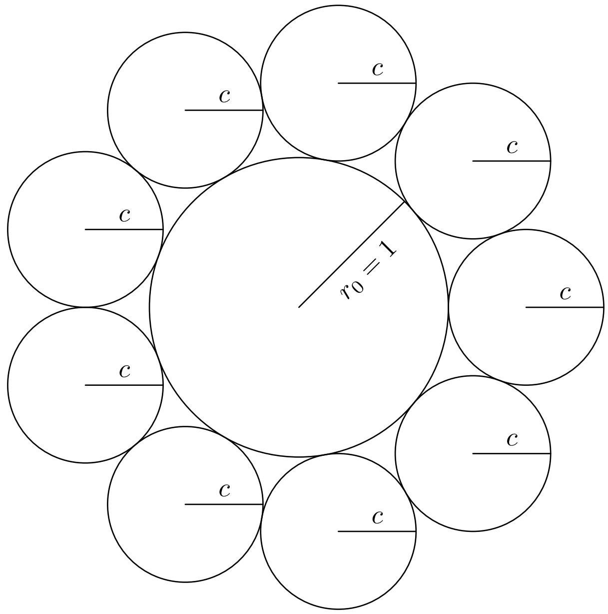

FIGURE 4.1. A flower with equal petal radii

one such large petal and label it P_1 and its radius r_1. Label the rest of the petals clockwise as $P_2, \ldots, P_k$ with their corresponding radii $r_2, \ldots, r_k$.

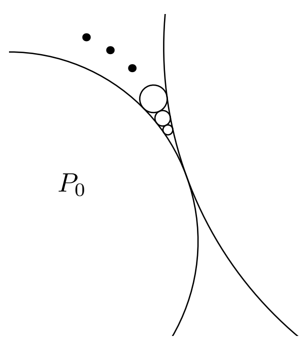

FIGURE 4.2. Circles drawn into the crevasse between P_0 and P_1

Next we examine P_2 and show that its radius r_2 is bounded below. For if not, then P_2 is drawn deeply into the crevasse between the central circle P_0 and the petal P_1. With P_2 deep in the crevasse, subsequent petals are drawn in as well and if r_2 is very small, then the chain of petals will never escape the crevasse and close up around the central circle. Hence there must be some lower bound for r_2.

Now we look at the crevasse between P_2 and P_0 and repeat the argument of the previous paragraph on P_3 to find that r_3 must also be bounded below. We continue around the flower discovering that there are lower bounds on each radius r_i. Since there are finitely many petals, there must be some positive number $\mathfrak{c}(k)$ which serves as a lower bound for each petal radius.

Finally, we scale the entire flower so that the central radius is r_0 and we are done. □

4.2 The Vertex Scaling Ring Lemma

The vertex scaling version of the ring lemma can be stated as follows. Note that the notation has been slightly changed from the original to better fit with the rest of this dissertation.

Recall that a *star-shaped set* $\Omega \subset \mathbb{R}^n$ is a set in $\mathbb{R}^n$ such that there is some point $s \in \Omega$ for which the geodesic segment connecting s to any other point $p \in \Omega$ is entirely contained in Ω. This is a slightly weaker notion than convexity.

Definition 18. *Let v_0 be an interior point of a star-shaped polygon P_n with n vertices $v_1, \dots, v_n$ labeled cyclically. The triangulation $\mathcal{T}$ of P_n with vertices $v_0, \dots, v_n$ and triangles $[v_0, v_i, v_{i+1}]$, one for each i with v_{n+1} taken to be v_1 is called a* star triangulation *of P_n.*

Lemma 19 (Lemma 3.3 of [LSW20])**.** *Let $(P_n, \mathcal{T})$ be a star triangulation of an N-gon with boundary vertices $v_1, \dots, v_N$ labeled cyclically and one interior vertex v_0 and $L : E(\mathcal{T}) \to \mathbb{R}_{>0}$ be a flat generalized PL metric on $\mathcal{T}$. There is a constant λ depending on L such that if $(P_N, \mathcal{T}, w * L)$ with $w : \{v_0, \dots, v_N\} \to \mathbb{R}$ is a generalized PL metric with zero curvature at v_0, then the ratio of edge lengths satisfies*

$$\frac{w * L(e_{i0})}{w * L(e_{i(i+1)})} \leq \lambda$$

for all indices i.

Proof. For ease of notation, let $L_{ij} := L(e_{ij})$ be the length (in L) of edge e_{ij} and define ℓ_{ij} by

$$\ell_{ij} := e^{w_i + w_j} L_{ij} = w * L(e_{ij}).$$

The key insight in this proof is that $e^{w_0}/e^{w_{i+1}}$ can be written in two ways:

$$\begin{aligned} \frac{e^{w_0}}{e^{w_{i+1}}} &= \frac{\ell_{0i}}{\ell_{i(i+1)}} \frac{L_{i(i+1)}}{L_{0i}} \\ &= \frac{\ell_{0(i+2)}}{\ell_{(i+1)(i+2)}} \frac{L_{(i+1)(i+2)}}{L_{0(i+2)}}. \end{aligned}$$

The triangle inequality tells us that

$$\ell_{0(i+2)} \leq \ell_{0(i+1)} + \ell_{(i+1)(i+2)}.$$

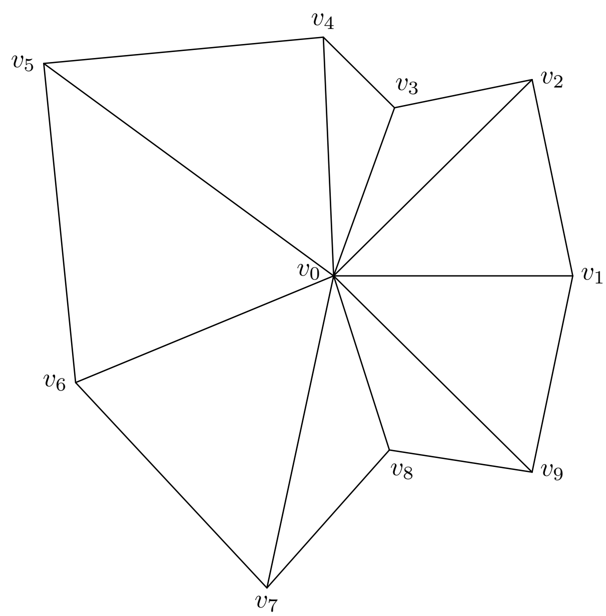

FIGURE 4.3. A star triangulation

We rearrange this inequality and divide it by $\ell_{(i+1)(i+2)}$, then use the above insight to get the following:

$$\begin{aligned}\frac{\ell_{0(i+1)}}{\ell_{(i+1)(i+2)}} &\geq \frac{\ell_{0(i+2)} - \ell_{(i+1)(i+2)}}{\ell_{(i+1)(i+2)}} \\ &= \frac{L_{0(i+2)}}{L_{(i+1)(i+2)}} \frac{e^{w_0}}{e^{w_{i+1}}} - 1 \\ &= \frac{L_{0(i+2)} L_{i(i+1)}}{L_{(i+1)(i+2)} L_{0i}} \frac{\ell_{0i}}{\ell_{i(i+1)}} - 1\end{aligned}$$

Now the L_{ij}'s are fixed, so we can define

$$m_i := \frac{L_{0(i+2)} L_{i(i+1)}}{L_{(i+1)(i+2)} L_{0i}}.$$

Without loss of generality, let $i = 1$ and assume towards a contradiction that there exists some sequence of discrete conformal factors $w^{(n)}$ such that

$$\frac{\ell_{01}(w^{(n)})}{\ell_{12}(w^{(n)})} \to \infty$$

as $n \to \infty$.

Then from the inequality above we see that

$$\frac{\ell_{02}}{\ell_{23}} \geq m_1 \frac{\ell_{01}}{\ell_{12}} - 1$$

and so $\frac{\ell_{02}(w^{(n)})}{\ell_{23}(w^{(n)})} \to \infty$ as well. Hence inductively, $\frac{\ell_{0j}(w^{(n)})}{\ell_{j(j+1)}(w^{(n)})} \to \infty$ for every $j = 1, 2, \ldots, N$.

But this means that in every triangle $[v_0, v_j, v_{j+1}]$, the angle $\gamma_{0,j(j+1)}$ at v_0 goes to zero as $n \to \infty$, which is a contradiction since the curvature at v_0 was assumed to be zero.

Thus there must be some upper bound on the ratio $\frac{\ell_{01}}{\ell_{12}}$ and further, since the same argument works when $i \neq 1$, we have an upper bound on $\frac{\ell_{0i}}{\ell_{i(i+1)}}$. □

4.3 Ring Lemma Assumption

The proofs of the circle packing and vertex scaling ring lemmas are very different, both in substance and in flavor. The circle packing ring lemma is entirely qualitative and uses only univalence (that is, that discs have disjoint interiors) while the vertex scaling ring lemma relies on the triangle inequality and some fairly careful algebra. In the circle packing case, the triangle inequality is automatically satisfied while in the vertex scaling case, there is no analogue to the idea of univalence.

These major differences suggest that even if a general ring lemma (like the following conjecture) is true, its proof may be elusive.

Conjecture 20. *Let (M, T, ℓ) be a piecewise flat surface in the discrete conformal structure $C_{\alpha,\eta}$ and let v_0 be a vertex in T with N neighbors. Then there is a constant $\lambda \in \mathbb{R}_{>0}$ depending on α, η, and N such that for every v_i neighboring v_0, $\frac{e^{f_i}}{e^{f_0}} \geq \lambda$.*

In what follows, we will usually take as an assumption that the above conjecture is true.

Chapter 5
Local Discrete Conformal Rigidity

Local Discrete Conformal Rigidity (LDCR) has its roots in the Hexagonal Packing Lemma, Lemma 85 in the Appendix. In this chapter, we state the Hexagonal Packing Lemma in the circle packing and vertex scaling cases and give a nearly complete proof in the circle packing case. This proof is a fairly illustrative example of a relatively standard type of diagonalization argument.

We also state a more general version of the Hexagonal Packing Lemma as a condition that we require in many of our main results. This more general condition does not rely on hexagonal combinatorics and it has the flavor of a condition on local rigidity, so we term this condition Local Discrete Conformal Rigidity (LDCR) rather than something like “The Generalized Hexagonal Packing Condition”.

5.1 The Circle Packing Hexagonal Packing Lemma

We take the statement (and proof) from [Ste05].

Theorem 21 (The Hexagonal Packing Lemma)**.** *There is a sequence $\{s_n\}$, decreasing to zero, with the following property. Let c_0 be a circle in a univalent Euclidean circle packing P and suppose the first n generations of circles about c_0 are combinatorially equivalent to n generations of the regular hexagonal packing around one of its circles. Then for any circle $c \in P$ tangent to c_0,*

$$\left| 1 - \frac{radius(c)}{radius(c_0)} \right| \leq s_n.$$

Proof. Consider a sequence $\{P_n\}$ of circle packings combinatorially equivalent to n generations of the regular hexagonal packing and assume further that each of these packings is translated and scaled so that the central circle c_0 has radius 1 and has its center at the origin.

By the Ring Lemma, the ratio of radii of two adjacent circles in any of these packings is bounded below by some constant k_6.

This means a few things. First, since each interior circle in a hexagonal packing has six neighbors, the bound given by the Ring Lemma is symmetric in the sense that if two neighboring circles c_v, c_w have radii r_v, r_w, then both of the ratios r_v/r_w and r_w/r_v are greater than k_6 and hence not only do we have a lower bound on the ratio of any two neighboring circles, but by using the lower bound on the reciprocal, we get an upper bound as well:

$$\frac{r_v}{r_w} > k_6 \qquad \Longleftrightarrow \qquad \frac{r_w}{r_v} < \frac{1}{k_6}.$$

We can also put upper and lower bounds on the ratios of radii of circles that are not necessarily adjacent, but belong to the same flower. Let c_0 be the center of a 6-flower with petals $c_1, c_2, \ldots, c_6$. Then for any $i, j \in \{1, 2, \ldots, 6\}$, we have that $r_i/r_0 > k_6$ and also that $r_0/r_j > k_6$. Hence we see that

$$\frac{r_i}{r_j} = \frac{r_i}{r_0}\frac{r_0}{r_j} > k_6^2$$

and for the same reason $r_j/r_i > k_6^2$. From these two inequalities we also get the corresponding upper bounds

$$\frac{r_i}{r_j} < \frac{1}{k_6^2} \qquad \text{and} \qquad \frac{r_j}{r_i} < \frac{1}{k_6^2}.$$

The next step is to use a diagonalization argument to show that there is a subsequence of packings $\{P_{n_k}\} \subset \{P_n\}$ that converges to the (infinite) regular hexagonal packing H. In order to show this convergence, we first have to be clear about exactly what a packing is and what it means for a sequence of packings to converge to some limit packing.

A circle packing P consists of two types of data, combinatorial and geometric. The combinatorial data is a simplicial complex K: a list of vertices, edges, and faces such that the boundary of any simplex is a union of simplices (of one dimension lower.) The vertices correspond to circles in the packing, each edge corresponds to a pair of tangent circles, and each face corresponds to a triple of mutually tangent circles. The geometric data is a list of radii R, one for each vertex in the complex K. These radii are subject to a "planarity" condition, which is most easily stated by saying that the sum of the angles around a vertex must be 2π. In other words, the discrete curvature at the vertex is 0. This ensures that the circles can actually be laid out (in the plane) in such a way that they are mutually disjoint and follow the prescribed tangency pattern.

So when we say a sequence of packings $\{P_n\}$ converges to a limit packing P, we have to talk about both combinatorial and geometric convergence.

We say that a sequence of complexes K_n *converges combinatorially* to the limit complex K if for every subcomplex $L \subset K$ which contains the vertex corresponding to the central circle c_0, there is some $N \in \mathbb{N}$ such that for every $n > N$, K_n has a subcomplex isomorphic to L. Intuitively, K_n and K "match" on ever-larger pieces of K as n goes to infinity. Since this combinatorial convergence is clear in our setting, we can say that $K_n \to K$ combinatorially.

Next we look at radii. We have already scaled each of our packings so that the central circle c_0 has radius 1 in all of them. We can further impose an order on the circles in each of our packings and index them by i such that circle c_i is labeled consistently in each packing that contains it.

Now we look at circle c_1. This circle is contained in the packing P_n for every $n \geq 1$, so we can look at the set of radii $\{r_1^{(n)}\}$ of c_1 in each P_n. We know that this set is bounded away from 0 and ∞ by the Ring Lemma argument above, so we have a set of real numbers contained in a bounded set. Hence there is a subsequence $\{r_1^{(n_{k_1})}\} \subset \{r_1^{(n)}\}$ converging to some number r_1. Let $\{P_{n_{k_1}}\} \subset \{P_n\}$ denote the subsequence of packings corresponding to the subsequence $\{r_1^{(n_{k_1})}\}$.

Next look at circle c_2. Let $\{r_2^{(n_{k_1})}\}$ be the sequence of radii of circle c_2 in the subsequence of packings $\{P_{n_{k_1}}\}$ constructed in the previous paragraph. Then again by the Ring Lemma argument above, $\{r_2^{(n_{k_1})}\}$ is contained in a bounded subset of $\mathbb{R}$ and hence there is some subsequence $\{r_2^{(n_{k_2})}\} \subset \{r_1^{(n_{k_1})}\}$ that converges to a number r_2. Again, let $\{P_{n_{k_2}}\}$ denote the subsequence of packings corresponding to the subsequence of radii $\{r_2^{(n_{k_2})}\}$.

We repeat this with every circle c_i, finding at every step that the radius of c_i is bounded away from 0 and ∞ and hence at every step we can pull out a subsequence of packings $\{P_{n_{k_i}}\}$ such that the sequence of radii of each circle (with index less than i) converges.

Finally, we choose our final convergent subsequence $\{P_{n_k}\}$ by diagonalization. We take the first packing in $\{P_{n_{k_1}}\}$, the second in $\{P_{n_{k_2}}\}$, the third in $\{P_{n_{k_3}}\}$ and so on. Then it is clear that at the level of combinatorics, $K_{n_k} \to K$. To show that the radii converge, it is enough to note that $\{P_{n_{k_j}}\} \subset \{P_{n_{k_i}}\}$ for $i < j$. In other words, later subsequences are subsequences of earlier ones, so, for example, the sequence of radii for circle c_1 corresponding to our final diagonalized sequence of packings, is a subsequence of the convergent (by construction) subsequence $\{r_1^{(n_{k_1})}\}$.

Hence we have a subsequence of packings that converges to a packing P which is combinatorially equivalent to the regular hexagonal packing in the plane. But the regular hexagonal packing is unique in the sense that only one set of radii satisfies the planarity condition, and that is the packing where every radius is equal. See Appendix 1 of [RS87] or §8.3 of [Ste05] for a proof of this surprisingly hard theorem.

To finish out the proof, let c be a circle in the flower around the central circle c_0 and let $r_c^{(n)}$ denote the radii of c in the packing P_n. Now the sequence of radii $\{r_c^{(n)}\}$ is bounded (by the Ring Lemma), so there are two possibilities. The first possibility is that this sequence of radii converges. In this case, every subsequence converges to 1 since we showed above that the subsequence of packings $\{P_{n_k}\}$ converges to the regular hexagonal packing. The second possibility is that the sequence of radii $\{r_c^{(n)}\}$ does not converge. If this is the case, then there must be subsequences that converge to different values. That is, there is at least one subsequence $\{r_c^{(n_j)}\} \subset \{r_c^{(n)}\}$ such that $r_c^{(n_j)} \to x$ for some $x \neq 1$. But this is impossible because this subsequence of radii $r_c^{(n_j)}$ corresponds to a subsequence of packings $\{P_{n_j}\}$ converging to the regular hexagonal packing and the regular hexagonal packing has all radii equal. Thus the sequence $\{r_c^{(n)}\}$ converges, as required. □

5.1.1 Generalizing to Packings Without Hexagonal Combinatorics

It is possible to weaken the hypotheses of the Hexagonal Packing Lemma in such a way that hexagonal combinatorics are no longer necessary. The proof of the Hexagonal Packing Lemma requires two things: first, we can use the Ring Lemma to bound radii of neighboring circles and second, the K_n's approach a unique limit packing. The Ring Lemma can be used to bound radii of neighboring circles on the condition that the degree of K_n is uniformly bounded and this bound does not depend on n. To show that the limit packing is unique, it suffices to show merely the existence of a limit packing since by the uniformization theorem, Theorem 89 in §B.2 of the Appendix, the maximal packing of a surface is unique. The following lemma assumes the existence of a limit packing in a slightly obfuscated form which we now briefly discuss.

Let K be an infinite complex that admits an exhaustion by a nested sequence of finite simply connected complexes K_j. Assume further that there is a distinguished vertex $v \in K_1$. Note that since the sequence $\{K_j\}$ is nested, $v \in K_j$ for all j. For each K_j, take $\mathcal{P}_j$ to be the maximal packing of the unit disc $\mathbb{D}$ with complex K_j. The existence of this packing is guaranteed by the hyperbolic version of the circle packing uniformization theorem, Proposition 6.1 of [Ste05], reproduced in the Appendix as Theorem 87.

It turns out that there are two options for the sequence of radii $\{R_j(v)\}$ corresponding to the distinguished vertex v. Either $R_j(v)$ decreases to zero as j goes to infinity or $R_j(v)$ decreases to some positive number R as $j \to \infty$. Surprisingly, the outcome is independent of the vertex v and the specific choice of exhausting sequence $\{K_j\}$, depending only on the infinite complex K. For a proof of this fact, see §8.1.3 of [Ste05].

Complexes K such that $R_j(v) \to 0$ are called *parabolic* and those for which $R_j(v) \to R > 0$ are called *hyperbolic*. Parabolic complexes have univalent packings in the plane (and none in the disc $\mathbb{D}$) while hyperbolic complexes have univalent packings in $\mathbb{D}$ (and none in the plane).

The vertex v in the discussion above is what is termed by Stephenson in [Ste05] an *α-vertex*. That is, an α-vertex is a distinguished vertex shared by all complexes under consideration.

Definition 22 (Definition 11.16 of [Ste05])**.** *A sequence $\{K_n\}$ of simply connected complexes is termed* asymptotically parabolic *if it is the case that whenever a subsequence $\{K_{n_j}\}$, regardless of specified α-vertices, converges to a combinatorial open disc K (that is, an infinite and simply connected complex), then K is parabolic.*

With the above definitions, we can now state the following lemma, which is a generalization of the Hexagonal Packing Lemma to circle packings with non-hexagonal combinatorics.

Lemma 23 (Lemma 20.3 of [Ste05]). *Let $\{K_n\}$ be a sequence of combinatorial closed discs (that is, finite and simply connected complexes) such that there exists a uniform bound d on deg(K_n) and the sequence $\{K_n\}$ is either a nested sequence which exhausts a parabolic combinatorial disc or is asymptotically parabolic. Then there exists a sequence $\{s_m\}$ of constants, decreasing to zero, with the following property: Suppose that for some $n, v \sim u$ are interior vertices of K_n whose combinatorial distances from ∂K_n are at least m and suppose that $P(R)$ and $\widetilde{P}(\widetilde{R})$ are two univalent, Euclidean circle packings for K_n. Then*

$$\left| \frac{\widetilde{R}(u)}{\widetilde{R}(v)} - \frac{R(u)}{R(v)} \right| \leq s_m.$$

In the above lemma, all that has changed is that now the domain circles do not necessarily have equal radii, so we need to compare the ratio of radii of image circles to the ratio of radii of domain circles, rather than simply comparing to unity. The proof of this lemma is basically identical to the standard hexagonal packing lemma.

There is one last version of this lemma, found in [HR93]. It is notationally a little more involved than the previous formulations, but the payoff is that this version of the lemma estimates how quickly the constants s_n go to zero. In fact, since the below lemma applies to packings with bounded degree, it is valid in the particular case of hexagonal combinatorics and so the hexagonal packing constants used in the proof of the Rodin-Sullivan theorem turn out to be of order $s_n = O(1/n)$. See also [He91], which shows that in the hexagonal case, $s_n \leq \alpha/n$ for some $\alpha > 0$ and, further, $s_n \geq 4/n$, so the estimate $s_n \leq \alpha/n$ is the best possible. It turns out that this estimate, $s_n \leq \alpha/n$ for some positive α, is sufficient to show that the ratio of radii function $r_\epsilon(z)$ to approach $|f'(z)|$, the modulus of the derivative of the Riemann map, as $\epsilon \to 0$, as we show in §B.3 of the Appendix.

Let n be an integer ≥ 2 and let P_n be a circle packing in $\mathbb{C}$ such that

1. The valence of P_n is bounded by k_0,
2. The radii of the circles of P_n are all bounded above by some positive r, and
3. there is some "center" circle c_0 of P_n such that the carrier of P_n contains a closed disc of radius $(2n+1)r$ which is concentric with c_0.

