

BOUND ON THE NUMBER OF SIMULTANEOUSLY BAND- AND
TIME-LIMITED QUASI-MONOCHROMATIC FREQUENCY MODES

by

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As members of the Dissertation Committee, we certify that we have read the final report prepared by Ali Cox entitled "Bound on the Number of Simultaneously Band- and Time-limited Quasi-Monochromatic Frequency Modes" and recommend that it be accepted as fulfilling the report requirement for the Degree of M.S. in Optical Sciences: Quantum Information Science and Engineering.

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I hereby certify that I have read this master's report prepared under my direction and recommend that it be accepted as fulfilling the report requirement.

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ABSTRACT

Based on insight from three seminal papers by Landau, Pollock and Slepian, which introduce the prolate-spheroidal wave functions as an optimal basis for almost completely capturing the degrees of freedom of a certain class of simultaneously time and band limited functions, we introduce an alternative set of narrow-band functions and discuss how the properties of the optimal prolate-spheroidal basis impose a bound on their number in terms of the time-bandwidth product.

Chapter 1 Main Chapter

1.1 Motivation

In many real world imaging tasks, a scene is probed with a signal of electromagnetic (EM) radiation whose frequency support lies in a finite band, and information about the scene must be inferred based on the signal within a certain finite duration of time. As a baseline, a complete characterization of the degrees freedom of such a restricted EM signal is a necessity for understanding the theoretical performance limits of a receiver processing the signal, whose goal is to extract information from the scene with which the signal interacts. This characterization is the subject of a sequence of 3 seminal papers [3], [1], and [2], which introduce the prolate spheroidal wavefunction basis as an answer to this question. While these functions form an orthonormal set, and thus can be used to construct valid EM field modes in which to describe the source signal, they do not in general, even for scenes composed of linear materials, maintain their functional forms after interacting with a target. This can be a complicating factor especially for quantum performance analyses, where the correlations between the field observables can be much more complex.

It is often desirable to describe a simultaneously band- and time-limited source signal as an excitation of orthogonal modes, where each mode has the additional property of being spectrally confined to a narrow neighborhood of a single frequency. This is because the interaction of light with the scene can be described by a convolution

process, and for a scene comprising of a linear material, the convolution of a single frequency EM signal results in a signal of the same frequency as the probe signal. In other words, up-to linear effects, the interaction of the source signal with the scene preserves orthogonality of frequency-modes, and hence the state of light at the receiver, represented in the frequency mode basis of the source modes, maintains the character of the original state of light exciting the source light, i.e. a separable state remains separable, and an entangled state maintains its entanglement structure.

A natural question that arises as a result of this realization is how many such single frequency modes can be fit in a source signal that is first band-limited to an interval of width W , then time-limited by a duration T ? In this report, we show one way of constructing such a set of modes, and apply properties of the prolate spheroidal wave function (PSWF) basis to put a bound on the size of the set.

We begin by reviewing the results of [3] and [2], followed by a discussion of the conditions underwhich these results apply to a band-limited optical signal having undergone a sharp temporal turn-on and turn-off, and end with a theorem. In the last section before concluding, we describe our construction of single frequency modes.

1.2 Prolate Spheroidal Wave Functions and the Time-Bandwidth Product

The PSWF basis is defined on the complete inner-product space $\mathcal{L}^2(\mathbb{R})$ on which we denote the \mathcal{L}^2 inner product by $\langle \cdot, \cdot \rangle_\infty$ and the induced \mathcal{L}^2 norm by $\|f\|_\infty =$

$\sqrt{\langle f, f \rangle_\infty}$. Let \mathbf{D} be the time-limiting operator on $\mathcal{L}^2(\mathbb{R})$ defined by its action on $k \in \mathcal{L}^2(\mathbb{R})$ given by

$$\mathbf{D}k(t) = \begin{cases} k(t), & t_0 \leq t \leq t_0 + T \\ 0, & \text{otherwise} \end{cases}.$$

Similarly, let \mathbf{B} denote the band limiting operator around center frequency ν_0 , defined on $\mathcal{L}^2(\mathbb{R})$ by its action:

$$\mathbf{B}k(t) = \frac{1}{2\pi} \int_{\nu_0 - W/2}^{\nu_0 + W/2} K(\omega) e^{i\omega t} d\omega.$$

We adopt the convention in [3] where uppercase letters are used for functions in their frequency domain. We define the action of the operator \mathbf{B} and \mathbf{D} on a frequency domain function F , as $\mathbf{O}F(\omega) = \mathcal{F}^{-1}[\mathbf{O}f](\omega)$ where \mathcal{F} is the Fourier transform operation and \mathbf{O} stands for either \mathbf{B} or \mathbf{D} . The operators \mathbf{D} and \mathbf{B} lead to the definition of the $\mathcal{L}^2(\mathbb{R})$ -subspaces $\mathcal{D} := \{k(t) \in \mathcal{L}^2(\mathbb{R}) \mid \mathbf{D}k(t) = k(t)\}$ of time-limited functions and $\mathcal{B} := \{k(t) \in \mathcal{L}^2(\mathbb{R}) \mid \mathbf{B}k(t) = k(t)\}$ of band-limited functions.

As pointed out in refs. [1, 2], a function cannot be simultaneously time and band-limited unless it is identically 0, i.e. $\mathcal{B} \cap \mathcal{D} = \{0\}$. Yet it is possible to define a nontrivial subset of \mathcal{D} that is approximately band-limited in the sense of having most of its energy contained within the frequency range $(\nu_0 - W/2, \nu_0 + W/2)$. For $\epsilon_W > 0$, we denote this set by $E(\epsilon_W)$, and define it as the set of unit norm functions $f \in \mathcal{D}$ satisfying

$$\int_{\nu_0 - W/2}^{\nu_0 + W/2} |F(\omega)|^2 d\omega \geq 1 - \epsilon_W^2. \quad (1.1)$$

Let $c = WT$. It follows from theorem 3 in [2] that the time and frequency translated prolate spheroidal wave functions $\{\Psi_j(c)\}_{j=0}^{\infty}$ defined as

$$\Psi_j(c, \omega) = e^{-i(t_0+T/2)\omega} \psi_j(c, \omega - \nu_0) \quad \forall \omega \in \mathbb{R}, \quad (1.2)$$

where the lower case $\{\psi_j(c)\}_{j=0}^{\infty}$ are the renormalized prolate spheroidal functions introduced in [3], span $E(\epsilon_W)$ up to a distance $12\epsilon_W^2$, when the basis is restricted to the first $\lfloor c \rfloor + 1$ functions. Note that the $\Psi_j(c)$ are time-limited functions, and satisfy the properties of the $\psi_j(c)$ presented in [3] with the roles of time and frequency swapped. In particular, they are eigenvectors of the operator \mathbf{DB} , with respective eigenvalues denoted $\{\lambda_j\}_{j=0}^{\infty}$, they are complete and orthogonal over $\omega \in (\nu_0 - W/2, \nu_0 + W/2)$ with respective norms equal to $\{\sqrt{\lambda_j}\}_{j=0}^{\infty}$, and they are orthonormal over the range $\omega \in \mathbb{R}$.

1.3 Application to an optical signal

We would like to capture the subset of $\mathcal{L}^2(\mathbb{R})$ corresponding to such a band-then-time limited signal with band W and duration T . This can be a temporal Electromagnetic (EM) signal that is generated by turning on a source of finite bandwidth W at some time $t = t_0$ and turning it off at time $t = t_0 + T$, hence it is a member of the space \mathbf{DB} . Let $g \in \mathcal{B}$ be normalized such that $\|\mathbf{D}g\|_{\infty} = 1$. Then by the completeness of $\{\Psi_j(c)\}_{j=0}^{\infty}$ in the frequency range $(\nu_0 - W/2, \nu_0 + W/2)$, there exists a set of complex coefficients $\{a_j(c)\}_{j=0}^{\infty}$ such that $G(\omega) = \sum_{j=1}^{\infty} a_j \mathbf{B}\Psi_j(c, \omega)$, implying $\sum_j \lambda_j^2 |a_j|^2 = 1$. Then Eq. 1.1 with $F = \mathbf{D}g$, giving the energy of the

function $\mathbf{D}g$ falling within the source frequency band, yields

$$\begin{aligned}\|\mathbf{B}\mathbf{D}g\|_{\infty}^2 &= \left\| \sum_{j=0}^{\infty} a_j \mathbf{B}\mathbf{D}\mathbf{B}\Psi_j(c) \right\|_{\infty}^2 \\ &= \left\| \sum_{j=0}^{\infty} \lambda_j a_j \mathbf{B}\Psi_j(c) \right\|_{\infty}^2 = \sum_{j=0}^{\infty} \lambda_j^3 |a_j|^2.\end{aligned}\quad (1.3)$$