Let P'_n be any other circle packing in $\mathbb{C}$ combinatorially equivalent to P_n. Suppose that c_0 is surrounded by circles $c_1, c_2, \ldots, c_k$ in P_n and that $c'_0, c'_1, \ldots, c'_k$ are the corresponding circles in P'_n. Let

$$d_1(P_n, P'_n) := \max \left\{ \frac{\text{radius}(c'_j)/\text{radius}(c'_l)}{\text{radius}(c_j)/\text{radius}(c_l)} \ : \ 0 \leq j, l \leq k \right\},$$

and let

$$s(P_n) := \sup_{P'_n} \left[d_1(P_n, P'_n) - 1 \right]. \tag{5.1}$$

Theorem 24 (Theorem 2.2 of [HR93])**.** *Let P_n be a circle packing such that the valence of P_n is bounded by k_0, the radii of circles in P_n are bounded above by some positive r, and there is some circle c_0 of P_n such that the carrier of P_n contains a closed disc of radius $(2n+1)r$ concentric with c_0. Then there is a constant C depending only on k_0 such that $s(P_n) \leq C/n$.*

There are a few interesting differences between this version of a generalized hexagonal packing lemma and Lemma 23. For one, Lemma 23 associates the constants s_m with a combinatorial distance, while He-Rodin here associate $s(P_n)$ with a particular packing P_n.

A second difference is that if we translate the He-Rodin version, (5.1), to notation more closely matching that of Lemma 23, we find that there is an extra ratio of radii on the outside of the inequality.

Let j, l be arbitrarily chosen between 0 and k and change the notation so that now $R(j) := \text{radius}(c_j)$ and $\widetilde{R}(j) := \text{radius}(c'_j)$. Switching the roles of j and l if necessary, we can assume without loss of generality that

$$\frac{\widetilde{R}(j)/\widetilde{R}(l)}{R(j)/R(l)} \geq 1.$$

Hence

$$\begin{aligned} s(P_n) &\geq |d_1(P_n, P'_n) - 1| \\ &\geq \left| \frac{\widetilde{R}(j)/\widetilde{R}(l)}{R(j)/R(l)} - 1 \right| \\ &= \left| \frac{\widetilde{R}(j)}{R(j)} \frac{R(l)}{\widetilde{R}(l)} - 1 \right|. \end{aligned}$$

Multiply through by $R(j)/R(l)$ to get

$$\left| \frac{\widetilde{R}(j)}{\widetilde{R}(l)} - \frac{R(j)}{R(l)} \right| \leq S(P_n) \frac{R(j)}{R(l)}.$$

However, since both these versions are dealing with packings with bounded degree, the ring lemma gives us uniform bounds on ratios $R(j)/R(l)$, so these two statements are, in fact, equivalent.

5.2 The Vertex Scaling Hexagonal Packing Lemma

There is also a vertex scaling version of the hexagonal packing lemma. Before we can state it, we need a few more definitions.

Let L be a lattice in the complex plane $\mathbb{C}$. Then there exists a Delaunay triangulation $\mathcal{T}_{st} = \mathcal{T}_{st}(L)$ of $\mathbb{C}$ with vertex set L such that $\mathcal{T}_{st}$ is invariant under the translation action of L.

Given the triangulation $\mathcal{T}_{st}$, we can define certain subcomplexes $\mathcal{B}_m(v)$ as follows: Let $B_m(v) = \{i \in V(\mathcal{T}_{st}) : d_c(i, v) \leq m\}$, where the distance $d_c(i, v)$ is the combinatorial distance between the vertices i and v. Note that $B_m(v)$ is a set of vertices. Let $\mathcal{B}_m(v)$ be the subcomplex of $\mathcal{T}_{st}$ whose simplices have vertices contained in $B_m(v)$.

We also need to define the phrase "embeddable development map", which Luo-Sun-Wu do in the following two definitions:

Definition 25 (Bottom of p 11 of [LSW20])**.** *If $(S, \mathcal{T}, l)$ is a flat generalized PL metric on a simply connected surface S, then a* developing map $\phi : (S, \mathcal{T}, l) \to \mathbb{C}$ *for $(\mathcal{T}, l)$ is an isometric immersion determined by $|\phi(v) - \phi(v')| = l(vv')$ for $v \sim v'$.*

Definition 26 (Definition 4.1 in [LSW20])**.** *A flat generalized PL metric on a simply connected surface $(X, \mathcal{T}, l)$ with developing map ϕ is said to be* embeddable *into $\mathbb{C}$ if for every simply connected finite subcomplex P of $\mathcal{T}$, there exists a sequence of flat PL metrics on P whose developing maps ϕ_n converge uniformly to $\phi|_P$ and $\phi_n : P \to \mathbb{C}$ is an embedding.*

Now we can state the vertex scaling version of the hexagonal packing lemma:

Lemma 27 (Lemma 4.7 of [LSW20])**.** *Take a standard hexagonal lattice $V = \mathbb{Z} + e^{2\pi/3}\mathbb{Z}$ and its associated standard hexagonal triangulation whose edge length function is $L : E \to \{1\}$. There is a sequence s_n of positive numbers decreasing to zero with the following property. For any integer n and vertex v, there exists $N = N(n, v)$ such that if $m \geq N$ and $(\mathcal{B}_m(v), w * L)$ is a flat Delaunay triangulated PL surface with embeddable developing map, then the ratio of the lengths of any to edges sharing a vertex in $\mathcal{B}_m(v)$ is at most $1 + s_n$.*

The proof of this lemma is more or less identical to that of the circle packing hexagonal packing lemma. We again use the (vertex scaling version of the) Ring Lemma to bound edge lengths away from 0 and ∞, giving us a convergent subsequence of vertex scaling metrics and then we use a uniqueness result (Theorem 4.3 of [LSW20]) to say that this convergent subsequence must converge to a known triangulation and hence the ratio of edge lengths must converge to 1.

Note that the vertex scaling hexagonal packing lemma is stated in terms of edge lengths while the circle packing hexagonal packing lemma is about radii. We saw the exact same thing in the statements and proofs of the Ring Lemma for circle packing and vertex scaling. For a general discrete conformal structure, we want bounds on ratios of the form e^{f_i}/e^{f_j}.

5.3 Local Discrete Conformal Rigidity Condition

For the rest of this section, let $\{T_n\}$ be a sequence of triangulations, $\{\mathcal{C}_n\}$ a sequence of discrete conformal structures on $\{T_n\}$, and $\{f_n\}$, $\{\overline{f}_n\}$ sequences of discrete conformal factors in the domain of $\{\mathcal{C}_n\}$. We also need the following definitions about combinatorial closed discs.

Definition 28. *A* combinatorial closed disc *is an abstract simplicial complex that triangulates a topological closed disc. That is, a complex that is finite, connected, simply connected, and has nonempty boundary.*

The above definition of combinatorial closed disc is slightly too general for our purposes, so we make the following more specific definition.

Definition 29. *A* combinatorial closed disc of generation m *is a combinatorial closed disc D_m with the property that there exists a vertex $v_0 \in D_m$ such that for every boundary vertex $w \in \partial D_m$, the combinatorial distance $d_{comb}(v_0, w) = m$. The vertex v_0 is referred to as the* center *of the combinatorial closed disc, or the disc can be said to be* centered at v_0.

We are usually not interested in purely abstract simplicial complexes, but instead simplicial complexes with assigned edge lengths. That is, we work with triangulations. When a subset of a triangulation is (combinatorially) isomorphic to a combinatorial closed disc, then we will call that subset a *realized combinatorial closed disc.*

Condition 30 (Local Discrete Conformal Rigidity)**.** *A sequence $\{(T_n, \mathcal{C}_n, f_n, \overline{f}_n)\}_{n=1}^{\infty}$ is said to* satisfy Local Discrete Conformal Rigidity *(LDCR) if there is a sequence of real numbers $\{s_m\}$ decreasing to zero such that for any vertex $v \in T_n$ that is the center of a realized combinatorial closed disc of generation m there exists a number $\mathcal{N}_m \in \mathbb{N}$ such that if $n \geq \mathcal{N}_m$ and w is a vertex adjacent to v then*

$$\left| \frac{e^{f_n(v)}}{e^{f_n(w)}} \frac{e^{\overline{f}_n(w)}}{e^{\overline{f}_n(v)}} - 1 \right| \leq s_m. \tag{5.2}$$

Note that we can always take a subsequence such that we can replace $\mathcal{N}_m$ with m.

We most frequently use the above assumption in the following context. Let M and N be diffeomorphic manifolds and let $M_n = (M, T_n, \mathcal{C}_n, f_n)$ and $N_n = (N, T_n, \mathcal{C}_n, \overline{f}_n)$ be piecewise flat manifolds with the same underlying triangulation and discrete conformal structure, but different discrete conformal factors and hence different edge lengths. Then we say that the sequence of pairs $\{(M_n, N_n)\}$ *satisfies LDCR* if $\{(T_n, \mathcal{C}_n, f_n, \overline{f}_n)\}$ satisfies $LDCR$.

We now prove an easy consequence of the LDCR condition. Define $H_n : V(T_n) \to \mathbb{R}$ by $H_n(v) = e^{\overline{f}_n(v)}/e^{f_n(v)}$. This function is a generalization of the ratio of radii function in circle packing, discussed in §B.3 of the Appendix.

Lemma 31. *Assume M, N are diffeomorphic manifolds and the sequences $M_n = \{M, T_n, \mathcal{C}_n, f_n\}$ and $N_n = \{N, T_n, C_n, f_n\}$ are such that $\{(M_n, N_n)\}$ satisfy Local Discrete Conformal Rigidity. Assume further that $v \sim w$ are adjacent vertices and each is the center of a realized combinatorial closed disc of generation at least m, where m is large enough so that $s_m \leq 1$. Then*

$$(1 - s_m)H_n(v) \leq H_n(w) \leq (1 + s_m)H_n(v) \tag{5.3}$$

and

$$(1 - 3s_m)H_n^2(v) \leq H_n^2(w) \leq (1 + 3s_m)H_n^2(v). \tag{5.4}$$

Proof. The first inequality is simply a rearrangement of (5.2). For the second inequality, we square (5.3) to get

$$(1 - s_m)^2 H_n^2(v) \leq H_n^2(w) \leq (1 + s_m)^2 H_n^2(v)$$

and then note that $1 - 3s_m \leq (1 - s_m)^2 \leq (1 + s_m)^2 \leq 1 + 3s_m$ for $s_m \leq 1$. □

Chapter 6

Convergence of Discrete Conformal Factors

6.1 Combinatorial closed discs and (ϑ, ϵ)-full triangulations

The next two lemmas allow us to relate combinatorial distances in realized combinatorial closed discs to geometric distances, at least for triangulations which are (ϑ, ϵ)-full and piecewise Euclidean. Recall that we say a simplex (σ, g) of dimension k with Riemannian metric g is (ϑ, ϵ)-*full* if every edge in σ has length less than or equal to ϵ and the (Riemannian) volume of σ satisfies

$$k\mathrm{vol}_g(\sigma) \geq \vartheta\epsilon^k.$$

A triangulation T is said to be (ϑ, ϵ)-full if each simplex in T is (ϑ, ϵ)-full.

Lemma 32. *Let Δ be a (ϑ, ϵ)-full Euclidean simplex of dimension k. Label the vertices of Δ as $\{v_0, v_1, \ldots, v_k\}$ and label the $(k-1)$-dimensional face opposite vertex v_i by $F_{k-1}(i)$. Let h_k be shortest distance between a vertex and the opposite face. That is,*

$$h_k = \min_i d(v_i, F_{k-1}(i)).$$

Then h_k satisfies

$$\vartheta\epsilon \leq h_k \leq \epsilon. \tag{6.1}$$

Proof. The longest edge in Δ being no larger than ϵ implies immediately that the height $h_k \leq \epsilon$.

For the other inequality, let $|F_{k-1}|$ denote the volume of the $(k-1)$-dimensional face whose distance from its opposite vertex is the smallest. Then the volume of the k-simplex Δ can be written as

$$\mathrm{vol}(\Delta) = \frac{1}{k}h_k|F_{k-1}|.$$

Since F_{k-1} is a $(k-1)$-simplex in its own right, we can write its volume as

$$|F_{k-1}| = \frac{1}{k-1}h_{k-1}|F_{k-2}|$$

and hence

$$\mathrm{vol}(\Delta) = \frac{1}{k(k-1)}h_k h_{k-1}|F_{k-2}|.$$

Proceeding in this manner, we find that

$$\mathrm{vol}(\Delta) = \frac{1}{k!} h_k h_{k-1} \dots h_2 |F_1|,$$

where $|F_1|$ is the length of an edge.

Since Δ was assumed (ϑ, ϵ)-full, $|F_1| \leq \epsilon$. Further, since the heights h_i are clearly no larger than the length of the longest side in their subsimplex, they too are such that $h_i \leq \epsilon$.

Since Δ was assumed (ϑ, ϵ)-full, $k!\mathrm{vol}(\Delta) \geq \vartheta\epsilon^k$, so

$$k!\mathrm{vol}(\Delta) = \left(\prod_{i=2}^{k} h_i \right) |F_1| \geq \vartheta\epsilon^k,$$

or, solving for h_k and using that $h_i \leq \epsilon$ and $|F_1| \leq \epsilon$:

$$h_k \geq \frac{\vartheta\epsilon^k}{|F_1| \prod_{i=2}^{k-1} h_i} \geq \frac{\vartheta\epsilon^k}{\epsilon^{k-1}} = \vartheta\epsilon.$$

□

Lemma 33. *Let D_m be a realized combinatorial closed disc of generation m centered at the vertex v such that every simplex in D_m is Euclidean and (ϑ, ϵ)-full. Then the distance $\delta = d_{g^\Delta}(v, \partial D_m)$ from v to the boundary of D_m satisfies*

$$\vartheta m\epsilon \leq \delta \leq m\epsilon. \tag{6.2}$$

Proof. Firstly, since every edge length in D_m is no greater than ϵ, clearly δ is no larger than ϵ multiplied by the generation. That is, $\delta \leq m\epsilon$.

We prove the left hand side of the inequality using induction.

When $m = 1$, D_1 consists of a central vertex v and a collection of simplices each sharing the vertex v. In this case, the distance from v to ∂D_1 is the length of a straight line connecting the vertex v to its opposing face in one of the simplices and hence by Lemma 32, $\delta \geq \vartheta\epsilon = \vartheta\epsilon m$ since $m = 1$.

Next assume that for every disc with $m-1$ generations the distance $\delta_{m-1} \geq \vartheta\epsilon(m-1)$. Take D_m to be a disc with m generations and let $D_{m-1} \subset D_m$ be the subcomplex of D_m with the same center but with $m-1$ generations.

Let γ be the shortest geodesic connecting ∂D_{m-1} and ∂D_m. Then there are two possibilities: either γ passes through a vertex of D_m or it does not. If it does, then γ is at least as long as a straight line connecting a vertex of a simplex to its opposite face, so the length $\ell(\gamma) \geq \vartheta\epsilon$ by Lemma 32.

If γ does not pass through any vertices, then since it is the shortest path between ∂D_{m-1} and ∂D_m, it must meet both ∂D_{m-1} and ∂D_m at right angles and hence

∂D_{m-1} and ∂D_m are parallel in some neighborhood around the endpoints of γ, and this neighborhood is determined by vertices in ∂D_{m-1} and ∂D_m. Hence we can translate γ until one of its endpoints is a vertex of either ∂D_{m-1} or ∂D_m and this translation will have the same length as the original. And so again, γ is at least as long as a straight line connecting a vertex to the opposing face in a (ϑ, ϵ)-full simplex and hence here too $\ell(\gamma) \geq \vartheta\epsilon$.

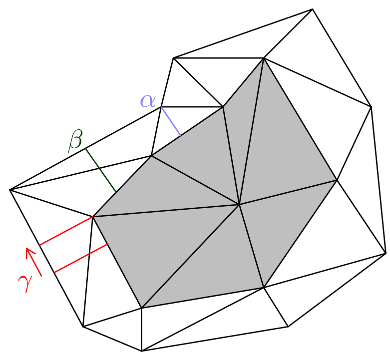

FIGURE 6.1. Geodesics connecting ∂D_m and ∂D_{m-1}.

Since the induction hypothesis was that D_{m-1} is such that the distance $\delta_{m-1} \geq \vartheta\epsilon(m-1)$, we have that δ_m satisfies

$$\delta_m \geq \delta_{m-1} + \ell(\gamma) \geq \vartheta\epsilon(m-1) + \vartheta\epsilon = \vartheta\epsilon m.$$

□

We use the next few lemmas to estimate the number of generations m of a closed combinatorial disc contained in a geodesic ball of radius R.

Lemma 34. *Let (M, g) be a Riemannian manifold and let $K \subset M$ be compact. Then there exists $R > 0$, depending only on K, such that for each point $p \in K$ there is a geodesic ball of radius R about p.*

Proof. For each point $x \in K$, there is a number $r_x \in \mathbb{R}_{>0}$ such that the ball $B_{2r_x}(x)$ of radius $2r_x$ about x is a convex geodesic ball. Cover K by balls of radius r_x about each x. Since K is compact there is a finite subcover $\{B_{r_i}(x_i)\}_{i=1}^N$, where $r_i := r_{x_i}$. Let $R := \min_i r_i$ be the minimum over i of the radii r_i.

Now take an arbitrary point $p \in K$. By assumption there is some i such that $p \in B_{r_i}(x_i)$. We will show that the ball $B_R(p)$ is contained in the convex geodesic ball $B_{2r_i}(x_i)$ and hence is itself geodesic. Let $y \in B_R(p)$. Then

$$\begin{aligned} d_g(y, x_i) &\leq d_g(y, p) + d_g(p, x_i) \\ &< R + r_i \\ &\leq 2r_i, \end{aligned}$$

where the last step follows since we have defined R as the minimum of the r_i's. So $d_g(y, x_i) < 2r_i$ and hence $y \in B_{2r_i}(x_i)$ for every $y \in B_R(p)$.

Thus $B_R(p) \subset B_{2r_i}(x_i)$ for some i for every p and hence there is a radius $R > 0$ such that for each point $p \in K$ the ball of radius R about p is a geodesic ball. □

Lemma 35. *Let (M, g) be a Riemannian manifold and let $K \subset M$ be a compact subset. Assume that M admits a triangulation T with geodesic edges such that the corresponding piecewise Euclidean manifold M_Δ is (ϑ, ϵ)-full, where $\epsilon \in (0, 1/(3\beta)]$ is small enough so that the Riemannian barycentric map $\Psi : M_\Delta \to M$ is well-defined. (The constant β is the same as in Theorem 15 and depends on the dimension and curvature of (M, g), as well as on ϑ.)*

Choose a vertex v such that v lies within the compact set K and let $R = R(K)$ be chosen as in Lemma 34. Let $D_m(v)$ be a realized closed combinatorial disc such that

1. *$D_m(v) \subset B_R(v)$, and*
2. *for any $q > m$, $D_q(v)$ has nonempty intersection with $M \setminus B_R(v)$.*

Then the following inequality holds:

$$m \geq \frac{R}{2\epsilon}.$$

Proof. Since we are assuming that $D_m(v)$ is such that no larger realized combinatorial closed disc $D_q(v)$ is completely contained in $B_R(v)$, we have in particular that $D_{m+1}(v)$ has nonempty intersection with the complement of $B_R(v)$. Take a point p in $D_{m+1}(v)$ such that p is not in $B_R(v)$ and let $a \in M_\Delta$ be the image of p under Ψ^{-1}. That is, $a := \Psi^{-1}(p)$.

Since Ψ and Ψ^{-1} are continuous everywhere and diffeomorphisms when restricted to (closed) simplices by Lemma 13, Proposition 72 in the Appendix tells us that

$$d_g(v, p) \leq \sqrt{1 + \beta\epsilon}\, d_{g^\Delta}(v, a) \leq (1 + \beta\epsilon) d_{g^\Delta}(v, a).$$

Since p is not in $B_R(v)$, the distance $d_g(v, p)$ must be no less than R, so

$$R \leq d_g(v, p) \leq d_{g^\Delta}(v, a)(1 + \beta\epsilon),$$

or, dividing through by $1 + \beta\epsilon$,

$$\frac{R}{1 + \beta\epsilon} \leq d_{g^\Delta}(v, a). \tag{6.3}$$

Further, by Lemma 33, $d_{g^\Delta}(v, a) \leq \epsilon(m + 1)$. If we combine these two estimates, we get that

$$\frac{R}{1 + \beta\epsilon} \leq d_{g^\Delta}(v, a) \leq \epsilon(m + 1),$$

or, dropping the middle term and rearranging,

$$R \le \epsilon(m+1) + \epsilon^2\beta(m+1). \tag{6.4}$$

It is easy to check that when ϵ is small,

$$\epsilon(m+1) + \epsilon^2\beta(m+1) \le 2m\epsilon. \tag{6.5}$$

Specifically, this is true when

$$\epsilon \le \frac{m-1}{\beta(m+1)}.$$

Finally, note that $(m-1)/\beta(m+1)$ increases with m, so

$$\frac{m-1}{\beta(m+1)} \ge \frac{2-1}{\beta(2+1)} = \frac{1}{3\beta}$$

for all $m \ge 2$. This means that as long as ϵ is taken to be less than or equal to $1/(3\beta)$, (6.5) is satisfied and hence $R \le 2m\epsilon$. Dividing through by 2ϵ gives the result. □

6.2 Ratio of exponential of discrete conformal factors converges

In this section we will prove that the ratio of exponentials of discrete conformal factors converges uniformly on compact subsets of a surface M to a continuous function. As a first step in that direction, we prove equicontinuity of a certain family of functions. We begin by defining a few maps and setting some notation.

First of all, we define a structure called a *generalized triangulated exhaustion* on a submanifold $\Omega \subset M$. This consists of two parts, a sequence $\{\Omega_n\}$ of compact subsets of M converging to Ω and a sequence of triangulations T_n, each of which triangulates the corresponding compact subset Ω_n. This sequence of compact subsets and triangulations of them correspond in the hexagonal circle packing case to the regions Ω_ϵ and the circle packing complexes on those regions. (See §B.1.)

Definition 36. *Let (M, g) be a complete Riemannian manifold and let $\Omega \subset M$ be an embedded submanifold, without boundary. Assume that there exists a sequence of compact submanifolds $\Omega_n \subset M$ such that the following properties are satisfied for each compact $K \subset \Omega$:*

1. *There exists $N \in \mathbb{N}$ such that $K \subset \Omega_n$ for every $n > N$.*

2. *For every open subset U containing Ω, there exists $Q \in \mathbb{N}$ such that $\Omega_n \subset U$ for every $n > Q$.*

3. *Each Ω_n is equipped with a triangulation T_n with geodesic edges and Riemannian barycentric map Ψ_n making each Euclidean simplex (Δ, g^{Δ}) that has at least one vertex in K a (ϑ, ϵ_n)-full simplex where $\{\epsilon_n\}$ is a sequence of real numbers decreasing to zero.*

In this case, we say that the sequence of pairs $\{(\Omega_n, T_n)\}$ is a generalized triangulated exhaustion *of Ω.*

Let M be a smooth Riemannian surface and let $\Omega \subset M$ be a submanifold that admits a generalized triangulated exhaustion $\{(\Omega_n, T_n)\}$. Suppose there are sequences of discrete conformal factors $\{f_n\}, \{\overline{f}_n\}$ such that $\{T_n, \mathcal{C}_n, f_n, \overline{f}_n\}$ satisfies LDCR (Condition 30) and, further, that the sequence $\{s_m\}$ is of order $1/m$. That is, assume there is a constant α such that $s_m \leq \alpha/m$ for all m.