Note that the quantity given by Eq. (1.3) can be arbitrarily small (implying $\mathbf{D}g \notin E(\epsilon_W)$ in general). This occurs, for example, if the a_j select a single term with a very high value j_0 of j in the sum in Eq. (1.3) so that $\lambda_{j_0} \ll 1$. However, even in this case, increasing T will inevitably make Eq. (1.3) approach 1. This can be understood by considering the dynamic decomposition of $G = \mathbf{B}\Psi_{j_0}(c_0)$, with c_0 fixed such that $j_0 > c_0$, into $\{\mathbf{B}\Psi_j(c)\}_j$ as the value of c increases above c_0 via an increase in T . Note that for any j and c , $\mathbf{B}\Psi_j(c)$ is an oscillatory function with exactly j zero crossings in a fixed band of width W and oscillation frequency and amplitude increasing symmetrically from the center towards the edges of the band. For $c < j$, the envelope of $\mathbf{B}\Psi_j(c)$ covers the entire band. It is only when $c > j$ that the spectral extent of the envelope of $\Psi_j(c)$ becomes less than W , and continues to shrink arbitrarily with increasing c . These properties of the PSWFs are illustrated by the plot of $\mathbf{B}\Psi_{50}(c)$ in Fig. 1.1. Thus, for $c < j_0$, the coefficients $a_j(c)$ are sharply peaked around $j = j_0$, since an oscillatory function with j_0 zero crossings and envelope extent W is best approximated by $\mathbf{B}\Psi_{j_0}(c)$: a function with those same properties. When c goes beyond j_0 , however, the spectral extent of $\mathbf{B}\Psi_{j_0}(c)$ shrinks, accompanied by an increase in its oscillation frequency, and a decrease in its functional resemblance to $\mathbf{B}\Psi_{j_0}(c_0)$. Then the optimal value of j

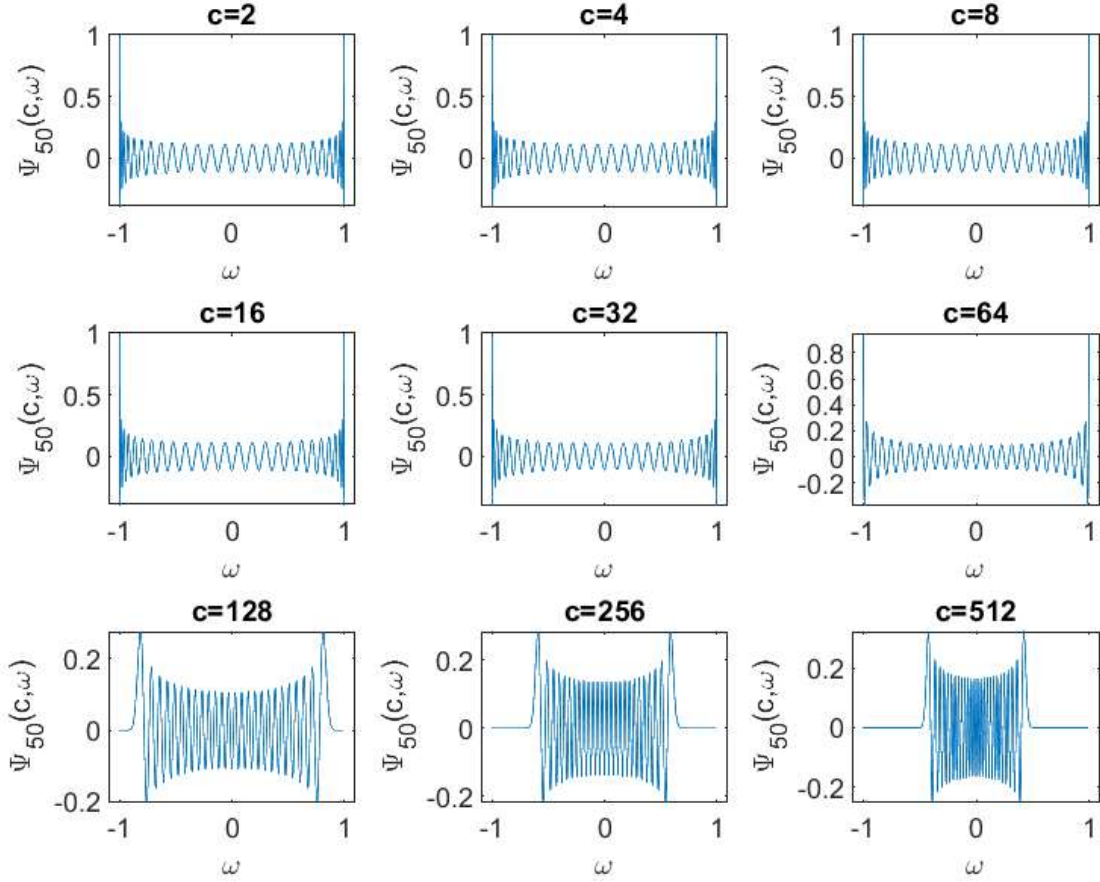


FIGURE 1.1: Plot of $\frac{1}{\sqrt{\lambda_{50}}} \mathbf{B} \Psi_{50}(c, \omega)$ for $W = 2, \nu_0 = 0$, and $t_0 = -T/2$ over the range $(\nu_0 - W/2, \nu_0 + W/2)$ for different values of $c = WT$. Note that the envelope covers the entire frequency range until $j > c$.

representing the oscillations of G in the central region of the band shifts to a value less than j_0 , despite higher values of j (corresponding to wider spectral envelopes) remaining necessary to continue accurately decomposing G across the entire band. Because of the rapidly increasing amplitude and oscillation frequency of $\Psi_j(c)$ at the edges of the band with increasing c and j , however, a strong suppression of the $a_j(c)$ begins at a value of j in the range $j_0 < j < c$, as $c \rightarrow \infty$. Then the weights $\lambda_j^2 |\alpha_j|^2$ strongly accentuate the contribution of λ_j for $j < c$ to the sum given by

Eq. (1.3), as illustrated in Fig. 1.2. Given that the λ_j are very close to 1 for $j < c$ and roughly exponentially decay for $j > c$, Eq. (1.1) is eventually satisfied for any $\epsilon_W > 0$, and particularly quickly for $\epsilon_W > 1 - \lambda_{\lfloor c \rfloor + 1}$. This qualitative argument

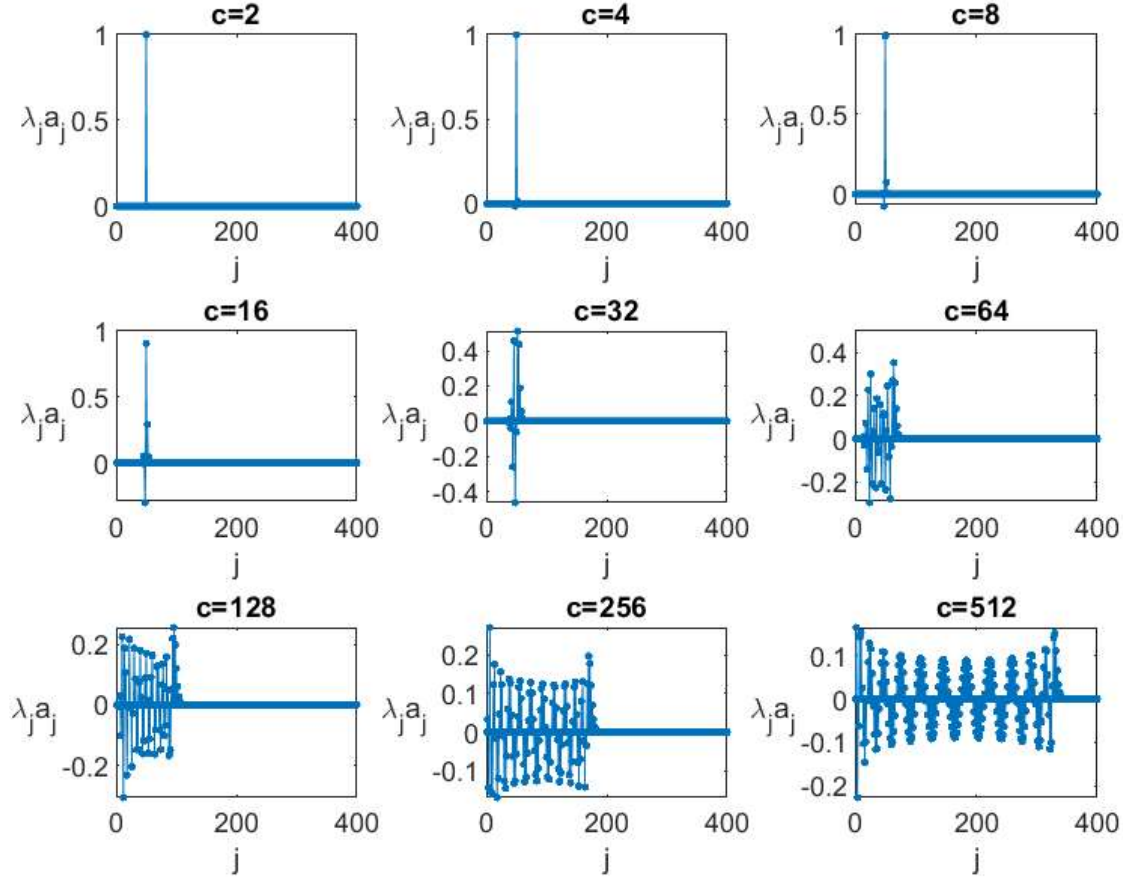


FIGURE 1.2: The decomposition profile of $\mathbf{B}\Psi_{50}(2)$ into the first 400 PSWF basis functions $(\mathbf{B}\Psi_n(c))_n$ for increasing values of c . Note that once $c > 50$, the cutoff threshold of the coefficients $\lambda_j a_j$ begin trailing significantly behind c . The fractional energy of $\mathbf{B}\Psi_{50}(2)$ falling within the frequency band of width W is given by the infinite series $\sum_{j=1}^{\infty} \lambda_j^3 |a_j|^2$, as per Eq. (1.3) evaluated with $g = \mathbf{B}\Psi_{50}(2)$.

for the T -eventual simultaneous band- and time-limited-ness of an initially band-limited function having undergone a hard temporal truncation by an interval of

duration T is made rigorous by the following theorem:

Theorem 1. *Let $W \in \mathbb{R}^+$, and let G be any function in $\mathcal{L}^2([-W/2, W/2])$ with unit norm. Then*

$$\lim_{T \rightarrow \infty} \|\mathbf{BDG}\|_{\infty} = 1. \quad (1.4)$$