Let $H_n : V(T_n) \to \mathbb{R}$ be the ratio of exponentials of discrete conformal factors in image and domain, $H_n(v) := e^{\overline{f}_n(v)}/e^{f_n(v)}$. Define $e^F_{\Delta,n} : \Omega_{\Delta,n} \to \mathbb{R}$ to be the linear interpolation of H_n^2. That is, $e^F_{\Delta,n}(a) = \sum_i \lambda^i H_n^2(v_i)$, where $(\lambda^0, \lambda^1, \lambda^2)$ are the barycentric coordinates of a. Finally, let $e^F_n : \Omega_n \to \mathbb{R}$ be the composition $e^F_n := e^F_{\Delta,n} \circ \Psi_n^{-1}$.

We prove the following proposition.

Proposition 37. *Let $\Omega \subset M$ be an embedded submanifold that admits a generalized triangulated exhaustion $\{(\Omega_n, T_n)\}$ and define $e^F_n : \Omega_n \to \mathbb{R}$ as above. Then under the condition that for each compact subset $K \subset \Omega$ there is a positive constant H_K such that $1/H_K \leq H_n(v) \leq H_K$ for every v and every n, the family $\{e^F_n\}$ is uniformly Lipschitz on compact subsets of Ω.*

The condition in the above proposition that $H_n(v) \leq H_K$ for every v and n is not a very strong condition, following directly from a discrete Schwarz Lemma. We discuss this condition in §10.6 below.

Before we prove Proposition 37, we need the following lemma.

Lemma 38. *For each n, let $\Omega_{\Delta,n} = (T_n, \mathcal{C}_n, f_n)$ and $\widetilde{\Omega}_{\Delta,n} = (T_n, \mathcal{C}_n, \widetilde{f}_n)$ be triangulated (ϑ, ϵ_n)-full piecewise flat surfaces sharing the same combinatorics and discrete conformal structure such that the sequence $\{(T_n, \mathcal{C}_n, f_n, \overline{f}_n)\}$ satisfies LDCR. Let $e^F_\sigma : \sigma \to \mathbb{R}$ be the restriction of $e^F_{\Delta,n}$ to a single (closed) simplex $\sigma = [v_0, v_1, v_2]$.*

Assume further that each vertex v_i of σ is the center of a (realized) closed combinatorial disc of generation m in T_n. Then the following inequality holds, where g^{Δ} is the metric $g^{\Delta}_{ij} = \langle v_i - v_0, v_j - v_0 \rangle_{\mathbb{R}^2}$ discussed in §3.2:

$$\left|de^F_\sigma\right|^2_{g^{\Delta}} \leq \left(\frac{6\alpha H_n^2(v_0)}{m\vartheta\epsilon_n}\right)^2. \tag{6.6}$$

Proof. A direct calculation shows that the inverse metric $\left(g^{\Delta}\right)^{-1}$ can be written as

$$\left(g^{\Delta}\right)^{-1} = \frac{1}{4A^2}\begin{pmatrix} \ell_{02}^2 & -\ell_{01}\ell_{02}\cos\phi_{12} \\ -\ell_{01}\ell_{02}\cos\phi_{12} & \ell_{01}^2 \end{pmatrix},$$

where ϕ_{12} is the angle between $v_1 - v_0$ and $v_2 - v_0$, and A is the area of σ.

If we want to use this form for the metric g^{Δ}, then we must change the coordinates we use for points in σ. We can rewrite e_{σ}^{F} to be a function from the unit simplex D instead of the standard simplex Δ simply by replacing λ^0 by $1 - \lambda^1 - \lambda^2$. In these coordinates, we have that

$$e_{\sigma}^{F}(\lambda^1, \lambda^2) = H_n^2(v_0) + \sum_{i=1}^{2} \lambda^i (H_n^2(v_i) - H_n^2(v_0)).$$

We can now calculate as follows, keeping in mind that every edge has length $\ell_{ij} \leq \epsilon_n$ and using the fact that $-2\cos\phi_{12} \leq 2$:

$$\begin{aligned} \left|de_{\sigma}^{F}\right|_{g^{\Delta}}^{2} &= \left(\frac{\partial e_{\sigma}^{F}}{\partial \lambda^1}\right)^2 \left(g^{\Delta}\right)^{11} + \left(\frac{\partial e_{\sigma}^{F}}{\partial \lambda^2}\right)^2 \left(g^{\Delta}\right)^{22} + 2\left(\frac{\partial e_{\sigma}^{F}}{\partial \lambda^1}\right)\left(\frac{\partial e_{\sigma}^{F}}{\partial \lambda^2}\right)\left(g^{\Delta}\right)^{12} \\ &\leq \frac{\epsilon_n^2}{4A^2}\left[(H_n^2(v_1) - H_n^2(v_0))^2 + (H_n^2(v_2) - H_n^2(v_0))^2 \right. \\ &\qquad \left. + 2\left(H_n^2(v_1) - H_n^2(v_0)\right)\left(H_n^2(v_2) - H_n^2(v_0)\right)\right]. \end{aligned}$$

By Lemma 31, we have that for $i = 1, 2$, $|H_n^2(v_i) - H_n^2(v_0)| \leq 3s_m H_n^2(v_0)$ since each v_i is the center of a (realized) closed combinatorial disc of generation m. Hence we have that

$$\left|de_{\sigma}^{F}\right|_{g^{\Delta}}^{2} \leq \frac{\epsilon_n^2 9 s_m^2 H_n^4(v_0)}{A^2}.$$

The simplex σ is (ϑ, ϵ_n)-full and a 2-simplex, so

$$2A \geq \vartheta \epsilon_n^2,$$

or, squaring and rearranging,

$$\frac{1}{A^2} \leq \frac{4}{\vartheta^2 \epsilon_n^4}.$$

Hence

$$\left|de_{\sigma}^{F}\right|_{g^{\Delta}}^{2} \leq \frac{36 s_m^2 H_n^4(v_0)}{\vartheta^2 \epsilon_n^2}.$$

Finally, we are assuming that $s_m \leq \alpha/m$, so

$$\left|de_{\sigma}^{F}\right|_{g^{\Delta}}^{2} \leq \frac{(36)(H_n^4(v_0))(\alpha^2/m^2)}{\vartheta^2 \epsilon_n^2} = \left(\frac{6\alpha H_n^2(v_0)}{m\vartheta\epsilon_n}\right)^2,$$

as required.

□

Now we can prove Proposition 37.

Proof of Proposition 37. Let $K \subset \Omega$ be compact and let $p, q \in K$. Take n large enough so that $K \subset \Omega_n$ for every n and define $a_n := \Psi_n^{-1}(p)$ and $b_n := \Psi_n^{-1}(q)$ to be the points in $\Omega_{\Delta,n}$ corresponding to points p, q in Ω. Then by definition,

$$e_n^F(p) = (e_{\Delta,n}^F \circ \Psi_n^{-1})(p) = e_{\Delta,n}^F(a_n)$$

and similarly, $e_n^F(q) = e_{\Delta,n}^F(b_n)$. Hence $\left|e_n^F(p) - e_n^F(q)\right| = \left|e_{\Delta,n}^F(a_n) - e_{\Delta,n}^F(b_n)\right|$.

Take a minimizing geodesic $\gamma : [0,1] \to M_\Delta$ connecting a_n and b_n and let $\{\sigma_i\}$ be an ordered list of simplices intersecting γ. Assume that unnecessary simplices have been removed from this list in the manner detailed at the beginning of the proof of Proposition 72 in the Appendix and just as in that proof, let $\{t_i\}_{i=1}^{Q-1}$ be the list of t-values where γ leaves one simplex and enters the next, with $t_0 := 0$ and $t_Q := 1$. Hence $a_n = \gamma(t_0)$ and $b_n = \gamma(t_Q)$.

Note that if a point x is in the intersection of two simplices σ and τ, then $e_{\Delta,n}^F(x)$ can be calculated in either σ or τ and the result is the same. That is, if $x \in \sigma \cap \tau$, then $e_{\Delta,n}^F(x) = e_\sigma^F(x) = e_\tau^F(x)$, so we can use the Triangle Inequality to perform the following calculation:

$$\begin{aligned} \left|e_n^F(p) - e_n^F(q)\right| &= \left|e_{\Delta,n}^F(a_n) - e_{\Delta,n}^F(b_n)\right| \\ &= \left|\sum_{i=0}^{Q} \left(e_{\sigma_i}^F(\gamma(t_i)) - e_{\sigma_i}^F(\gamma(t_{i+1}))\right)\right| \\ &\leq \sum_{i=0}^{Q} \left|e_{\sigma_i}^F(\gamma(t_i)) - e_{\sigma_i}^F(\gamma(t_{i+1}))\right|. \end{aligned} \tag{6.7}$$

Each term of (6.7) can be calculated on a single simplex, so we can use the Mean Value Theorem on each term separately and then sum at the end. Since e_σ^F is linear on σ, the differential de_σ^F is constant, so the Mean Value Theorem says that if x and y are points in σ, then

$$\left|e_\sigma^F(x) - e_\sigma^F(y)\right|_{\mathbb{R}} \leq \left|de_\sigma^F\right|_{g^\Delta} d_{g^\Delta}(x, y). \tag{6.8}$$

Lemma 38 gives us an upper bound on $|de_\sigma^F|_{g^\Delta}$ and we now use this upper bound with (6.7) to get

$$\begin{aligned} \left|e_n^F(p) - e_n^F(q)\right| &\leq \sum_{i=0}^{Q} \left|e_{\sigma_i}^F(\gamma(t_i)) - e_{\sigma_i}^F(\gamma(t_{i+1}))\right| \\ &\leq \sum_{i=0}^{Q} \left|de_{\sigma_i}^F\right|_{g^\Delta} d_{g^\Delta}(\gamma(t_i), \gamma(t_{i+1})) \\ &\leq \alpha \sum_{i=0}^{Q} \frac{H_n^2(v_i)}{m_i \vartheta \epsilon_n} d_{g^\Delta}(\gamma(t_i), \gamma(t_{i+1})), \end{aligned}$$

where the vertex v_i is one of the vertices of σ_i and m_i is the number of generations of the largest realized closed combinatorial disc about v_i that is contained in Ω.

Next let $R = R(K)$ be the value guaranteed by Lemma 34 such that for each point $p \in K$ there is a geodesic ball of radius R about p completely contained in Ω. Now by Lemma 35, we have that $m_i \geq R/(2\epsilon_n)$ and we are assuming that $H_n(v_i) \leq H_K$ for every vertex v_i contained in K, so we have the following:

$$\begin{aligned}\left|e_n^F(p) - e_n^F(q)\right| &\leq 6\alpha \sum_{i=0}^{Q} \frac{H_n^2(v_i)}{m_i \vartheta \epsilon_n} d_{g^\triangle}(\gamma(t_i), \gamma(t_{i+1})) \\ &\leq \frac{12\alpha H_K^2}{R\vartheta} \sum_{i=0}^{Q} d_{g^\triangle}(\gamma(t_i), \gamma(t_{i+1})) \\ &= \frac{12\alpha H_K^2}{R\vartheta} d_{g^\triangle}(a_n, b_n),\end{aligned}$$

where the last equality follows since γ is a minimizing geodesic from a_n to b_n.

Finally, Proposition 72 says that $d_{g^\triangle}(a_n, b_n) \leq (1 + \beta\epsilon) d_g(p, q)$, so

$$\left|e_n^F(p) - e_n^F(q)\right| \leq \frac{12H_K^2 a}{\vartheta R} d_{g^\triangle}(a_n, b_n) \leq \frac{12H_K^2 a}{\vartheta R}(1 + \beta\epsilon) d_g(p, q)$$

and hence every e_n^F is Lipschitz with the same constant $L \leq 12\alpha H_K^2(1+\beta)/(\vartheta R)$ and the family $\{e_n^F\}$ is uniformly equicontinuous. □

Finally, we have an easy corollary of the previous result.

Corollary 39. *There exists a subsequence $\{e_{n_k}^F\} \subset \{e_n^F\}$ and a positive continuous function $e^F : \Omega \to \mathbb{R}$ such that $e_{n_k}^F \to e^F$ uniformly on compact subsets of Ω.*

Proof. This proof is an application of the Arzelá-Ascoli Theorem, Theorem 64 in the Appendix.

By Proposition 37, the collection $\{e_n^F\}$ is equicontinuous on compact subsets, so all that remains is to show that $\{e_n^F\}$ is pointwise bounded. That is, that for any $p \in \Omega$, $|e_n^F(p)| \leq C$ for some constant C that does not depend on n. But this follows immediately from the assumption that $H_n(v) = e^{\overline{f}_n(v)}/e^{f_n(v)} \leq H_K$ for every v and every n.

Hence if K is an arbitrary compact subset of Ω, then there is a subsequence of $\{e_n^F\}$ that converges uniformly on K.

All that remains is a diagonalization argument to show that there is a subsequence $\{e_{n_k}^F\}$ that is convergent on *every* compact subset of Ω.

Take a countable exhaustion of Ω by compact subsets. That is, let $\{K_i\}_{i=1}^{\infty}$ be compact subsets of Ω such that $K_i \subset K_{i+1}$ and for each point $p \in \Omega$, there is a number $Q \in \mathbb{N}$ such that $p \in K_i$ for every $i \geq Q$.

Let $\{e^F_{1,n}\} \subset \{e^F_n\}$ be a subsequence that converges uniformly on K_1. Then take a further subsequence $\{e^F_{2,n}\} \subset \{e^F_{1,n}\}$ which converges uniformly on K_2. Proceed in this manner indefinitely.

Finally, build a new subsequence $\{e^F_i\}$ as follows. Let $e^F_1 = e^F_{1,1}$, the first element in the first subsequence, $e^F_2 = e^F_{2,2}$, the second element in the second, and so on. Then since $\{e^F_i\} \subset \{e^F_{1,n}\}$, the subsequence $\{e^F_i\}$ converges in K_1. Similarly $\{e^F_i\}_{i=2}^{\infty} \subset \{e^F_{2,n}\}$ and hence $\{e^F_i\}$ converges in K_2 as well. By the same argument, $\{e^F_i\}$ converges uniformly on each of the countably many subsets $\{K_i\}$, and hence on any compact subset of Ω. $\square$

Chapter 7

Estimates on the metric distortion of discrete conformal mappings

We begin by showing that squared edge lengths in domain and image are relatively close.

Lemma 40. *Let $\{(M_n, g_n^\Delta)\}$ and $\{(N_n, h_n^\Delta)\}$ be sequences of discrete conformal manifolds with discrete conformal factors $f_n, \overline{f}_n$ respectively, and assume that the sequences $\{(M_n, N_n)\}$ satisfy LDCR and that the ring lemma holds. Then there exists a constant C such that for every edge e_{vw}, where the vertices v and w and the centers of realized combinatorial closed discs of generation m, the following holds:*

$$H_n^2(v)\ell_{vw}^2(1 - Cs_m) \leq \overline{\ell}_{vw}^2 \leq H_n^2(v)\ell_{vw}^2(1 + Cs_m).$$

Proof. First write

$$\begin{aligned} \overline{\ell}_{vw}^2 &= \alpha_v e^{2\overline{f}_n(v)} + \alpha_w e^{2\overline{f}_n(w)} + 2\eta_{vw} e^{\overline{f}_n(v)+\overline{f}_n(w)} \\ &= H_n^2(v)\left(\alpha_v e^{2f_n(v)} + \alpha_w e^{2f_n(w)} \frac{e^{2f_n(v)}}{e^{2\overline{f}_n(v)}}\frac{e^{2\overline{f}_n(w)}}{e^{2f_n(w)}} + 2\eta_{vw} e^{f_n(v)+f_n(w)} \frac{e^{\overline{f}_n(w)}}{e^{f_n(w)}}\frac{e^{f_n(v)}}{e^{\overline{f}_n(v)}}\right) \end{aligned} \tag{7.1}$$

Since we assumed that the sequence of discrete conformal manifolds satisfies LDCR, we have by Lemma 31 that

$$1 - s_m \leq \frac{e^{f_n(v)}}{e^{\overline{f}_n(v)}}\frac{e^{\overline{f}_n(w)}}{e^{f_n(w)}} \leq 1 + s_m \tag{7.2}$$

and

$$1 - 3s_m \leq \frac{e^{2f_n(v)}}{e^{2\overline{f}_n(v)}}\frac{e^{2\overline{f}_n(w)}}{e^{2f_n(w)}} \leq 1 + 3s_m. \tag{7.3}$$

Now we use (7.2) and (7.3) to estimate the third and second terms of (7.1) respectively:

$$\begin{aligned} &H_n^2(v)\left(\alpha_v e^{2f_n(v)} + (1 - 3s_m)\alpha_w e^{2f_n(w)} + (1 - s_m)\left(2\eta_{vw} e^{f_n(v)+f_n(w)}\right)\right) \\ &\qquad \leq \overline{\ell}_{vw}^2 \\ &\qquad \leq H_n^2(v)\left(\alpha_v e^{2f_n(v)} + (1 + 3s_m)\alpha_w e^{2f_n(w)} + (1 + s_m)\left(2\eta_{vw} e^{f_n(v)+f_n(w)}\right)\right). \end{aligned}$$

Since $1 - 3s_m$ is smaller than both $1 - s_m$ and 1 while $1 + 3s_m$ is larger than both $1 + s_m$ and 1, we have that

$$H_n^2(v)(1 - 3s_m)\ell_{vw}^2 \leq \bar{\ell}_{vw}^2 \leq H_n^2(v)(1 + 3s_m)\ell_{vw}^2,$$

as required. □

Lemma 41. *Let $\{(M_n, g_n^\Delta)\}$ and $\{(N_n, h_n^\Delta)\}$ be as in Lemma 40 and suppose $e_{\Delta,n}^F : M_n \to \mathbb{R}$ is defined as in Chapter 6. That is, for any $a \in M_n$ with barycentric coordinates $(\lambda^0, \ldots, \lambda^k)$, define $e_{\Delta,n}^F(a) = \sum_{i=0}^k \lambda^i H_n^2(v_i)$. Let v_i be any vertex of the simplex containing a and let m be such that the realized closed combinatorial disc of generation m about a is contained in M_n. Suppose further that m is large enough such that $s_m \leq 1/6$.*

Then

$$(1 - 3s_m)H_n^2(v_i) \leq e_{\Delta,n}^F(a) \leq (1 + 3s_m)H_n^2(v_i) \tag{7.4}$$

and

$$(1 - 6s_m)e_{\Delta,n}^F(a) \leq H_n^2(v_i) \leq (1 + 6s_m)e_{\Delta,n}^F(a). \tag{7.5}$$

Proof. Without loss of generality, choose a vertex in the simplex containing a and label this vertex v_0.

For the first inequality, (7.4), we calculate as follows, making use of Lemma 31 for the last step.

$$\begin{aligned}
\left|e_{\Delta,n}^F(a) - H_n^2(v_0)\right| &= \left|\sum_{i=0}^k \lambda^i H_n^2(v_i) - H_n^2(v_0)\right| \\
&= \left|H_n^2(v_0)\sum_{i=1}^k(-\lambda^i) + \sum_{i=1}^k \lambda^i H_n^2(v_i)\right| \\
&= \left|\sum_{i=1}^k \lambda^i (H_n^2(v_i) - H_n^2(v_0))\right| \\
&\leq \sum_{i=1}^k \lambda^i \left|H_n^2(v_i) - H_n^2(v_0)\right| \\
&\leq 3s_m H_n^2(v_0).
\end{aligned}$$

This shows (7.4). For (7.5), we look separately at the left and right sides of (7.4), as follows.

From the left hand side of (7.4), we have that

$$(1 - 3s_m)H_n^2(v_0) \leq e_{\Delta,n}^F(a),$$

or, dividing through by $1-3s_m$,

$$H_n^2(v_0) \leq \frac{e^F_{\Delta,n}(a)}{1-3s_m} \leq e^F_{\Delta,n}(a)(1+6s_m), \tag{7.6}$$

since $1/(1-3s_m) \leq 1+6s_m$ for $0 < s_m \leq 1/6$.

Similarly, dividing the right hand side of (7.4) by $1+3s_m$ gives

$$H_n^2(v_0) \geq \frac{e^F_{\Delta,n}(a)}{1+3s_m} \geq e^F_{\Delta,n}(1-3s_m) \geq e^F_{\Delta,n}(1-6s_m), \tag{7.7}$$

since $1/(1+3s_m) \geq 1-3s_m \geq 1-6s_m$ for all $s_m > 0$.

Combining (7.6) and (7.7) gives the result. □

Theorem 42. *Let $\{(M_n, g_n^\Delta)\}$ and $\{(N_n, h_n^\Delta)\}$ be as above, with the additional restrictions that they are (ϑ,ϵ)-full and have dimension two. Let $e^F_{\Delta,n}$ be as in Lemma 41. Let $a \in \sigma^\Delta$ be a point lying in a simplex σ^Δ and assume that each vertex in σ^Δ is the center of a realized closed combinatorial disc of generation m, where m is taken to be large enough so that $s_m \leq 1/6$. Then there exists a constant C such that for any vector $X \in T_aM_n$,*

$$(1-Cs_m)|X|^2_{e^F_{\Delta,n}g_n^\Delta} \leq |X|^2_{\phi_n^* h_n^\Delta} \leq (1+Cs_m)|X|^2_{e^F_{\Delta,n}g_n^\Delta}.$$

Proof. We will be using Lemmas 69 and 70 from the Appendix to prove this. Lemma 69 gives us a lower bound on the eigenvalues of $e^F_{\Delta,n}g^\Delta$ which we use along with Lemma 31 to bound $|e^F_{\Delta,n}g^\Delta_{ij} - h^\Delta_{ij}|$. With that bound in hand, Lemma 70 gives the result.

By Lemma 69, we have that eigenvalues λ_i for $e^F_{\Delta,n}g^\Delta$ are all bounded below by

$$\frac{\vartheta^2\epsilon^2 e^F_{\Delta,n}}{4} \leq \lambda_i,$$

since the eigenvalues of $e^F_{\Delta,n}g^\Delta$ are precisely the eigenvalues of g^Δ multiplied by $e^F_{\Delta,n}$.