Proof. Without loss of generality, let $t_0 = -T/2$, and $\nu_0 = 0$. Using the convolution theorem, the time-truncation of G can be expressed as

$$\mathbf{DG} = \frac{1}{2\pi} \mathcal{F}[\mathbb{1}_{(-T/2, T/2)}] * G, \quad (1.5)$$

where $\mathbb{1}_I$ is the indicator function taking value 1 on an interval I and value 0 outside of I . Its Fourier transform is given by $\mathcal{F}[\mathbb{1}_{(-T/2, T/2)}](\omega) = T \operatorname{sinc}(\frac{T}{2}\omega)$. Let $a = T/2$. Expanding Eq. (1.5) inside the integral norm in Eq. (1.4) and squaring yields

$$\begin{aligned} \lim_{T \rightarrow \infty} \|\mathbf{BDG}\|_{\infty}^2 &= \lim_{a \rightarrow \infty} \int_{-W/2}^{W/2} d\omega \frac{a^2}{\pi^2} \left| \int_{-W/2}^{W/2} d\omega' \operatorname{sinc}(a(\omega - \omega')) G(\omega') \right|^2 \\ &= \lim_{a \rightarrow \infty} \frac{a}{\pi^2} \iint_{-\frac{W}{2}}^{\frac{W}{2}} d\omega' d\omega'' G(\omega') \overline{G(\omega'')} \\ &\quad \times \int_{a(-\frac{W}{2}-\omega')}^{a(\frac{W}{2}-\omega')} \operatorname{sinc}(u) \operatorname{sinc}(u + a(\omega' - \omega'')) du, \quad (1.6) \end{aligned}$$

where the final line is the result of swapping the order of integration after expanding the square-integral, and applying the change-of-variables $u = a(\omega - \omega')$. Note that in the limit of $a \rightarrow \infty$, the evaluation of the u -integral depends critically on whether or not $\omega' = \pm \frac{W}{2}$. To avoid case-wise evaluation, the ω' - and ω'' -integrals

in Eq. (1.6) can be expressed as a limit as well:

$$\lim_{T \rightarrow \infty} \|\mathbf{BDG}\|_{\infty}^2 = \lim_{a \rightarrow \infty} \lim_{b \rightarrow 0_+} \mathcal{I}(a, b), \quad (1.7)$$

where

$$\begin{aligned} \mathcal{I}(a, b) = & \frac{a}{\pi^2} \iint_{-\frac{W}{2}+b}^{\frac{W}{2}-b} d\omega'' d\omega' \overline{G(\omega'')} G(\omega') \\ & \times \int_{a(-\frac{W}{2}-\omega')}^{a(\frac{W}{2}-\omega')} \text{sinc}(u) \text{sinc}(u + a(\omega' - \omega'')) du. \end{aligned} \quad (1.8)$$

Now for any a , 1) $\lim_{b \rightarrow 0_+} \mathcal{I}(a, b)$ exists, since direct evaluation results in the squared \mathcal{L}^2 -norm of an \mathcal{L}^2 -function, as can be seen in its re-packed form on the first line of Eq. (1.6). Let $L(a)$ denote this limit. Moreover, 2) the convergence of $\lim_{b \rightarrow 0} \mathcal{I}(a, b)$ is uniform in a , by the following argument. For a fixed a , the $\omega' - \omega''$ -integrand is a $\mathcal{L}^1([-W/2, W/2]^2)$ function, since it is the product of two $\mathcal{L}^2([-W/2, W/2])$ functions, and a bounded continuous $\mathcal{L}^2([-W/2, W/2]^2)$ function (the u -integral). Then there exists $k > 0$ such that for any $\varepsilon > 0$, if $|b| < C(a)\varepsilon^k$, then $|\mathcal{I}(a, b) - L(a)| < \varepsilon$, where $C(a)$ is a possibly a -dependent but bounded coefficient. (It is straightforward to show that if no such k existed to satisfy the ε - δ conditional bound, then the integrand cannot be a member of \mathcal{L}^1) Let k_0 be such a k . Then the a -dependent coefficient can be removed by choosing $k = k_0 + 1$. In other words, the convergence condition of $\mathcal{I}(a, b)$ to $L(a)$ can be made a -independent, and hence it is uniform.

The two observations listed above are the sufficient conditions for changing the order of the a - and b -limits, by the Moore-Osgood theorem. Thus, Eq. (1.7) can be

written as

$$\begin{aligned}
\lim_{T \rightarrow \infty} \|\mathbf{BDG}\|_{\infty}^2 &= \lim_{a \rightarrow \infty} \lim_{b \rightarrow 0_+} \mathcal{I}(a, b) \\
&= \frac{1}{\pi^2} \lim_{b \rightarrow 0_+} \int \int_{-\frac{W}{2}+b}^{\frac{W}{2}-b} d\omega'' d\omega' \overline{G(\omega'')} G(\omega') \lim_{a \rightarrow \infty} a \int_{-\infty}^{\infty} \text{sinc}(u) \text{sinc}(u + a(\omega' - \omega'')) du \\
&= \frac{1}{\pi^2} \lim_{b \rightarrow 0_+} \int \int_{-\frac{W}{2}+b}^{\frac{W}{2}-b} d\omega'' d\omega' \overline{G(\omega'')} G(\omega') \lim_{a \rightarrow \infty} a \pi \text{sinc}(a(\omega' - \omega'')) \\
&= \frac{1}{\pi^2} \lim_{b \rightarrow 0_+} \int \int_{-\frac{W}{2}+b}^{\frac{W}{2}-b} d\omega'' d\omega' \overline{G(\omega'')} G(\omega') \pi^2 \delta(\omega' - \omega'') \\
&= \int_{-W/2}^{W/2} |G(\omega')|^2 d\omega' = \|\mathbf{BG}\|_{\infty}^2 = 1. \quad (1.9)
\end{aligned}$$

Taking the root of both sides yields the result stated in the theorem. \square

For the given source bandwidth W , we assume that T is sufficiently large, so that $\mathbf{Dg} \in E(\epsilon_W)$, by theorem 1. Thus, the condition of theorem 3 of [2] is satisfied, and hence the projection of the temporal mode profile of the signal onto the first $\lfloor c \rfloor + 1$ functions given by Eq. (1.2) suffices to accurately describe the profile up to a distance-squared of $12\epsilon_W^2$:

$$\begin{aligned}
&\min_{H \in \text{Sp}(\{\Psi_j(c)\}_{j=0}^{\lfloor c \rfloor})} \|\mathbf{Dg} - h\|_{\infty}^2 \\
&= \left\| \mathbf{DG} - \sum_{j=0}^{\lfloor c \rfloor} \langle \mathbf{DG}, \Psi_j(c) \rangle_{\infty} \Psi_j(c) \right\|_{\infty}^2 < 12\epsilon_W^2. \quad (1.10)
\end{aligned}$$

Let \mathfrak{H} denote the total energy of the source signal. The implication of Eq. (1.10) for the dimension of the EM radiation field emitted by the source is that any of the infinitely many independent physical degrees of freedom specifying the

source field can be approximated to within $\epsilon_W \sqrt{12\mathfrak{H}}$ of its true value by linear combinations of the values of quadratures of EM field modes whose temporal component is in $\{\Psi_j(c)\}_{j=0}^{\lfloor c \rfloor}$. Whereas the true value itself can be on the order of $\sqrt{\mathfrak{H}}$. But ϵ_W can be made arbitrarily small by increasing the signal duration T . Thus the quadratures that don't correspond to the modes whose temporal component is among $\{\Psi_j(x, y)\}_{j=0}^{\lfloor c \rfloor}$ hold vanishing information about the signal and can be ignored.