Next, we want to bound $|e^F_{\Delta,n}g^\Delta_{ij} - h^\Delta_{ij}|$. Note that by definition of these metrics, the diagonal terms g^Δ_{ii} and h^Δ_{ii} are simply ℓ^2_{0i} and $\overline{\ell}^2_{0i}$ respectively, while the off-diagonal terms $g^\Delta_{ij}, i \neq j$ and are given by

$$\begin{aligned} g^\Delta_{ij} &= \langle p_i - p_0, p_j - p_0 \rangle \\ &= \ell_{0i}\ell_{0j}\cos(\theta_{ij}) \\ &= \frac{1}{2}\left(\ell^2_{0i} + \ell^2_{0j} - \ell^2_{ij}\right) \end{aligned}$$

and similarly,

$$h^\Delta_{ij} = \frac{1}{2}\left(\overline{\ell}^2_{0i} + \overline{\ell}^2_{0j} - \overline{\ell}^2_{ij}\right).$$

We begin by bounding expressions of the form $\left|e^F_{\Delta,n}\ell^2_{ij} - \bar{\ell}^2_{ij}\right|$. This bound immediately gives us bounds on the diagonal terms $\left|e^F_{\Delta,n}g^{\Delta}_{ii} - h^{\Delta}_{ii}\right|$ and we use the triangle inequality to bound the off-diagonal terms $\left|e^F_{\Delta,n}g^{\Delta}_{ij} - h^{\Delta}_{ij}\right|$ for $i \neq j$.

We start by combining Lemmas 40 and 41 to get

$$(1-3s_m)(1-6s_m)e^F_{\Delta,n}\ell^2_{ij} \leq \bar{\ell}^2_{ij} \leq (1+3s_m)(1+6s_m)e^F_{\Delta,n}\ell^2_{ij},$$

or, more compactly

$$\left|e^F_{\Delta,n}\ell^2_{0i} - \bar{\ell}^2_{0i}\right| \leq 18s_m e^F_{\Delta,n}\ell^2_{0i}, \tag{7.8}$$

since $1-18s_m \leq (1-3s_m)(1-6s_m)$ and $(1+3s_m)(1+6s_m) \leq 1+18s_m$ for $0 < s_m \leq 1/6$.

Now for the off-diagonal terms, we have the following calculation:

$$\begin{aligned}
\left|e^F_{\Delta,n}g_{ij} - h_{ij}\right| &= \left|\frac{1}{2}\left(e^F_{\Delta,n}\ell^2_{0i} - \bar{\ell}^2_{0i} + e^F_{\Delta,n}\ell^2_{0j} - \bar{\ell}^2_{0j} - e^F_{\Delta,n}\ell^2_{ij} + \bar{\ell}^2_{ij}\right)\right| \\
&\leq \frac{1}{2}\left(\left|e^F_{\Delta,n}\ell^2_{0i} - \bar{\ell}^2_{0i}\right| + \left|e^F_{\Delta,n}\ell^2_{0j} - \bar{\ell}^2_{0j}\right| + \left|e^F_{\Delta,n}\ell^2_{ij} - \bar{\ell}^2_{ij}\right|\right) \\
&\leq \frac{1}{2}\left(18s_m e^F_{\Delta,n}\ell^2_{0i} + 18s_m e^F_{\Delta,n}\ell^2_{0j} + 18s_m e^F_{\Delta,n}\ell^2_{ij}\right) \\
&\leq 27s_m e^F_{\Delta,n}\epsilon^2,
\end{aligned}$$

where the last inequality follows from the assumption that every edge has length no greater than ϵ.

Let $\mu := \frac{216s_m}{\vartheta^2}$. Note that μ depends on m, the generation of the closed combinatorial discs around each vertex of σ^{Δ}. Then since the smallest eigenvalue of $e^F_{\Delta,n}g^{\Delta}$ is $\lambda_{\min} \geq e^F_{\Delta,n}\vartheta^2\epsilon^2/4$, we have that

$$\begin{aligned}
\frac{\mu\lambda_{\min}}{2} &\geq \frac{216s_m}{2\vartheta^2}\frac{e^F_{\Delta,n}\vartheta^2\epsilon^2}{4} \\
&= 27s_m e^F_{\Delta,n}\epsilon^2.
\end{aligned}$$

Since we showed above that $\left|e^F_{\Delta,n}g_{ij} - h_{ij}\right|$ is bounded above by this same expression, we have that

$$\left|e^F_{\Delta,n}g_{ij} - h_{ij}\right| \leq \mu\frac{\lambda_{\min}}{2}$$

and hence we can apply Lemma 70 to get that

$$\begin{aligned}
\left||X|^2_{e^F_{\Delta,n}g^{\Delta}} - |X|^2_{\phi^*_n h^{\Delta}}\right| &\leq \mu|X|^2_{e^F_{\Delta,n}g^{\Delta}} \\
&= \frac{216s_m}{\vartheta^2}|X|^2_{e^F_{\Delta,n}g^{\Delta}}.
\end{aligned}$$

Letting $C := \frac{216}{\vartheta^2}$ and rewriting as a two-sided bound gives the result. $\square$

Chapter 8

Convergence of the sequence of mappings

We begin by recalling the definition of a generalized triangulated exhaustion.

Definition 43. *Let (M, g) be a complete Riemannian manifold and let $\Omega \subset M$ be an embedded submanifold, without boundary. Assume that there exists a sequence of compact submanifolds $\Omega_n \subset M$ such that the following properties are satisfied for each compact $K \subset \Omega$:*

1. *There exists $N \in \mathbb{N}$ such that $K \subset \Omega_n$ for every $n > N$.*
2. *For every open subset U containing Ω, there exists $Q \in \mathbb{N}$ such that $\Omega_n \subset U$ for every $n > Q$.*
3. *Each Ω_n is equipped with a triangulation T_n with geodesic edges and Riemannian barycentric map Ψ_n making each Euclidean simplex (Δ, g^{Δ}) that has at least one vertex in K a (ϑ, ϵ_n)-full simplex where $\{\epsilon_n\}$ is a sequence of real numbers decreasing to zero.*

In this case, we say that the sequence of pairs $\{(\Omega_n, T_n)\}$ is a generalized triangulated exhaustion *of Ω.*

The first two conditions in the above definition serve to pick out a sequence of compact manifolds. Our main results are about convergence in compact subsets and these $\{\Omega_n\}$ exhaust those compact subsets. It would be slightly easier to simply take $\{\Omega_n\}$ to be an exhaustion of Ω by compact sets and in practice this is often exactly what we do. However, there are certain contexts where it makes sense to have more freedom in our sequence of compact subsets. For example, in [Gli16], compact manifolds are taken that are larger than the limit manifold. Our definition allows for this freedom.

For each n, let $\Omega_{\Delta,n} := \Psi_n^{-1}(\Omega_n)$ denote the piecewise flat manifold corresponding to Ω_n. We will frequently drop the n from the subscript when no confusion arises from doing so.

Definition 44. *Let (M, g) and (N, h) be complete Riemannian surfaces and let $\Omega \subset M$ and $\widetilde{\Omega} \subset N$ be embedded submanifolds such that Ω and $\widetilde{\Omega}$ are diffeomorphic. Let $\{(\Omega_n, T_n)\}$ and $\{(\widetilde{\Omega}_n, T_n)\}$ be generalized triangulated exhaustions of Ω and $\widetilde{\Omega}$ respectively such that in both cases the triangulations T_n are combinatorially identical.*

Let $\{f_n\}$ and $\{\widetilde{f}_n\}$ be sequences of discrete conformal factors for $\Omega_{\Delta,n}$ and $\widetilde{\Omega}_{\Delta,n}$ respectively such that for each n the piecewise flat manifolds $(\Omega_n, T_n, \ell(f_n))$ and $(\widetilde{\Omega}_n, T_n, \ell(\widetilde{f}_n))$

are discrete conformal under the discrete conformal structure $\mathcal{C}_n$ *with discrete conformal map* ϕ_n.

Assume that the following conditions hold:

1. *The sequence* $\{T_n, \mathcal{C}_n, f_n, \widetilde{f}_n\}$ *satisfies the LDCR condition, Condition 30.*
2. *The piecewise flat surfaces* $(\Omega_n, T_n, \ell(f))$ *and* $(\widetilde{\Omega}_n, T_n, \ell(\widetilde{f}))$ *satisfy the Ring Lemma condition, Condition 20.*
3. *For each compact* $K \subset \Omega$*, the ratio of discrete conformal factors* $e^{\widetilde{f}_n(v)}/e^{f_n(v)}$ *has a uniform upper bound* $H = H(K)$ *depending only on the compact set* K.
4. *There is some point* $x \in \Omega$ *such that the image set* $\{\Phi_n(x)\}$ *is contained in some compact subset* $V \subset N$.

In this case, we say that $\{(\Omega_n, \widetilde{\Omega}_n, T_n, f_n, \widetilde{f}_n)\}$ *is an* admissible sequence for $(\Omega, \widetilde{\Omega})$.

If in addition the LDCR constants s_m *are such that there is some positive constant* α *such that* $s_m \le \alpha/m$*, then we say that* $\{(\Omega_n, \widetilde{\Omega}_n, T_n, f_n, \widetilde{f}_n)\}$ *is a* proper *admissible sequence for* $(\Omega, \widetilde{\Omega})$.

Definition 45. *Let* $(\Omega, \widetilde{\Omega})$ *have an admissible sequence* $\{(\Omega_n, \widetilde{\Omega}_n, T_n, f_n, \widetilde{f}_n)\}$ *with discrete conformal maps* $\{\phi_n : \Omega_{\Delta,n} \to \widetilde{\Omega}_{\Delta,n}\}$. *For each* n *define* $\Phi_n : \Omega_n \to \widetilde{\Omega}_n$ *as the composition* $\Phi_n := \widetilde{\Psi}_n \circ \phi_n \circ \Psi_n^{-1}$, *where* $\Psi_n, \widetilde{\Psi}_n$ *are the Riemannian barycentric maps on* $\Omega_{\Delta,n}, \widetilde{\Omega}_{\Delta,n}$ *respectively. This map* Φ_n *is called a* barycentric discrete conformal map.

$$\begin{array}{ccc} (\Omega_n, g) & \xrightarrow{\Phi_n} & (\widetilde{\Omega}_n, h) \\ \uparrow{\scriptstyle \Psi_n} & & \uparrow{\scriptstyle \widetilde{\Psi}_n} \\ (\Omega_{\Delta,n}, g^{\Delta}), & \xrightarrow{\phi_n} & (\widetilde{\Omega}_{\Delta,n}, h^{\Delta}) \end{array}$$

FIGURE 8.1. Defining Φ_n

Lemma 46. *The barycentric discrete conformal map* Φ_n *is a diffeomorphism on each closed simplex* σ.

Proof. By Lemma 13, both $\widetilde{\Psi}_n$ and Ψ_n^{-1} are diffeomorphisms on closed simplices, so all that remains is to note that ϕ_n is piecewise linear and hence clearly smooth even on the boundary of a simplex. □

Define the set $E \subset M$ to be $E := \bigcup_n E(T_n)$, the union over all n of (closed) edges in the triangulations T_n. Note that for each n, the set of edges $E(T_n)$ is a set of measure zero (in M) and since there are countably many such subsets, the countable union E is also a set of measure zero.

Proposition 47. *Assume M and N are Riemannian surfaces and let $\Omega \subset M$ and $\widetilde{\Omega} \subset N$ be diffeomorphic embedded submanifolds with admissible sequence $\{(\Omega_n, \widetilde{\Omega}_n, T_n, f_n, \widetilde{f}_n)\}$.*

Take $Q \in \mathbb{N}$ large enough so that the point p lies within Ω_n for all $n > Q$ and assume that for each such n, the closest vertex v to p is the center of a realized closed combinatorial disc of radius m in T_n. Then there is some constant C for which the following estimate holds:

$$(1-\beta\epsilon_n)^2(1-Cs_m)e_n^F|X|_g^2 \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon_n)^2(1+Cs_m)e_n^F|X|_g^2, \tag{8.1}$$

where β is the Riemannian barycentric constant from Lemma 16, $\{\epsilon_n\}$ is the sequence of maximum edge lengths for the generalized triangulated exhaustions $\{(\Omega_n, T_n)\}$ of Ω and $\{(\widetilde{\Omega}_n, T_n)\}$ of $\widetilde{\Omega}$, and $\{s_m\}$ is the sequence of LDCR constants.

Proof. For this proof we will repeatedly "peel back" the functions that together make up Φ_n, using the estimates we have built up along the way.

First we rewrite $|X|^2_{\Phi_n^* h}$ as

$$\begin{aligned} |X|^2_{\Phi_n^* h} &= \left| \left(\widetilde{\Psi}_n \circ \phi_n \circ \Psi_n^{-1} \right)_* X \right|_h^2 \\ &= \left| \left(\phi_n \circ \Psi_n^{-1} \right)_* X \right|^2_{\widetilde{\Psi}_n^* h} \end{aligned}$$

and then use (3.7) from Theorem 16 to get that

$$(1-\beta\epsilon) \left| \left(\phi_n \circ \Psi_n^{-1} \right)_* X \right|^2_{h^\Delta} \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon) \left| \left(\phi_n \circ \Psi_n^{-1} \right)_* X \right|^2_{h^\Delta}. \tag{8.2}$$

Next we rewrite $|(\phi_n \circ \Psi_n^{-1})_* X|^2_{h^\Delta}$ as

$$\left| \left(\phi_n \circ \Psi_n^{-1} \right)_* X \right|^2_{h^\Delta} = \left| \left(\Psi_n^{-1} \right)_* X \right|^2_{\phi_n^* h^\Delta}$$

and use Theorem 42 to estimate the right hand side. When combined with (8.2), this shows that

$$(1-\beta\epsilon)(1-Cs_m) \left| \left(\Psi_n^{-1} \right)_* X \right|^2_{e_\Delta^F g^\Delta} \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon)(1+Cs_m) \left| \left(\Psi_n^{-1} \right)_* X \right|^2_{e_\Delta^F g^\Delta}. \tag{8.3}$$

Recall that we have defined the function $e_n^F \in C^0(M)$ to be $e_n^F := e_\Delta^F \circ \Psi_n^{-1}$. With this definition, it is easy to see that

$$\begin{aligned} \left(\Psi_n^{-1} \right)^* \left(e_\Delta^F g^\Delta \right) &= \left(e_\Delta^F \circ \Psi_n^{-1} \right) \left(\Psi_n^{-1} \right)^* g^\Delta \\ &= e_n^F \left(\Psi_n^{-1} \right)^* g^\Delta. \end{aligned} \tag{8.4}$$

This calculation allows us to rewrite $|(\Psi_n^{-1})_* X|^2_{e^F_\Delta g^\Delta}$ as

$$\begin{aligned}\left|\left(\Psi_n^{-1}\right)_* X\right|^2_{e^F_\Delta g^\Delta} &= |X|^2_{\left(\Psi_n^{-1}\right)^*\left(e^F_\Delta g^\Delta\right)} \\ &= |X|^2_{e^F_n\left(\Psi_n^{-1}\right)^* g^\Delta} \\ &= e^F_n |X|^2_{\left(\Psi_n^{-1}\right)^* g^\Delta}\end{aligned}$$

and we can estimate this last expression using (3.8) from Theorem 16. We find that

$$(1-\beta\epsilon)e^F_n|X|^2_g \leq e^F_n|X|^2_{\left(\Psi_n^{-1}\right)^* g^\Delta} \leq (1+\beta\epsilon)e^F_n|X|^2_g$$

which when substituted into (8.3) gives

$$(1-\beta\epsilon)^2(1-Cs_m)e^F_n|X|^2_g \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon)^2(1+Cs_m)e^F_n|X|^2_g.$$

□

With the above proposition in hand, we can prove the following theorem, which is the main result of this dissertation.

Theorem 48. *Let $\{(\Omega_n, \widetilde{\Omega}_n, T_n, f_n, \widetilde{f}_n)\}$ be an admissible sequence for $(\Omega, \widetilde{\Omega})$ and let $\{\Phi_n\}$ be the corresponding sequence of barycentric discrete conformal maps. Then the family $\{\Phi_n\}$ has a subsequence that converges uniformly on compact subsets of Ω. Furthermore, if the admissible sequence is proper, then there exists a positive continuous function e^F such that $\Phi_n^* h \to e^F g$ in L^∞ on compact subsets of Ω, and hence convergence is to a conformal map.*

Proof. We begin by showing that the family $\{\Phi_n\}$ is equicontinuous on compact subsets of Ω. Let $K \subset \Omega$ compact and let $L \in \mathbb{N}$ be large enough so that $K \subset \Omega_n$ for all $n \geq L$. Take $p, q \in K$. Then by Proposition 47, we have that

$$|X|_{\Phi_n^* h} \leq (1+\beta\epsilon_n)\sqrt{1+Cs_m}\sqrt{e^F_n}|X|_g.$$

Since $\{\epsilon_n\}$ is a decreasing sequence we can easily find $\epsilon > 0$ such that $\epsilon_n < \epsilon$ for every $n \geq L$. Further, we can assume that there is some m such that every vertex in K is the center of a combinatorial closed disc of at least m generations in Ω_n for every n. Hence s_m does not depend on n. Finally, since we are assuming $H_n(v) \leq H_K$ for every vertex v in K, we have that $e^F_n(p) \leq H_K^2$ for every point $p \in K$. Hence

$$|X|_{\Phi_n^* h} \leq (1+\beta\epsilon)\sqrt{1+Cs_m}H_K|X|_g. \tag{8.5}$$

Now each Φ_n is a homeomorphism on K and Lemma 46 shows that Φ_n is a diffeomorphism when restricted to each (closed) simplex. Hence we can use Proposition 72 from the Appendix along with (8.5) to see that

$$d_h(\Phi_n(p), \Phi_n(q)) \leq H_K(1+\beta\epsilon)\sqrt{1+Cs_m}d_g(p,q).$$

This means that each Φ_n is Lipschitz with the same Lipschitz constant and so the family $\{\Phi_n\}$ is equicontinuous on K.

Next we use the Arzelà-Ascoli Theorem, Proposition 65 in the Appendix, to show that $\{\Phi_n\}$ has a convergent subsequence. We have already shown that the family $\{\Phi_n\}$ is equicontinuous, so next we need to show that for each $p \in \Omega$ the image set $S(p) = \{\Phi_n(p) : p \in \Omega_n\}$ is contained in a compact set. To do so, let $x \in \Omega$ be the point such that $\{\Phi_n(x)\}$ is contained in some compact $V \subset N$. This point was assumed to exist in the definition of admissible sequence.

Let $p \in \Omega$ and take K to be a compact set in Ω containing p. Then for $n, m > L$, we can calculate as follows:

$$\begin{aligned} d_h(\Phi_n(p), \Phi_m(p)) &\leq d_h(\Phi_n(p), \Phi_n(x)) + d_h(\Phi_n(x), \Phi_m(x)) + d_h(\Phi_m(x), \Phi_m(p)) \\ &\leq H_K(1+\beta)\sqrt{1+C}d_g(p,x) + \mathrm{diam}(V) + H_K(1+\beta)\sqrt{1+C}d_g(x,p) \\ &= 2H_K(1+\beta)\sqrt{1+C}d_g(p,x) + \mathrm{diam}(V) \\ &\leq A, \end{aligned}$$

where $A > 0$ does not depend on n. Hence the set $S(p)$ is contained in the closed ball $\overline{B_A(y)}$, which is a closed and bounded subset of a complete manifold and hence compact by Lemma 74 in the Appendix.

Thus we have that the family $\{\Phi_n\}$ is equicontinuous and pointwise bounded on each compact $K \subset \Omega$ and hence has a convergent subsequence on each such K. Take an exhaustion of Ω by compact subsets $K_1 \subset K_2 \subset \ldots$ and use a diagonalization argument to build a subsequence $\{\Phi_{n_i}\}$ such that Φ_{n_i} converges uniformly on each K_j and hence the subsequence $\{\Phi_{n_i}\}$ converges on any compact subset.

Finally we show that $\Phi_n^* h \to e^F g$ in L^∞ on compact subsets of Ω.

Let $K \subset \Omega$ be compact and take $L \in \mathbb{N}$ to be large enough so that $K \subset \Omega_n$ for every $n > L$. Note that by discarding elements of $\{\Omega_n\}$, we may assume without loss of generality that $\Omega_n \subset \Omega_{n+1}$ for every n. Take $R > 0$ such that $B_R(p)$ is a geodesic ball of radius R contained in Ω_L and hence $B_R(p)$ is also contained in Ω_n for $n > L$. Such an R exists by Lemma 34 along with our earlier assumption that the Ω_n's are nested.

Let X be a vector field. Then by Proposition 47, the following inequality holds for any point in $\Omega \setminus E$, where each vertex v in T_n is the center of a closed combinatorial disc of m generations

$$(1-\beta\epsilon_n)^2 (1 - Cs_m) e_n^F |X|_g^2 \leq |X|^2_{\Phi_n^* h} \leq (1+\beta\epsilon_n)^2 (1 + Cs_m) e_n^F |X|_g^2. \tag{8.6}$$

By assumption $\epsilon_n \to 0$ as $n \to \infty$, so both $(1-\beta\epsilon_n)^2$ and $(1+\beta\epsilon_n)^2$ approach 1 as $n \to \infty$. Corollary 39 says that $e_n^F \to e^F$ uniformly as $n \to \infty$. All that remains is to show that as $n \to \infty$, $1 - Cs_m$ and $1 + Cs_m$ both approach 1.

To do this, first recall that we are assuming that $s_m \leq \alpha/m$ and that every point in K is the center of a geodesic ball of radius R contained in Ω_n for each n larger than L. By Lemma 35, we have that

$$\frac{1}{m} \leq \frac{2\epsilon_n}{R},$$

so

$$\begin{aligned} s_m &\leq \frac{\alpha}{m} \\ &\leq \frac{2\alpha\epsilon_n}{R} \end{aligned}$$

and hence

$$1 - \frac{2\alpha C\epsilon_n}{R} \leq 1 - Cs_m \leq 1 + Cs_m \leq 1 + \frac{2\alpha C\epsilon_n}{R}.$$

Since $\epsilon_n \to 0$ as $n \to \infty$, we have that $1 - Cs_m$ and $1 + Cs_m$ approach 1 as $n \to \infty$.

We have shown that on $K \setminus E$, both the left hand and right hand sides of (8.6) converge uniformly to $e^F|X|_g^2$, so clearly the middle term, $|X|^2_{\Phi_n^* h}$ must as well. Hence by Proposition 68 in §A.2, $\Phi_n^* h \to e^F g$ in L^∞. □

Chapter 9

Application to Hexagonal Combinatorics on Flat Manifolds

In this chapter, we apply our main result, Theorem 48, to the Rodin-Sullivan context. This amounts showing that there is a proper admissible sequence for $(\Omega, \mathbb{D})$, as in Definition 44.

Let $(\mathbb{C}, g)$ be the complex plane with the usual Euclidean metric g and let $\Omega \subset \mathbb{C}$ be a simply connected bounded region in the plane. For each $\epsilon > 0$ let Ω_ϵ be the domain of the circle packing map f_ϵ, as in §B.1 of the Appendix and let T_ϵ be the corresponding (hexagonal, equilateral) triangulation. Recall that $|T_\epsilon|$ denotes the carrier of the triangulation T_ϵ.