1.4 Construction of the Frequency Modes

Our construction of frequency modes is based on packing of Gaussian peaks into the frequency interval $(\nu_0 - W/2, \nu_0 + W/2)$. Let $G_{\vec{\omega}, \sigma}(\omega) = \frac{1}{(\pi\sigma^2)^{1/4}} e^{-\frac{(\omega - \bar{\omega})^2}{2\sigma^2}} e^{-i\omega(t_0 + T/2)}$. Let M' be a non-negative integer, $\vec{\omega} = (\bar{\omega}_j)_{j=0}^{M'} \in \mathbb{R}^{M'}$ and $\vec{\sigma} = (\sigma_j)_{j=0}^{M'} \in (\mathbb{R}^+)^{M'}$. Let \mathcal{G} denote the Gram-Schmidt orthonormalization operator acting on a sequence of functions in $\mathcal{L}^2(\mathbb{R})$ by subtracting from each function in the sequence its projection onto the space spanned by all previous functions, doing so in the order of the sequence and normalizing all functions so that they have unit \mathcal{L}^2 -norms. We denote the action of \mathcal{G} on a sequence a as $\mathcal{G}[a]$. We denote the action of \mathcal{G} on a sequence a as $\mathcal{G}[a]$, and the m^{th} member of the resulting sequence as $\mathcal{G}[a]_m$. Consider the $(\lfloor c \rfloor + 1)$ -element sequence $s(\vec{\sigma}, \vec{\omega})$ of functions whose j^{th} element is given by

$$s_j(\vec{\sigma}, \vec{\omega}) = \begin{cases} G_{\bar{\omega}_j, \sigma_j}, & j < M' \\ \Psi_j(c), & M' \leq j \leq \lfloor c \rfloor \end{cases}$$

for $j = 0, 1, \dots, \lfloor c \rfloor$. Let \mathfrak{C} denote a constraint function on $\vec{\sigma}$ and $\vec{\omega}$ given by:

$$\mathfrak{C}(\vec{\sigma}, \vec{\omega}) = \begin{cases} 1, & f \in E(\epsilon_W) \quad \forall f \in \mathcal{G} [\mathbf{D}_s(\vec{\sigma}, \vec{\omega})] \\ 0, & \text{otherwise} \end{cases}. \quad (1.11)$$

To pack as many quasi-monochromatic modes as possible, we maximize M' subject to the constraint given by Eq. (1.11):

$$M := \max_{\mathfrak{C}(\vec{\sigma}, \vec{\omega})=1} M'. \quad (1.12)$$

The resulting optimal Gaussian means and variances are

$$(\vec{\sigma}^*, \vec{\omega}^*) := \operatorname{argmax}_{\mathfrak{C}(\vec{\sigma}, \vec{\omega})=1} M', \quad (1.13)$$

where M' is understood to be a function of the means $\vec{\omega}$ and variances $\vec{\sigma}$, - it is the dimension of the mean and variance vectors. Finally, we define the temporal modes of the source signal as the time-limited and orthogonalized Gaussian envelope modes with the optimal parameters provided by Eq. (1.13):

$$f_m := \mathcal{G} [\mathbf{D}G_{\omega_m^*, \sigma_m^*}] \quad (1.14)$$

where $m \in \{0, 1, \dots, \max(\lfloor c \rfloor, M)\}$.

Note that the constraint of Eq. (1.11) on the maximization given by Eq. (1.12) governs the final configuration of temporal modes in two important ways: First, it ensures that the resulting modes are mostly contained within the range $\omega \in$

$(\nu_0 - W/2, \nu_0 + W/2)$, by definition of $E(\epsilon_W)$ membership. Secondly, it prevents two means $\bar{\omega}_j$ and $\bar{\omega}_k$ for $j \neq k$ from approaching significantly. This is thanks to the fact that Gram-Schmidt orthogonalization of two closely-centered Gaussians in the frequency band allowed by \mathbf{B} results in a function with a significant portion of its energy lying outside of the band. The variances $\vec{\sigma}$ are restricted to be no less than $\mathcal{O}(T^{-1})$, as too narrow a peak risks being excluded from $E(\epsilon_W)$ because of the frequency-spreading effect of \mathbf{D} on non-time limited functions playing into the definition of the constraint \mathcal{C} .