Lemma 49. *The sequence $\{(|T_\epsilon|, T_\epsilon)\}$ is a generalized triangulated exhaustion of Ω.*

Proof. First of all, $(\mathbb{C}, g)$ is a complete Riemannian manifold and Ω is an embedded submanifold. Further, each $|T_\epsilon|$ is clearly compact.

Since the radii of circles in Ω_ϵ goes to 0 as ϵ does, every compact subset $K \subset \Omega$ is eventually contained in $|T_\epsilon|$ for all ϵ small enough. Further, each $|T_\epsilon|$ lies within Ω, and hence necessarily within any open set $U \supset \Omega$. This shows the first two conditions of Definition 44.

Finally we note that since every triangle in T_ϵ is equilateral with edge length 2ϵ, each triangle is $(\sqrt{3}/2, 2\epsilon)$-full and hence the third condition is satisfied as well and $\{(|T_\epsilon|, T_\epsilon)\}$ is a generalized triangulated exhaustion of Ω. □

Recall that the circle packing maps f_ϵ are normalized so that the image carriers $|T'_\epsilon|$ are all contained within the unit disc $\mathbb{D}$. The next lemma will show that there is a proper admissible sequence for $(\Omega, \mathbb{D})$.

A word of caution regarding notation is necessary here. In the rest of the dissertation, ϵ is usually an edge length, whereas in the Rodin-Sullivan context, ϵ is a radius. That is, edges in T_ϵ have length 2ϵ. Also, in the rest of the dissertation, $f_n(v)$ usually denotes the discrete conformal factor corresponding to the vertex v. In this section, f_ϵ denotes a discrete conformal map and the discrete conformal factors are log radii. That is, $f_n(v) = \log r(v)$, where $r(v)$ is the radius corresponding to the vertex v. In this chapter we will avoid using the notation $f_n(v)$, instead writing $\log r(v)$ so as to minimize any possible confusion arising from f_ϵ and f_n referring to very different functions.

The discrete conformal factors in the domain triangulation T_ϵ are all 0 since T_ϵ is an equilateral triangulation. We will denote radii in the image triangulation by $r'_\epsilon(v)$, where v is a vertex in the image triangulation T'_ϵ.

Before we can prove Lemma 51, we first must prove the following proposition.

Proposition 50. *Given a compact set $K \in \Omega$ there exists a constant C depending only on K such that for every vertex v that is adjacent to a vertex in K, the radius $r'_\epsilon(v)$ in the image of the circle packing map satisfies the following estimate, where R'_ϵ is the largest radius of any circle in the image packing that corresponds to a vertex adjacent to a vertex in K:*

$$r'_\epsilon(v) \geq CR'_\epsilon.$$

Proof. We argue similarly to the proof of Proposition 1 in [Rod89].

Let d denote the distance from K to the boundary of Ω, $d = d(K, \partial\Omega)$. For a given radius ϵ there is a natural number $N \in \mathbb{N}$ such that for any vertex $w \in K$, w can be surrounded by N generations of the regular hexagonal packing contained in Ω_ϵ. Let $H_{\epsilon,\delta}$ denote the hexagon determined by the convex hull of the centers of the circles of generation N about w, where δ is the "radius" of the hexagon $H_{\epsilon,\delta}$. That is, $\delta = 2N\epsilon$.

It follows that $\delta \leq d$ and furthermore, if we take ϵ small enough, we can ensure that $\delta \geq d/2$. To see this first note that $\delta \geq d - 2\epsilon$, for if not we can simply take N larger. If we take

$$\epsilon < \frac{1 - 1/\sqrt{3}}{2},$$

then $\delta \geq d/\sqrt{3}$. Now a hexagon with radius δ lies inside a ball of radius δ, but also the ball of radius $\delta\sqrt{3}/2$ is contained in the hexagon. Since $\delta \geq d/\sqrt{3}$, we have that

$$\frac{\sqrt{3}}{2}\delta \geq \frac{\sqrt{3}}{2}\frac{d}{\sqrt{3}} = \frac{d}{2}$$

and hence the ball of radius $d/2$ is contained in the hexagon of radius δ.

Next let $r'_\epsilon(w)$ be the radius corresponding to the vertex w in the image packing Ω'_ϵ and let r' be the radius of any other vertex in the image $f_\epsilon(H_{\epsilon,\delta})$. By the argument in [Rod89], there is some constant B such that

$$r' \geq e^{-2B\delta/d} r'_\epsilon(w) \geq e^{-2B} r'_\epsilon(w), \tag{9.1}$$

where the last inequality follows since $\delta \geq d/2$.

Now consider a vertex v adjacent to some vertex in K such that its radius in the image Ω'_ϵ is maximal, so $r'_\epsilon(v) = R'_\epsilon$. Take w to be any other vertex in K. The distance in Ω between the vertices v and w is less than the diameter D of Ω, so v and w can be connected by a path lying in overlapping hexagons $H_{\epsilon,\delta}$, where the number of such hexagons is no larger than D/δ. Since $\delta \geq d/2$, the number of hexagons is no larger than

$$\frac{D}{\delta} \leq \frac{2D}{d}.$$

Using (9.1) along the path of hexagons, we see that

$$r' \geq e^{-4DB/d} R'_\epsilon.$$

Letting $C := e^{-4DB/d}$ gives the result. □

Lemma 51. *The sequence $\{(|T'_\epsilon|, T_\epsilon)\}$ is a generalized triangulated exhaustion of $\mathbb{D}$.*

Proof. First of all, note that the second condition in Definition 36 is trivially satisfied since the carrier $|T'_\epsilon|$ is contained in $\mathbb{D}$ for every ϵ.

To show the first condition holds we use the Length-Area Lemma, Lemma 83 in §B.1 of the Appendix to show that the radii of boundary circles converge uniformly to 0, at which point it is clear that any compact $K \subset \mathbb{D}$ is contained in $|T'_\epsilon|$ for all ϵ sufficiently small. For the full argument showing the uniform convergence of radii of boundary circles, see page 252 of [Ste05].

Next fix a compact subset $V \subset \mathbb{D}$ and take ϵ small enough so that $V \subset |T'_\epsilon|$ for every ϵ. We want to show that every triangle in $|T'_\epsilon|$ that has at least one vertex in K satisfies the fullness condition.

Note that since the inverse circle packing maps f_ϵ^{-1} are continuous, the pre-image $f_\epsilon^{-1}(V)$ is compact. Define $K := f_\epsilon^{-1}(V)$. By Proposition 50, we know that if $\sigma = [v_0, v_1, v_2] \subset |T'_\epsilon|$ is a triangle with one of its vertices contained in V then the radii $r'_\epsilon(v_i)$ corresponding to the vertices of σ satisfy $r'_\epsilon(v_i) \geq CR'_\epsilon$.

At this point we can use Heron's formula to estimate the area A of σ as follows:

$$\begin{aligned} A &= \sqrt{r'_\epsilon(v_0) r'_\epsilon(v_1) r'_\epsilon(v_2) \left(r'_\epsilon(v_0) + r'_\epsilon(v_1) + r'_\epsilon(v_2)\right)} \\ &\geq \sqrt{3} C^2 R'^2_\epsilon. \end{aligned}$$

Since R'_ϵ is the largest radius of any vertex adjacent to a vertex in V, we have shown that each triangle σ that has at least one vertex in V is $(\sqrt{3}C^2, 2R'_\epsilon)$-full (recall Definition 14) and hence the sequence $\{(|T'_\epsilon|, T_\epsilon)\}$ is a generalized triangulated exhaustion of $\mathbb{D}$. □

Now that we have shown that there is a generalized triangulated exhaustion of $\mathbb{D}$, we next show that there is a proper admissible sequence for $(\Omega, \mathbb{D})$.

Lemma 52. *There is a proper admissible sequence $\{(|T_\epsilon|, |T'_\epsilon|, T_\epsilon, 0, \log r'_\epsilon)\}$ for $(\Omega, \mathbb{D})$.*

Proof. First of all, Ω and $\mathbb{D}$ are diffeomorphic embedded submanifolds of the complex plane $\mathbb{C}$, which is a Riemannian surface. We have already shown that $\{(|T_\epsilon|, T_\epsilon)\}$ and

$\{(|T'_\epsilon|, T_\epsilon)\}$ are generalized triangulated exhaustions of Ω and $\mathbb{D}$ respectively, with the same underlying (combinatorial) triangulation T_ϵ.

Note that $\{(T_\epsilon, C_{1,1}, 0, \log r'_\epsilon)\}$ satisfies the LDCR condition by the Hexagonal Packing Lemma and for each ϵ the piecewise flat surfaces $(|T_\epsilon|, T_\epsilon, 2\epsilon)$ in the domain and $(|T'_\epsilon|, T_\epsilon, \ell(\log r'_\epsilon))$ in the image both satisfy the circle packing version of the Ring Lemma, and hence obviously Condition 20.

The third condition in Definition 44 holds in this context by Corollary 59 in the next chapter. This result follows from a discrete analogue of the Schwarz Lemma.

The fourth condition follows immediately from the normalization assumption on the circle packing maps f_ϵ. Explicitly, the distinguished point z_0 is mapped under f_ϵ to a point within the flower of the image circle c'_0, which is centered at the origin. Using the Hexagonal Packing Lemma and the Ring Lemma, we could find a (closed) ball $\overline{B}_{R_\epsilon}(0)$ about the origin whose radius R_ϵ decreases with ϵ such that $f_\epsilon(z_0)$ lies within this closed ball. However, it is simpler to note that $f_\epsilon(z_0)$ will always lie within the unit disc $\mathbb{D}$ and the closure $\overline{\mathbb{D}}$ is compact in $\mathbb{C}$. Hence the image set $\{f_\epsilon(z_0)\}$ is contained in a compact subset of $\mathbb{C}$, as required.

At this point we have shown that the sequence $\{(|T_\epsilon|, |T'_\epsilon|, T_\epsilon, 0, \log r'_\epsilon)\}$ is an admissible sequence for $(\Omega, \mathbb{D})$. In addition, we want this admissible sequence to be *proper*. That is, we want the LDCR constants s_m to be such that $s_m \leq A/m$ for some positive constant A independent of m. Both Aharonov in [Aha94] and He in [He91] prove this result, with quite different proofs. □

Since we have shown that circle packing maps on Ω induce a proper admissible sequence for $(\Omega, \mathbb{D})$, the following corollary is a special case of the main result, Theorem 48.

Corollary 53. *Let $\Omega \subset \mathbb{C}$ be a simply connected bounded region in the plane and let $\{f_\epsilon\}$ be the sequence of circle packing maps into the unit disc $\mathbb{D}$. Then $\{f_\epsilon\}$ has a subsequence that converges uniformly on compact subsets of Ω to the Riemann mapping $f : \Omega \to \mathbb{D}$.*

Chapter 10
Future Work

The main theorems in this dissertation, Proposition 47 and Theorem 48, give a general proof of a convergence result under some fairly restrictive assumptions. As such, the most obvious path to strengthening these theorems is to try to remove some of the assumptions, ideally by proving them.

We here give as complete a list as possible of the assumptions we make for our main theorems, discussing each in turn.

10.1 Dimension

Our main theorems assume that we are working with *surfaces*, that is, Riemannian manifolds of dimension two.

Many of the proofs in this dissertation do not rely on the relevant manifolds being surfaces. We have tried to make explicit which results hold for higher dimensional manifolds and which do not.

Note that in dimensions higher than two, Karcher simplices may not be totally geodesic, even if their edges are geodesic segments. This can lead to these simplices being a little strange in dimensions higher than two. Specifically, Karcher simplices are convex hulls of their vertices if and only if the simplex is totally geodesic. None of our results require totally geodesic simplices, but this issue can make thinking about higher dimensions intuitively difficult.

One place we explicitly use the dimension of our manifold is in calculating the inverse metric g_{Δ}^{-1}. In two dimensions, this inverse is extremely easy to calculate, while in higher dimensions, it is more difficult In principle it should be possible to calculate this inverse in general, regardless of dimension. We would need to show that there is some estimate equivalent to (6.6) for higher dimensions, but there is no obvious reason such a thing would not exist.

The dimension of the manifold is perhaps most important in the Ring Lemma and the LDCR assumptions. It is an open question whether higher dimensional analogues of these two lemmas are true.

With regards to the Ring Lemma, one method of proof that only works for the circle packing case uses the Descartes Circle Theorem. (See Appendix B of [Ste05] for this argument.) The Descartes Circle Theorem can be proven in higher dimensions using linear algebra, as in C, so it might be that at least the circle packing Ring Lemma is true in a higher dimensional context.

As for the LDCR assumption, it is hard to say whether it holds in higher dimensions or not. The proof of LDCR in the cases for which we know it holds relies on uniqueness of an infinite packing or lattice but very little is known about even 3-dimensional (sphere) packings.

Another potentially serious problem with working in higher dimensions is that conformal maps on higher dimensional Riemannian manifolds tend to lose some of the flexibility we see on surfaces. There is a theorem in smooth conformal geometry that any conformal map on $\mathbb{R}^n$ for $n > 2$ is a Möbius transformation. See §3.8 of [GMP17] for a proof of this fact. Hence if we have some notion of a sequence of discrete conformal mappings on a manifold of dimension greater than two and we want this sequence to limit to a smooth conformal mapping (as happens in dimension two), then the limit of the sequence is necessarily a Möbius transformation. This suggests that the sequence has significantly less freedom than in the two-dimensional case.

10.2 The Mesh

This might just be the hardest batch of assumptions to either prove or check. We try to list them carefully.

Let Ω be a submanifold of (M, g). We want to know whether Ω admits a generalized triangulated exhaustion $\{(\Omega_n, T_n)\}$. It is easy to show the existence of at least one sequence of compact sets Ω_n that approach Ω in the sense that conditions 1 and 2 of Definition 36 are satisfied. For example, an exhaustion of Ω by compact subsets satisfies both of these conditions, the second one trivially.

The third condition, however, is harder to satisfy. For one thing, the fullness criterion on the triangulation T_n necessitates that the shortest edge in T_n can be no shorter than $\vartheta\epsilon_n$, as was shown in Lemma 32. In fact, a necessary but insufficient condition for T_n to be a (ϑ, ϵ_n)-full triangulation is that the ratio of any two edge lengths ℓ_{ij}/ℓ_{ab} must satisfy the following two-sided bound:

$$\vartheta \leq \frac{\ell_{ij}}{\ell_{ab}} \leq \frac{1}{\vartheta}.$$

Note that this bound is independent of n.

Proposition 54. *A two-dimensional Euclidean triangulation T is (ϑ, ϵ)-full if and only if every interior angle $\gamma_{i,jk}$ is bounded away from 0 and π and the ratio of any two edge lengths ℓ_{ij} and ℓ_{ab} satisfies the following inequality for some positive C:*

$$C \leq \frac{\ell_{ij}}{\ell_{ab}} \leq \frac{1}{C}. \tag{10.1}$$

Proof. Let the triangulation T be (ϑ, ϵ)-full and let $\ell_{ij}, \ell_{\alpha\beta} \in E(T)$ be edges. By Lemma 32, we have that

$$\vartheta\epsilon \leq \ell_{ij}, \ell_{\alpha\beta} \leq \epsilon,$$

which shows immediately that

$$\vartheta = \frac{\vartheta\epsilon}{\epsilon} \leq \frac{\ell_{ij}}{\ell_{\alpha\beta}} \leq \frac{\epsilon}{\vartheta\epsilon} = \frac{1}{\vartheta}$$

and hence (10.1) holds with $C = \vartheta$.

Next we show that in a (ϑ, ϵ)-full triangulation, interior angles $\gamma_{i,jk}$ are bounded away from 0 and π. To do so, let $\sigma = [v_0, v_1, v_2]$ be a triangle in T. Then the area of σ is $\text{Area}(\sigma) = (\ell_{01}\ell_{02}\sin\gamma_{0,12})/2$, where ℓ_{01}, ℓ_{02} are the lengths of the edges $e_{01} = [v_0, v_1]$ and $e_{02} = [v_0, v_2]$ respectively and $\gamma_{0,12}$ is the angle between e_{01} and e_{02}.

Since σ is (ϑ, ϵ)-full, its area satisfies

$$2\text{Area}(\sigma) \geq \vartheta\epsilon^2$$

and the edges lengths ℓ_{01} and ℓ_{02} are both less than or equal to ϵ, so we have

$$\begin{aligned} \epsilon^2 \sin\gamma_{0,12} &\geq \ell_{01}\ell_{02}\sin\gamma_{0,12} \\ &= 2\text{Area}(\sigma) \\ &\geq \vartheta\epsilon^2, \end{aligned}$$

so $\sin\gamma_{0,12} \geq \vartheta$ and hence $\gamma_{0,12} \in [\arcsin(\vartheta), \pi - \arcsin(\vartheta)]$.

To prove the other direction, assume that every interior angle $\gamma_{i,jk}$ lies within the interval $[A, \pi - A]$ for some A between 0 and π and the ratio of any two edge lengths satisfies (10.1) for some positive C. We want to show that there is some constant ϑ such that each simplex σ in the triangulation T is (ϑ, ϵ)-full.

Take ϵ to be the length of the longest edge in T and let $\sigma = [v_0, v_1, v_2]$ be a triangle in T. The area of σ is $(\ell_{01}\ell_{02}\sin\gamma_{0,12})/2$ and by assumption both ℓ_{01} and ℓ_{02} are bounded below by $C\epsilon$. further, $\gamma_{0,12}$ is bounded below by A, again by assumption. Thus we have

$$2\text{Area}(\sigma) = \ell_{01}\ell_{02}\sin\gamma_{0,12} \geq C^2\epsilon^2 A$$

and σ is (ϑ, ϵ)-full where $\vartheta = C^2 A$. □

We suspect that in general it is possible to find generalized triangulated exhaustions that satisfy the fullness criterion, at least for reasonably well-behaved manifolds. The success of polygonal meshes in computer graphics is good evidence for the truth of this condition, at least for reasonable surfaces embedded in $\mathbb{R}^3$. For more in-depth discussion of using these kinds of meshes in a practical setting, see the books [JGHW18] and [GY08]. In practice, it is likely going to be easiest to find a generalized triangulated exhaustion $\{(\Omega_n, T_n)\}$ by looking at the particular submanifold Ω and finding a sequence of triangulations specifically suited to it.

Once we have a generalized triangulated exhaustion on Ω, we can define discrete conformal maps ϕ_n on the corresponding piecewise flat manifolds $\Omega_{\Delta,n}$. For a given discrete conformal map ϕ_n, it is always possible to construct the image piecewise flat manifold $\widetilde{\Omega}_{\Delta,n} = \phi_n(\Omega_{\Delta,n})$, but we need more from $\widetilde{\Omega}_{\Delta,n}$ than just its existence.

We first need that $\widetilde{\Omega}_{\Delta,n}$ is (ϑ, ϵ_n)-full, which is not at all obvious and may not even be true in general. In the proof of the Rodin-Sullivan Theorem, we use the Hexagonal Packing Lemma to argue that as radii in the domain go to zero, triangles in the image become arbitrarily close to equilateral. An argument similar to this is likely always required to show that for n sufficiently large, simplices in the image are close enough to their corresponding simplices in the domain to be (ϑ, ϵ_n)-full.

We also need that there is an image submanifold $\widetilde{\Omega} \subset N$, where (N, h) is a Riemannian manifold and $\Omega, \widetilde{\Omega}$ are diffeomorphic, and a sequence of triangulations $\widetilde{T}_n$ such that each $\widetilde{T}_n$ is the Riemannian barycentric map $\widetilde{\Psi}$ and has as its underlying simplicial complex the piecewise flat manifold $\widetilde{\Omega}_{\Delta,n}$.

The existence of such triangulations $\widetilde{T}_n$ is not obvious. In the case of tangential circle packing, for example, this is the result of the uniformization theorem, Theorem 89. The proof of this theorem is long and difficult, spanning an entire chapter of [Ste05].

In general, more uniformization results are needed for discrete conformal geometry.

10.3 Discrete Conformal Structures

In this dissertation, we define discrete conformal structures by parametrizing them by $\alpha \in \mathbb{R}^{|V|}$ and $\eta \in \mathbb{R}^{|E|}$. Most of the time, specific discrete conformal structures are studied, so a particular paper will study one of: tangential circle packing, vertex scaling, or circle packings with separations or overlaps. One of the main advantages of our approach is that we can generalize from the most relevant examples.

However, this general framework of discrete conformal structures as we have defined them is not the only possible generalization. It turns out that if we consider two adjacent triangles embedded in $\mathbb{C}$ and examine the cross-ratio of their four vertices under various discrete conformal maps, we find that vertex scaling maps preserve the length-cross-ratio while circle packing maps preserve the angle of a certain cross-ratio, although the points involved are not the centers of circles. Bücking uses this insight in [Büc19] to find a parameter ϑ which interpolates between maps that preserve the magnitude of the cross ratio and maps which preserve the argument of the cross-ratio.

We give the definition here, changing notation as required to better match the rest of the dissertation.

Let

$$Q([v_i, v_j] := \operatorname{cr}(v_i, v_l, v_j, v_k) = \frac{(v_i - v_l)(v_j - v_k)}{(v_l - v_j)(v_k - v_i)},$$

where the vertices v_i, v_l, v_j, v_k are arranged as in Figure 10.1.

Similarly, let $\widetilde{Q}([v_i, v_j])$ be the same quantity but in the image instead of the domain. Now we are ready for the definition.

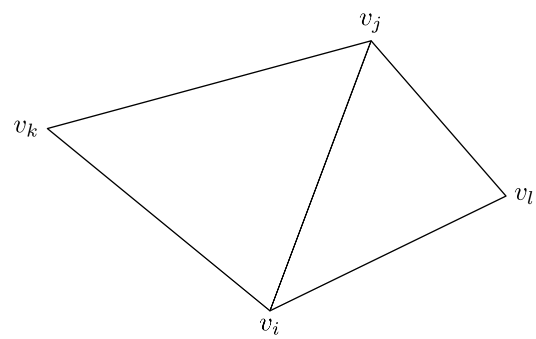

FIGURE 10.1. Configuration of the four vertices v_i, v_l, v_j, v_k.

Definition 55 (Definition 2.1 of [Büc19])**.** *The discrete immersion $F : T \to \mathbb{C}$ with image triangulation $\widetilde{T}$ is called* discrete ϑ-conformal *for a constant $\vartheta \in [0, \pi/2]$ if for all interior edges $[v_i, v_j]$ with adjacent triangles $[v_i, v_l, v_j]$ and $[v_i, v_j, v_k]$ the following equality holds*

$$\mathrm{Re}[e^{-i\vartheta} \log(Q([v_i, v_j]))] = \mathrm{Re}[e^{-i\vartheta} \log(\widetilde{Q}([v_i, v_j]))],$$

where the values of the logarithm are taken to be in $\mathbb{R} + i(0, 2\pi)$.

When $\vartheta = 0$, the length-cross-ratios in domain and image agree, and hence the two triangulations are related by a vertex scaling. When $\vartheta = \pi/2$, the arguments of the cross-ratios for adjacent triangles agree, and hence if circumcircles are added to each triangle, corresponding circle patterns can be found in image and domain. Recall that the original triangulation does not correspond to the pattern in the usual way, but instead each face of the triangulation is circumscribed by one of the circles.

Another approach to bringing several discrete conformal structures into one overarching framework is given in [BL23]. In this approach, discrete conformal structures are conceptualized as triangulations with *decorations*, that is, choices of circles at each vertex.

More formally, consider the following definitions.

Definition 56. *Let P be a euclidean polygon. A* decoration *of P is a choice of circle about each of its vertices.*

Definition 57. *Let (M, T, ℓ) be a piecewise flat manifold. A* decoration *of (M, T, ℓ) is a choice of decoration of each triangle of T such that it is consistent along edges of pairs of neighboring faces. The circles of the decoration are called* vertex-circles.

Clearly the vertex-circles are completely determined by their radii, so we can say that (M, T, ℓ, r) is a *decorated PL surface*, where r is the vector of radii. Next we define what it means for two decorated PL surfaces to be discretely conformally equivalent.

Definition 58. *Let* (M, T, ℓ, r) *and* $(M, T, \widetilde{\ell}, \widetilde{r})$ *be two decorated PL surfaces that are topologically and combinatorially equivalent. These two decorated PL surfaces are* discretely conformally equivalent *if there is some* $u \in V(T)^*$ *such that*

$$\widetilde{r}_i = e^{u_i} r_i \quad \text{and} \tag{10.2}$$

$$\widetilde{\ell}_{ij}^2 = \left(e^{2u_i} - e^{u_i+u_j}\right) r_i^2 + e^{u_i+u_j} \ell_{ij}^2 + \left(e^{2u_j} - e^{u_i+u_j}\right) r_j^2 \tag{10.3}$$

for every edge e_{ij}*.*

When r_i and r_j are both positive, this is an inversive distance discrete conformal structure and if r_i and r_j are both zero, this is a vertex scaling discrete conformal structure.

All three of these general frameworks (that is, the framework studied in this dissertation, the framework studied by Bücking in [Büc19], and the framework studied by Lutz and Bobenko in [BL23]) allow for proofs that are valid for multiple discrete conformal structures at once. We make no claims as to the superiority of any of the three frameworks. All three are interesting and have different flavors to their definitions and proofs. It seems likely that all three will lead to better understanding of discrete conformal structures in general.

10.4 Piecewise Hyperbolic or Spherical Manifolds

In this dissertation, we approximated smooth surfaces using piecewise flat ones. We mostly did this for convenience, because hyperbolic and spherical discrete conformal structures are more complicated than their Euclidean counterparts. However, most manifolds are not flat and it seems likely that, for example, a hyperbolic manifold might be better approximated by a piecewise hyperbolic manifold than by a piecewise flat one.

In the case of a piecewise constant (nonzero) curvature manifold, one concern is that we would have to replace the Riemannian barycentric maps Ψ_n with a map from a simplex in a constant curvature space to the manifold. These new maps would likely have to be a composition of a map $x : \Delta \to M$ and the inverse of a map $y : \Delta \to \mathbb{G}$, where $\mathbb{G}$ is either the hyperbolic plane $\mathbb{H}$ or the sphere $\mathbb{S}$. That is, $\Psi_n := x \circ y^{-1}$. Our metric estimates would likely get slightly worse as a result (more factors of $(1 - \beta\epsilon_n)$ or $(1 + \beta\epsilon_n)$), but beyond that, not much would change.

Another concern is that in order to prove Theorem 42, the metric estimate on the discrete conformal maps ϕ_n, we use the flat metric $g_{ij}^{\Delta} = \langle p_i - p_0, p_j - p_0 \rangle$. This metric is not the right metric for a hyperbolic or spherical triangle. In fact, the whole proof of the metric estimate in Theorem 42 relies on results about Euclidean simplices and Euclidean inner products which would need to be proven separately in other geometries.

In principle, the argument in this dissertation would work just as well with piecewise hyperbolic or spherical manifolds, the proofs would just be more complicated and the estimates slightly worse. However, it is not at all clear that the added complication of working with piecewise hyperbolic or spherical manifolds is worth it, especially since hyperbolic and spherical triangles approach Euclidean triangles as their edge lengths decrease.

10.5 The LDCR Assumption and the Ring Lemma

The fact that both of these lemmas are true in both circle packing and vertex scaling discrete conformal structures suggests that they are probably true in general.

The ring lemma is particularly intriguing. There does seem to be some underlying geometric truth to it. It makes intuitive sense that the ratio of lengths of outside edges to lengths of spokes would have a lower bound depending on how many spokes there are. Of course, the ratio of exponentials of discrete conformal factors usually is not exactly a ratio of edge lengths, but they more or less encode the same information.

But the proofs of the ring lemma in the two cases of circle packing and vertex scaling are wildly different and even seem to use fundamentally different geometric properties. The proof of the circle packing ring lemma uses the concept of univalence of a packing, that is, that the intersections of the interiors of discs in the packing are empty. The vertex scaling ring lemma uses the triangle inequality for its proof. Somehow the geometric property that we want to measure corresponds to disjoint disc interiors for circle packing and a satisfied triangle inequality in vertex scaling. It is hard to say what it might be in the general case.

The Local Discrete Conformal Rigidity assumption is more technical but maybe more straightforward because of it. As long as we have the right kind of rigidity result and a ring lemma, LDCR follows from a diagonalization argument like that in Section 5.1. Another example of this kind of argument is the proof of Lemma 20.3 of [Ste05].

10.6 Bound on Ratio of Discrete Conformal Factors

In this section we briefly discuss the bound $H_K \geq H_n(v) \geq 1/H_K$ for every vertex v in a given compact set K. The upper bound on $H_n(v)$ follows directly from a discrete Schwarz Lemma analogue and the lower bound follows from simply switching the role of image and domain.

Recall that the smooth Schwarz Lemma says that a conformal self-map of the unit disc $\mathbb{D}$ is a contraction. The discrete analogue will say that the ratio of exponentials of discrete conformal factors at the origin is bounded above. Since in tangential circle packing the ratio of exponentials of discrete conformal factors (ratio of radii in this case) approaches the magnitude of the derivative (see §B.3), the Discrete Schwarz Lemma approaches the classical Schwarz Lemma in the limit, at least in the circle packing case.

We here state the circle packing versions of the Discrete Schwarz Lemma and the upper bound on $H_n(v)$, proving the latter, before proceeding to the general case.

Theorem 59 (Circle Packing Schwarz Lemma, Thm 5.1 of [Rod87])**.** *There is an absolute constant α with the following property. Let HCP_m be m generations of the regular hexagonal circle packing. Let D be the smallest disc which contains HCP_m. Let HCP'_m be any circle packing combinatorially equivalent to HCP_m and also contained in D. Then*

$$R'_0 \leq \alpha R_0, \tag{10.4}$$

where R_0 and R'_0 are the radii of the generation zero circles in HCP_m and HCP'_m respectively.

Corollary 60 (Theorem 6.2 of [Rod87])**.** *Let $K \subset \Omega$ be compact. There is a constant H_K with the following property. Let $\epsilon > 0$ be sufficiently small and let $c \mapsto c'$ be the circle packing isomorphism of an ϵ-circle packing approximation Ω_ϵ of Ω onto a suitably normalized circle packing D_ϵ of the unit disc $\mathbb{D}$. Then $R(c')/R(c) \leq H_K$ for all circles c of Ω_ϵ which intersect K.*

Proof. Consider a circle c in Ω_ϵ such that $c \cap K \neq \emptyset$. Let m be maximal with respect to the property that Ω_ϵ contains a copy of HCP_m centered at c and take Δ to be the smallest disc containing this HCP_m.

The circle packing isomorphism $\Omega_\epsilon \to D_\epsilon$ gives us a corresponding HCP'_m contained in the unit disc $\mathbb{D}$. Let λ be the radius of the disc Δ and rescale the unit disc $\mathbb{D}$ so it has the same radius as Δ. Applying Theorem 59 gives

$$\lambda R(c') \leq \alpha R(c).$$

Next we can assume ϵ is as small as we like, so in particular we can take $\epsilon < 1/4d(K, \mathbb{C} \setminus \Omega)$. With this bound on ϵ, λ is bounded below by

$$\frac{1}{2} d(K, \mathbb{C} \setminus \Omega) \leq \lambda,$$

for if not, then $\lambda < d(K, \mathbb{C} \setminus \Omega)/2$ and so

$$\begin{aligned} \lambda + 2\epsilon &< \frac{1}{2} d(K, \mathbb{C} \setminus \Omega) + 2\epsilon \\ &< d(K, \mathbb{C} \setminus \Omega). \end{aligned}$$

But this is a contradiction because we assumed that Δ is the smallest disc containing HCP_m. Hence $\lambda \geq d(K, \mathbb{C} \setminus \Omega)/2$ and the ratio of radii becomes

$$\frac{R(c')}{R(c)} \leq \frac{\alpha}{\lambda} \leq \frac{2\alpha}{d(K, \mathbb{C} \setminus \Omega)}$$

and we have the result, with $H_K := 2\alpha/d(K, \mathbb{C} \setminus \Omega)$. □

The equivalent of the circle packing discrete Schwarz Lemma would be the following conjecture.

Condition 61 (Discrete Schwarz Lemma)**.** *There is a constant α independent of v and m with the following property. Let $D_m(v)$ be a realized closed combinatorial disc of generation m and let $R > 0$ be the radius of the smallest geodesic ball $B_R(v)$ containing $D_m(v)$. Let Φ be a barycentric discrete conformal map scaled so that the image $\Phi(D_m(v))$ is contained in a geodesic ball $B_R(\Phi(v))$ with the same radius R and centered at $\Phi(v)$. Then*

$$e^{\overline{f}(v)} \leq \alpha e^{f(v)}$$

where $\overline{f}(v)$ and $f(v)$ are discrete conformal factors at v in $\Phi(D_m(v))$ and $D_m(v)$ respectively.

Assuming the above condition is true, the following corollary is an easy consequence.

Corollary 62. *Let $K \subset \Omega$ be compact and choose R such that $B_R(p)$ is a geodesic ball for every $p \in K$. Let v be a vertex in K and assume m is maximal with respect to the property that there exists a realized closed combinatorial disc $D_m(v)$ of generation m centered at v with $D_m(v)$ completely contained in $B_R(v)$. Then $H_n(v) \leq \alpha/R$, where α and R are independent of m.*

Proof. Rescaling the image $\phi(D_m(v))$, we have by the Discrete Schwarz Lemma Condition that $Re^{\overline{f}_n(v)} \leq ae^{f_n(v)}$. Hence $H_n(v) = e^{\overline{f}_n(v)}/e^{f_n(v)} \leq \alpha/R$, as required. □

The above corollary immediately implies that $H_n(v)$ is bounded above by a constant H_K depending only on the compact set K, which is what we need in order for the third condition on an admissible sequence (Definition 44) to hold.

10.7 The LDCR Constants s_m

In order to prove that e_n^F converges to some continuous function e^F, we needed to assume that the LDCR constants s_m are of order $1/m$. In tangential circle packing with bounded valence we know this is true. However, these constants do not seem to be much studied in other discrete conformal structures.

Are they still of order $1/m$ in other contexts? We conjecture that they are, mainly because we believe that discrete conformal maps converge to smooth conformal maps and $s_m = O(1/m)$ is integral in proving the last piece of Theorem 48, that is, that Φ_n is "nearly conformal" in the sense that there is some positive function e^F such that the metric $\Phi_n^* h$ approaches the metric $e^F g$ uniformly on compact subsets.

Appendix A

Background in Analysis and Differential Geometry

A.1 The Arzelà-Ascoli Theorem

The Arzelà-Ascoli Theorem is a standard result in analysis. We state it here for completeness, following the treatment in [Rud86].

Definition 63. *Let $\mathcal{F}$ be a collection of complex functions on a metric space X with metric ρ.*

We say that $\mathcal{F}$ is equicontinuous *if to every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $f \in \mathcal{F}$ and for all pairs of points x, y with $\rho(x, y) < \delta$. (In particular, every $f \in \mathcal{F}$ is uniformly continuous.)*

We say that $\mathcal{F}$ is pointwise bounded *if to every $x \in X$ corresponds an $M(x) < \infty$ such that $|f(x)| \leq M(x)$ for every $f \in \mathcal{F}$.*

Theorem 64 (Theorem 11.28 of [Rud86])**.** *Suppose that $\mathcal{F}$ is a pointwise bounded equicontinuous collection of complex functions on a metric space X, and that X contains a countable dense subset E. Then every sequence $\{f_n\}$ has a subsequence that converges uniformly on every compact subset of X.*

Notice that this theorem is only stated for complex-valued functions over a metric space. This is not the context in which we use it, so here we give a proof of the more general form that is used in the proofs of Corollaries 48 and 39.

Proposition 65. *Let M and N be smooth manifolds and let $\mathcal{F}$ be an equicontinuous collection of mappings $f : M \to N$ such that for any point $p \in M$, the set $\{f(p) : f \in \mathcal{F}\}$ is contained in a compact subset of N. Then every sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence that converges uniformly on every compact subset of M.*

Proof. The first half of this proof is a classic diagonalization argument.

Let $E \subset M$ be countable and dense. Such a subset exists because M is a manifold. Enumerate E as $E = \{x_1, x_2, \dots\}$. Consider $\{f_n(x_1)\}$. We assumed this set is contained in a compact subset of N, so there is a subsequence $\{f_{n_1}(x_1)\} \subset \{f_n(x_1)\}$ that converges in N. Pick out the corresponding subsequence of functions $\{f_{n_1}\}$, and consider the subsequence of points $\{f_{n_1}(x_2)\}$. Again, $\{f_{n_1}(x_2)\}$ is a subset of a compact set, so we have another convergent subsequence $\{f_{n_2}(x_2)\} \subset \{f_{n_1}(x_2)\}$ and this gives us another subsequence of functions $\{f_{n_2}\} \subset \{f_{n_1}\}$.

We proceed in this manner, at each step looking at the sequence $\{f_{n_k}(x_{k+1})\}$ of images and extracting a convergent subsequence which then gives us a sequence of functions from which we can extract the next subsequence of images.

Finally we are left with a nested sequence of subsequences of functions $\{f_n\} \supset \{f_{n_1}\} \supset \{f_{n_2}\} \supset \ldots$ from which we will draw our final convergent subsequence f_{r_k} as follows.

Let f_{r_1} be the first element of $\{f_{n_1}\}$, f_{r_2} the second element of $\{f_{n_2}\}$, and, in general, f_{r_k} the kth element of $\{f_{n_k}\}$. Then since $\{f_{r_k}(x_i)\}$ converges for every x_i as long as $i \geq k$, the sequence $\{f_{r_k}\}$ converges on all of E.

For the second half of the proof, we use the equicontinuity of $\mathcal{F}$ as well as the denseness of E in M.

Let $K \subset M$ be a compact subset and choose $\epsilon > 0$. By equicontinuity of $\mathcal{F}$, there exists a $\delta > 0$ such that if $p, q \in M$ with $d_M(p,q) < \delta$, then $d_N(f(p), f(q)) < \epsilon/3$ for every $f \in \mathcal{F}$.

Take a finite open cover of K by balls of radius $\delta/2$. We can do this because K is compact. Since there are finitely many such balls, we can label them as $B_1, B_2, \ldots, B_L$. Further, E is dense in M, so for each B_i there exists at least one point $p_i \in B_i \cap E$. We already showed that $\lim_{k\to\infty} f_{r_k}(p_i)$ exists for each i (since $p_i \in E$), so there exists a number $N \in \mathbb{N}$ such that $d_N(f_{r_m}(p_i), f_{r_n}(p_i)) < \epsilon/3$ for $m, n > N$.

Let $x \in K$ and let $\epsilon > 0$. Choose δ small enough so that $d_N(f(x), f(y)) < \epsilon/3$ for every y with $d_M(x,y) < \delta$ and choose N large enough so that for every $x_i \in E$, $d_N(f_{r_m}(x_i), f_{r_n}(x_i)) < \epsilon/3$ for $m, n > N$.

Since $x \in K$ and $\{B_i\}_{i=1}^{L}$ covers K, $x \in B_i$ for some i and furthermore, $d(x, p_i) < \delta$ since B_i has radius $\delta/2$ and $p_i, x \in B_i$.

We can calculate:

$$\begin{aligned} d(f_{r_m}(x), f_{r_n}(x)) &\leq d(f_{r_m}(x), f_{r_m}(p_i)) + d(f_{r_m}(p_i), f_{r_n}(p_i)) + d(f_{r_n}(p_i), f_{r_n}(x)) \\ &< \qquad \epsilon/3 \qquad\qquad\qquad + \epsilon/3 \qquad\qquad\qquad + \epsilon/3 \\ &= \epsilon \end{aligned}$$

□

A.2 Convergence in L^∞

We follow the notation and treatment in [Fol99].

Definition 66. *Let f be a measurable function on a set X with measure μ. The* essential supremum of $|f|$, $||f||_\infty$ *is defined to be*

$$||f||_\infty = \inf \{a \geq 0 \, : \, \mu(\{x : |f(x)| > a\}) = 0\},$$

with the convention that $\inf \emptyset = \infty$*.*

We now define

$$L^\infty = \{f : X \to \mathbb{C} : f \text{ is measurable and } ||f||_\infty < \infty\},$$

with the usual convention that two functions that are equal a.e. define the same element of L^∞.

Proposition 67. $||\cdot||_\infty$ *is a norm on* L^∞.

Proposition 68. $||f_n - f||_\infty \to 0$ *if and only if there exists a measurable set* E *such that* $\mu(E^c) = 0$ *and* $f_n \to f$ *uniformly on* E.

A.3 Estimates on Euclidean Inner Products

Lemma 69 (Lemma 3 of [DGW16])**.** *Let* $p_0, \ldots, p_n \in \mathbb{R}^m$ *be the vertices of a* (ϑ, ϵ)*-full Euclidean* n*-simplex and let* $g^\Delta_{ij} = \langle p_i - p_0, p_j - p_0 \rangle_{\mathbb{R}^m}$ *denote the pullback of its metric to the unit simplex* D*. Then the eigenvalues* λ_k *of* g^Δ *satisfy*

$$\vartheta \epsilon n^{1-n} \leq \sqrt{\lambda_k} \leq \epsilon n.$$

Proof. Let A be the matrix with columns $p_i - p_0$. Notice that $g^\Delta = A^t A$ and hence the eigenvalues of g^Δ are the squared singular values of the matrix A. (See, for example, Theorem 2.6.3 in [HJ13].)

Let $||\cdot||_2$ be the matrix norm induced by the Euclidean norm on $\mathbb{R}^n$. We first show that $||A||_2 \leq n\epsilon$.

It is well-known that for any Euclidean n-simplex s, the radius r of its insphere satisfies

$$\text{vol}_n(s) = \frac{r}{n}\text{vol}_{n-1}(\partial s). \tag{A.1}$$

This is because an n-simplex s^n can be partitioned into $n+1$ simplices (of the same dimension), where each of these smaller simplices has as its base a face in the boundary ∂s^n and the altitudes of each of these smaller n-simplices are all radii of the insphere.

The following formula can be used to find the n-volume of a n-simplex given an embedding of it into $\mathbb{R}^{n+1}$. See [Ste66] for a proof and more discussion.

$$\text{vol}_n(s) = \frac{(-1)^n}{n!} \det \begin{bmatrix} v_1 - v_0 & \ldots & v_n - v_0 \end{bmatrix} \tag{A.2}$$

Next we find the n-volume of the unit simplex D^n and the $(n-1)$-volume of its boundary ∂D^n. Using straightforward induction arguments and some linear algebra, as well as the formula (A.2), we find that

$$\text{vol}_n(D^n) = \frac{1}{n!} \qquad \text{and} \qquad \text{vol}_{n-1}(\partial D^n) = \frac{n + \sqrt{n}}{(n-1)!}. \tag{A.3}$$

Combining (A.1) and (A.3), we see that the radius of the insphere of D^n is

$$r = \frac{1}{n + \sqrt{n}} \leq \frac{1}{2n}.$$

Since the insphere of a simplex is completely contained in the simplex, any vector $v \in TD^n$ with length less than or equal to $2r$ can be represented as $p - q$ with $p, q \in D^n$. In particular, let $|v| = 1/n$ and let $p, q \in D$ be such that $v = p - q$.

The image of v under A, $Av = Ap - Aq$, must lie entirely in the simplex s, so $|Ap - Aq| \leq \mathrm{diam}(s)$. For Euclidean simplices, their diameter is simply the length of their longest edge, so

$$||A||_2 \leq \frac{|Av|}{|v|} \leq \frac{\epsilon}{1/n} = n\epsilon.$$

Next let $A = U\Sigma V^*$ be the singular value decomposition of A. Then U and V are unitary matrices, so

$$||A||_2 = ||U^* AV||_2 = ||\Sigma||_2.$$

Note that every eigenvalue of a matrix has absolute value less than the norm of the matrix, for any matrix norm. Hence in particular, $|\sigma_i| \leq ||\Sigma||_2$ for any eigenvalue σ_i of Σ.

Now since $||\Sigma||_2 = ||A||_2 \leq n\epsilon$, the eigenvalues σ_i of Σ are all such that $|\sigma_i| \leq n\epsilon$. But the eigenvalues σ_i are exactly the singular values of A, so g^Δ has as eigenvalues $\lambda_k = \sigma_k^2 \leq n^2\epsilon^2$. This gives us the upper bound on the largest eigenvalue.

For the lower bound on small eigenvalues, first note that since $\lambda_{\max} \leq (n\epsilon)^2$,

$$\lambda_{\min}(n\epsilon)^{2n-2} \geq \lambda_{\min}\lambda_{\max}^{n-1}.$$

Every eigenvalue λ_k is less than or equal to $\lambda_{\max}$ and there are at most n eigenvalues, one of which is $\lambda_{\min}$. Hence

$$\lambda_{\min}\lambda_{\max}^{n-1} \geq \Pi_{i=1}^n \lambda_i = \det g^\Delta.$$

Finally, we need to show that $\det g^\Delta \geq \vartheta^2\epsilon^{2n}$, which is a straightforward calculation using (A.2) and remembering that the simplex was assumed to be (ϑ, ϵ)-full.

□

Lemma 70 (Lemma 6 of [DGW16])**.** *Let g and $\overline{g}$ be inner products on $\mathbb{R}^n$ such that all eigenvalues of g (with respect to the Euclidean inner product) are larger than $\lambda_{min} > 0$ and $|g_{ij} - \overline{g}_{ij}| \leq \mu n^{-1}\lambda_{min}$. Then $|(g - \overline{g})(v, v)| \leq \mu|v|_g^2$.*

Proof. We calculate as follows:

$$\begin{aligned}
|(g-\overline{g})(v,v)| &= \left|(g-\overline{g})_{ij}v^i v^j\right| \\
&\le |(g-\overline{g})_{ij}|\,|v^i||v^j| \\
&\le \sum_{i,j} \mu n^{-1}\lambda_{min}|v^i||v^j| \\
&\le \sum_{i,j} \mu n^{-1}\lambda_{min}\frac{1}{2}\left(|v^i|^2+|v^j|^2\right) \\
&= \mu\lambda_{min}\sum_i |v^i|^2 \\
&\le \mu g(v,v).
\end{aligned}$$

□

A.4 Integrating Pullback Metrics

A.4.1 Manifolds with Boundary

If $A \subset \mathbb{R}^n$ is an arbitrary subset (not necessarily open), then a function $F : A \to \mathbb{R}^m$ is said to be *smooth on A* if it admits a smooth extension to an open neighborhood of each point, or more precisely, if for every $x \in A$, there exists an open subset $U_x \subset \mathbb{R}^n$ containing x and a smooth function $\widetilde{F} : U_x \to \mathbb{R}^m$ that agrees with F on $U_x \cap A$.