Now, note that all modes defined by Eq. (1.14) are in $E(\epsilon_W)$. Then by Eq. (1.10),

$$\text{Sp} \left(\{f_m\}_{m=1}^M \right) \underset{\epsilon_W}{\subseteq} \text{Sp} \left(\{\Psi_m(c)\}_{m=0}^{\lfloor c \rfloor} \right), \quad (1.15)$$

where we define the approximate linear space containment $V_1 \underset{\epsilon_W}{\subseteq} V_2$ between two subspaces V_1 and V_2 to mean that vectors in V_1 can diverge from V_2 by an angle of at most $\arcsin(\sqrt{12}\epsilon_W)$:

$$\max_{\vec{v} \in V_1} \min_{\vec{u} \in V_2} \sqrt{1 - \left| \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{v}\| \|\vec{u}\|} \right|^2} < \sqrt{12}\epsilon_W.$$

Now suppose for the sake of argument that $M > \lfloor c \rfloor$. Then Eq. (1.15) implies that Eq. (1.10) holds with the $\Psi_j(c)$ replaced by f_j , and $12\epsilon_W^2$ on the RHS replaced by $\sin^2 \left(2 \arcsin(\sqrt{12}\epsilon_W) \right) < 24\epsilon_W^2$. Let $k > \lfloor c \rfloor$. Then it follows that

$$\min_{h \in \text{Sp}(\{f_j\}_{j=0}^{\lfloor c \rfloor})} \|f_k - h\|^2 = \|f_k\|^2 < 24\epsilon_W^2,$$

which is a contradiction since all f_k for any $k \in \{1, \dots, M\}$ have unit norm. Thus,

M cannot exceed $\lfloor c \rfloor = \lfloor WT \rfloor$. In fact, it is likely that $M < \lfloor c \rfloor$, since equality still implies that Eq. (1.10) holds with $\text{Sp} \left(\{\Psi_j(c)\}_j^{\lfloor c \rfloor} \right)$ replaced by $\text{Sp} \left(\{f_j\}_j^{\lfloor c \rfloor} \right)$, which, in the case that $M = \lfloor c \rfloor$, contains no prolate spheroidal wave functions. But it was shown in ref. [2] (via the sampling function basis in theorem 10) that for an arbitrary functional basis, the RHS of Eq. (1.10) can scale slower than ϵ_W^2 as $\epsilon_W \rightarrow 0$. In fact the prolate spheroidal wave functions maximize the power of ϵ_W on the RHS. Even though $M \leq \lfloor c \rfloor$, it is evident that M must scale proportionally with $\lfloor c \rfloor$: Consider the (sub-optimal) packing of the $G_{\bar{\omega}_j, \sigma_j}$ where $\sigma_j = \sigma_k$ for all pairs $j, k \in \mathbb{N}$, the $\bar{\omega}_k$ are spaced in the interval $\nu_0 \pm W/2$ evenly and sparsely enough relative to σ_k such that the effect of \mathcal{G} is negligible, and σ_j and M is chosen such that $\mathfrak{C}(\vec{\sigma}, \vec{\omega}) = 1$ with $\epsilon_W = 1 - \lambda_0$. The asymptotic proportionality $\lfloor c \rfloor \in \mathcal{O}(M)$ as $c \rightarrow \infty$ in this sub-optimal packing of spectral wave packets is a consequence of the reciprocity of variances of Gaussian waveforms and their spectra. In reality, the basis $\{f_j\}_{j=1}^M$ is optimized to include a maximum number M of quasi monochromatic frequency modes, optimally spaced and squeezed within the frequency band of the light source. So even if $M < \lfloor c \rfloor$, we expect M to be close to $\lfloor c \rfloor$. As the signal time T grows, the line width of each f_m shrinks to accommodate an increasing number M of Gaussian envelope modes in a fixed frequency band of width W without significantly overlapping, so that the F_m become ever better approximations of Dirac-delta distributions. The exact value of the limit $\lim_{T \rightarrow \infty} \frac{M}{\lfloor TW \rfloor}$ for the frequency mode construction leading to Eqs. (1.12) and (1.13) is left as an open question.

1.5 Conclusion

We have provided a maximal construction of time-band-limited quasi-monochromatic functions based on optimal packing of Gaussian envelopes into a frequency band of W while satisfying an energy-containment requirement to be considered time- and band-limited. We have also argued that a function basis constructed in such a way describes an optical signal generated from an optical source of bandwidth W turned on for a duration T if T is sufficiently large.

Using properties of the prolate spheroidal wave functions, we have argued that there can be at most $\lfloor c \rfloor = \lfloor WT \rfloor$ of these functions, although a tighter bound in terms of c likely exists, but is left as an open problem to find.

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