Diffeomorphisms on arbitrary subsets are then defined in the obvious way. That is, given arbitrary subsets $A, B \in \mathbb{R}^n$ a *diffeomorphism from A to B* is a smooth bijective map $f : A \to B$ with smooth inverse.

Now let M be a topological manifold with boundary. That is, M is a second-countable Hausdorff space in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$. An open subset $U \subset M$ together with a map $\phi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a *chart for M*.

We can make the distinction between an *interior chart*, which is a chart for which $\phi(U)$ is an open subset of $\mathbb{R}^n$ and a *boundary chart*, which is a chart for which $\phi(U)$ is an open subset of $\mathbb{H}^n$ and $\phi(U) \cap \partial\mathbb{H}^n \neq \emptyset$.

Combining what we know about smooth functions on (possibly not open) subsets of $\mathbb{R}^n$ with our definitions about topological manifolds with boundary, we can now talk about smooth manifolds with boundary.

Let $U \subset \mathbb{H}^n$ be open relatively to $\mathbb{H}^n$ and let $F : U \to \mathbb{R}^k$ be a smooth map. Then for every $x \in U$, there exists a neighborhood U_x containing x and open in $\mathbb{R}^n$ and a smooth map $\widetilde{F} : U_x \to \mathbb{R}^k$ such that $\widetilde{F}$ agrees with F on $U_x \cap \mathbb{H}^n$.

Since $\widetilde{F} : U_x \to \mathbb{R}^k$ is smooth in the usual sense, that means that its partial derivatives $\partial\widetilde{F}/\partial x^i$ are continuous on U_x. Let p approach x along a path contained in $U_x \cap \operatorname{int} \mathbb{H}^n$. Then since the functions $\partial \widetilde{F}/\partial x^i$ are continuous and $F = \widetilde{F}$ on $U_x \cap \operatorname{int} \mathbb{H}^n$,

$$\left.\frac{\partial \widetilde{F}}{\partial x^i}\right|_x = \lim_{p \to x} \left.\frac{\partial \widetilde{F}}{\partial x^i}\right|_p = \lim_{p \to x} \left.\frac{\partial F}{\partial x^i}\right|_p .$$

Hence partial derivatives of $\widetilde{F}$ at points $x \in U \cap \partial \mathbb{H}^n$ are determined by partial derivatives of F on $U \cap \operatorname{int} \mathbb{H}^n$. In particular, they are independent of the choice of extension.

Finally we can say that if M is a topological manifold with boundary, then a *smooth structure on M* is a maximal smooth atlas, where charts in the atlas are smooth in the sense just defined.

A.4.2 Pullback Metrics on Smooth Manifolds

This section takes definitions from [Lee18] and [Lee12].

Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map and A a covariant k-tensor field on N. For every $p \in M$, we define a tensor $dF_p^*(A) \in T^k(T_p^*M)$, called the *pointwise pullback of A by F at p*, by

$$dF_p^*(A)(v_1, \ldots, v_k) = A(dF_p(v_1), \ldots, dF_p(v_k))$$

for $v_1, \ldots, v_k \in T_pM$ and we define the *pullback of A by F* to be the tensor field F^*A on M defined by

$$(F^*A)_p = dF_p^*(A_{F(p)}).$$

As long as F is a smooth map and A is a smooth tensor field, F^*A is also a smooth tensor field.

Riemannian metrics are smooth covariant 2-tensor fields, so if g is a Riemannian metric on N and $F : M \to N$ is smooth, then the pullback F^*g is a smooth tensor field on M. This pullback might not be positive definite, but whenever it is, it called the *pullback metric* determined by F. The following proposition gives a criterion for when this is true.

Proposition 71. *Suppose $F : M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a Riemannian metric on M if and only if F is a smooth immersion.*

Proof. First assume that F is a smooth immersion. Then its differential is injective at every point on M and hence dF_pX_p is only the zero vector when X_p is. Thus at a point $p \in M$,

$$F^*g_p(X_p, X_p) = g_p(dF_pX_p, dF_pX_p) \geq 0$$

for all X_p, with equality precisely when X_p is the zero vector.

Next assume that F^*g is positive definite and let $X_p \in T_pM$ be such that $dF_pX_p = 0$. We want to show that X_p is the zero vector. Since $dF_pX_p = 0$, we have that $g_p(dF_pX_p, dF_pX_p) = 0$

$$\begin{aligned} 0 &= g_p(dF_pX_p, dF_pX_p) \\ &= F^*g_p(X_p, X_p), \end{aligned}$$

but F^*g was assumed to be positive definite, so X_p must be the zero vector. □

A.4.3 Pullback Metrics and Manifolds with Boundary

Proposition 72. *Let (M, g) and (N, h) be homeomorphic manifolds and let $F : M \to N$ be a homeomorphism. Suppose that (M, g) admits a finite triangulation T with geodesic edges and that $F|_\sigma$ is a diffeomorphism when restricted to any (closed) simplex $\sigma \in T$, treating σ as a manifold with boundary.*

Suppose also that on each simplex σ, $F|_\sigma^* h$ is close to g in the sense that for any $X \in T\sigma$,

$$|X|_{F|_\sigma^* h} \leq C|X|_g \tag{A.4}$$

for some constant C which does not depend on σ.

Then for any points $p, q \in M$,

$$d_h(F(p), F(q)) \leq C d_g(p, q).$$

Proof. Let $\gamma : [0, 1] \to M$ be a minimizing geodesic with $\gamma(0) = p$ and $\gamma(1) = q$ and let $\{\sigma_i\}$ be the (finite) collection of simplices which have nonempty intersection with γ. Each σ_i intersects γ either in a point or in a curve segment.

If $\sigma_i \cap \gamma$ is a point, then that point lies in some other simplex σ_j such that $\sigma_j \cap \gamma$ is a curve segment. Hence simplices whose intersections with γ are points can be safely ignored.

If $\sigma_i \cap \gamma$ is a curve segment, then either that curve segment lies completely in a subsimplex of σ_i or it passes through the interior of σ_i. If $\sigma_i \cap \gamma$ is completely contained in a subsimplex and that subsimplex is not on the boundary of the triangulation T, then the subsimplex is shared by at least two simplices. Choose one arbitrarily and discard the rest.

After discarding simplices as detailed above, we are left with a list $\{\sigma_i\}_{i=1}^Q$ of simplices that we can order according to the order in which γ passes through them. Assume this has been done and let $\{t_i\}_{i=1}^{Q-1}$ be the list of t-values where γ leaves one simplex and enters the next. Let $t_0 = 0$ and $t_Q = 1$.

Now $F(\gamma)$ is a piecewise smooth curve in N connecting $F(p)$ and $F(q)$, so the distance $d_h(F(p), F(q)) \leq \ell(F(\gamma))$, and we can estimate as follows:

$$\begin{aligned} d_h(F(p), F(q)) &\leq \ell(F(\gamma)) \\ &= \sum_{i=0}^{Q-1} \int_{t_i}^{t_{i+1}} \left|(F \circ \gamma)'(t)\right|_h dt \\ &= \sum_{i=0}^{Q-1} \int_{t_i}^{t_{i+1}} \left|\gamma'(t)\right|_{F^*h} dt \\ &\leq \sum_{i=0}^{Q-1} \int_{t_i}^{t_{i+1}} C|\gamma'(t)|_g \, dt \\ &= C \int_0^1 |\gamma'(t)|_g \, dt \\ &= C d_g(p, q). \end{aligned}$$

□

A.5 Complete Manifolds and the Heine-Borel Property

Recall the Hopf-Rinow Theorem:

Theorem 73 (Theorem 6.19 of [Lee18])**.** *A connected Riemannian manifold is metrically complete if and only if it is geodesically complete.*

Recall also that a metric space X is said to *have the Heine-Borel property* if every closed and bounded subset of X is compact.

We now prove that connected complete manifolds have the Heine-Borel property.

Lemma 74. *Every closed and bounded subset of a connected, complete Riemannian manifold is compact.*

Proof. Let (M, g) be a connected, complete Riemannian manifold and let $K \subset M$ be a closed and bounded subset. Let $p \in K$. Since K is bounded, there is some $R > 0$ such that for any $q \in K$, $d_g(p, q) \leq R$. Further, M is (geodesically) complete, so there is a minimizing geodesic $\gamma : [0, 1] \to M$ from p to q.

Since M is (geodesically) complete, we have in particular that the exponential map at p, $\exp_p$ is defined on all of T_pM. Hence the geodesic γ can be written as $\gamma(t) = \exp_p(tV)$, where $V = \gamma'(0) \in T_pM$. The vector V has length $|V|_g$ exactly equal to the distance $d_g(p, q)$ and hence $V \in \overline{B_R(0)} = \{v \in T_pM : |v|_g \leq R\}$, which is a closed Euclidean ball and hence clearly compact.

Now $q = \exp_p(V) \in \exp_p\left(\overline{B_R(0)}\right)$ and this is true for any q in the compact set K, so in fact, $K \subset \exp_p\left(\overline{B_R(0)}\right)$. Since $\exp_p$ is a diffeomorphism, it is in particular continuous, and hence the image of the compact ball $\overline{B_R(0)}$ is also compact and hence K is a closed subset of a compact set and thus compact itself. □

A.6 The Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem is one of the most important theorems about Riemannian surfaces. Here is the statement:

Theorem 75 (Theorem 9.7 of [Lee18]). *If (M, g) is a smoothly triangulated compact Riemannian 2-manifold, then*

$$\int_M K\,dA = 2\pi\chi(M),$$

where K is the Gaussian curvature of g and dA is its Riemannian density.

The usual proof of the Gauss-Bonnet Theorem is by way of the Gauss-Bonnet Formula:

Theorem 76 (Theorem 9.3 of [Lee18]). *Let (M, g) be an oriented Riemannian 2-manifold. Suppose γ is a positively oriented curved polygon in M, and Ω is its interior. Then*

$$\int_\Omega K\,dA + \int_\gamma \kappa_N\,ds + \sum_{i=1}^{k} \varepsilon_i = 2\pi, \tag{A.5}$$

where K is the Gaussian curvature of g, dA is its Riemannian volume form, $\varepsilon_1, \ldots, \varepsilon_k$ are the exterior angles of γ, and the second integral is taken with respect to arc length.

The integrand κ_N in (A.5) is the *signed curvature* of γ and is defined by

$$\kappa_N := \langle D_t\dot{\gamma}(t), N(t)\rangle,$$

where $N(t)$ is the inward pointing normal to $\partial\Omega$. Note that κ_N is only defined at points where γ is smooth.

As with so much of smooth manifold theory, the Gauss-Bonnet Theorem has an analogue in the discrete setting. Before we can state it, we need to define discrete curvature. Note that the following definition is only valid for two-dimensional manifolds. In higher dimensions, curvature is concentrated at co-dimension two manifolds. For an example of higher dimensional discrete curvatures, see [Gli11] and [CGY10].

Definition 77. *Let (M, T, ℓ) be a piecewise constant curvature manifold. Then the* discrete curvature K_i *at vertex v_i is equal to*

$$K_i = 2\pi - \sum_{j,k} \gamma_{i,jk}, \tag{A.6}$$

where $\gamma_{i,jk}$ is the interior angle at v_i in the triangle $[v_i, v_j, v_k]$.

Theorem 78 (Discrete Gauss-Bonnet Theorem). *Let (M, T, ℓ) be a compact, connected piecewise constant curvature surface, with curvature $\lambda \in \mathbb{R}$. Then the total curvature satisfies*

$$\sum_{i=1}^{N} K_i = 2\pi\chi(M) - \lambda Area(M). \tag{A.7}$$

Proof. Let θ_i^{jk} be the interior angle of the triangle $\sigma = \{i, j, k\}$ at vertex v_i. Then since σ is isometric to a geodesic triangle embedded in some smooth manifold with constant curvature λ, we can use the Gauss-Bonnet Formula (A.5) to relate the area A_{ijk} to the sum of the interior angles in the triangle:

$$\begin{aligned}
&\int_\sigma \lambda \, dA + \int_\gamma \kappa_N \, ds + 3\pi - (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) = 2\pi \\
&\implies \lambda A_{ijk} + 0 + 3\pi - (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) = 2\pi \\
&\implies (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) = \lambda A_{ijk} + \pi.
\end{aligned}$$

Now we add together the vertex curvatures at each vertex:

$$\begin{aligned}
\sum_{i=1}^{N} K_i &= \sum_{i=1}^{N} (2\pi - \sum_{j,k} \theta_i^{jk}) \\
&= 2\pi N - \sum_{i,j,k} \theta_i^{jk} \\
&= 2\pi N - \sum_{\sigma} (\theta_i^{jk} + \theta_j^{ik} + \theta_k^{ij}) \\
&= 2\pi N - \pi|F| - \lambda \text{Area}(M) \\
&= 2\pi\chi(M) - \lambda \text{Area}(M),
\end{aligned}$$

where the last equality follows because on a triangulated, closed surface, every edge belongs to two faces and every face includes three edges, so $2|E| = 3|F|$. □

Appendix B

Circle packing and the Rodin-Sullivan Theorem

B.1 The Rodin-Sullivan Theorem

In 1985, William Thurston gave a talk entitled "The Finite Riemann Mapping Theorem" at Purdue University. In the talk, based on his previous work on orbifolds in [Thu80], Thurston conjectured that a certain sequence of circle packing maps converges to the Riemann map, appropriately normalized. This conjecture became the Rodin-Sullivan Theorem when Rodin and Sullivan proved it in their 1987 paper [RS87].

We here carefully state the result and summarize the proof. This theorem (and its proof) is a blueprint for several similar theorems about other discrete conformal structures. It also serves as the main motivating example for this dissertation, as can be seen explicitly in Chapter 9 where we reprove the Rodin-Sullivan Theorem using our method.

To begin with, we first define some terminology and notation. Throughout what follows, we assume that we are always working in an open connected subset of either the plane or the 2-sphere. The definitions in this section are taken directly from [RS87].

Definition 79. *Let Ω be a region in the plane or the 2-sphere. A* circle packing in Ω *is a collection of closed disks contained in Ω and having disjoint interiors.*

Each circle packing has an associated triangulation T constructed by connecting the centers of every pair of adjacent circles with geodesic segments. The vertices of T are the centers of the circles, the edges are the geodesic segments connecting neighboring circle centers, and the faces are the triangles formed by these, one triangle for every triple of mutually tangent circles.

Definition 80. *Let T be a triangulation in a manifold M. Its* carrier*, denoted $|T| \subset M$, is the geometric complex in M that is the image under T of the underlying simplicial complex K.*

Definition 81. *A* circle packing of the sphere *is one whose associated triangulation is a triangulation of the sphere.*

If one of the disks in a circle packing of the sphere is the exterior of the unit disk (under stereographic projection) then the remaining circles are said to be a circle packing of the unit disk.

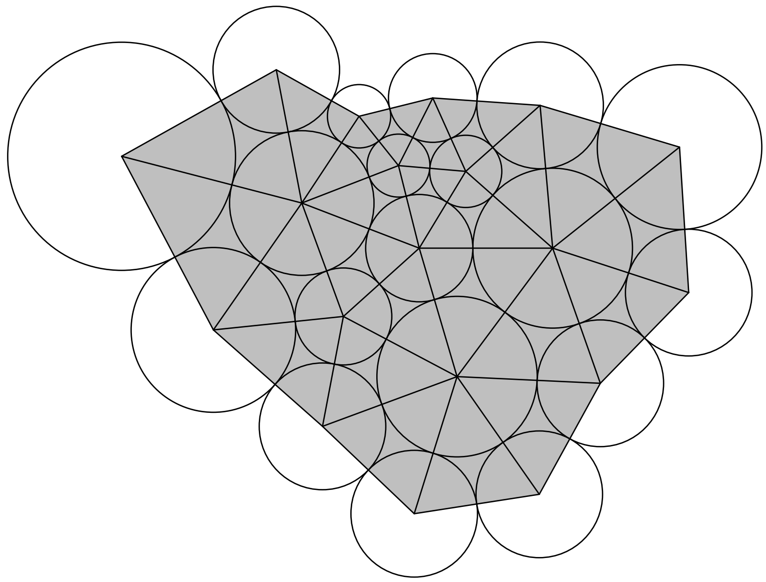

FIGURE B.1. The carrier of a circle packing

Note that by the above definitions, there is a difference between a circle packing *in* the sphere as in Definition 79 or a circle packing *of* the sphere as in Definition 81. The former is any packing that lives on the sphere (as long as it packs a connected, open subset of the sphere) and the latter refers only to those packings that cover the entire sphere in the sense that every point on the sphere is in one of the faces of the associated triangulation.

A similar distinction exists between a circle packing *in* the unit disk versus a circle packing *of* the unit disk. The former is any packing that lives in an open, connected subset of the disk while the latter refers only to a very specific class of packings, those that are packings of the sphere that stereographically project to the plane as in Definition 81.

There is an equivalent definition for a packing of the unit disk given in [Ste05]. Stephenson talks about packings that "fill" the unit disk. By this he means packings such that if we think of the unit disk as the hyperbolic plane, then every boundary circle is a horocycle. This condition is exactly equivalent to the condition that if we treat the boundary of the unit disk as a circle in our packing, then the packing lifts (under stereographic projection) to a packing of the sphere (or, as Stephenson would say, a packing that *fills* the sphere).

Luckily we are rarely interested in packings that do not "fill" the spaces they reside in, so the distinction between a packing *in* a space and a packing *of* a space will almost never lead to confusion.

Let c be a circle in a circle packing. The *flower* centered at c is the closed set consisting of c and its interior, all circles tangent to c and their interiors, and the interiors of all the triangular interstices formed by these circles.

A finite sequence of circles from a circle packing is called a *chain* if each circle except the last is tangent to its successor and the chain is a *cycle* if the first and last circles are tangent.

Next we explain the set up for the Rodin-Sullivan theorem.

Let Ω be a simply connected bounded region in the plane with two distinguished points z_0 and z_1. Let $\epsilon > 0$ and let H_ϵ be the regular hexagonal packing of the plane by circles of radius ϵ.

We will specify a certain finite subset Ω_ϵ of H_ϵ to be the domain of the circle packing map f_ϵ defined below. Let c_0 be a circle in H_ϵ whose flower contains the point z_0. The packing Ω_ϵ is the union $\Omega_\epsilon = I_\epsilon \cup B_\epsilon$, where I_ϵ is the set of *inner circles* and B_ϵ is the set of *border circles*.

To define the set of inner circles, I_ϵ, first let c_0 be a circle in H_ϵ whose flower contains z_0. Then form chains of circles starting from c_0 such that the flowers of every circle on every chain is completely contained in Ω. The set of circles belonging to at least one such chain is I_ϵ.

The set of border circles is the set of circles which are not themselves in I_ϵ but which are tangent to at least one member of I_ϵ. Note that the circles in B_ϵ form a cycle which we call the *border*.

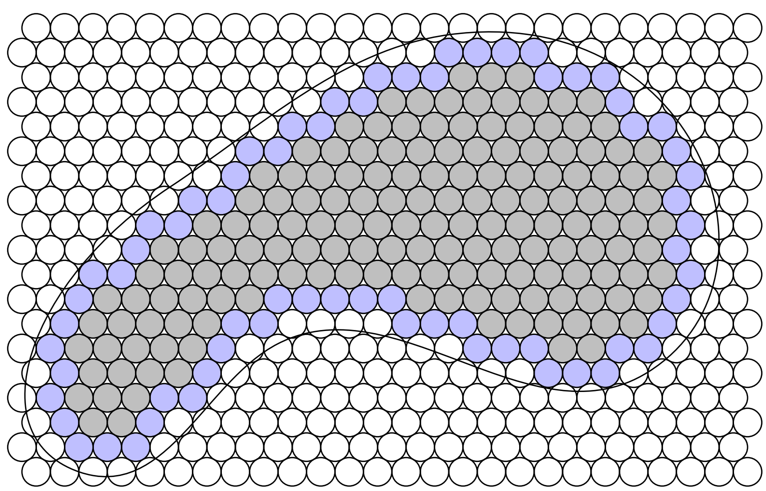

FIGURE B.2. The packing Ω_ϵ. Interior circles are grey, boundary circles are blue.

Let T_ϵ be the triangulation in the region Ω that corresponds to the circle packing Ω_ϵ and complete T_ϵ to a topological triangulation T^*_ϵ of the 2-sphere by adding a vertex at ∞ along with disjoint Jordan arcs connecting ∞ to the centers of the border circles.

Then by the uniformization theorem for circle packings of the sphere, Theorem 86 in the next section, we have a circle packing of the sphere whose triangulation is isomorphic to T^*_ϵ with an isomorphism preserving the orientation of the sphere. This packing is unique up to Möbius transformations.

We do two normalizations on this. First we require that the vertex ∞ of T_ϵ^* corresponds to the disk in the packing (of the sphere) that corresponds to the exterior of the unit disk and hence if we remove the disk corresponding to ∞ from the packing, we are left with a packing of the unit disk which we will label Ω'_ϵ. The second normalization is to use a Möbius transformation that fixes the unit disk to move the circles c'_0 and c'_1 (the image circle corresponding to a circle $c_1 \in \Omega_\epsilon$ whose flower contains z_1) to the origin and along the positive x-axis respectively.

At this point we have a sequence of correspondences between circle packings in Ω and circle packings in $\mathbb{D}$. This sequence of correspondences induces a sequence of piecewise linear maps $f_\epsilon : |T_\epsilon| \to |T'_\epsilon|$, where $|T_\epsilon|$ is the carrier of T_ϵ and $|T'_\epsilon|$ is the carrier of T'_ϵ. It is exactly this sequence of maps, $\{f_\epsilon\}$ which converges to the Riemann map as $\epsilon \to 0$.

The theorem as stated in Rodin-Sullivan ([RS87]) is the following:

Theorem 82. *The isomorphism $\Omega_\epsilon \to \Omega'_\epsilon$ of circle packings determines an approximate mapping which, as $\epsilon \to 0$, converges to a conformal homeomorphism of Ω with the unit disk.*

We outline the proof below, but first we will write out three geometric lemmas which are really the heart of the proof.

First we have the Length-Area Lemma, which relates the radius of a circle c to a quantity involving the number of chains separating c from the origin. The proof of this lemma is an easy application of the Cauchy-Schwarz inequality and can be found in full on page 250 of [Ste05].

Lemma 83 (Length-Area Lemma)**.** *Let c be a circle in a circle packing in the unit disk. Let $S_1, S_2, \ldots, S_k$ be k disjoint chains which separate c from the origin and from a point on the boundary of the disk. Denote the combinatorial lengths of these chains by $n_1, n_2, \ldots, n_k$. Then*

$$radius(c) \leq \frac{1}{\sqrt{n_1^{-1} + n_2^{-1} + \cdots + n_k^{-1}}}.$$

Next we have the Ring Lemma and the Hexagonal Packing Lemma, which were discussed at length in Chapters 4 and 5 respectively.

Lemma 84 (Ring Lemma)**.** *There is a constant r depending only on n such that if n circles surround the unit disk (i.e., they form a cycle externally tangent to the unit disk) then each circle has radius at least r.*

Lemma 85 (Hexagonal Packing Lemma)**.** *There is a sequence s_n, decreasing to zero, with the following property. Let c_0 be a circle in a finite packing P of circles in the plane and suppose the packing P around c_0 is combinatorially equivalent to n generations of the regular hexagonal circle packing about one of its circles. Then the* *ratio of radii of any two circles in the flower around c_0 differs from unity by less than s_n.*

Sketch of the proof of Theorem 82. First we check that the domain $|T_\epsilon|$ and range $|T'_\epsilon|$ converge to the whole region Ω and the unit disk $\mathbb{D}$ respectively. For $|T_\epsilon| \to \Omega$, simply note that $\Omega = \bigcup_\epsilon |T_\epsilon|$ and any compact subset of Ω is contained in every $|T_\epsilon|$ with sufficiently small ϵ. To show the analogous convergence of $|T'_\epsilon| \to \mathbb{D}$, we use the Length-Area Lemma to show that the radii of border circles of Ω'_ϵ converge uniformly to 0 as $\epsilon \to 0$, at which point it is clear that again, $\mathbb{D} = \bigcup_\epsilon |T'_\epsilon|$ and any compact subset $K \subset \mathbb{D}$ is contained in every $|T'_\epsilon|$ for small enough ϵ.

Next the Ring Lemma tells us that every angle in every triangle in the image triangulation T'_ϵ is bounded away from zero, independently of ϵ. Since the maps $f_\epsilon : |T_\epsilon| \to |T'_\epsilon|$ map equilateral triangles to triangles that never differ too much from equilateral, there is some $\kappa > 1$ such that f_ϵ is κ-quasiconformal for all ϵ.

Using well-known results from quasiconformal geometry, we get that the family $\{f_\epsilon\}$ is equicontinuous on compact subsets of Ω and hence $\{f_\epsilon\}$ is a normal family and has a subsequence converging to some κ-quasiconformal map $f : \Omega \to D$. Then we use some results from analysis to check that f is one-to-one and onto.

Finally, the Hexagonal Packing Lemma is used to show that as $\epsilon \to 0$, the radii of the image circles all approach the same value and so the triangles in the image triangulation T'_ϵ become arbitrarily close to equilateral. This implies that the quasiconformal distortion of f_ϵ decreases to 1 as ϵ goes to 0 and hence the limit function f is 1-quasiconformal. That is, the limit function f is conformal. □

B.2 Circle Packing Uniformization

In the circle packing context, uniformization results are fairly robust. When taken together the following four theorems say that every combinatorial surface (that is, a simplicial 2-complex that triangulates some oriented Riemann surface) has a unique maximal circle packing and this packing lives on a canonical Riemann surface determined by the topology of the combinatorial surface K.

The first of these uniformization theorems, Theorem 86, has a long history and at least three independently discovered proofs. For a short discussion of this history, see Notes I of [Ste05].

Theorem 86 (Proposition 7.1 of [Ste05])**.** *Let K be a combinatorial sphere. Then there exists an essentially unique univalent circle packing $\mathcal{P}_K$ for K of the Riemann sphere $\mathbb{P}$.*

Theorem 87 (Proposition 6.1 of [Ste05])**.** *Let K be a combinatorial closed disc (that is, simply connected, finite, and with nonempty boundary). Then there exists an essentially unique univalent circle packing $\mathcal{P}_K \subset \mathbb{D}$ for K such that every boundary circle is a horocycle.*

Theorem 88 (Proposition 8.1 of [Ste05]). *Let K be a combinatorial open disc (hence, infinite, simply connected, without boundary). Then there exists an essentially unique univalent circle packing $\mathcal{P}_K$ for K whose carrier fills either the hyperbolic plane $\mathbb{D}$ or the Euclidean plane $\mathbb{C}$.*

Theorem 89 (Proposition 9.1 of [Ste05]). *Let K be a multiply connected combinatorial surface. Then there exist a Riemann surface $\mathcal{S}_K$ and a circle packing $\mathcal{P}_K$ for K in the associated intrinsic metric on $\mathcal{S}_K$ such that $\mathcal{P}_K$ is univalent and fills $\mathcal{S}_K$. The Riemann surface $\mathcal{S}_K$ is unique up to conformal equivalence and $\mathcal{P}_K$ is unique up to conformal automorphisms of $\mathcal{S}_K$.*

An important note is that these uniformization theorems do *not* say that any Riemann surface S can be packed. Rather, given a *topological* surface, a triangulation can be found of it and then there is *some* Riemann surface homeomorphic to the starting topological surface on which a packing of the triangulation lives.

B.3 Ratio of Radii and the Derivative

By the Rodin-Sullivan Theorem (and He-Schramm's extension), circle packing maps from Jordan domains to the unit disc converge to a conformal homeomorphism, specifically, the Riemann map. In fact, the situation is even a little bit better. It turns out that the ratio of radii of image to domain circles converges to the modulus of the derivative of the Riemann map, as we show in this section.

B.3.1 Sufficiency of $s_m = O(1/m)$

For the easiest case, hexagonal circle packing, it was proven fairly early that the ratio of radii function converges to the modulus of the derivative under the assumption that the hexagonal packing constants s_m are of order $1/m$.

For this proof we closely follow [Rod89] and our notation will be similar to that used in the proof of the Rodin-Sullivan Theorem in §B.1 above.

Let Ω_ϵ again be the domain packing of a region Ω and $\Omega'_\epsilon \subset \mathbb{D}$ the image packing. Let $|T_\epsilon|, |T'_\epsilon|$ be the carriers of the packings Ω_ϵ and Ω'_ϵ respectively. Then the circle packing isomorphisms $\{c \mapsto c'\}$ extend to piecewise linear mappings $f_\epsilon : |T_\epsilon| \to |T'_\epsilon|$ and, as $\epsilon \to 0$, these mappings converge uniformly on compact subsets of Ω to the Riemann mapping f, suitably normalized.

Let z be the center of a circle in Ω_ϵ and let

$$r_\epsilon(z) := \frac{\text{radius}(c')}{\text{radius}(c)}.$$

Extend r_ϵ to be a map on the whole carrier $|T_\epsilon|$, so $r_\epsilon : |T_\epsilon| \to \mathbb{R}$.

Theorem 90 (Proposition 1 of [Rod89])**.** *Let $r_\epsilon : |T_\epsilon| \to \mathbb{R}$ be the ratio of radii function just defined and let $f : \Omega \to \mathbb{D}$ be the properly normalized Riemann mapping function of Ω. A sufficient condition that r_ϵ converges to $|f'|$ uniformly on compact subsets of Ω as $\epsilon \to 0$ is that $s_m = O(1/m)$ as $m \to \infty$.*

Proof. Fix $z \in \Omega$. For $\epsilon, \delta > 0$ define $H_{\epsilon,\delta}$ to be a hexagon of diameter 2δ centered at the ϵ circle closest to z. Call this circle c_z. As $\epsilon \to 0$ let $H_{\epsilon,\delta}$ approach a limiting hexagon $H_{0,\delta}$ centered at z. Now since the circle packing maps f_ϵ approach f as $\epsilon \to 0$,

$$|f'(z)|^2 = \lim_{\delta \to 0} \frac{|f(H_{0,\delta})|}{|H_{0,\delta}|} = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \frac{|f_\epsilon(H_{\epsilon,\delta})|}{|H_{\epsilon,\delta}|}, \tag{B.1}$$

where $|H_{\epsilon,\delta}|$ and $|f_\epsilon(H_{\epsilon,\delta})|$ denote the areas of $H_{\epsilon,\delta}$ and its image $f_\epsilon(H_{\epsilon,\delta})$.

Now let $H_{\epsilon,\delta}$ be the convex hull of the centers of N generations of circles in Ω_ϵ. Then since each circle has radius ϵ, the "radius" δ of $H_{\epsilon,\delta}$ is $\delta = 2N\epsilon$. Assume the distance $d(z, \partial\Omega)$ from z to $\partial\Omega$ is at least d, where d is much larger than δ which in turn is much larger than ϵ.

Then the packing Ω_ϵ contains at least $\lfloor d/(2\epsilon) \rfloor - 1$ generations of the hexagonal packing centered about c_z. If c is any other circle in Ω_ϵ that intersects $H_{\epsilon,\delta}$, then c has at least $\lfloor d/(2\epsilon) \rfloor - (N+1)$ generations of the hexagonal circle packing surrounding it.

Now since $\delta = 2N\epsilon$, $d/(2\epsilon) = (Nd)/\delta$ and so the number of generations surrounding c can be written as follows:

$$\begin{aligned} \left\lfloor \frac{d}{2\epsilon} \right\rfloor - (N+1) &= \left\lfloor \frac{Nd}{\delta} \right\rfloor - (N+1) \\ &= \left\lfloor \frac{Nd - \delta(N+1)}{\delta} \right\rfloor \\ &= \left\lfloor \frac{N(d-\delta)}{\delta} \right\rfloor - 1. \end{aligned}$$

But δ was assumed much smaller than d and, further, both are fixed for the moment while N is going to infinity. Hence in the limit, the number of generations about any circle c meeting $H_{\epsilon,\delta}$ is simply $\lfloor (Nd)/\delta \rfloor$.

Next we estimate the area of $f_\epsilon(H_{\epsilon,\delta})$.

Let R be the radius of the generation zero circle Ω in $f_\epsilon(H_{\epsilon,\delta})$. Every circle in Ω'_ϵ which lies in $f_\epsilon(H_{\epsilon,\delta})$ is a generation zero circle of a hexagonal configuration of $\lfloor dN/\delta \rfloor$ generations of circles of Ω'_ϵ and we can use the circle packing constants s_n to estimate the radius r of any of these circles as follows.

Let c be a circle in Ω_ϵ which lies inside $f_\epsilon(H_{\epsilon,\delta})$ and let c have radius r. Take a chain of circles connecting c and the generation zero circle Ω of $f_\epsilon(H_{\epsilon,\delta})$. This chain of circles lies completely in $f_\epsilon(H_{\epsilon,\delta})$ and hence each circle in the chain can be considered to be the generation zero circle of a hexagonal packing with $\lfloor dN/\delta \rfloor$ generations. Number these circles, with c_1 being adjacent to Ω, c_2 adjacent to c_1 and so on, until we get to the last circle c.

Now we can apply the hexagonal packing estimate to this chain of circles. First, since c_1 is a generation one circle of a hexagonal packing about Ω with $k := \lfloor dN/\delta \rfloor$ generations, we have that

$$1 - s_k \leq \frac{r_1}{R} \leq 1 + s_k.$$

Next, c_2 is a generation one circle of a hexagonal packing about c_1, so

$$1 - s_k \leq \frac{r_2}{r_1} \leq 1 + s_k.$$

Combining these two inequalities gives

$$(1 - s_k)^2 \leq \frac{r_2}{R} \leq (1 + s_k)^2.$$

Then c_3 is a generation one circle in a hexagonal packing with k generations about c_2, so $1 - s_k \leq r_3/r_2 \leq 1 + s_k$ and hence $(1 - s_k)^3 \leq r_3/R \leq (1 + s_k)^2$.

Continue in this manner until we reach the final circle c. This circle is at most N generations from the starting circle Ω, so

$$(1 - s_k)^N \leq \frac{r}{R} \leq (1 + s_k)^N.$$

Multiplying through by R tells us that the radius r of any circle in Ω'_ϵ lying in $f_\epsilon(H_{\epsilon,\delta})$ is at least $R(1 - s_k)^N$ and at most $R(1 + s_k)^N$.

Next we use these estimates on radii to bound the area of $f_\epsilon(H_{\epsilon,\delta})$. Note that $f_\epsilon(H_{\epsilon,\delta})$ completely contains a regular hexagon with "radius" $RN(1 - s_k)^N$ and is completely contained in another with "radius" $RN(1 + s_k)^N$. Hence we have that

$$\frac{3\sqrt{3}(NR(1 - s_k)^N)^2}{2} \leq |f_\epsilon(H_{\epsilon,\delta})| \leq \frac{3\sqrt{3}(NR(1 + s_k)^N)^2}{2}.$$

Now the area $|H_{\epsilon,\delta}| = 3\sqrt{3}\delta^2/2 = 6\sqrt{3}(N\epsilon)^2$ since it is a regular hexagon with side length $\delta = 2N\epsilon$, so we can divide the above inequality by $|H_{\epsilon,\delta}|$ to get

$$\frac{R^2(1 - s_k)^{2N}}{4\epsilon^2} \leq \frac{|f_\epsilon(H_{\epsilon,\delta})|}{|H_{\epsilon,\delta}|} \leq \frac{R^2(1 + s_k)^{2N}}{4\epsilon^2}.$$

Since we are taking a limit as $\epsilon \to 0$, we may as well drop the 4's in the denominator. We also replace k by its value $k = \lfloor dN/\delta \rfloor$ to get

$$\left(\frac{R}{\epsilon}\right)^2 (1 - s_{\lfloor dN/\delta \rfloor})^{2N} \leq \frac{|f_\epsilon(H_{\epsilon,\delta})|}{|H_{\epsilon,\delta}|} \leq \left(\frac{R}{\epsilon}\right)^2 (1 + s_{\lfloor dN/\delta \rfloor})^{2N}. \tag{B.2}$$

Now if we fix δ then as $\epsilon \to 0$, $N \to \infty$. Further, we are assuming that s_m is of order $O(1/m)$, so $0 \le s_m \le B/m$ for some constant B. Hence

$$\begin{aligned}\lim_{\epsilon \to 0}(1 + s_{\lfloor dN/\delta \rfloor})^{2N} &\le \lim_{N\to\infty}\left(1 + \frac{B}{dN/\delta}\right)^{2N} \\ &= \lim_{N\to\infty}\left(1 + \frac{B\delta}{dN}\right)^{2N} \\ &= e^{2B\delta/d}.\end{aligned}$$

The same calculation, *mutatis mutandi* gives

$$\lim_{\epsilon\to 0}(1 - s_{\lfloor dN/\delta \rfloor})^{2N} \ge e^{-2B\delta/d}.$$

Now taking a limit in (B.2) and using the previous two statements gives

$$\limsup_{\epsilon\to 0}\left(\frac{R}{\epsilon}\right)^2 e^{-2B\delta/d} \le \frac{|f(H_{0,\delta})|}{|H_{0,\delta}|} \le \liminf_{\epsilon\to 0}\left(\frac{R}{\epsilon}\right)^2 e^{2B\delta/d}. \tag{B.3}$$

Finally, let $\delta \to 0$ in (B.3). The middle term becomes $|f'(z)|^2$ and we have that $\limsup_{\epsilon\to 0}(R/\epsilon) \le |f'(z)| \le \liminf_{\epsilon\to 0}(R/\epsilon)$, so $\lim_{\epsilon\to 0}(R/\epsilon)$ exists and is equal to $|f'(z)|$. Since the functions r_ϵ are piecewise linear, $\lim_{\epsilon\to 0} r_\epsilon(z)$ must be the same as $\lim_{\epsilon\to 0} R/\epsilon$ and hence $\lim_{\epsilon\to 0} r_\epsilon(z) = |f'(z)|$, as required. □

As we noted in §5.1, He and Rodin showed in [HR93] that s_m is indeed of order $1/m$ as long as the packing has bounded valence, which a hexagonal packing clearly does. Hence we truly have that the ratio of radii function approaches the magnitude of the derivative of the Riemann mapping. That is, $r_\epsilon(z) \to |f'(z)|$.

Appendix C

Proving the Descartes Circle Theorem with Linear Algebra

The proof of the circle packing ring lemma is entirely qualitative. Nevertheless, it is actually quite easy to calculate the minimum radius for petals in an n-flower by examining certain extremal flowers and using Descartes' Circle Theorem. The details of this calculation can be found in Appendix B of [Ste05].

The Descartes Circle Theorem says that for any quad of four tangent circles, their radii $r_0, \ldots, r_3$ satisfy

$$\left(\sum_{i=0}^{3} \frac{1}{r_i}\right)^2 = 2\sum_{i=0}^{3} \frac{1}{r_i^2}.$$

It turns out there is an easy generalization of this theorem to higher dimensions and also, with a little bit of work, it is even possible to find an analogue for circles (or spheres) in hyperbolic and spherical space.

Most of the following can be found in [Wil81] in a more general framework. Sections 3, 9, and 13 are especially relevant.

We are going to write n-spheres as points in $\mathbb{R}^{n+2}$, as follows. Let $\mathcal{C}$ be a n-sphere, $\mathcal{C} = \{x \in \mathbb{R}^{n+1} \ : \ |x-a| = r\}$. We will represent $\mathcal{C}$ by the $(n+2)$-vector

$$C = \frac{1}{2r}\left(2a, |a|^2 - r^2 - 1, |a|^2 - r^2 + 1\right).$$

Next we define an inner product $*$ on $\mathbb{R}^{n+2}$ by

$$X * W := \sum_{i=1}^{n+1} x^i w^i - x^{n+2} w^{n+2}.$$

This is the usual Lorentzian inner product. With this inner product, n-spheres are normalized so that $C * C = 1$.

We can also look at what happens when we take the inner product $C_1 * C_2$ for two circles. We get

$$\begin{aligned} C_1 * C_2 &= \frac{1}{4r_1r_2}\big(4a_1 \cdot a_2 + (|a_1|^2 - r_1^2 - 1)(|a_2|^2 - r_2^2 - 1) \\ &\qquad\qquad - (|a_1|^2 - r_1^2 + 1)(|a_2|^2 - r_2^2 + 1)\big) \\ &= \frac{1}{2r_1r_2}\left(r_1^2 + r_2^2 - |a_1 - a_2|^2\right) \\ &= -\sigma_{12}, \end{aligned}$$

where $\sigma_{12} := \frac{|a_1-a_2|^2-r_1^2-r_2^2}{2r_1r_2}$ is the inversive distance between the n-spheres C_1, C_2. Hence we can say that $C_1 * C_2 = -1$ if and only if C_1 and C_2 are externally tangent.

Now we are in a position to turn the Descartes Circle Theorem into solving a linear system. Let $\{C_i\}$ be a collection of n spheres, each of dimension $n-3$ and embedded in $\mathbb{R}^{n-2}$. Then we can represent them by vectors in $\mathbb{R}^n$ and we assume they are mutually externally tangent, so $C_i * C_j = -1$ for every $i \neq j$.

Since these spheres are linearly independent vectors in $\mathbb{R}^n$ (with respect to the standard euclidean inner product) and there are n of them, we can write any other vector in $\mathbb{R}^n$ as a linear combination of them. In particular, define $E := (0, \ldots, 0, -1, -1)$. (See Section 12 of [Wil81] for an explanation of where this E comes from. For us here it is simply a convenience.) Then we can write

$$E = \sum_{i=0}^{n-1} x^i C_i$$

for some $x = (x^0, \ldots, x^{n-1}) \in \mathbb{R}^n$. Taking a star product of E with one of the circles C_i gives

$$\begin{aligned} E * C_i &= x^i C_i * C_i + \sum_{j \neq i} x^j C_i * C_j \\ &= x^i - \sum_{j \neq i} x^j. \end{aligned}$$

On the other hand, explicitly calculating $E * C_i$ yields

$$E * C_i = \frac{1}{2r_i}\left(-(|a_i|^2 - r_i^2 - 1) + |a_i|^2 - r_i^2 + 1\right) \tag{C.1}$$

$$= \frac{1}{r_i}. \tag{C.2}$$

For notational convenience, define $b_i := 1/r_i$. This is sometimes called the *bend* of a sphere.

Equating the two representations for $E * C_i$ above and replacing $1/r_i$ with b_i, we see that

$$x^i - \sum_{j \neq i} x^j = b_i \tag{C.3}$$

for every i. Hence we have a system of n linear equations which can be represented by the matrix equation $Ax = b$, where A is the $n \times n$ matrix with 1's along the main diagonal and -1's everywhere else and $b = (b_0, \ldots, b_{n-1})^T$. The matrix A is non-singular for every $n > 2$, and in fact its inverse is

$$A^{-1} = \frac{1}{2(n-2)} \begin{pmatrix} n-3 & -1 & \ldots & -1 \\ -1 & n-3 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-3 \end{pmatrix}.$$

Hence we can easily solve the equation $Ax = b$ for x to get

$$x^i = \frac{1}{2(n-2)}\left((n-3)b_i - \sum_{j\neq i} b_j\right). \tag{C.4}$$

Next we examine $E * E$. On the one hand, an extremely easy direct calculation gives

$$E * E = 0 + \cdots + (-1)(-1) - (-1)(-1) = 0$$

and on the other hand, writing $E = \sum_{i=0}^{n-1} x^i C_i$ gives

$$\begin{aligned} E * E &= \sum_{i=0}^{n-1} x^i (E * C_i) \\ &= \sum_{i=0}^{n-1} x^i b_i, \end{aligned}$$

where the last equality follows from (C.2).

Now if we use (C.4) to rewrite x^i in the above expression, we get

$$\begin{aligned} 0 &= E * E \\ &= \frac{1}{2(n-2)} \sum_{i=0}^{n-1} b_i \left((n-3)b_i - \sum_{j\neq i} b_j\right). \end{aligned}$$

Multiplying by $2(n-2)$ and rearranging gives

$$2\sum_{i\neq j} b_i b_j = (n-3)\sum_{i=0}^{n-1} b_i^2. \tag{C.5}$$

Finally, we can write out $\left(\sum b_i\right)^2$ and simplify using (C.5):

$$\begin{aligned} \left(\sum_{i=0}^{n-1} b_i\right)^2 &= \sum_{i=0}^{n-1} b_i^2 + 2\sum_{i\neq j} b_i b_j \\ &= \sum_{i=0}^{n-1} b_i^2 + (n-3)\sum_{i=0}^{n-1} b_i^2 \\ &= (n-2)\sum_{i=0}^{n-1} b_i^2. \end{aligned}$$

If we let $n = 4$ and replace b_i with $1/r_i$, then this last equality is clearly the original Descartes Circle Theorem.

This procedure also works in hyperbolic or spherical space, with a few changes. Circles must be represented differently and instead of the vector $E = (0, \ldots, -1, -1)$, we use either $S = (0, \ldots, 0, -1)$ or $H = (0, \ldots, -1, 0)$. See Section 12 of [Wil81] for more.

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